THERE IS NO CLASSIFICATION OF THE DECIDABLY PRESENTABLE STRUCTURES

MATTHEW HARRISON-TRAINOR

Abstract. A computable structure $\mathcal{A}$ is decidable if, given a formula $\varphi(\bar{x})$ of elementary first-order logic, and a tuple $\bar{a} \in \mathcal{A}$, we have a decision procedure to decide whether φ holds of $\bar{a}$. We show that there is no reasonable classification of the decidably presentable structures. Formally, we show that the index set of the computable structures with decidable presentations is Σ^1_1-complete. We also show that for each n the index set of the computable structures with n-decidable presentations is Σ^1_1-complete.

1. Introduction

In effective mathematics, the object with which we are concerned are *computable structures*. A mathematical structure—a set together with operations and relations on that set—is *computable* if the set and the operations and relations on it are all computable. For example, a computable field is one where the domain is a computable set and the operations of addition and multiplication are computable. In a computable structure, we can effectively answer quantifier-free questions, such as, for elements a, b, and c of a field, whether $a + b = b \cdot c$.

There are many other questions about a structure that we might want to answer in a computable way. For example, in a field, we might want to be able to decide whether a given polynomial has a root. In general, this is undecidable, but sometimes, such as for algebraically closed fields, this can be done. In fact, given a computable algebraically closed field, as a result of quantifier elimination we can decide the answer to any question that can be formulated in elementary first-order logic, i.e., as a logical formula using $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\forall$, and $\exists$. In general, we say that a computable structure is *decidable* if there is a method to compute, given elements $a_1, \ldots, a_n$ and a formula φ of elementary first-order logic, whether φ holds of $a_1, \ldots, a_n$. Every computable presentation of an algebraically closed field is decidable. There are other examples where certain computable presentations of a structure are decidable, but other computable presentations of the same structure are not. For example, the standard computable presentation of the linear order $(\mathbb{N}, <)$ is decidable. However, there are also other computable presentations of $(\mathbb{N}, <)$ in which the successor relation is not computable, and hence this copy is not decidable.

This paper is about the problem of characterizing those computable structures which have a decidable presentation. This problem was probably first stated by Goncharov, and has more recently been posed for example by Bazhenov at the 2015 Mal'cev Meeting and Fokina at the 2016 ASL meeting in Storrs, CT. We will show that there is no such characterization.

More formally, our main theorem is as follows. Fix an effective list of the diagrams of the (partial) computable structures.

Theorem 1.1. *The index set*

$$I_{d\text{-}pres} = \{i \mid \text{the } i\text{th computable structure is decidably presentable}\}$$

The author was partially supported by an NSERC PGS-D award and a Banting fellowship. Much of this work was completed while the author was at the University of California, Berkeley.

is Σ^1_1*-complete.*

This theorem is proved in Section 6.

As a result, there is no possible reasonable characterization of the computable structures with decidable presentations. What we mean is that there is no simpler way to check whether a computable structure $\mathcal{A}$ has a decidable presentation than to ask: *Does there exist a decidable structure* $\mathcal{B}$ *and a classical isomorphism between* $\mathcal{A}$ *and* $\mathcal{B}$? This requires searching through all possible isomorphisms, of which there may be continuum-many, between $\mathcal{A}$ and $\mathcal{B}$. (Contrast this with a very naive, and incorrect, candidate for a characterization: A computable structure $\mathcal{A}$ has a decidable copy if and only if there is a computable listing of the types it realizes. In this case, we must look through the countably many possible computable listings of types, and check whether they list the types in $\mathcal{A}$. This requires only quantifiers over natural numbers, and objects which can be coded by natural numbers.[1]) If there were a simpler characterization of the computable structures with decidable presentations, then one would expect that characterization to yield a simpler way of checking whether a computable structure has a decidable presentation.

A similar approach was taken in [DKL⁺15], where it was shown that there is no reasonable characterization of computable categoricity, and in [DM08], where it was shown that there is no reasonable classification of torsion-free abelian groups. This approach originated with [GN02a]. See also [LS07, Fok07, CFG⁺07, FGK⁺15, GBM15a, GBM15b].

1.1. n**-decidable structures.** One can also ask whether a computable structure has an n-decidable copy. An *n-decidable structure* is a structure in which we can decide whether a formula φ, with n alternations of quantifiers, holds of a tuple $a_1, \ldots, a_\ell$. For each n, there are n-decidable structures which are have no $n+1$-decidable copies, and there is a structure which has n-decidable copies for all n, but no decidable copy [CM98]. We say that a structure is *n-presentable* is it has an n-decidable copy. There is no simpler characterization of the n-presentable structures.

Theorem 1.2. *For each* $n \in \omega$*, the index set*

$$I_{n\text{-}pres} = \{i \mid \text{the } i\text{th computable structure is } n\text{-presentable}\}$$

is Σ^1_1*-complete.*

The proof of Theorem 1.2 in the case $n = 1$ will be simpler than the proof of Theorem 1.1, and so we will begin by proving Theorem 1.2 in Section 4. In the construction for $n \geq 2$ and for Theorem 1.1, we must guess at Σ^0_2 facts; along the true stages, the construction is essentially the same as that of Theorem 1.2.

1.2. **Decidable presentability in familiar classes.** What if we are interested in a specific class of structures, such as fields, graphs, or groups? Hirschfeldt, Khoussainov, Shore, and Slinko [HKSS02] showed that many classes of structures—such as graphs and groups—are universal in the sense that given any structure $\mathcal{A}$ in any language, we can find a structure in that particular class (i.e., we can find a group, graph, etc.) which shares many of the same computability-theoretic properties as $\mathcal{A}$. Miller, Poonen, Schoutens, and Shlapentokh [MPSS] recently showed that the class of fields is also universal. One might hope to transform the structures built for Theorem 1.1 into structures in these various classes, and show that, for example, the index set of the decidable graphs is Σ^1_1. Unfortunately, decidability is not one of the properties maintained by these constructions.

Nevertheless, one can search for a construction which maintains decidability. Motivated by this, and other questions, in [BHT] the author and Bazhenov show that there is a computable operator

[1]For some restricted classes of structures, such a characterization might be possible. For example, Andrews [And14] showed that if $\mathcal{M}$ is a model of a decidable ω-stable theory with countably many countable models, then $\mathcal{M}$ has a decidable copy if and only if all of the types realized in $\mathcal{M}$ are recursive.

Φ such that given a structure $\mathcal{A}$, $\Phi(\mathcal{A})$ is a graph whose isomorphism type depends only on the isomorphism type of $\mathcal{A}$, and one can compute the elementary diagram of $\Phi(\mathcal{A})$ from that of $\mathcal{A}$ and vice versa. From this we get:

Theorem 1.3. *The index set of the decidably presentable graphs are Σ^1_1-complete.*

It follows that one cannot characterize which graphs are decidably presentable.

For other structures, it is still unknown whether there is a transformation that preserves decidability.

Question 1.4. Can one characterize the decidably presentable groups or fields?

One expects that one cannot characterize the decidably presentable structures in these classes.

Other familiar classes of structures, such as linear orders and boolean algebras, are not universal. It is possible that such classes admit a characterization of the decidably presentable structures in that class. For linear orders in particular, this question has already been raised:

Question 1.5 (Moses, see [CLLS00])**.** Can one characterize the linear orderings which have a decidable copy?

We believe that the Friedman-Stanley [FS89] transformation T of structures into linear orders preserves decidability, in the sense that $\mathcal{A}$ is decidable if and only if $T(\mathcal{A})$ is decidable. It would follow that the answer to this question is "no".

Torsion-free abelian groups are another class of structures which is not universal. However, it is still an open question whether or not torsion-free abelian groups are Borel complete.

Question 1.6. Can one characterize the torsion-free abelian groups which have a decidable copy?

1.3. **Structures all of whose copies are decidable.** In this paper, we consider structures which have one computable copy which is decidable. One could also consider structures all of whose computable copies are decidable. We call such a structure *intrinsically decidable*. One can, as usual, also define a notion of relative intrinsic decidability: A structure is *relatively intrinsically decidable* if, for every isomorphic copy $\mathcal{A}$ of that structure, the elementary diagram of $\mathcal{A}$ is computable in $deg(\mathcal{A})$. By the uniform version of a theorem of Ash, Knight, Manasse, and Slaman [AKMS89], and independently Chisholm [Chi90], a computable structure $\mathcal{A}$ is relatively intrinsically decidable if and only if it has a sort of quantifier elimination: Every elementary first-order definable subset of $\mathcal{A}$ is (uniformly) definable by a computable infinitary Σ_1 formula, and also by a computable infinitary Π_1 formula. One expects there to be structures which are intrinsically decidable but not relatively intrinsically decidable, as there are, for example, structures which are computably categorical but not relatively computably categorical [Gon77]. Note that deciding whether a structure is relatively intrinsically decidable is arithmetic; however, one might guess that intrinsic decidability is actually Σ^1_1 complete.

Question 1.7. What is the complexity of the index set of the computable structures all of whose computable copies are decidable?

2. Some useful lemmas

2.1. **A sequence of structures.** It is well-known that there are computable structures $\mathcal{C}_\infty$ such that the index set of the computable structures which are isomorphic to $\mathcal{C}_\infty$ is Σ^1_1-complete. A small modification of the same argument, which we will repeat below in brief, shows that the same is true of decidable structures: There is a decidable structure $\mathcal{C}_\infty$ such that the index set of the decidable structures which are isomorphic to $\mathcal{C}_\infty$ is Σ^1_1-complete. We will use these structures in the constructions for Theorems 1.1 and 1.2.

To build the structure $\mathcal{C}_\infty$ we will use the following lemma, which is probably folklore; similar results appear in, for example, [Ash91].

Lemma 2.1. *Given a computable linear order $\mathcal{L}$, we can, uniformly in $\mathcal{L}$, build a decidable copy of $\omega^\omega \cdot (1+\mathcal{L})$.*

Proof. It is well-known that there is a decidable copy, which we will call $\mathcal{W}$, of ω^ω; we may also choose $\mathcal{W}$ so that $\mathcal{W}+\mathcal{W}$ is decidable. Define $\mathcal{A} = \mathcal{W}\cdot(1+\mathcal{L})$. We represent elements of $\mathcal{A}$ as pairs (l,w) with $l \in 1+\mathcal{L}$ and $w \in \mathcal{W}$, ordered lexicographically starting with l. We claim that $\mathcal{A}$ is decidable.

Indeed, given a tuple $\bar{a}$, break up $\bar{a}$ into tuples $\bar{a}_1,\ldots,\bar{a}_n$ where each element of $\bar{a}_i$ is of the form (l_i,w) for some $w\in\mathcal{W}$, and $l_1<\cdots<l_n$. Let $\bar{a}_i$ consist of the elements $a_1^1<\cdots<a_1^{m_i}$, and let w_i^j be such that $a_i^j=(l_i,w_i^j)$. Then (see Corollary 13.39 of [Ros82]) the complete type of $\bar{a}$ is determined effectively by the elementary first-order theories of the intervals

$$(-\infty,a_1^1],[a_1^1,a_1^2],\ldots,[a_1^{m_1},a_2^1],[a_2^1,a_2^2],\ldots,[a_2^{m_2},a_3^1],\ldots,[a_n^{m_n},\infty).$$

Each interval $[a_i^j,a_i^{j+1}]$ has the same order type as $[w_i^j,w_i^{j+1}]$ which is decidable, as it is a definable subset of $\mathcal{W}$. The order type of $[a_i^{m_i},a_{i+1}^1]$ is $\omega^\omega\cdot[l_i,l_{i+1})+w_{i+1}^1$, which has the same theory as $\omega^\omega+w_{i+1}^1$ (see Theorem 6.21 of [Ros82]); this theory is decidable. The interval $(-\infty,a_1^1]$ has the same theory as either w_1^1 (if l_1 is smaller than $\mathcal{L}$) or $\omega^\omega+w_1^1$ (if $l_1\in\mathcal{L}$). Finally, the interval $[a_n^{m_n},\infty)$ has the same theory as ω^ω. Thus the type of $\bar{a}$ is computable in $\mathcal{A}$, and so $\mathcal{A}$ is decidable. □

Lemma 2.2. *Let S be a Σ_1^1 set. There is a decidable structure $\mathcal{C}_\infty$ and a uniformly decidable sequence of structures $(\mathcal{C}_n)_{n\in\omega}$ such that $\mathcal{C}_n\cong\mathcal{C}_\infty$ if and only if $n\in S$. All of these structures are in the same language.*

Proof. Harrison [Har68] constructed a computable linear order $\mathcal{H}$ of order type $\omega_1^{CK}(1+\mathbb{Q})$. From [CDH08, Lemma 5.2] or [GN02b, Theorem 4.4(d)], we get a computable sequence of computable linear orders $(\mathcal{L}_n)_{n\in\omega}$ such that $\mathcal{L}_n$ is isomorphic to $\mathcal{H}$ if and only if $n\in S$; moreover, if $n\notin S$, then $\mathcal{L}_n$ is well-founded. Then letting $\mathcal{C}_n$ be a decidable copy of $\omega^\omega\cdot(1+\mathcal{L}_n)$, we get a uniformly decidable sequence of structures $(\mathcal{C}_n)_{n\in\omega}$. (We take $\mathcal{C}_\infty$ to be a decidable copy of $\mathcal{H}$, which is isomorphic to $\omega^\omega\cdot(1+\mathcal{H})$.) If $\mathcal{L}_n$ was well-founded, so is $\mathcal{C}_n$, and if $\mathcal{L}_n$ was isomorphic to $\mathcal{H}$, then so is $\mathcal{C}_n$. □

2.2. **Building decidable structures from disjoint unions.** In this section, we will prove four lemmas about constructing a decidable structure by taking disjoint unions of other decidable structures. We will use these lemmas during the construction.

