Logic, Graph Theory, and Complexity II

Computing Crossing Numbers

Martin Grohe

University of Edinburgh

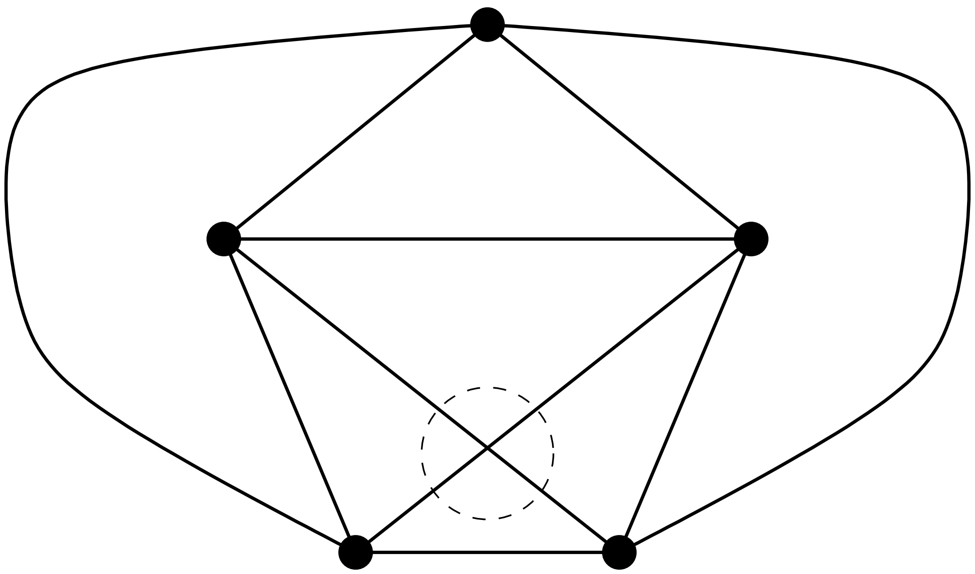
Crossing Numbers

Definition

The **crossing number** of a graph is the minimum number of edge crossings that are needed to draw the graph onto the plane.

Example

The crossing number of $\mathcal{K}_5$ *is 1.*



The algorithmic problem

CROSSING NUMBER	
Input:	Graph $\mathcal{G}$, non-negative integer k.
Problem:	Decide if $\mathcal{G}$ has crossing number at most k (and find a drawing with at most k crossings if this is the case).

Theorem (Garey, Johnson 1982)

CROSSING NUMBER *is NP-complete.*

The parameterized problem

For $k \geq 0$:

k-CROSSING NUMBER	
Input:	Graph $\mathcal{G}$.
Problem:	Decide if $\mathcal{G}$ has crossing number at most k (and find a drawing with at most k crossings if this is the case).

Theorem (Hopcroft, Tarjan 1974)
0-CROSSING NUMBER *(i.e. planarity testing) is solvable in linear time.*

Corollary
*For every $k \geq 0$, k-*CROSSING NUMBER *is solvable in time $O(n^{2k+1})$.*

Our Theorem

Theorem

*For every $k \geq 0$, k-*CROSSING NUMBER *is solvable in quadratic time.*

Remark

Actually, we prove a **uniform** version of the theorem: CROSSING NUMBER is **fixed-parameter tractable** with exponent 2 (and parameter dependence at most 5-fold exponential).

Digression: Computing the genus of a graph

GENUS	
Input:	Graph $\mathcal{G}$, non-negative integer k.
Problem:	Decide if $\mathcal{G}$ has (orientable) genus at most k.

For $k \geq 0$:

k-GENUS	
Input:	Graph $\mathcal{G}$.
Problem:	Decide if $\mathcal{G}$ has (orientable) genus at most k.

Computing the Genus (cont'd)

Theorem (Thomassen 1988)
GENUS *is NP-complete.*

Theorem (Filotti, Miller, Reif 1979)
For every $k \geq 0$, k-GENUS *is solvable in time* $n^{O(k)}$.

