Computable Structures of High Scott Rank

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$\mathcal{L}_{\omega_1\omega}$ is the infinitary logic which allows countable conjunctions and disjunctions.

There is a hierarchy of $\mathcal{L}_{\omega_1\omega}$-formulas based on their quantifier complexity. We denote these by $\Sigma^{\mathtt{in}}_\alpha$ and $\Pi^{\mathtt{in}}_\alpha$.

A formula is $\Sigma^{\mathtt{in}}_\alpha$ if it is a disjunction of $\Pi^{\mathtt{in}}_\beta$ formulas for $\beta < \alpha$.
A formula is $\Pi^{\mathtt{in}}_\alpha$ if it is a conjunction of $\Sigma^{\mathtt{in}}_\beta$ formulas for $\beta < \alpha$.

We will mostly consider computable $\mathcal{L}_{\omega_1\omega}$-formulas.
We denote these by $\Sigma^{\mathtt{c}}_\alpha$ and $\Pi^{\mathtt{c}}_\alpha$.

Example

There is a Π_2^c formula which describes the class of torsion groups. It consists of the group axioms together with:

$$(\forall x) \bigvee_{n \in \mathbb{N}} nx = 0.$$

Example

There is a Π_1^c formula which describes the dependence relation on triples x, y, z in a $\mathbb{Q}$-vector space:

$$\bigwedge_{(a,b,c) \in \mathbb{Q}^3 \setminus \{(0,0,0)\}} ax + by + cz = 0$$

Example

There is a Σ_3^c sentence which says that a $\mathbb{Q}$-vector space has finite dimension:

$$\bigvee_{n \in \mathbb{N}} (\exists x_1, \ldots, x_n)(\forall y)\ y \in \operatorname{span}(x_1, \ldots, x_n).$$

Example

There is a Π_3^c sentence which says that a $\mathbb{Q}$-vector space has infinite dimension:

$$\bigwedge_{n \in \mathbb{N}} (\exists x_1, \ldots, x_n)\ \operatorname{Indep}(x_1, \ldots, x_n).$$

Let $\mathcal{A}$ be a countable structure.

Theorem (Scott 1965)

There is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that:

$$\mathcal{B} \text{ countable}, \mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

φ is a *Scott sentence* of $\mathcal{A}$.

Example

$(\omega, <)$ has a Π^c_3 Scott formula consisting of the Π^c_2 axioms for linear orders together with:

$$\forall y_0 \bigvee\!\!\!\bigvee_{n \in \omega} \exists y_n < \cdots < y_1 < y_0 \left[\forall z \; (z > y_0) \vee (z = y_0) \vee (z = y_1) \vee \cdots \vee (z = y_n) \right].$$

Definition (Scott rank)

$SR(\mathcal{A})$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence.

Theorem (Montalbán 2015)

Let $\mathcal{A}$ be a countable structure, and α a countable ordinal. TFAE:

- *$\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence.*
- *Every automorphism orbit in $\mathcal{A}$ is $\Sigma^{\mathtt{in}}_{\alpha}$-definable without parameters.*
- *$\mathcal{A}$ is uniformly (boldface) $\mathbf{\Delta}^0_{\alpha}$-categorical without parameters.*

Let $\mathcal{A}$ be a computable structure.

Theorem (Nadel 1974)

$\mathcal{M}$ has Scott rank $\leq \omega_1^{CK} + 1$.

$SR(\mathcal{A}) < \omega_1^{CK}$ if $\mathcal{A}$ has a computable Scott sentence.

$SR(\mathcal{A}) = \omega_1^{CK}$ if each automorphism orbit is definable by a computable formula, but $\mathcal{A}$ does not have a computable Scott sentence.

$SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is an automorphism orbit which is not defined by a computable formula.

Theorem (Harrison 1968)

There is a computable linear order $\mathcal{H}$ with order type $\omega_1^{CK}(1+\mathbb{Q})$.

$\mathcal{H}$ has no hyperarithmetic descending sequences.

We call this the Harrison order.

Take an element a which is in the non-standard part of $\mathcal{H}$.

The orbit of a is not definable by a computable $\mathcal{L}_{\omega_1\omega}$ formula.

If it was, then the orbit would be hyperarithmetic, and we could compute a hyperarithmetic descending sequence.

So the Harrison order has Scott rank $\omega_1^{CK}+1$.

How do you build a computable structure of Scott rank ω_1^{CK}?

Theorem (Makkai 1981)

There is a Δ^0_2 structure of Scott rank ω_1^{CK}.

Theorem (Knight, Millar ~2005?)

There is a computable structure of Scott rank ω_1^{CK}.

I will talk about a later construction of Calvert, Knight, and Millar (2006).

The structure will be an infinitely branching rooted tree.
Assign to each node in a tree its tree rank:

- $rk(x) = 0$ if x is a leaf.
- $rk(x)$ is otherwise the least ordinal (or possibly ∞) greater than the ranks of the children of x.

If $rk(x) = \infty$, then there is a path through x.

Definition

A tree T is *thin* if there is a computable ordinal bound on the ordinal tree ranks at each level of the tree.

Definition

A tree T is *homogenous* if:

- Whenever x has a successor of rank α, it has infinitely many successors of rank α.
- If some element at level n has a successor of rank α, every element at level n with rank $> \alpha$ has a successor of rank α.

Theorem (Calvert, Knight, Millar 2006)

There is a computable thin homogeneous tree with no bound on the ordinal tree ranks at all levels.

It has Scott rank ω_1^{CK}.

Until recently, these were essentially all of the examples we had.

Because there are so few examples of computable structures of high Scott rank, there are many general questions about them that we don't know the answer to.