Lemma 2.3. *Let $\mathcal{A}_1,\ldots,\mathcal{A}_k$ be decidable (resp. 1-decidable) structures. Then the disjoint union $\mathcal{B}$ of $\mathcal{A}_1,\ldots,\mathcal{A}_k$, with relations $R_1,\ldots,R_k$ picking out the domains of $\mathcal{A}_1,\ldots,\mathcal{A}_k$ respectively, is also decidable (resp. 1-decidable). This is uniform.*

Proof. We give the proof for decidable structures; the 1-decidable case is similar. It suffices to show that $\mathcal{B}$ is decidable with respect to the many-sorted logic with sorts defined by $R_1,\ldots,R_k$. The many-sorted logic has quantifiers which range only over a single sort R_i, and the relations of a structure $\mathcal{A}_i$ are restricted to the sort R_i. Indeed, it is easy to translate any formula in the single-sorted language of $\mathcal{B}$ to an equivalent formula in the many-sorted language. In what follows, by an $\mathcal{A}_i$-formula we mean a formula involving only the sort $\mathcal{A}_i$.

We can easily argue by induction on formulas that each formula φ in the many-sorted language of $\mathcal{B}$ is equivalent to a boolean combination of $\mathcal{A}_i$-formulas. For example, if $\varphi\equiv(\exists x\in R_p)\psi$, and (placing the boolean combination equivalent to ψ in disjunctive normal form)

$$\psi\equiv\bigvee_{i=1}^{r}\bigwedge_{j=1}^{k}\theta_{i,j}$$

where $\theta_{i,j}$ is a $\mathcal{A}_j$-formula, we get that

$$\varphi \equiv \bigvee_{i=1}^{r} \bigwedge_{j=1}^{k} \theta'_{i,j}$$

where $\theta'_{i,p} = (\exists x \in R_p)\theta_{i,p}$ and $\theta'_{i,j} = \theta_{i,j}$ if $j \neq p$.

Then given a formula φ in the many-sorted language of $\mathcal{B}$, write φ as a boolean combination of $\mathcal{A}_i$-formulas:

$$\varphi \equiv \bigvee_{i=1}^{r} \bigwedge_{j=1}^{k} \theta_{i,j}$$

where $\theta_{i,j}$ is a $\mathcal{A}_j$-formula. We can decide the truth of each $\theta_{i,j}$ as each $\mathcal{A}_j$ is decidable, and hence we can decide the truth of φ. □

A slightly more complicated argument proves the following similar lemma.

Lemma 2.4. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be decidable structures. Then the disjoint union $\mathcal{B}$ of $\mathcal{A}_1, \ldots, \mathcal{A}_k$, with an equivalence relation E whose equivalence classes pick out the structures $\mathcal{A}_i$, is also decidable. This is uniform.*

Proof sketch. The structure $\mathcal{B}$ is effectively bi-interpretable, using first-order formulas, with the structure from the previous lemma after naming one element from each of the k equivalence classes. □

Our third lemma allows us to take the disjoint union of infinitely many structures, as long as they are all elementarily equivalent.

Lemma 2.5. *Let $(\mathcal{A}_i)_{i\in\omega}$ be a sequence of uniformly decidable structures. Suppose that for each i and j, $\mathcal{A}_i \equiv \mathcal{A}_j$. Let $\mathcal{B}$ be the disjoint union of the $\mathcal{A}_i$, with an equivalence relation E whose equivalence classes pick out the structures $\mathcal{A}_i$. Then $\mathcal{B}$ is decidable. This is uniform.*

Proof. View the structures as relational structures. Given a formula $\varphi(x_1, \ldots, x_\ell)$ and $a_1, \ldots, a_\ell$, we need to decide whether $\mathcal{B} \models \varphi(a_1, \ldots, a_\ell)$. Let n be the quantifier depth of φ. Let $\mathcal{B}^*$ be substructure of $\mathcal{B}$ which consists of those structures $\mathcal{A}_i$ containing $a_1, \ldots, a_\ell$ and n other structures $\mathcal{A}_i$. We claim that $\mathcal{B} \models \varphi(a_1, \ldots, a_\ell)$ if and only if $\mathcal{B}^* \models \varphi(a_1, \ldots, a_\ell)$. Since $\mathcal{B}^*$ is decidable, uniformly in n, by the previous lemma, we can decide whether $\mathcal{B} \models \varphi(a_1, \ldots, a_\ell)$. Thus $\mathcal{B}$ is decidable, and this is uniform.

To see that $\mathcal{B} \models \varphi(a_1, \ldots, a_\ell)$ if and only if $\mathcal{B}^* \models \varphi(a_1, \ldots, a_\ell)$, we can play the Ehrenfeucht-Fraššé game with depth n. Denote by $\mathcal{M} \overset{r}{\sim} \mathcal{N}$ that Duplicator has a winning strategy for the Ehrenfreucht-Fraššeé game with r moves, i.e., that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same formulas with quantifier depth r. We want to show that $(\mathcal{B}^*; a_1, \ldots, a_\ell) \overset{n}{\sim} (\mathcal{B}; a_1, \ldots, a_\ell)$. To prove this, it is more convenient to prove a stronger claim

Claim 2.6. *Given r and m with $r + m \leq n + \ell$, tuples $\bar{x}_1 \in A_{j_1}, \ldots, \bar{x}_m \in A_{j_m}$, all in $\mathcal{B}^*$, and $\bar{y}_1 \in A_{k_1}, \ldots, \bar{y}_m \in A_{k_m}$ (with no repetition among the lists of the structures), $(\mathcal{B}^*; \bar{x}_1, \ldots, \bar{x}_m) \overset{r}{\sim} (\mathcal{B}; \bar{y}_1, \ldots, \bar{y}_m)$ if and only if for each i, $(\mathcal{A}_{j_i}; \bar{x}_i) \overset{r}{\sim} (\mathcal{A}_{k_i}; \bar{y}_i)$.*

From this, if we take $r = n$ and (rearranging $a_1, \ldots, a_\ell$) take

$$(a_1, \ldots, a_\ell) = (\bar{x}_1, \ldots, \bar{x}_m) = (\bar{y}_1, \ldots, \bar{y}_m)$$

with $j_i = k_i$ for all i, then we immediately get that $(\mathcal{B}^*; a_1, \ldots, a_\ell) \overset{n}{\sim} (\mathcal{B}; a_1, \ldots, a_\ell)$ as desired. So the proof of the claim will finish the proof of the lemma.

Proof of claim. The proof of this claim is by induction on r. For $r = 0$, $\bar{x}_1, \ldots, \bar{x}_m$ satisfy the same atomic formulas in $\mathcal{B}^*$ as $\bar{y}_1, \ldots, \bar{y}_m$ do in $\mathcal{B}$ if and only if for each i, $\bar{x}_i \in \mathcal{A}_{j_i}$ satisfies the same atomic formulas in $\mathcal{A}_{j_i}$ as $\bar{y}_i$ does in $\mathcal{A}_{k_i}$. Given $r > 0$, it is clear that if $(\mathcal{B}^*; \bar{x}_1, \ldots, \bar{x}_m) \overset{r}{\sim} (\mathcal{B}; \bar{y}_1, \ldots, \bar{y}_m)$ then for each i, $(\mathcal{A}_{j_i}; \bar{x}_i) \overset{r}{\sim} (\mathcal{A}_{k_i}; \bar{y}_i)$. For the other direction, suppose that for each i, $(\mathcal{A}_{j_i}; \bar{x}_i) \overset{r}{\sim} (\mathcal{A}_{k_i}; \bar{y}_i)$. Given $y' \in \mathcal{B}$, we must find $x' \in \mathcal{B}^*$ such that $(\mathcal{B}^*; \bar{x}_1, \ldots, \bar{x}_m, x') \overset{r-1}{\sim} (\mathcal{B}; \bar{y}_1, \ldots, \bar{y}_m, y')$. (The other case—finding $y' \in \mathcal{B}$ given $x' \in \mathcal{B}^*$—is similar and actually easier.)

Case 1. If $y' \in A_{k_i}$ for some $i = 1, \ldots, m$, then since $(\mathcal{A}_{j_i}; \bar{x}_i) \overset{r}{\sim} (\mathcal{A}_{k_i}; \bar{y}_i)$, there is $x' \in A_{j_i}$ such that $(\mathcal{A}_{j_i}; \bar{x}_i x') \overset{r-1}{\sim} (\mathcal{A}_{k_i}; \bar{y}_i y')$. Thus, by the induction hypothesis, $(\mathcal{B}^*; \bar{x}_1, \ldots, \bar{x}_m, x') \overset{r-1}{\sim} (\mathcal{B}; \bar{y}_1, \ldots, \bar{y}_m, y')$.

Case 2. Otherwise, let k_{m+1} be such that $y' \in A_{k_{m+1}}$. Since $r + m \leq n + \ell$, we can choose j_{m+1} different from $j_1, \ldots, j_m$ such that $A_{j_{m+1}}$ is included in $\mathcal{B}^*$. Since $A_{k_{m+1}} \equiv A_{j_{m+1}}$, we can find $x' \in A_{j_{m+1}}$ such that $(A_{k_{m+1}}, y') \overset{r-1}{\sim} (A_{j_{m+1}}, x')$. We then have, with $\bar{x}_{m+1} = x'$ and $\bar{y}_{m+1} = y'$, that $(\mathcal{A}_{j_i}; \bar{x}_i) \overset{r-1}{\sim} (\mathcal{A}_{k_i}; \bar{y}_i)$ for $i = 1, \ldots, m+1$ and that $(r-1) + (m+1) \leq n + \ell$. So $(\mathcal{B}^*; \bar{x}_1, \ldots, \bar{x}_m, x') \overset{r-1}{\sim} (\mathcal{B}; \bar{y}_1, \ldots, \bar{y}_m, y')$ by the induction hypothesis. □

Our final lemma replaces the assumption of elementary equivalence in the previous lemma with the assumption that the languages of the $\mathcal{A}_i$ are disjoint.

Lemma 2.7. *Let $(\mathcal{A}_i)_{i \in \omega}$ be a sequence of uniformly decidable (resp. 1-decidable) structures in disjoint languages $(\mathcal{L}_i)_{i \in \omega}$. Let $\mathcal{B}$ be structure in the language $\{P_i \mid i \in \omega\} \cup \bigcup_i \mathcal{L}_i$ which is the disjoint union of the $\mathcal{A}_i$, with unary predicates $(P_i)_{i \in \omega}$ picking out the domains of the $\mathcal{A}_i$. Then $\mathcal{B}$ is decidable (resp. 1-decidable). This is uniform.*

Proof. Given a formula $\varphi(\bar{x})$, for some n, φ uses only the symbols in $\{P_i \mid i < n\} \cup \bigcup_{i<n} \mathcal{L}_i$. Then by Lemma 2.3, the reduct of $\mathcal{B}$ to the language $\{P_i \mid i < n\} \cup \bigcup_{i<n} \mathcal{L}_i$ is decidable. (Note that this reduct is not the disjoint union of $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$, but the disjoint union of those structures together with a pure set whose domain is defined by $\neg(P_0 \vee \cdots \vee P_{n-1})$.) Thus we can decide whether or not φ holds of a tuple from $\mathcal{B}$. The proof for 1-decidable structures is similar. □

3. Labeling

3.1. **The intuitive idea of the labeling strategy.** In this section we will describe a labeling strategy which was used, among other things, to build structures of finite computable dimension ([Gon80]) and to show that the index set of computably categorical structures is Σ^1_1-complete ([DKL+15]). The technique was probably first used by Selivanov [Sel76]. The context is that we are building a structure $\mathcal{A}$ while watching an opponent's structure $\mathcal{B}$. We are trying to meet various requirements (which depend on the specifics of the construction), while also ensuring that either $\mathcal{A}$ is not isomorphic to $\mathcal{B}$, or if they are isomorphic, then they are computably isomorphic by a particular isomorphism f built during the construction. We want to give our opponent two choices: either they let us build $\mathcal{A}$ not isomorphic to $\mathcal{B}$, or they allow f to be an isomorphism. Of course, we cannot hope to build $\mathcal{A}$ always not isomorphic to $\mathcal{B}$, because $\mathcal{B}$ can just copy $\mathcal{A}$. On the other hand, it seems difficult to make f an isomorphism, as it could be that at some point we define $f(a) = b$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and then our opponent builds $\mathcal{B}$ so that f is no longer an isomorphism. One way of stopping this from happening is by making $\mathcal{A}$ rigid, so that each element of $\mathcal{A}$ has only one possible image in $\mathcal{B}$. But often this does not allow us to meet our other requirements. The beauty of the labeling construction is that we can make $\mathcal{A}$ rigid at every finite stage, but not rigid in the limit (if $\mathcal{A}$ and $\mathcal{D}$ are isomorphic).

While building $\mathcal{A}$, we will be able to attach labels to elements; we can think of the elements as being labeled by elements of ω. The exact mechanics of the labels are not too important—the formalities will be described later in this section—but for, say, a graph, one can think of an element as being labeled by n if it has attached to it a loop of length n. So we can always add labels to elements (by attaching a new loop), but we can never remove labels.

We think of $\mathcal{A}$ as being built by stages, with finitely many elements at each stage. During the construction, we will be acting for some other requirements to accomplish some specific goal; moreover, in a specific construction, we might have to adapt the strategy with the labels in order to meet these other requirements. We will just describe the basic strategy with the labels, under the assumption that some other strategy is influencing the construction at the same time.