Theorem (Robertson, Seymour 1995)
For every $k \geq 0$, k-GENUS *is solvable in cubic time.*

Theorem (Mohar 1996)
For every $k \geq 0$, k-GENUS *is solvable in linear time.*

Our quadratic time algorithm for k-Crossing Number

- The class of graphs of crossing number at most k is not closed under taking minors.
- Nevertheless, our algorithm heavily depends on the structure theory for graphs with excluded minors and algorithmic ideas developed in this context.

An Outline

We fix $k \geq 1$ (for the rest of the talk).

Input: Graph $\mathcal{G}$

if genus$(\mathcal{G}) > k$ **then**
    reject
else
    while tw$(\mathcal{G}) > w(k)$ **do**
        reduce size of $\mathcal{G}$ without changing the crossing number
    solve problem on remaining graph $\mathcal{G}$ of tree-width $\leq w(k)$.

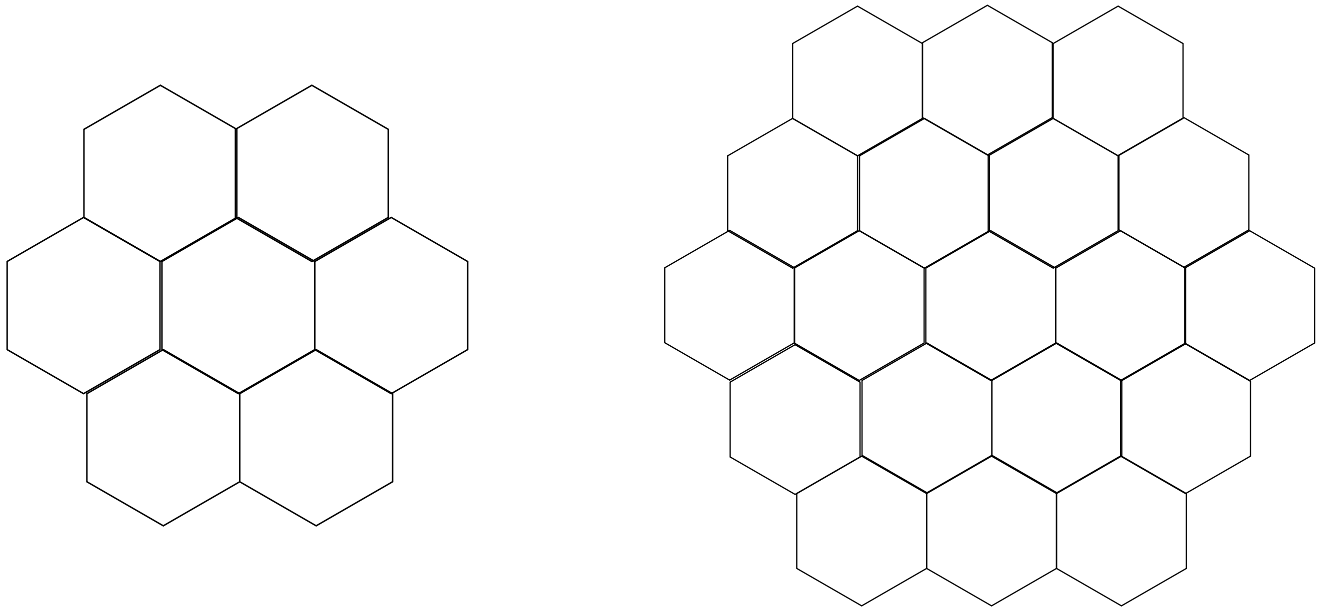
Forbidden Crossings

k-FCN

Input: Graph $\mathcal{G} = (V, E)$ and subset $F \subseteq E$.

Problem: Decide if $\mathcal{G}$ has a drawing in the plane with at most k edge crossings, none of which involves an edge in F.

The Excluded Grid Theorem



Grids of radius 2 and 3.

Theorem (Robertson and Seymour 1986)

For every $r \geq 1$ there exists a w such that every graph of tree-width $> w$ has a subgraph homeomorphic to the grid of radius r.