I'm going to talk about two recent constructions of new models of high Scott rank:

- Structures of Scott rank ω_1^{CK} and $\omega_1^{CK}+1$ which are not computably approximable.
- A structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

The latter is joint work with Greg Igusa and Julia Knight.

The Harrison linear order is approximated by the computable ordinals:

For every computable sentence φ true of the Harrison linear order, there is a computable ordinal α such that $(\alpha, <) \vDash \varphi$.

So the Harrison linear order is a "limit" of the computable ordinals.

Let T be the computable tree of Scott rank ω_1^{CK} from the previous slides.

Theorem (Calvert, Knight, Millar 2006)

There is a sequence T_α of computable trees such that $SR(T_\alpha) < \omega_1^{CK}$ and $T_\alpha \equiv_\alpha T$.

So T is a limit of computable structures of low Scott rank in the same way.

Definition

$\mathcal{A}$ is computably approximable if every computable infinitary sentence φ true in $\mathcal{A}$ is also true in some computable $\mathcal{B} \ncong \mathcal{A}$ with $SR(\mathcal{B}) < \omega_1^{CK}$.

The Harrison linear order and the homogenous thin tree are both computably approximable.

Question (Calvert, Knight 2006)

Is every computable model of high Scott rank computably approximable?

I was initially interested in a different question.

Let φ be a sentence of $\mathcal{L}_{\omega_1\omega}$.

Definition

The *Scott spectrum* of φ is the set

$$\mathrm{SS}(T) = \{\alpha \in \omega_1 \mid \alpha \text{ is the Scott rank of a countable model of } T\}.$$

Question

Classify the Scott spectra.

Definition

Let L be a linear order.

- wfp(L) is the well-founded part of L.
- wfc(L) is L with the non-well-founded part collapsed to a single element.

If $\mathcal{C}$ is a class of linear orders, we can apply to operations to each member of $\mathcal{C}$ to get wfp$(\mathcal{C})$ and wfc$(\mathcal{C})$.

Example

- $\mathrm{wfp}(\omega_1^{CK}(1+\mathbb{Q})) = \omega_1^{CK}$
- $\mathrm{wfc}(\omega_1^{CK}(1+\mathbb{Q})) = \omega_1^{CK}+1$

Theorem (ZFC + PD)

The Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences are exactly the sets of the form:

- $\mathrm{wfp}(\mathcal{C})$,
- $\mathrm{wfc}(\mathcal{C})$, *or*
- $\mathrm{wfp}(\mathcal{C}) \cup \mathrm{wfc}(\mathcal{C})$

where $\mathcal{C}$ is a $\boldsymbol{\Sigma}^1_1$ class of linear orders.

The construction, from $\mathcal{C}$, of an $\mathcal{L}_{\omega_1\omega}$-sentence does not use PD, and:

- We can get a $\Pi_2^{\mathtt{in}}$ sentence.
- If the class $\mathcal{C}$ is lightface, then we get a $\Pi_2^{\mathtt{c}}$ sentence.
- The Harrison linear order, with each element named by a constant, forms a Σ^1_1 class with a single member. Recall that $\mathrm{wfp}(\mathcal{H}) = \{\omega_1^{CK}\}$ and $\mathrm{wfc}(\mathcal{H}) = \{\omega_1^{CK} + 1\}$.

Theorem (H-T.)

There is a computable model $\mathcal{A}$ *of Scott rank* $\omega_1^{CK}+1$ *and a* Π_2^c *sentence* ψ *such that:*

- $\mathcal{A} \models \psi$
- $\mathcal{B} \models \psi \Longrightarrow SR(\mathcal{B}) = \omega_1^{CK}+1$.

The same is true for Scott rank ω_1^{CK}.

Corollary

There are computable models of Scott rank ω_1^{CK} *and* $\omega_1^{CK}+1$ *which are not computably approximable.*

Definition

Given a model $\mathcal{A}$, we define the computable infinitary theory of $\mathcal{A}$,

$$Th_\infty(\mathcal{A}) = \{\varphi \text{ a computable } \mathcal{L}_{\omega_1\omega} \text{ sentence} \mid \mathcal{A} \vDash \varphi\}.$$

Let T be the computable thin homogeneous tree of Scott rank ω_1^{CK}.

Since it is thin, for each level n of the tree and ordinal α, there is a computably formula which says whether there is a node of rank α at level n.

Because it is homogeneous, it is determined up to automorphism by which tree ranks appear at each level.

So $Th_\infty(T)$ is $\aleph_0$-categorical.

Question (Millar, Sacks 2008)

Is there a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical?

Any other models of the same theory would necessarily be non-computable and of Scott rank at least $\omega_1^{CK} + 1$.

Theorem (Millar, Sacks 2008)

There is a structure $\mathcal{A}$ of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

$\mathcal{A}$ is not computable, but $\omega_1^{\mathcal{A}} = \omega_1^{CK}$. ($\mathcal{A}$ lives in a fattening of $\mathcal{L}_{\omega_1^{CK}}$.)

Freer generalized this to arbitrary admissible ordinals.

Theorem (H-T., Igusa, Knight)

There is a computable structure of Scott rank ω_1^{CK} whose computable infinitary theory is not $\aleph_0$-categorical.

The structure is a set of finite ascending sequences in the Harrison linear order.

We “disguise” these sequences by making it take about α quantifier alternations to decide whether the *n*th entry of a sequence is α.

The non-prime models of the infinitary theory of this structures have additional infinite ascending sequences which are cofinal in the well-founded part of the Harrison linear order.

Such sequences cannot be described by an infinitary sentence.

Thanks!