To begin, give each new element of $\mathcal{A}$ two new labels which are different from the labels attached to every other element of $\mathcal{A}$. One of the labels will be the "primary" label and the other the "secondary" label. We will maintain the property that every element of $\mathcal{A}$ has two labels—one primary and one secondary—which no other element of $\mathcal{A}$ shares. We will also maintain a "bag" of labels with which every element is labeled. So at any stage, each element has two labels which are unique to it, and then all of the labels in the bag which are shared by every other element.

At this point, we can wait without building more of $\mathcal{A}$ until $\mathcal{B}$ looks like $\mathcal{A}$, in the sense that the elements of $\mathcal{B}$ satisfy the same atomic relations as the elements of $\mathcal{A}$; in particular, this means that they are labeled in the same way. If this never happens, then $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic. If at some stage s_0 the structures $\mathcal{A}$ and $\mathcal{B}$ look the same, then since each element of $\mathcal{A}$ has a label which no other element of $\mathcal{A}$ has (and so the same is true of $\mathcal{B}$), there is a unique isomorphism between $\mathcal{A}$ and $\mathcal{B}$ at stage s_0 which maps each element of $\mathcal{A}$ to the element of $\mathcal{B}$ with the same labels. Define f on the elements of $\mathcal{A}$ in accordance with this isomorphism. Add each of the primary labels to the bag, and label each element of $\mathcal{A}$ with every label from the bag. The secondary labels become the primary labels, and we give each element a new secondary label which no other element shares.

Now once again $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic, and so we can wait until they again look isomorphic. If this never happens, then as before we are happy. The other possibility is that at some stage s_1 the two structures again look the same. Since $\mathcal{B}$ looks the same as $\mathcal{A}$, for each element of $\mathcal{A}$ there is a corresponding element of $\mathcal{B}$ with the same labels, and so once again there is a unique isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Let $a \in \mathcal{A}$ be in the domain of f, with $f(a) = b$. At stage s_1, a is labeled by its primary label, which at stage s_0 was its secondary label, and no other element of $\mathcal{A}$ has the same label. At stage s_0, b also had this label, and so it is the only element of $\mathcal{B}$ at stage s_1 which has this label (because $\mathcal{A}$ has only one element with this label, and $\mathcal{A}$ and $\mathcal{B}$ look the same). So it must be that the unique isomorphism between $\mathcal{A}$ and $\mathcal{B}$ at stage s_1 maps a to b; our opponent has not been able to cause f to be wrong. Thus we have shown that the unique isomorphism between $\mathcal{A}$ and $\mathcal{B}$ at stage s_1 extends f. We can extend f in agreement with this unique isomorphism. Then, once again: we label each element of $\mathcal{A}$ with each primary label, putting those labels in the bag; the secondary labels become primary labels; and we give each element a new secondary label.

Continuing in this way, we have two possible outcomes:

(1) There are only finitely many stages at which $\mathcal{A}$ and $\mathcal{B}$ look the same. The outcome is that they are not isomorphic.
(2) There are infinitely many stages at which $\mathcal{A}$ and $\mathcal{B}$ look the same. The outcome is that we have built a computable isomorphism $f\colon \mathcal{A} \to \mathcal{B}$. Moreover, all of the labels end up in the bag, and all elements of $\mathcal{A}$ have the same labels.

In outcome (2), the labels have "disappeared" in the sense that they don't help to distinguish between elements of $\mathcal{A}$. Thus while $\mathcal{A}$ looks rigid at every finite stage, at the end of the construction it is not rigid.

3.2. **The formal details behind the labels.** Now we will describe formally how we will attach labels. We will want to be able to attach labels to a 1-decidable structure in such a way that the structure stays 1-decidable. We want to be able to add labels in a c.e. way—that is, so that at any stage, we can add a label to a node—so that the resulting structure, with the labels attached, is also 1-decidable, and so that in the 1-diagram of an isomorphic copy of $\mathcal{A}$, we can enumerate the labels. We will essentially use a Marker extension [Mar89]. Fix an infinite computable set $\mathcal{L}$ of labels. Given a sequence of subsets $X = (X_\ell)_{\ell\in\mathcal{L}}$ of $\mathcal{A}$, we want to define a three-sorted structure $\mathcal{A}^X$, whose first sort is just the structure $\mathcal{A}$. The sets X_ℓ should be thought of as the set of elements assigned the label ℓ. We want the following three lemmas to be true:

Lemma 3.1. *Let $\mathcal{A}$ be a structure and let $X = (X_\ell)_{\ell\in\mathcal{L}}$ be subsets of $\mathcal{A}$. The sets X_ℓ are definable in $\mathcal{A}^X$ by $\exists\forall$ formulas, and these formulas are uniform in ℓ and independent of $\mathcal{A}$ or X.*

Lemma 3.2. *Let $\mathcal{A}$ be a computable structure and let $X = (X_\ell)_{\ell\in\mathcal{L}}$ be a computable sequence of codes for c.e. subsets of $\mathcal{A}$. Then, uniformly in X and in the atomic diagram of $\mathcal{A}$, we can build a computable copy of $\mathcal{A}^X$.*

Lemma 3.3. *Let $\mathcal{A}$ be a 1-decidable structure and let $X = (X_\ell)_{\ell\in\mathcal{L}}$ be a computable sequence of codes for c.e. subsets of $\mathcal{A}$. Then, uniformly in X and in the 1-diagram of $\mathcal{A}$, we can build a 1-decidable copy of $\mathcal{A}^X$.*

The first lemma says that we can recover the labels in a Σ^0_2 way from a computable structure, or in a c.e. way from a 1-decidable structure. The second lemma says that we can label the elements of a computable structure in a c.e. way. The final lemma says that if we have a 1-decidable structure $\mathcal{A}$, we can label its elements in a c.e. way while maintaining 1-decidability. When we think of building a structure $\mathcal{A}$ with labels, instead of building a copy of $\mathcal{A}^X$, we will instead build $\mathcal{A}$ and c.e. sets X_ℓ of labeled elements. By Lemmas 3.2 and 3.3, this is sufficient whether we are building a computable structure or a 1-decidable structure. Similarly, using Lemma 3.1, if we are diagonalizing against a 1-decidable structure $\mathcal{A}^X$, we will instead think of ourselves as diagonalizing against a 1-decidable structure $\mathcal{A}$ and c.e. sets X_ℓ of labeled elements.

Definition of $\mathcal{A}^X$. As we said before, $\mathcal{A}^X$ will be a three-sorted structure. We will refer to the sorts as $\mathcal{A}$, $\mathcal{S}_1$, and $\mathcal{S}_2$. The language of $\mathcal{A}^X$ will be the language of $\mathcal{A}$ augmented with functions $f\colon\mathcal{S}_1\to\mathcal{A}$ and $g\colon\mathcal{S}_2\to\mathcal{S}_1$, a unary relation $U^\ell\subseteq\mathcal{S}_1$ for each $\ell\in\omega$, and a unary relation $R\subseteq\mathcal{S}_2$.

For each element x of $\mathcal{A}$, there will be infinitely many elements y of the second sort $\mathcal{S}_1$ with $f(y)=x$. These will be partitioned into infinitely many disjoint sets U^ℓ for $\ell\in\omega$. Each element of $\mathcal{S}_1$ will be the pre-image, under f, of some $x\in\mathcal{A}$.

For each element y of $\mathcal{S}_1$, there will be infinitely many elements $z\in\mathcal{S}_2$ with $g(z)=y$, and each element of $\mathcal{S}_2$ will be the pre-image, under g, of some $y\in\mathcal{S}_1$.

For every $x\in\mathcal{A}$, there will be infinitely many $y\in f^{-1}(x)\cap U^\ell$ such that there are infinitely many $z\in g^{-1}(y)$ with $R(z)$, and infinitely many $z\in g^{-1}(y)$ with $\neg R(z)$. If $x\notin X_\ell$, this will be the case for all $y\in f^{-1}(x)\cap U^\ell$, but if $x\in X_\ell$, then there will also be infinitely many $y\in f^{-1}(x)\cap U^\ell$ such that for all $z\in g^{-1}(y)$, $R(z)$. See Figure 1.

Now we prove the three lemmas stated above.

Proof of Lemma 3.1. The set X_ℓ is definable as the subset of the first sort of $\mathcal{A}^X$ defined by

$$(\exists y\in\mathcal{S}_1)\left[f(y)=x\wedge U^\ell(y)\wedge(\forall z\in\mathcal{S}_2)(g(z)=y\to R(z))\right]. \qquad\square$$

Proof of Lemma 3.2. The copy of $\mathcal{A}^X$ we build will have the computable copy of $\mathcal{A}$ in the first sort, the second sort will contain elements (x,ℓ,s,t), and the third sort will contain elements (x,ℓ,s,t,u).

We define

$$\begin{aligned} U^{\ell} &= \{(x,\ell,s,t) \in S_1\} \\ f&:\mathcal{S}_2 \to \mathcal{S}_1 \text{ defined by } (x,\ell,s,t,u) \mapsto (x,\ell,s,t) \\ g&:\mathcal{S}_1 \to \mathcal{A} \text{ defined by } (x,\ell,s,t) \mapsto x. \end{aligned}$$

It only remains to define the relation R. Given s, t, and u, we will have $R(x,\ell,s,t,u)$ if and only if u is even or if u is odd and x enters X_ℓ exactly at stage s. $\square$

Proof of Lemma 3.3. We can build a 1-decidable copy of $\mathcal{A}^X$ by putting the 1-decidable copy of $\mathcal{A}$ in the first sort, and defining the second and third sorts as in the previous lemma. Given a tuple $\bar{a} \in \mathcal{A}^X$ and an existential formula $(\exists \bar{y})\varphi(\bar{x},\bar{y})$, we want to decide whether $\mathcal{A}^X \models (\exists \bar{y})\varphi(\bar{a},\bar{y})$. First, we may rewrite φ in the language where we replace the language of $\mathcal{A}$ with the predicates

$$P^{\theta(x_1,\dots,x_n)} = \{(a_1,\dots,a_n) \in \mathcal{A}^n : \mathcal{A} \models \theta(a_1,\dots,a_n)\}$$

where θ is an existential formula in the language of $\mathcal{A}$. Next, we may assume that φ is a conjunction of atomic formulas.

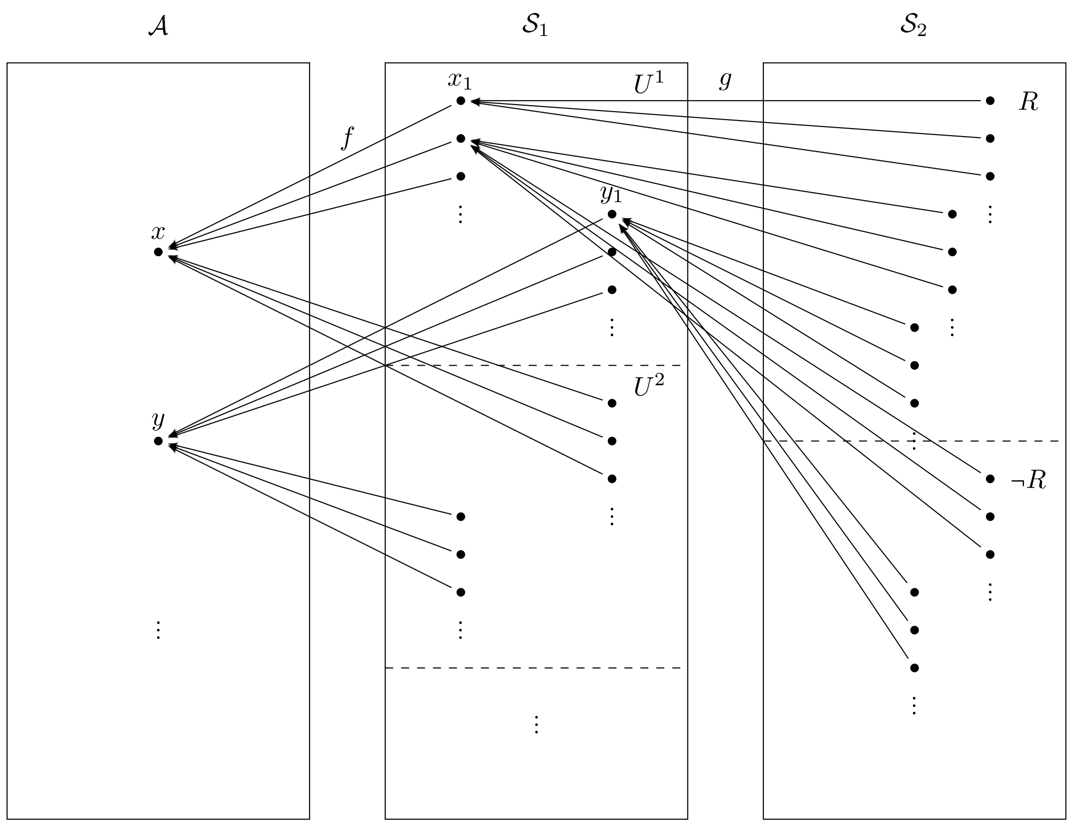


Figure 1. A diagram of how the labeling works. Note that x is labeled 1, because there is an element $x_1 \in U^1$ such that $f(x_1) = x$, and for all $z \in \mathcal{S}_2$ with $g(z) = x_1$, $R(z)$. If all of the elements of $f^{-1}(y) \cap U^1$ looked like y_1, then y would not be labeled 1.