An Algorithmic Version of the Excluded Grid Theorem

Theorem

(Robertson and Seymour 1995, Bodlaender 1996,
Perković and Reed 2000)

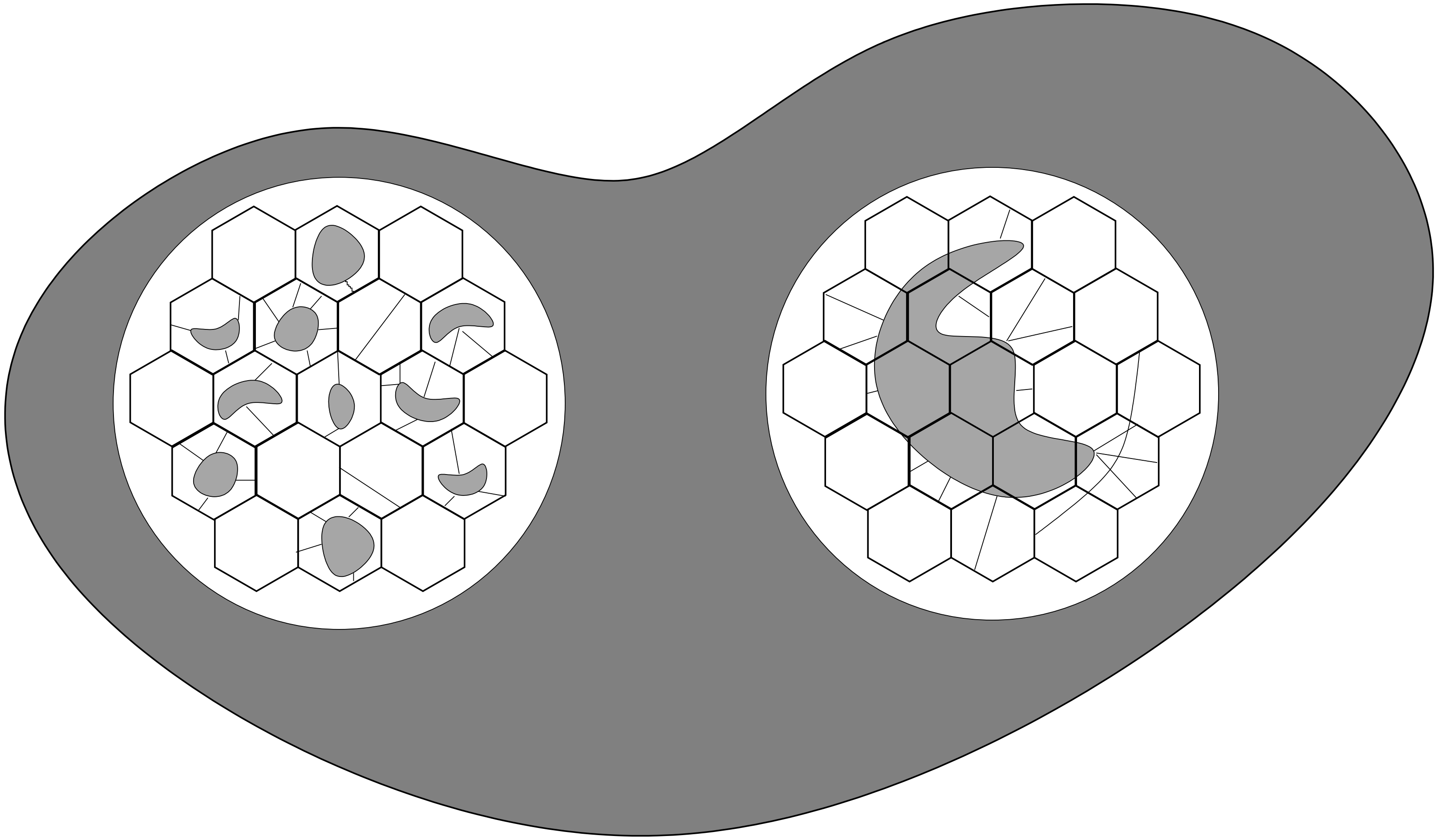
For every $r \geq 1$ *there exists a* $w = w(r)$ *and a linear time algorithm that, given a graph* $\mathcal{G}$*,*

- *either recognizes that the tree-width of* $\mathcal{G}$ *is* $\leq w$*,*
- *or computes a subgraph of* $\mathcal{G}$ *homeomorphic to the grid of radius* r*.*

Flat grids

Definition

A grid $\mathcal{H}$ in a graph $\mathcal{G}$ is **flat**, if $\mathcal{H}$ together with all components attached to $\mathcal{H}$ is a planar graph.