We will show that $(\exists \bar{y})\varphi(\bar{x},\bar{y})$ is equivalent, in $\mathcal{A}^X$, to a quantifier-free formula $\psi(\bar{x})$ in an expanded language with the predicate

$$Q = \{(x,\ell,s,t) \in \mathcal{S}_1 : x \notin X_{\ell,\text{at}s}\}$$

which is only allowed to appear positively. (Here, $X_{\ell,\text{at}s}$ is the set of elements entering X_ℓ exactly at the stage s.) Note that the predicates Q and P^θ are computable in $\mathcal{A}^X$, and so we can decide whether $\mathcal{A}^X \vDash \psi(\bar{a})$, and hence whether $\mathcal{A}^X \vDash (\exists \bar{y})\varphi(\bar{a},\bar{y})$.

Arguing by induction, it suffices to show that if $\varphi(x_1,\ldots,x_n)$ is a quantifier-free formula in which Q appears only positively, then $(\exists x_n)\varphi(x_1,\ldots,x_n)$ is equivalent in $\mathcal{A}^X$ to a formula $\psi(x_1,\ldots,x_{n-1})$ in which Q appears only positively.

Since every element of $\mathcal{A}$ is the image of an element of $\mathcal{S}_1$ under g, and every element of $\mathcal{S}_1$ is the image of an element of $\mathcal{S}_2$ under f, we may assume that $x_1,\ldots,x_n$ are from the sort $\mathcal{S}_2$. We may write $\varphi(x_1,\ldots,x_n)$ in the following form:

$$\begin{aligned} P^{\theta(y_1,\ldots,y_n)}(f(g(x_1)),\ldots,f(g(x_n))) \wedge \left[\bigwedge_{i\in I(Q)} Q(g(x_i))\right] \\ \wedge \left[\bigwedge_{i\in I(U_\ell)} U^\ell(g(x_i))\right] \wedge \left[\bigwedge_{i\in I(\neg U^\ell)} \neg U^\ell(g(x_i))\right] \wedge \left[\bigwedge_{i\in I(R)} R(x_i)\right] \wedge \left[\bigwedge_{i\in I(\neg R)} \neg R(x_i)\right] \\ \wedge \left[\bigwedge_{\{i,j\}\in J_1^=} x_i = x_j\right] \wedge \left[\bigwedge_{\{i,j\}\in J_1^\neq} x_i \neq x_j\right] \\ \wedge \left[\bigwedge_{\{i,j\}\in J_2^=} g(x_i) = g(x_j)\right] \wedge \left[\bigwedge_{\{i,j\}\in J_2^\neq} g(x_i) \neq g(x_j)\right] \\ \wedge \left[\bigwedge_{\{i,j\}\in J_3^=} f(g(x_i)) = f(g(x_j))\right] \wedge \left[\bigwedge_{\{i,j\}\in J_3^\neq} f(g(x_i)) \neq f(g(x_j))\right]. \end{aligned}$$

So that we can refer to it later, let $\chi(x_1,\ldots,x_n)$ be the conjunction of all of these conjuncts other than $P^\theta(f(g(x_1)),\ldots,f(g(x_n)))$. We may assume that φ is looks consistent in the sense that $I(U^\ell)$ and $I(\neg U^\ell)$ are disjoint, $I(R)$ and $I(\neg R)$ are disjoint, and so on.

Case 1. If $\{n,i\} \in J_1^=$ for some i, then $(\exists x_n)\varphi(x_1,\ldots,x_n)$ is clearly equivalent to $\varphi(x_1,\ldots,x_{n-1},x_i)$.

Case 2. Otherwise, if $\{n,i\} \in J_2^=$ for some i, then $(\exists x_n)\varphi(x_1,\ldots,x_n)$ is equivalent to

$$P^\theta(f(g(x_1)),\ldots,f(g(x_{n-1})),f(g(x_i))) \wedge Q(g(x_i)) \wedge \chi'(x_1,\ldots,x_{n-1})$$

if $n \in I(\neg R)$, and

$$P^{\theta(y_1,\ldots,y_n)}(f(g(x_1)),\ldots,f(g(x_{n-1})),f(g(x_i))) \wedge \chi'(x_1,\ldots,x_{n-1})$$

otherwise, where $\chi'(x_1,\ldots,x_{n-1})$ is $\chi(x_1,\ldots,x_n)$ with $g(x_n)$ replaced by $g(x_i)$ everywhere, and any term involving only x_n (but not $g(x_n)$, or $f(g(x_n))$) deleted.

Case 3. Otherwise, if $\{n,i\} \in J_3^=$ for some i, then $(\exists x_n)\varphi(x_1,\ldots,x_n)$ is equivalent to

$$P^{\theta(y_1,\ldots,y_n)}(f(g(x_1)),\ldots,f(g(x_{n-1})),f(g(x_i))) \wedge \chi'(x_1,\ldots,x_{n-1})$$

where $\chi'(x_1,\ldots,x_{n-1})$ is $\chi(x_1,\ldots,x_n)$ with $f(g(x_n))$ replaced by $f(g(x_i))$ everywhere, and any term involving only x_n or $g(x_n)$ (but not $f(g(x_n))$) deleted.

Case 4. Otherwise, $(\exists x_n)\varphi(x_1,\ldots,x_n)$ is equivalent to

$$P^{(\exists y_n)\theta(y_1,\ldots,y_n)}(f(g(x_1)),\ldots,f(g(x_{n-1}))) \wedge \chi'(x_1,\ldots,x_{n-1})$$

where $\chi'(x_1,\dots,x_{n-1})$ is $\chi(x_1,\dots,x_n)$ with any term involving x_n, $g(x_n)$, or $f(g(x_n))$ deleted. $\square$

4. 1-presentable structures

4.1. **Overview of the construction.** In this section, we will prove the case $n = 1$ of Theorem 1.2: The index set of 1-presentable structures is Σ^1_1-complete. The general case is essentially the same, but restricting to the case $n = 1$ will make the proof more readable, and, in fact, the case $n \geq 2$ will follow from the proof of Theorem 1.1. (See Section 6.4.)

Fix a Σ^1_1 set S. We must build a uniformly computable sequence of computable structures $(\mathcal{M}_n)_{n\in\omega}$ such that $\mathcal{M}_n$ is 1-presentable if and only if $n \in S$. We will accomplish this by building another computable sequence of 1-decidable structures $(\mathcal{M}^-_n)_{n\in\omega}$ such that if $n \in S$, then $\mathcal{M}_n \cong \mathcal{M}^-_n$ while diagonalizing against 1-decidable structures in the case $n \in S$.[2]

Fix, as in Lemma 2.2, decidable structures $\mathcal{C}_n$ and $\mathcal{C}_\infty$ such that $\mathcal{C}_n \cong \mathcal{C}_\infty$ if and only if $n \in S$. We will use $\mathcal{C}_n$ and $\mathcal{C}_\infty$ in the construction of $\mathcal{M}_n$. Also fix a computable listing $(\mathcal{D}_i)_{i\in\omega}$ of the (possibly partial) 1-diagrams of the 1-decidable structures.

The structures $\mathcal{M}_n$ will be the disjoint unions of infinitely many structures $(\mathcal{A}_{n,i})_{i\in\omega}$, each distinguished in $\mathcal{M}_n$ by some unary relation P_i. The languages of the $\mathcal{A}_i$ will be disjoint (but essentially the same, i.e., disjoint copies of the same language). Similarly, the $\mathcal{M}^-_n$ will be disjoint unions of 1-decidable structures $(\mathcal{A}^-_{n,i})_{i\in\omega}$. We may assume that each of the structures $\mathcal{D}_i$ is a partial structure of this form. There are two properties that we want from the construction of $\mathcal{A}_{n,i}$:

(1) If $n \in S$, then for each i, $\mathcal{A}_{n,i} \cong \mathcal{A}^-_{n,i}$.
(2) If $n \notin S$ and $\mathcal{D}_i$ is a 1-decidable structure, then $\mathcal{A}_{n,i}$ will not be isomorphic to the structure with domain P_i in the 1-decidable structure $\mathcal{D}_i$.

Thus, if $n \in S$, then $\mathcal{M}_n \cong \mathcal{M}^-_n$ and so $\mathcal{M}_n$ has a 1-decidable presentation; $\mathcal{M}^-_n$ is 1-decidable by Lemma 2.7 because the languages of the $\mathcal{A}^-_i$ are disjoint. On the other hand, if $n \notin S$, then $\mathcal{M}_n$ is not 1-presentable as it cannot be isomorphic to any 1-decidable structure $\mathcal{D}_i$.

The structure $\mathcal{A}_{n,i}$ will be a labeled structure in the sense of Section 3. We want to think of building a structure $\mathcal{A}$ by constructing a structure and putting labels on its elements, rather than dealing with the intricacies of how the labels are encoded; towards that, we make the following non-standard definition.[3]

Definition 4.1. Let $\mathcal{L}$ be a language and let $(\ell_k)_{k\in\omega}$ and L be additional unary predicates. A presentation $\mathcal{A}$ of a structure in the language $\mathcal{L} \cup \{L\} \cup \{\ell_k \mid k \in \omega\}$ is *computable* (resp. *1-decidable*) if $\mathcal{A} \restriction \mathcal{L}$, the reduct of $\mathcal{A}$ to $\mathcal{L}$, is computable (resp. 1-decidable) in the usual sense, and the relations L and ℓ_k are uniformly c.e.

For each n and i, we will construct a structure $\mathcal{A}$ as a computable structure in the sense of Definition 4.1, and $\mathcal{A}^-$ as a 1-decidable structure in this sense, while diagonalizing against a partial 1-decidable structure $\mathcal{D}$. $\mathcal{D}$ will be the 1-decidable (in the sense of Definition 4.1) structure obtained by Lemma 3.1 from the structure with domain P_i in $\mathcal{D}_i$. $\mathcal{A}_{n,i}$ will be the computable structure (in the normal sense) $(\mathcal{A} \restriction \mathcal{L})^{\{L,\ell_k | k\in\omega\}}$; it is computable (in the normal sense) by Lemma 3.2 because $\mathcal{A}$ is computable (in the sense of Definition 4.1). Similarly, $\mathcal{A}^-_{n,i}$ will be the 1-decidable structure (in the normal sense) $(\mathcal{A}^- \restriction \mathcal{L})^{\{L,\ell_k | k\in\omega\}}$; it is 1-decidable by Lemma 3.3.

For the remainder of the construction, we can fix n and i as long as we work uniformly in n and i. We must construct $\mathcal{A}$ and $\mathcal{A}^-$ computable and 1-decidable respectively such that:

(1) If $n \in S$, then for each i, $\mathcal{A} \cong \mathcal{A}^-$.
(2) If $n \notin S$ and $\mathcal{D}$ is a 1-decidable structure, then $\mathcal{A}$ will not be isomorphic to $\mathcal{D}$.

---

[2]We thank Richard Shore for pointing out this way of thinking of things.

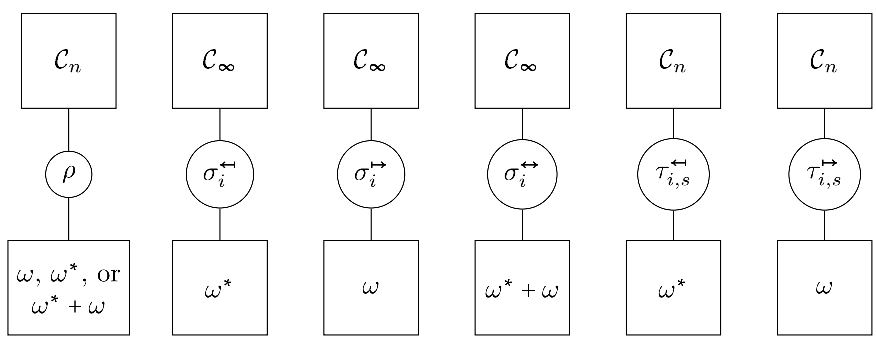
[3]This kind of non-standard definition is also made, for example, in Definition 2.3 of [DKL+15].

4.2. **The intuitive picture.** We will now describe the language and general form of $\mathcal{A}$. There will be a set N of *nodes*. To each node ν, we attach two other structures: a structure in the language of Lemma 2.2 (which is just the language of linear orders) with domain T_ν and a linear order with domain W_ν. T_ν will be isomorphic to either $\mathcal{C}_n$ or $\mathcal{C}_\infty$, and W_ν will be isomorphic to one of ω, ω^*, or $\omega^* + \omega$. We call T_ν the *tag* of ν, and we say that the elements of T_ν are the *T-elements* of ν. To each node ν, we associate the structure consisting of T_ν and W_ν. We call this structure the *ν-component* of $\mathcal{A}$.

We will begin the construction at stage $s = 0$. To start, put into $\mathcal{A}$ the distinguished node ρ, whose component we call the *special component*, and the other nodes $(\sigma_i^{\mapsfrom})$, $(\sigma_i^{\mapsto})$, and $(\sigma_i^{\leftrightarrow})$, and $(\tau_{i,0}^{\mapsfrom})$ and $(\tau_{i,0}^{\mapsto})$. At later stages of the construction, we will add new nodes $(\tau_{i,s}^{\mapsfrom})$ and $(\tau_{i,s}^{\mapsto})$ for other values of s.

For the node ρ: Let T_ρ contain a copy of $\mathcal{C}_n$, and let W_ρ begin with a single element. For each node $\sigma_i^{\square}$: Let $T_{\sigma_i^{\square}}$ contain a copy of $\mathcal{C}_\infty$, and let $W_{\sigma_i^{\square}}$ contain a linear order which depends on $\square$: for $\square = \mapsfrom$, set $W_{\sigma_i^{\mapsfrom}} = \omega^*$; for $\square = \mapsto$, set $W_{\sigma_i^{\mapsto}} = \omega$; and for $\square = \leftrightarrow$, set $W_{\sigma_i^{\leftrightarrow}} = \omega^* + \omega$. The nodes $\tau_{i,0}^{\square}$ will be the same as the nodes $\sigma_i^{\square}$, except that $T_{\tau_{i,0}^{\square}}$ will contain a copy of $\mathcal{C}_n$ instead of $\mathcal{C}_\infty$. It will be important for the proof of Lemma 4.8 that these copies of $\mathcal{C}_n$ (and similarly $\mathcal{C}_\infty$) are identical, in the sense that the map which takes the first element of one copy of $\mathcal{C}_n$ to the first element of another copy, and so on, is an isomorphism.

The structure $\mathcal{A}^-$ will be built in exactly the same way, except that it will not include the special component ρ. ($\mathcal{A}^-$ must be 1-decidable; $\mathcal{C}_n$ and $\mathcal{C}_\infty$ are already decidable, and we use 1-decidable copies of ω, ω^*, and $\omega^* + \omega$. We argue in Lemma 4.6 that because of this $\mathcal{A}^-$ is 1-decidable.)



Our strategy is to have one of the following outcomes, depending on whether or not $n \in S$:

- If $n \in S$, then the special component will be isomorphic to infinitely many other components. (The fact that $n \in S$ means that $\mathcal{C}_n$ and $\mathcal{C}_\infty$ are isomorphic, and so the special component is tagged in the same way as the components $\sigma_i^{\square}$, and so it is possible for these components to be isomorphic.) Thus $\mathcal{A}$ and $\mathcal{A}^-$ are isomorphic, and $\mathcal{A}$ is 1-presentable.
- If $n \notin S$, then the special component will not be isomorphic to any other component of $\mathcal{A}$. We will try to identify the corresponding component ν in $\mathcal{D}$, and make W_ρ different from W_ν using the fact that W_ν is 1-decidable. We use the labeling strategy to try to identify the component of $\mathcal{D}$ which corresponds to the special component of $\mathcal{A}$.

Now we will describe in more detail how we obtain these outcomes.

To the nodes ν, and to the T-elements, we attach labels in the sense described in Section 3. We have infinitely many labels ℓ_k and a distinguished label L. The elements of W_ν will not get labels. The idea is to use the strategy described in Section 3 with the labels ℓ_k to force $\mathcal{D}$ to either not be isomorphic to $\mathcal{A}$, or to guess at a partial isomorphism between $\mathcal{A}$ and $\mathcal{D}$ which is defined on the nodes and T-elements. What will happen is that if, at some stage, we guess that a node ν of $\mathcal{A}$

maps to a node μ of $\mathcal{D}$, then there will be one of three outcomes: either $\mathcal{A}$ is not isomorphic to $\mathcal{D}$ and the labels are froze, or either ν or μ is "killed" in a way to be described later (which will mean that neither is the special component), or ν is tagged in the same way as μ—that is, their tags are either both isomorphic to C_n or both isomorphic to C_∞. This will be used to guess at the special component of $\mathcal{D}$. We will say later what the label L is for.

We have essentially four different possible outcomes of the construction, depending on whether $n \in S$ or $n \notin S$, and whether or not $\mathcal{D}$ "looks like" $\mathcal{A}$ infinitely often, in the sense described in Section 3. By "looks like", we mean that the labels of the nodes and of the tags in $\mathcal{A}$ look like those in $\mathcal{D}$; we do not require that the sets W_ν look the same. It follows from the discussion in Section 3 that if $\mathcal{D}$ does not look like $\mathcal{A}$ infinitely often, then we freeze the labels and $\mathcal{D}$ is not isomorphic to $\mathcal{A}$. On the other hand, because we are not labeling the elements of the sets W_ν, it is possible for $\mathcal{D}$ to "look like" $\mathcal{A}$ infinitely often and yet not be isomorphic to it.

If $n \notin S$, we need to make $\mathcal{A}$ not isomorphic to the 1-decidable structure $\mathcal{D}$. If $\mathcal{D}$ does not look like $\mathcal{A}$ infinitely often, then $\mathcal{D}$ is not isomorphic to $\mathcal{A}$ and we are done. The other possibility is that $\mathcal{D}$ looks like $\mathcal{A}$ infinitely often. We will start by thinking about what would happen if we build $\mathcal{A}$ with just the nodes ρ and $\sigma_i^\square$ (and $\mathcal{A}^-$ with just the nodes $\sigma_i^\square$), without the nodes $\tau_{i,s}^\square$. In this case, we build a map f which is a partial isomorphism between the nodes and tags of $\mathcal{A}$ and the nodes and tags of $\mathcal{D}$. We build f by matching an element of $\mathcal{A}$ to the element of $\mathcal{D}$ with the same labels. At some stage s, we define $f(\rho)$, and we will have that the tag of ρ is isomorphic to the tag of $f(\rho)$, so that $f(\rho)$ must be tagged with C_n. Moreover, since at the moment we are omitting the $\tau_{i,s}^\square$-components, each other node of $\mathcal{A}$, and hence of $\mathcal{D}$, is tagged with C_∞. Now $W_{f(\rho)}$ is a 1-decidable linear order, so we can computably build W_ρ to be different from it (by having a least element if and only if $W_{f(\rho)}$ does not; $W_{f(\rho)}$ cannot change its mind once it has a least element, while we can as we are just building a computable structure). Since we are in the case $n \notin S$, C_n and C_∞ are not isomorphic, and so ρ and $f(\rho)$ are the only nodes of $\mathcal{A}$ and $\mathcal{D}$ respectively which are tagged with C_n. So any isomorphism between $\mathcal{A}$ and $\mathcal{D}$ would have to map ρ to $f(\rho)$. But no such map extends to an isomorphism, as W_ρ and $W_{f(\rho)}$ are not isomorphic. Hence $\mathcal{A}$ and $\mathcal{D}$ are not isomorphic. Essentially, because of the labels we could identify the special component of $\mathcal{D}$, and then because our opponent was building a 1-decidable structure and we just had to be computable, we could make our structure different. We will return to consider what happens in this case when we include the nodes $\tau_{i,s}^\square$ after we deal with the other cases.

Next let us think about the case $n \in S$, and $\mathcal{D}$ looks like $\mathcal{A}$ infinitely often. Since $n \in S$, C_n and C_∞ are isomorphic. We will ensure that the special ρ-component looks like some $\sigma_i^\square$-component. We do this by ensuring that W_ρ is one of ω, ω^*, or $\omega^* + \omega$, and that (because $\mathcal{D}$ looks like $\mathcal{A}$ infinitely often) all of the nodes and T-elements have all of the same labels. Because there are infinitely many components isomorphic to the ρ-component, we can remove the ρ-component from $\mathcal{A}$ without changing the isomorphism type of $\mathcal{A}$; thus $\mathcal{A}$ and $\mathcal{A}^-$ are isomorphic. $\mathcal{A}^-$ will be 1-decidable because because we completely decide what each of its components looks like at the start of the construction. So we get a 1-decidable copy of $\mathcal{A}$.

The final case is when $n \in S$, but $\mathcal{D}$ does not look like $\mathcal{A}$ infinitely often. This is the case where we need the nodes $\tau_{i,s}^\square$. The problem is that we want to do the same thing as in the previous paragraph, but because we stop adding new labels after some point, every node ρ or $\sigma_i^\square$ will have its own label that is not shared by any other node. So no two of these components look the same. What we will do is that at every stage s where $\mathcal{D}$ looks like $\mathcal{A}$, we will add infinitely many new nodes $\tau_{i,s}^\square$ to $\mathcal{A}$ (and $\mathcal{A}^-$). Each $\tau_{i,s}^\square$ will be tagged by $\mathcal{C}_n$, just as ρ is, and we will add labels to the $\tau_{i,s}^\square$-components in exactly the same way that the ρ-component is labeled. Because we choose whether $W_{\tau_{i,s}^\square}$ is ω or ω^* at this point, we can make each of these new components 1-decidable. In this way, if s is the last stage at which $\mathcal{D}$ looks like $\mathcal{A}$, then the ρ-component is isomorphic to each $\tau_{i,s}^{\square}$-component (with $\square = \mapsto$ or $\square = \leftarrow$ depending on whether W_ρ is ω or ω^*). Then once again $\mathcal{A}$ and $\mathcal{A}^-$ are isomorphic.

Now we need to go back to the case $n \notin S$ and $\mathcal{D}$ looks like $\mathcal{A}$ infinitely often. We now have many components which might look like the ρ-component, which messes up the strategy that we were going to use. This is where we use the L label that we have not yet used. Whenever $\mathcal{D}$ looks like $\mathcal{A}$ at stage s, we label each node $\tau_{i,s'}^{\square}$, $s' < s$, with L. This"kills" them so that we no longer have to worry about them. If $\mathcal{D}$ looks like $\mathcal{A}$ infinitely often, then every single $\tau_{i,s}^{\square}$ is at some point labeled L, but ρ is not; so once again, the special component is unique and has only one possible image under isomorphism in $\mathcal{D}$. There is now an additional consideration, which is that at any stage, there may be many components of $\mathcal{D}$ which look like the special component (whereas before there was only one, which is how we defined $f(\rho)$). We will guess that the special component is the least one of these. If $\mathcal{A}$ and $\mathcal{D}$ are going to be isomorphic, then there can only be one component of $\mathcal{D}$ which infinitely often looks like the special component: every other component which, at some point, looks like the special component will get killed by being labeled L (after which we never think that it looks like the special component). So eventually we will start to guess correctly at the special component of $\mathcal{D}$. It is possible that in $\mathcal{D}$, every component that looks like the special component is later killed by being labeled L, or that there are two or more components which look like the special component and which are not killed. But if any of these things happen, then $\mathcal{A}$ and $\mathcal{D}$ are not isomorphic and so we are happy in any case.

4.3. **The formal construction of $\mathcal{A}$.** For the entire construction, we build $\mathcal{A}^-$ in the same way as $\mathcal{A}$, except that $\mathcal{A}^-$ does not have the special component ρ.

At stage $s = 0$, put into $\mathcal{A}$ the distinguished node ρ, and the other nodes $(\sigma_i^{\leftarrow})$, $(\sigma_i^{\mapsto})$, and $(\sigma_i^{\leftrightarrow})$, and $(\tau_{i,0}^{\leftarrow})$ and $(\tau_{i,0}^{\mapsto})$. At later stages of the construction, we will add new nodes $(\tau_{i,s}^{\leftarrow})$ and $(\tau_{i,s}^{\mapsto})$ for other values of s. Set W_ν and T_ν as described in the previous section.

Assign, to each of the nodes ρ and $\sigma_i^{\square}$, and to each of their T-elements, two unique labels ℓ_k. Label the $\tau_{i,0}^{\square}$ in the same way as ρ. It will always be true at each stage s that every node ρ and $\sigma_i^{\square}$ and each of their T-elements will have two unique labels that distinguish them from every other such element. No nodes will be labeled L at this point. The bag begins empty.

Certain stages will be *expansionary stages*. The expansionary stages are those where we get more evidence that $\mathcal{A}$ is isomorphic to $\mathcal{D}$. The stage 0 is an expansionary stage by definition. At each expansionary stage s, we will have a number $\mathrm{scope}(s)$ which measures how much of the structures $\mathcal{A}$ and $\mathcal{D}$ we are looking at. Begin with $\mathrm{scope}(0) = 0$.

At each stage s, we will have a *target*, $\mathrm{target}(s)$, for ρ. The target is a node of $\mathcal{D}$ which we think is the image, under isomorphism, of ρ. We will try to make W_ρ different from the target. We do this by choosing a *direction*, $\mathrm{direction}(s)$, for ρ at stage s, which is either *left* or *right*. If the direction is left, then we are trying to build W_ρ to be a copy of ω^*; if it is right, then we are trying to build a copy of ω. We will update the target and direction only at expansionary stages. At every stage, expansionary or not, we will add a single element to W_ρ depending on the direction at that stage. Thus W_ρ will end up being isomorphic to ω, ω^*, or $\omega^* + \omega$.

Construction at stage s. At stage s, so far we have built $\mathcal{A}[s-1]$. The first thing we do at stage s is to decide whether the stage s is expansionary. Let s^* be the last expansionary stage. Stage s is expansionary if there are:

(1) nodes $\nu_0, \dots, \nu_r$ of $\mathcal{A}[s-1]$, containing among them the first $\mathrm{scope}(s^*)$ nodes of $\mathcal{A}[s-1]$;
(2) T-elements $\bar{a}_0 \in T_{\nu_0}, \dots, \bar{a}_r \in T_{\nu_r}$, containing among them the first $\mathrm{scope}(s^*)$ elements of each of these components;
(3) nodes $\mu_0, \dots, \mu_r$ of $\mathcal{D}[s]$, containing among them the first $\mathrm{scope}(s^*)$ nodes of $\mathcal{D}[s]$; and

(4) T-elements $\bar{d}_0 \in T_{\mu_0}, \dots, \bar{d}_r \in T_{\mu_r}$, containing among them the first scope(s^*) elements of each of these components

such that

- the atomic types of $\nu_0, \dots, \nu_r; \bar{a}_0, \dots, \bar{a}_r$ in $\mathcal{A}[s-1]$ and $\mu_0, \dots, \mu_r; \bar{d}_0, \dots, \bar{d}_r$ in $\mathcal{D}[s]$ are the same, and
- each of the elements from $\nu_0, \dots, \nu_r; \bar{a}_0, \dots, \bar{a}_r$ has the same labels in $\mathcal{A}[s-1]$ as the corresponding elements from $\mu_0, \dots, \mu_r; \bar{d}_0, \dots, \bar{d}_r$ have in $\mathcal{D}[s]$.