Flat Grids in Graphs of Bounded Genus

Lemma (Thomassen 1997)

For all $k, r \geq 0$ *there exists an* $s = s(r, k)$ *such that:*
If $\mathcal{G}$ *is a graph of genus* $\leq k$, *and* $\mathcal{H} \subseteq \mathcal{G}$ *is homeomorphic to a grid of radius* s, *then there exists a* $\mathcal{K} \subseteq \mathcal{H}$ *homeomorphic to a grid of radius* r *such that* $\mathcal{K}$ *is flat in* $\mathcal{G}$.

Finding Flat Grids

We let $r = 4k + 3$, $s = s(k, r)$, $w = w(s)$.

Lemma

There exists a linear time algorithm that, given a graph $\mathcal{G}$:

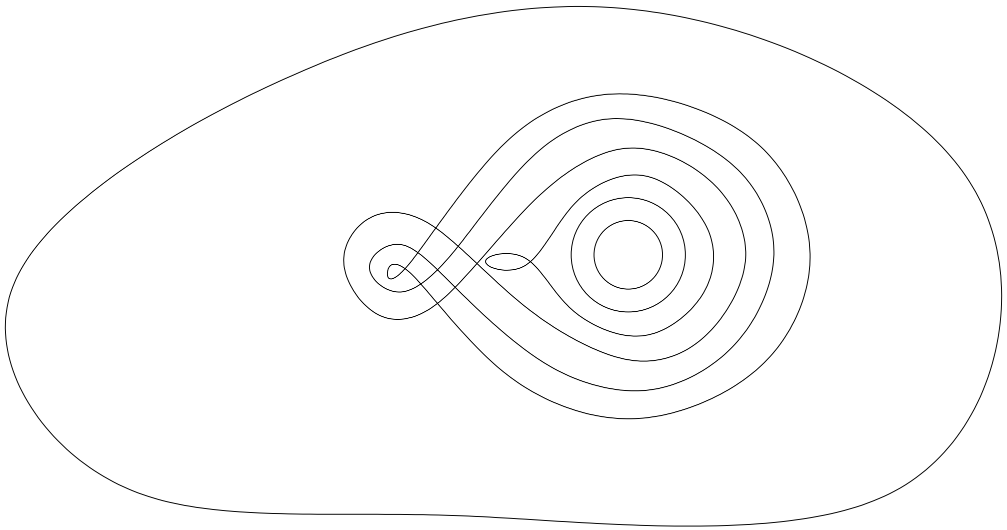
- *either decides that* crossing-number$(\mathcal{G}) > k$*,*
- *or decides that* tw$(\mathcal{G}) \leq w$*,*
- *or computes a $\mathcal{K} \subseteq \mathcal{G}$ homeomorphic to a grid of radius r that is flat in $\mathcal{G}$.*

Lemma

Let

- $\mathcal{G}$ *graph of crossing number* $\leq \mathrm{k}$*,*
- Δ *drawing of* $\mathcal{G}$ *with the minimum number of crossings,*
- $\mathcal{K} \subseteq \mathcal{G}$ *homeomorphic to a grid of radius* r*, flat in* $\mathcal{G}$*.*

Then no edge of one of the innermost cycles of $\mathcal{K}$ *is involved in a crossing of* Δ*.*



The Reduction Step

Definition

A k**-good drawing of** $\mathcal{G}$ **with respect to** F is a drawing of $\mathcal{G}$ with at most k crossings that involve none of the edges in F.