Otherwise, stage s is not expansionary. If stage s is expansionary, let scope$(s) \geq$ scope$(s^*) + 1$ be large enough that $\nu_0, \dots, \nu_r$ are among the first scope(s) nodes of $\mathcal{A}$, $\bar{a}_0, \dots, \bar{a}_r$ are among the first scope(s) elements of their components, $\mu_0, \dots, \mu_r$ are among the first scope(s) nodes of $\mathcal{D}$, and $\bar{d}_0, \dots, \bar{d}_r$ are among the first scope(s) elements of their components. The only place that we use the 1-decidability of $\mathcal{D}$ here is to use Lemma 3.2 to enumerate the labels on elements of $\mathcal{D}$.

If stage s is expansionary, then continue by *updating the target* followed by *renewing labels* as described below. If the stage s is not expansionary, the target and direction are the same as they were at the last expansionary stage. At all stages, expansionary or not, we finish by adding a new element to W_ρ. If direction(s) = right, add the new element to the right of all existing elements. Otherwise, if direction(s) = left, add the new element to the left of the existing ones. In this way we obtain the structure $\mathcal{A}[s]$.

Updating the target. In $\mathcal{D}[s]$, find the least node, if one exists, which is labeled exactly by the labels of ρ (and so not by L). Set target(s) to be this node. (If no such element exists, target(s) is undefined and direction(s) = right.)

Now, look at the linear order $W_{\text{target}(s)}$. If it has a greatest element (i.e., an element which the 1-diagram of $\mathcal{D}[s]$ says is the greatest element), set direction(s) = right. Otherwise, set direction(s) = left.

Renewing labels. Recall that s^* was the previous expansionary stage. First, apply the label L to each node τ_{i,s^*} and its T-elements. Second, each of the nodes ρ and $\sigma_i^\square$ and their T-elements have a primary and secondary label which are not in the bag. Add each of the primary labels to the bag. The secondary labels becomes the primary labels. Then, label each of these elements with each label from the bag along with a new unique secondary label.

Build new nodes $\tau_{i,s}^{\leftarrow}$ and $\tau_{i,s}^{\mapsto}$ tagged with copies of $\mathcal{C}_n$. Attach a copy of ω^* or ω to each of these nodes respectively. Label these nodes and their T-elements in the same way that ρ and its T-elements are currently labeled.

4.4. **The verification.** The possibilities to consider are:

- $n \in S$ and there are finitely many expansionary stages;
- $n \in S$ and there are infinitely many expansionary stages;
- $n \notin S$ and there are finitely many expansionary stages;
- $n \notin S$ and there are infinitely many expansionary stages.

In all cases, W_ρ is isomorphic to either ω, ω^*, or $\omega^* + \omega$.

Lemma 4.2. *W_ρ is isomorphic to either ω, ω^*, or $\omega^* + \omega$. These three cases correspond, respectively, to having* direction(s) = right *for all but finitely many s,* direction(s) = left *for all but finitely many s, and* direction(s) = right *and* direction(s) = left *for infinitely many s each.*

Proof. At each stage s we add a single element to W_ρ on either the left or right hand side, depending on the direction. ∎

Note that the direction can only change at an expansionary stage, so that if there are only finitely many expansionary stages, W_ρ is isomorphic to either ω or ω^*. This is why we only add nodes $\tau_{i,s}^{\mapsto}$ and $\tau_{i,s}^{\leftarrow}$, but not $\tau_{i,s}^{\leftrightarrow}$. The only way for W_ρ to be isomorphic to $\omega^* + \omega$ is if the target changes infinitely many times; this might happen if the opponent keeps labeling the target with L, killing it. But then the opponent will not build a component isomorphic to ρ.

First, consider the case where there are finitely many expansionary stages. The first thing we note is that $\mathcal{A}$ and $\mathcal{D}$ cannot be isomorphic (which is exactly what we want when $n \notin S$).

Lemma 4.3. *If $\mathcal{A}$ is isomorphic to $\mathcal{D}$, then there are infinitely many expansionary stages.*

Proof. Suppose to the contrary that there is a last expansionary stage s^*, and that $\mathcal{A}$ is isomorphic to $\mathcal{D}$ at the end of the construction, say by an isomorphism f. Then after stage s^*, we never add any more nodes into $\mathcal{A}$, and we never add any new labels to any elements. Let $\mu_0, \ldots, \mu_r$ be the first scope(s^*) nodes of $\mathcal{A}$ together with the inverse images, under f, of the first scope(s^*) nodes of $\mathcal{D}$. Let $\bar{a}_0 \in T_{\mu_0}, \ldots, \bar{a}_r \in T_{\mu_r}$ be the first scope(s^*) elements of these components, together with the inverse images, under f, of the first scope(s^*) elements of $T_{f(\mu_0)}, \ldots, T_{f(\mu_r)}$. Then, for sufficiently large s, $\mu_0, \ldots, \mu_r; \bar{a}_0, \ldots, \bar{a}_r$ and $f(\mu_0), \ldots, f(\mu_r); f(\bar{a}_0), \ldots, f(\bar{a}_r)$ have the same labels in $\mathcal{A}[s-1]$ and $\mathcal{D}[s]$ respectively. Such a stage s is expansionary. □

If s is the last expansionary stage, then the node ρ will be labeled in a different way from each of the nodes $\sigma_i^{\square}$. But the ρ-component and its tag will be labeled in the same was as the $\tau_{i,s}^{\square}$-components because we never add any new labels after stage s. Each node $\tau_{i,s^*}^{\square}$ for $s^* < s$ will be killed by being labeled L, but the nodes $\tau_{i,s}^{\square}$ will never be killed.

Now consider the case where there are infinitely many expansionary stages. We renew the labels infinitely many times. This will cause all of the nodes ρ or $\sigma_i^{\square}$ and their T-elements to be labeled in exactly the same way: each is labeled with all of the labels in the bag. Moreover, at each expansionary stage s, we kill each node $\tau_{i,s^*}^{\square}$, $s^* < s$, by labeling it L, after which we add some new nodes $\tau_{i,s}^{\square}$. So if there are infinitely many expansionary stages, each such node is eventually killed by being labeled L. The nodes ρ and $\sigma_i^{\square}$ are never labeled L.

Lemma 4.4. *If there are infinitely many expansionary stages, then every node ρ or $\sigma_i^{\square}$ and their T-elements have exactly the same labels as each other. Each $\tau_{i,s}^{\square}$ is labeled L.*

Now we have to think about whether $n \in S$ or $n \notin S$. If $n \in S$, then $\mathcal{C}_n$ and $\mathcal{C}_\infty$ are isomorphic. If there are finitely many expansionary stages, and s is the last expansionary stage, then because the ρ-component and its tag are be labeled in the same was as the $\tau_{i,s}^{\square}$-components, the ρ-component is isomorphic to the $\tau_{i,s}^{\square}$-components (for the appropriate value of $\square$, depending on whether W_ρ is ω or ω^*). On the other hand, if there are infinitely many expansionary stages, then the nodes ρ and $\sigma_i^{\square}$ are all labeled in the same way, namely with all of the labels in the bag; as $\mathcal{C}_n$ and $\mathcal{C}_\infty$ are isomorphic, the ρ-component is isomorphic to the $\sigma_i^{\square}$-components (for the appropriate value of $\square$, depending on whether W_ρ is ω, ω^*, or $\omega^* + \omega$). In any case, the ρ-component is isomorphic to infinitely many other components of $\mathcal{A}$. Thus $\mathcal{A}$ is isomorphic to $\mathcal{A}^-$, which is 1-decidable.

Lemma 4.5. *If $n \in S$, then $\mathcal{A}$ is isomorphic to $\mathcal{A}^-$.*

Proof. Since $n \in S$, $\mathcal{C}_n \cong \mathcal{C}_\infty$. We claim that if we run the construction without building the node ρ and its component, we get a structure $\mathcal{A}^-$ which is 1-decidable and isomorphic to $\mathcal{A}$. To see that $\mathcal{A}^-$ is isomorphic to $\mathcal{A}$, there are two cases. First, if there are infinitely many expansionary stages then, by Lemma 4.3, ρ and its T-elements, and each node $\sigma_i^{\square}$ and their T-elements, all have the same labels. Also, W_ρ is isomorphic to one of ω, ω^*, or $\omega^* + \omega$. So ρ and its component is actually isomorphic to each of the $\sigma_i^{\square}$ and their components for the appropriate choice of $\square$. Since there are infinitely many such nodes, removing ρ does not change the isomorphism type.

On the other hand, if there are only finitely many expansionary stages, then W_ρ is isomorphic to either ω or ω^*. Let s^* be the last expansionary stage. After that stage, we never add any more labels. Then ρ and its component is isomorphic to each of the $\tau^{\square}_{i,s^*}$ and their components for some $\square \in \{\leftarrow\!\!\mapsfrom, \mapsto\}$. $\square$

Lemma 4.6. *$\mathcal{A}^-$ is 1-decidable.*

Proof. It suffices to show that the reduct of $\mathcal{A}^-$ to the language without the labels is 1-decidable (in fact this reduct is decidable), from which it will follow that $\mathcal{A}^-$ itself, with the labels, is 1-decidable. The rest of the proof of this lemma will be in this smaller language without the labels.

Whenever we add a new node ν to $\mathcal{A}^-$, we immediately decide whether $T_\nu = \mathcal{C}_n$ or $T_\nu = \mathcal{C}_\infty$, and whether W_ν is isomorphic to ω, ω^*, or $\omega^* + \omega$. These structures—$\mathcal{C}_n$, $\mathcal{C}_\infty$, ω, ω^*, and $\omega^* + \omega$—all have decidable presentations. So the structure which is the disjoint union of T_ν and W_ν is decidable, uniformly in ν, by Lemma 2.3. Since this disjoint union is essentially (i.e., up to effective bi-interpretability using finitary Δ_0 formulas) the ν-component, the ν-component is decidable.

By Lemma 2.5, the following five structures are decidable:

(1) The disjoint union of the $\sigma^{\square}_i$-components, for a fixed $\square \in \{\mapsfrom, \mapsto, \leftrightarrow\}$.
(2) The disjoint union of the $\tau^{\square}_{i,s}$-components, for a fixed $\square \in \{\mapsfrom, \mapsto\}$.

Then by Lemma 2.4, the disjoint union of these five structures is also decidable. This is effectively bi-interpretable, using finitary Δ_0 formulas, to $\mathcal{A}^-$, which is thus decidable. $\square$

The harder case is when $n \notin S$. If there are are finitely many expansionary stages, then we proved above that $\mathcal{A}$ and $\mathcal{D}$ are not isomorphic, which is what we want. If there are infinitely many expansionary stages, then ρ is the only node which is both tagged $\mathcal{C}_n$ and not labeled L; each node $\sigma^{\square}_i$ is tagged $\mathcal{C}_\infty$, and each node $\tau^{\square}_{i,s}$ is labeled L. We will argue below that $\mathcal{A}$ cannot be isomorphic to $\mathcal{D}$, because if $\mathcal{D}$ also has a unique node ν which is both tagged $\mathcal{C}_n$ and not labeled L, then for sufficiently large stages of the construction, ν will be the target, and so we will build W_ρ to be different from W_ν.

Now we get into the more technical part of the verification. We will need two lemmas that say that the labels do what we want, essentially formalizing the discussion from Section 3.1. The new detail is that we have the label L. The first lemma essentially says that if an element from $\mathcal{A}$ and an element from $\mathcal{D}$ are labeled in the same way at some stage, then they must be labeled in the same way at all later stages if $\mathcal{A}$ and $\mathcal{D}$ are going to be isomorphic, except in the case that one of those elements gets labeled L (in which case that element is part of a $\tau^{\square}_{i,s}$-component which has been killed).

Lemma 4.7. *Let s be an expansionary stage and suppose that $a \in \mathcal{A}$ and $d \in \mathcal{D}$ are nodes which are among the first* $\mathrm{scope}(s)$ *nodes of of $\mathcal{A}$ and $\mathcal{D}$ respectively (or T-elements which are among the first* $\mathrm{scope}(s)$ *elements of their components, and are associated to nodes which are among the first* $\mathrm{scope}(s)$ *nodes), and so that a has the same labels in $\mathcal{A}[s-1]$ as d does in $\mathcal{D}[s]$. Then for any expansionary stage $s^* \geq s$, either a and d have the same labels in $\mathcal{A}[s^*-1]$ and $\mathcal{D}[s^*]$ respectively, or one of them is labeled L.*

Proof. It suffices to show that if $s^* \geq s$ is an expansionary stage at which a and d have the same labels in $\mathcal{A}[s^*-1]$ and $\mathcal{D}[s^*]$ respectively (and are not labeled L), and $s^{**} > s^*$ is the next expansionary stage, then either a and d have the same labels in $\mathcal{A}[s^{**}-1]$ and $\mathcal{D}[s^{**}]$ or one of them is labeled L.

Let ℓ_{k_1} be the primary label of a in $\mathcal{A}[s^*-1]$, and let ℓ_{k_2} be its secondary label. Then by assumption, d is also labeled ℓ_{k_1} and ℓ_{k_2} in $\mathcal{D}[s^*]$. If a does not get labeled L at stage s^*, then a is either in the ρ-component or one of the $\sigma^{\square}_i$-components. So during stage s^*, ℓ_{k_2} becomes the primary label of a, and a gets a new secondary label ℓ_{k_3}. Now at all stages t, $s^* < t < s^{**}$, we do not add any labels to elements of $\mathcal{A}$. In $\mathcal{A}[s^{**}-1]$, the only elements labeled ℓ_{k_2} are either labeled the same way as a, or labeled L. Since s^{**} is an expansionary stage, and d is among the first $\operatorname{scope}(s) < \operatorname{scope}(s^{**})$ nodes of $\mathcal{D}$ if it is a node (or the first $\operatorname{scope}(s)$ elements of its component, which is among the first $\operatorname{scope}(s)$ components of $\mathcal{D}$, if d is a T-element), there is an element $a' \in \mathcal{A}[s^{**}-1]$ which is labeled in the same way as d. As d is labeled ℓ_{k_2}, a' is labeled ℓ_{k_2}, and so they must both be labeled in the same way as a, or be labeled L. □

Next, we want to say that if $\mathcal{A}$ and $\mathcal{D}$ are going to be isomorphic, then we can effectively guess at which element from $\mathcal{D}$ corresponds to the special component ρ. Recall that if $\mathcal{A}$ and $\mathcal{D}$ are isomorphic, then there are infinitely many expansionary stages, and (if $n \notin S$) that ρ is the only node of $\mathcal{A}$ which is never labeled L and which is tagged $\mathcal{C}_n$. So the element ν from $\mathcal{D}$ which corresponds to ρ should be one which, at some stage, has the same labels as ρ, and which is never labeled L at some later stage. What needs to be proved is that such a ν will be tagged with $\mathcal{C}_n$ and not $\mathcal{C}_\infty$.

Lemma 4.8. *Suppose that $\mathcal{A}$ and $\mathcal{D}$ are in fact isomorphic. Let s be an expansionary stage, and let ν and μ be nodes of $\mathcal{A}$ and $\mathcal{D}$ respectively, which are among the first* $\operatorname{scope}(s)$ *nodes of those structures, and assume that neither are ever labeled L. If, at stage s, ν and μ are labeled in the same way in $\mathcal{A}[s-1]$ and $\mathcal{D}[s]$ respectively, then $T_\nu \subseteq \mathcal{A}$ and $T_\mu \subseteq \mathcal{D}$ are isomorphic.*

At each expansionary stage s, the target is a node ν from $\mathcal{D}$ which is labeled in the same way at stage s as ρ. So this lemma says that either the target ν later gets killed by being labeled L, or ν is never killed and T_ν and T_ρ are isomorphic.

Proof of Lemma 4.8. Let s be an expansionary stage. In the proof of the lemma, we will frequently use the fact that if two elements a and a' are labeled in the same way at stage s (or at any stage after s but before the next expansionary stage) but not equal, it must be for one of the following reasons:

- a and a' are both in the ρ-component or some $\tau_{i,s}^{\square}$-component; these components are tagged in identical ways, and a and a' are corresponding elements of their respective components.
- a and a' are both $\tau_{i,s^*}^{\square}$-components, for some $s^* < s$, and thus labeled L at stage s; these components are tagged in identical ways, and a and a' are corresponding elements of their respective components.

The reader should keep this fact in mind.

Let $s_0 = s, s_1, s_2, \dots$ list the expansionary stages after s. By the previous lemma, at each expansionary stage s_i, ν and μ are labeled in the same way in $\mathcal{A}[s_i - 1]$ and $\mathcal{D}[s_i]$ respectively.

Since $\mathcal{A}$ and $\mathcal{D}$ are isomorphic, by Lemma 4.3 there are infinitely many expansionary stages. Given i, define a partial isomorphism $f_i\colon T_\nu \to T_\mu$, as follows. Put a T-element a, which is among the first $\operatorname{scope}(s_i)$ elements of T_ν, into the domain of f_i if there is d a T-element of μ, which is among the first $\operatorname{scope}(s_i)$ elements of T_μ, such that a and d have the same labels in $\mathcal{A}[s_i - 1]$ and $\mathcal{D}[s]$ respectively. In this case, set $f_i(a) = d$. (Note that there can be at most one such d for a given a, as no two elements of the same component of $\mathcal{A}[s_i - 1]$ are labeled in the same way.)

Claim 4.9. *If $i < i'$, then $f_i \subseteq f_{i'}$.*

Suppose that $f_i(a) = d$. Then a and d are labeled in the same way in $\mathcal{A}[s_i - 1]$ and $\mathcal{D}[s_i]$ respectively, and are among the first $\operatorname{scope}(s_i)$ elements of T_ν and T_μ respectively. Since ν and μ are never labeled L, neither are a and d at the expansionary stage $s_{i'}$; we will not label a by L, and if d was labeled L, then $s_{i'}$ could not be an expansionary stage. So by the previous lemma, at the stage $s_{i'}$, a and d are labeled in the same way. Thus we will define $f_{i'}(a) = d$.

Let $f = \bigcup_{i \in \omega} f_i$.

Claim 4.10. *f is one-to-one.*

If f was not one-to-one, then for some i, we would have $f_i(a_1) = f_i(a_2) = d$. So then, in $\mathcal{A}[s_i - 1]$, a_1 and a_2 are labeled in the same way; but they are both in the same component, and so this cannot happen.

Claim 4.11. *f is total and onto.*

To see that f is total, fix $a \in T_\nu$. For some sufficiently large i, a will be among the first scope(s_i) elements of T_ν. Then, at the next expansionary stage s_{i+1}, there will have to be some $\mu'; d'$ corresponding (in the sense that they witness that s_{i+1} is a true stage) to $\nu; a$ and ν' corresponding to μ. Now since ν and μ are labeled in the same way, and μ and ν' are labeled in the same way, ν and ν' are labeled in the same way in $\mathcal{A}[s_{i+1} - 1]$. From the construction, we see that T_ν and $T_{\nu'}$ are *identical* copies of either $\mathcal{C}_n$ or $\mathcal{C}_\infty$, where by identical we mean that they are isomorphic via a map taking the first element of one to the first element of the other, the second element to the second, and so on. (The nodes ν and ν' might be, for example, ρ and $\tau^{\mapsto}_{j,s_i}$, or τ_{j,s_i} and τ_{j',s_i}; in the construction, we gave these identical tags.) Thus there is $a' \in T_{\nu'}$ which corresponds to $a \in T_\nu$, and since a is among the first scope(s_i) of T_ν, a' is among the first scope(s_i) elements of $T_{\nu'}$. Also, ν' is among the first scope(s_i) nodes of $\mathcal{A}$. Thus there is $d \in T_\mu$ which is labeled in the same way as a', which is labeled in the same way as a; hence we would set $f_{i+1}(a) = d$.

To see that f is onto, a similar but not identical argument works. Fix $d \in T_\mu$. For some sufficiently large i, d will be among the first scope(s_i) elements of T_μ. Then, at the next expansionary stage s_{i+1}, there will have to be some $\nu'; a'$ corresponding to $\mu; d$ and μ' corresponding to ν. Now since ν and μ are labeled in the same way, and μ and ν' are labeled in the same way, ν and ν' are labeled in the same way in $\mathcal{A}[s_{i+1} - 1]$. From the construction, we see that T_ν and $T_{\nu'}$ are identically either copies of $\mathcal{C}_n$ or $\mathcal{C}_\infty$. Thus there is $a \in T_\nu$ which corresponds to $a' \in T_{\nu'}$. Then d is labeled the same way as a', which is labeled in the same way as a; hence we would set $f_{i+1}(a) = d$.

Claim 4.12. *f is an isomorphism.*

It suffices to show that each f_i is a partial isomorphism. At stage s_i, let $a_0, \ldots, a_r$ be the elements in the domain of f_i, and let $d_0 = f_i(a_0), \ldots, d_r = f_i(a_r)$. Since s_i is an expansionary stage, there must be elements $a_0', \ldots, a_r'$ of $\mathcal{A}[s_i - 1]$ which are labeled in the same way, and have the same atomic type as $d_0, \ldots, d_r$ in $\mathcal{D}[s_i]$. But then $a_0', \ldots, a_r'$ are labeled in the same way, in $\mathcal{A}[s_i - 1]$, as $a_0, \ldots, a_r$. We can see from the construction that $a_0, \ldots, a_r$ and $a_0', \ldots, a_r'$ must then have the same atomic type in $\mathcal{A}[s_i - 1]$. (It is possible that $a_0, \ldots, a_r$ are not equal to $a_0', \ldots, a_r'$, for example if the former are in T_ρ and the latter are in $T_{\tau^{\mapsto}_{0,s_{i-1}}}$.) Hence f_i is a partial isomorphism.

This finishes the proof of the lemma. □

Finally, we have to prove that the construction does what we want in the case $n \notin S$.

Lemma 4.13. *If $n \notin S$, then $\mathcal{A}$ is not isomorphic to $\mathcal{D}$.*

Proof. Suppose to the contrary that $\mathcal{A}$ was isomorphic to $\mathcal{D}$ via an isomorphism f. Then by Lemma 4.3 there are infinitely many expansionary stages.

Note that ρ is the only node of $\mathcal{A}$ which is both tagged $\mathcal{C}_n$ and not labeled L: since $\mathcal{C}_n$ and $\mathcal{C}_\infty$ are not isomorphic, no node $\sigma_i^{\square}$ is tagged $\mathcal{C}_n$, and since there are infinitely many expansionary stages, each $\tau_{i,s}^{\square}$ is labeled L.

Let $d_0, d_1, d_2, \ldots$ list the elements of $\mathcal{D}$, and let $d_i = f(\rho)$. Let t be a stage after which each of $d_0, \ldots, d_{i-1}$, if it is the image, under f, of a node $\tau_{i,s}^{\square}$ or one of its T-elements, is labeled L; thus, if one of these elements ever becomes labeled L, it does so by stage t. Suppose that t is also large enough that ρ and d_i are among the first scope(t) nodes of $\mathcal{A}$ and $\mathcal{D}$ respectively. We claim that for all expansionary stages $s > t$, target$(s) = d_i$.

Suppose to the contrary that there is an expansionary stage s at which target$(s) \neq d_i$. Since ρ is among the first scope(t) nodes of $\mathcal{A}$, there is at least one $d_j \in \mathcal{D}[s]$ among the first scope(s) nodes of $\mathcal{D}$ which has the same labels as ρ at stage s; since target$(s) \neq d_i$, there is one such $d_j \neq d_i$.

Then either d_i and ρ are labeled differently at stage s, or there is a node d_j, $j < i$, among the first scope(s) nodes of of $\mathcal{D}$, which is labeled in the same way as d_i at stage s (and hence both are labeled in the same way as ρ).

In the first case—if d_i and ρ are labeled differently at stage s—then there is another node $\nu \neq \rho$ of $\mathcal{A}[s-1]$, which is among the first scope(s) nodes of $\mathcal{A}$, which is labeled in the same way as d_i is in $\mathcal{D}[s]$. Note that d_i is not labeled L, as $f(\rho) = d_i$. Also, ν is never labeled L, since it is labeled differently from ρ at stage s (but not labeled L) and so must be one of the nodes $\sigma_i^\square$. So by Lemma 4.8, T_{d_i} is isomorphic to T_ν. Since ν is of the form $\sigma_i^\square$, T_ν is isomorphic to $\mathcal{C}_\infty$. This is a contradiction, as $d_i = f(\rho)$ and T_ρ is isomorphic to $\mathcal{C}_n$.

In the second case—if there is a node d_j, $j < i$, among the first scope(s) nodes of $\mathcal{D}$, which is labeled in the same way as d_i at stage s—by Lemma 4.8, T_{d_j} and T_{d_i} are both isomorphic to $T_\rho = \mathcal{C}_n$. By choice of t, they are also never labeled L. But then $\mathcal{D}$ cannot be isomorphic to $\mathcal{A}$, as ρ is the only node ν of $\mathcal{A}$ not labeled L and with T_ν isomorphic to $\mathcal{C}_n$.

So for all expansionary stages $s > t$, target$(s) = d_i$. If $W_{f(\rho)} = \omega^*$, then at some point the greatest element of $W_{f(\rho)}$ is enumerated into $\mathcal{D}$, and the 1-diagram says that this is the greatest element. Then, from some sufficiently large expansionary stage on, the direction is always right. Thus $W_\rho = \omega$. On the other hand, if $W_{f(\rho)} = \omega$ or $\omega^* + \omega$, then there is never a greatest element of $W_{f(\rho)}$, and so the direction is always left. Then $W_\rho = \omega^*$. In all cases, W_ρ is not isomorphic to $W_{f(\rho)}$, a contradiction. □

Lemmas 4.5, 4.6, and 4.13 are exactly what we wanted from the construction as described in Section 4.1. This completes the proof of Theorem 1.2. (Recall that the $\mathcal{A}$ from the construction was really just one component $\mathcal{A}_{n,i}$ of the structure $\mathcal{M}_n$ that we were building in Section 4.1. We advise re-reading Section 4.1 at this point to recall how everything fits together.)

5. New kinds of labels

The rest of the paper is devoted to proving Theorem 1.1. The system of labeling which we used previously no longer works with decidable structures, because in a decidable structure we can compute whether or not an element has a particular label, rather than this being c.e. Instead of labeling elements with something that is first-order definable, we will now label them by the existence of a non-principal type, which is a Σ^0_2 fact over the elementary diagram. Then, when examining the decidable structure $\mathcal{D}$ against which we are diagonalizing, we must guess at the labels (whereas before, the labels were c.e.).