$(\mathcal{G}, F)$ is k**-good** if there is a k-good drawing of $\mathcal{G}$ with respect to F.

Reduction Step

Linear Time transformation

$$(\mathcal{G}, F) \quad \mapsto \quad (\mathcal{G}', F')$$

such that $|\mathcal{G}'| < |\mathcal{G}|$ and

$$(\mathcal{G}, F)\ k\text{-good} \iff (\mathcal{G}', F')\ k\text{-good}$$

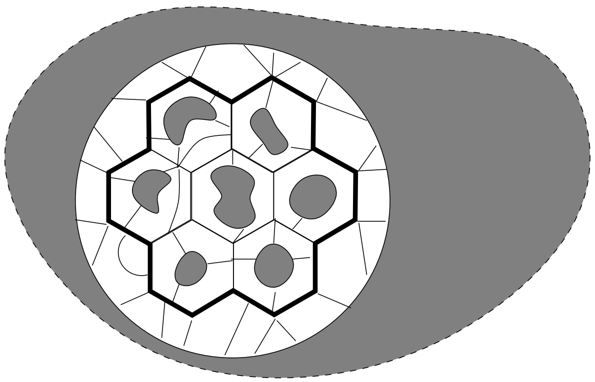
(for graphs $\mathcal{G}$ of genus $\leq k$ and tree-width $> w$).

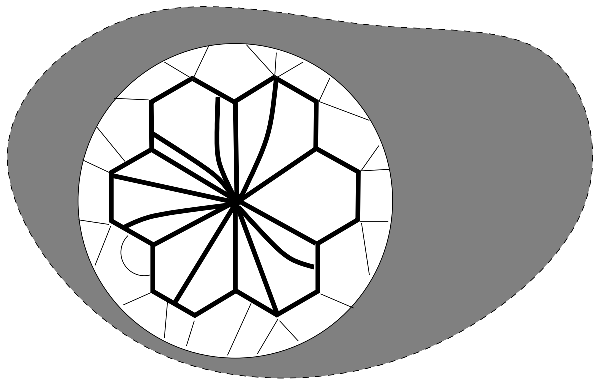
The Reduction Step (cont'd)

Input: Graph $\mathcal{G} = (V, E)$ (of genus $\leq k$ and tree-width $> w$),
set $F \subseteq E^{\mathcal{G}}$ of "forbidden edges".

1. Compute flat grid $\mathcal{K}$ of radius r in $\mathcal{G}$.
2. Compute graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by contracting the innermost cycle of $\mathcal{K}$ and everything attached to it to a single vertex v.
3. Compute $F' := F \cup \{$all edges incident with v and the edges on the second cycle of $\mathcal{K}\}$.

Output: $\mathcal{G}', F'$.





Corollary

There is a quadratic time transformation

$$(\mathcal{G}, F) \quad \mapsto \quad (\mathcal{G}', F')$$

such that the tree-width of $\mathcal{G}'$ is $\leq w$ and

$$(\mathcal{G}, F)\ k\text{-}\textit{good} \iff (\mathcal{G}', F')\ k\text{-}\textit{good}\ .$$

It remains to solve the problem k-FCN on graphs of tree-width $\leq w$.

k-FCN
Input: Graph $\mathcal{G} = (V, E)$ and subset $F \subseteq E$.
Problem: Decide if $(\mathcal{G}, F)$ is k-good.

k-FCN on Graphs of Bounded Tree-Width

Definability Lemma

For every $k \geq 1$ there exists an MSO-formula χ_k such that for every graph $\mathcal{G}$ and subset $F^{\mathcal{G}} \subseteq E^{\mathcal{G}}$:

$$(\mathcal{G}, F^{\mathcal{G}}) \text{ satisfies } \chi_k \iff (\mathcal{G}, F^{\mathcal{G}}) \text{ is } k\text{-good.}$$

Recall Courcelle's theorem:

Theorem (Courcelle 1990)

Let $w \geq 1$. Then for every MSO-sentence φ there is linear time algorithm deciding whether a given structure of tree-width at most w satisfies φ.

Applying Courcelle's theorem to the formula χ_k of the lemma yields:

Corollary

Let $w \geq 1$. Then the restriction of k-FCN to input graphs of tree-width at most w is solvable in linear time.

Proof of the Definability Lemma

Graphs are encoded as incidence structures:

$$\mathcal{G} = (G, V^{\mathcal{G}}, E^{\mathcal{G}}, I^{\mathcal{G}}),$$

where

- $V^{\mathcal{G}}$ is the vertex set,
- $E^{\mathcal{G}}$ is the edge set,
- $I^{\mathcal{G}} \subseteq V^{\mathcal{G}} \times E^{\mathcal{G}}$ is the incidence relation.

This allows us to quantify over both vertices and edges.

Let F be another unary relation symbol (for the forbidden edges).
The vocabulary of our formulas will be

$$\{V, E, I, F\}.$$

The Definability Lemma will be proved by induction on k.

Proof of the Definability Lemma – Base Step

Lemma

There is an MSO-formula $\pi(X, x, y)$ *saying that* X *is the vertex set of a path from* x *to* y*.*

Lemma

For every graph $\mathcal{H}$ *there is an MSO-sentence* $\mu_{\mathcal{H}}$ *saying that a graph has a subgraph homeomorphic to* $\mathcal{H}$*.*

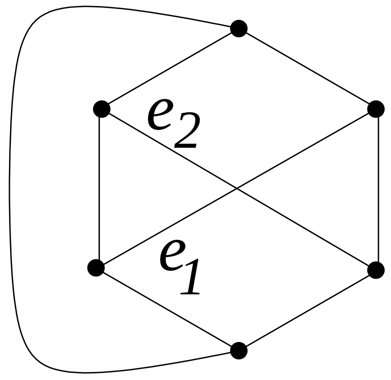
Lemma

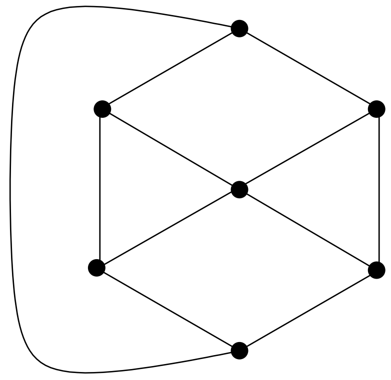
There is an MSO-sentence saying that a graph is planar.

Proof: Use Kuratowski's theorem and the previous lemma.

Defining Crossings

For a graph $\mathcal{G}$ and edges e_1, e_2, let $\mathcal{G}^{e_1 \times e_2}$ be obtained from $\mathcal{G}$ by crossing e_1 and e_2.





Lemma

For every MSO-sentence φ there exists an MSO-formula $\varphi^*(y_1, y_2)$ such that for all graphs $\mathcal{G} = (G, V^{\mathcal{G}}, E^{\mathcal{G}})$, edge sets $F^{\mathcal{G}} \subseteq E^{\mathcal{G}}$ and edges $e_1, e_2 \in E^{G} \setminus F$ that do not have an endpoint in common we have:

$$(\mathcal{G}, F^{\mathcal{G}}) \models \varphi^*(e_1, e_2) \iff (\mathcal{G}^{e_1 \times e_2}, F^{\mathcal{G}}) \models \varphi.$$

Proof of the Definability Lemma – Inductive Step

Observation

$(\mathcal{G}, F)$ *is* k*-good* $\iff$ $(\mathcal{G}, F)$ *is* $(k-1)$*-good or there exist edges* $e_1, e_2 \in E \setminus F$ *with no endpoint in common such that* $(\mathcal{G}^{e_1 \times e_2}, F)$ *is* $(k-1)$*-good.*

We let

$$\chi_k = \chi_{k-1} \vee \exists y_1 \exists y_2 \Big(Ey_1 \wedge Ey_2 \\ \wedge \forall v_1 \forall v_2 \big((Iv_1 e_1 \wedge Iv_2 e_2) \to v_1 \neq v_2 \big) \\ \wedge \chi^*_{k-1}(y_1, y_2) \Big)$$