Because of this, we must also introduce Marker extensions. The new labels and the Marker extensions we use will be described in this section.

5.1. Σ^0_2 **labeling of decidable structures.** This subsection will be analogous to Section 3. Once again, fix an infinite computable set $\mathcal{L}$ of labels. Given a decidable structure $\mathcal{A}$ and a sequence $X = (X_\ell)_{\ell \in \mathcal{L}}$ of subsets of $\mathcal{A}$, we want to build a two-sorted structure $\mathcal{A}^X$, whose first sort is just the structure $\mathcal{A}$, which codes X in a Σ^0_2 way over the elementary diagram of $\mathcal{A}$. The following three lemmas describe what we want from $\mathcal{A}^X$.

Lemma 5.1. *Let $\mathcal{A}$ be a structure and let $X = (X_\ell)_{\ell \in \mathcal{L}}$ be a sequence of Σ^0_2 subsets of $\mathcal{A}$. The sets X_ℓ are definable in $\mathcal{A}^X$ by computable formulas of the form $\exists x \bigwedge_{i \in I} \psi_i(x, \cdot)$, with the ψ_i quantifier-free. These formulas are computable uniformly in ℓ, and are independent of $\mathcal{A}$ or X.*

Lemma 5.2. *Let $\mathcal{A}$ be a computable structure and let $X = (X_\ell)_{\ell \in \mathcal{L}}$ be a uniform sequence of indices for Σ^0_2 subsets of $\mathcal{A}$. Then, uniformly in X and in the atomic diagram of $\mathcal{A}$, we can build a computable copy of $\mathcal{A}^X$.*

Lemma 5.3. *Let $\mathcal{A}$ be a decidable structure and let $X = (X_{\ell_i})_{i \in \omega}$ be a uniform sequence of indices for Σ^0_2 subsets of $\mathcal{A}$. Then, uniformly in X and in the elementary diagram of $\mathcal{A}$, we can build the elementary diagram of a decidable copy of $\mathcal{A}^X$.*

The first lemma says that we can recover the labels in a Σ^0_2 way from a computable copy of $\mathcal{A}^X$. The second lemma says that we can apply labels to a computable structure in a Σ^0_2 way. The final lemma says that we can also apply labels to a decidable structure in a Σ^0_2 way. With the labels of Section 3, we thought about building $\mathcal{A}^X$ by building $\mathcal{A}$ and c.e. sets X_ℓ of labeled elements. With the new labels, we will think of building $\mathcal{A}^X$ by building $\mathcal{A}$ and Σ^0_2 sets X_ℓ of labeled elements. Similarly, before we thought of ourselves as diagonalizing against a 1-decidable structure $\mathcal{A}$ and c.e. sets X_ℓ of labeled elements; now, if we are diagonalizing against a decidable structure $\mathcal{A}^X$, we will instead think of ourselves as diagonalizing against a decidable structure $\mathcal{A}$ and Σ^0_2 sets X_ℓ of labeled elements.

Definition of $\mathcal{A}^X$. $\mathcal{A}^X$ will again be two-sorted, with the first sort consisting of $\mathcal{A}$. We will call the second sort $\mathcal{S}$. The language of $\mathcal{A}^X$ will be the language of $\mathcal{A}$ augmented with a function $f\colon \mathcal{S} \to \mathcal{A}$, a unary predicate $U^\ell \subseteq \mathcal{S}$ for each label ℓ, and infinitely many unary relations $R_i \subseteq \mathcal{S}$, $i \in \omega$.

The second sort $\mathcal{S}$ will be partitioned into the pre-images $f^{-1}(x)$ of the elements $x \in \mathcal{A}$, and each fibre $f^{-1}(x)$ will be partitioned into infinitely many disjoint sets U^ℓ. If $i < i'$, and $R_{i'}$ holds of an element, then R_i will hold of that element, and for each x, ℓ, i there will be infinitely many elements of $f^{-1}(x) \cap U^\ell$ satisfying R_j for $j < i$ but not R_i. There is a unique non-principal type p_ℓ in $f^{-1}(x) \cap U_\ell$ of an element satisfying R_i for all i.

We will define the relations R_i such that, given $x \in \mathcal{A}$ and ℓ, if $x \in X_\ell$ then there is a single realization of the non-principal type p_ℓ in $f^{-1}(x) \cap U^\ell$, and otherwise there will be no realizations of p_ℓ in $f^{-1}(x) \cap U^\ell$.

Proof of Theorem 5.1. The set X_ℓ is definable as the subset of the first sort of $\mathcal{A}^X$ defined by

$$(\exists y)\left[f(y) = x \wedge U^\ell(y) \wedge \bigwedge_i R_i(y) \right]. \qquad \square$$

Proof of Theorem 5.2. Let X_ℓ be defined by

$$x \in X_\ell \iff (\exists y)\left[(x,y) \in X_\ell^\Pi\right]$$

where X_ℓ^Π is Π^0_1 and, if $x \in X_\ell$, then there is a unique y witnessing this. We can find such a set X_ℓ^Π uniformly in a Σ^0_2 index for X_ℓ.

The copy of $\mathcal{A}^X$ we build will have the decidable copy of $\mathcal{A}$ in the first sort, and the second sort will contain elements (x, ℓ, s, t) and (x, ℓ, ∞, t) with x from the first sort and ℓ, s, and t in ω. We will have $f(x,\ell,s,t) = f(x,\ell,\infty,t) = x$ and $U^\ell(x,m,s,t)$ if and only if $m = \ell$. Given s, t, and i, we will have $R_i(x,\ell,s,t)$ if and only if $s < i$. We will have $R_i(x,\ell,\infty,t)$ if and only if $(x,t) \in X_\ell^\Pi$ at stage i. This defines a computable copy of $\mathcal{A}^X$. $\square$

Proof of Theorem 5.3. We can build a decidable copy of $\mathcal{A}^X$ by putting the decidable copy of $\mathcal{A}$ in the first sort, and defining the second sort as in the previous lemma. This copy of $\mathcal{A}^X$ is decidable.

For each ℓ, let $\mathcal{A}^X[\ell]$ be the reduct of $\mathcal{A}^X$ which discards all of the predicates R_i except for $R_0, \ldots, R_\ell$. We claim that $\mathcal{A}^X[\ell]$ is decidable uniformly in ℓ. From this it will follow that $\mathcal{A}^X$ is decidable.

These reducts are quite simple structures: Given $x \in \mathcal{A}$, there are infinitely many elements y of $f^{-1}(x)$, each of which each have, for each $0 \le i \le \ell+1$, infinitely many elements in $g^{-1}(y)$ with R_j for $j < i$ but not R_i. Thus any two such elements y are isomorphic. A simple argument, in the style of Lemma 3.3 (or Lemma 5.10 to follow) but without having to introduce the predicate Q, shows that every formula is equivalent in $\mathcal{A}^X[\ell]$ to a quantifier-free formula in the language with the additional predicate

$$P^{\theta(y_1,\ldots,y_n)}(x_1,\ldots,x_n) = \{(a_1,\ldots,a_n) \in \mathcal{A}^n \colon \mathcal{A} \models \theta(a_1,\ldots,a_n)\}$$

where θ is any formula in the language of $\mathcal{A}$. □

5.2. **The guesses.** In this section, fix a (possibly partial) decidable structure $\mathcal{D}$, and a computable sequence $X = (X_\ell)_{\ell\in\mathcal{L}}$ of indices of Σ^0_2 subsets of $\mathcal{D}$, just as one might obtain from a decidable copy of $\mathcal{D}^X$ as in Lemma 5.1. (Even if $\mathcal{D}$ is a partial structure, we can still obtain a sequence of Σ^0_2 sets in this way.) We will describe a way of guessing at membership in the sets X_ℓ. We will define, at each stage s, a guess X^s_ℓ at X_ℓ. We think of the elements of X^s_ℓ as being *labeled ℓ at stage s*. Of course, X^s_ℓ may be mistaken about X_ℓ, so we will also have a notion of true stage. If s is a true stage, then any element labeled ℓ at stage s will actually be labeled ℓ (i.e., $X^s_\ell \subseteq X^s$), and any element actually labeled ℓ will be labeled ℓ at sufficiently high true stages ($X_\ell = \bigcup_{\text{true stages } s} X^s_\ell$).

Since X_ℓ is Σ^0_2, we can write

$$x \in X_\ell \Longleftrightarrow (\exists n)(\forall m)\,[(x,n,m) \in X^{\mathsf{c}}_\ell]$$

for some uniformly computable predicates X^{c}_ℓ. Fix an enumeration of the tuples (x,ℓ,n), where $x \in \mathcal{D}$, ℓ is a label, and $n \in \omega$. Assume that in this enumeration, if (x,ℓ,n) comes before (x,ℓ,n'), then $n < n'$.

In defining the sets X^s_ℓ and the true stages, we will use sets G_s which keep track of witnesses. G_s will be a finite set of tuples (x,ℓ,n). For each $(x,\ell,n) \in G_s$, we will have that for all $m < s$, $(x,n,m) \in X^{\mathsf{c}}_\ell$; the converse will not necessarily be true. If, for all $m < s$, we have $(x,n,m) \in X^{\mathsf{c}}_\ell$, and n is the least such witness, then we say that *x appears to be labeled ℓ at stage s with witness n.* Note that if, at some stage, x appears to be labeled ℓ with witness n, and then at some later stage, x does not appear to be labeled ℓ with witness n, then x can never again appear to be labeled ℓ with witness n. It is, however, possible for x to not appear to be labeled ℓ with witness n, then later to appear to be labeled ℓ with witness n, and then later to again not appear to be labeled ℓ with witness n.

Begin with $G_0 = \varnothing$. At stage s, we will have defined G_{s^*} for $s^* < s$. We must now define G_s. If there is some $(x,\ell,n) \in G_{s-1}$ so that x does not appear to be labeled ℓ at stage s with witness n, then G_{s-1} is mistaken. In this case, let $t < s$ be greatest such that for each $(x,\ell,n) \in G_t$, x appears to be labeled ℓ at stage s with witness n, and let $G_s = G_t$. Otherwise, if there are no mistakes to correct, let (x,ℓ,n) be least (in our fixed enumeration) such that $(x,\ell,n) \notin G_{s-1}$ but x appears to be labeled ℓ at stage s with witness n. Let $G_s = G_{s-1} \cup \{(x,\ell,n)\}$. (If no such tuple exists, let $G_s = G_{s-1}$.) Note that there is no other $m \neq n$ with $(x,\ell,m) \in G_{s-1}$.

We will use standard notation to talk about the true path. Write $s \le_0 t$ if and only if $s \le t$, and $s \le_1 t$ if $s \le t$ and $G_s \subseteq G_t$.

Lemma 5.4. *If $s < t < u$, and $s \le_1 u$, then $s \le_1 t$.*

Proof. Suppose to the contrary that $s \not\le_1 t$, so that $G_s \nsubseteq G_t$. We may assume that t is the least such. So $G_s \subseteq G_{t-1}$. Since $G_s \nsubseteq G_t$, we can see from the definition of G_t that there is $(x,\ell,n) \in G_s$ so that x does not appear to be labeled ℓ at stage t with witness n. By choice of t, at stage $t-1$, x

appeared to be labeled ℓ with witness n. So, at stage u, that x cannot appear to be labeled ℓ with witness n, and so (x,ℓ,n) cannot be in G_u. So $s \not\leq_1 u$. $\square$

We say that a stage s is a *true stage* if, for all $t > s$, $s \leq_1 t$.

Lemma 5.5. *There are infinitely many true stages.*

Proof. Assume that there is a greatest true stage s. There is some least t such that $s+1 \not\leq_1 t$. Since s is a true stage, $G_s \subseteq G_{s+1}, G_t$. By choice of t, $G_{s+1} \notsubseteq G_t$; by the minimality of t, $G_{s+2},\dots,G_{t-1} \notsubseteq G_t$ as well. Then we see from the construction that $G_t = G_s$. Thus $t \leq_1 u$ for all $u > t$, contradicting the choice of s. $\square$

We call the sequence $s_0 < s_1 < s_2 < \cdots$ of true stages the *true path* of the construction.

Lemma 5.6. *If s is a true stage, and $t \leq_1 s$, then t is also a true stage.*

Proof. Suppose that $t \leq_1 s$. Then, by Lemma 5.4, $t \leq_1 s^*$ for all s^* with $t \leq s^* \leq s$; and since $s \leq_1 s^*$ for all $s^* \geq s$, $t \leq_1 s^*$ for all $s^* \geq t$. $\square$

Define $X_\ell^s = \{x \mid (\exists n)\,(x,\ell,n) \in G_s\}$. Note that if $s \leq_1 t$, then $X_\ell^s \subseteq X_\ell^t$. The next lemma will show that the set X_ℓ is the union, along the true stages, of the sets X_ℓ^s.

Lemma 5.7. $X_\ell = \bigcup_{i\in\omega} X_\ell^{s_i}$.

Proof. Note that if $x \notin X_\ell$, then for all n, there is m such that $(x,n,m) \notin X_\ell^{\mathrm{c}}$. Fix n, and let m be such that $(x,n,m) \notin X_\ell^{\mathrm{c}}$. Thus, for all stages $s > m$, $(x,\ell,n) \notin G_s$; so, for any true stage t, $(x,\ell,n) \notin G_t$. Since this is true for all n, $x \notin X_\ell^s$ for any true stage s.

On the other hand, suppose that $x \in X_\ell$, but for all true stages s, $x \notin X_\ell^s$. Since $x \in X_\ell$, for some n, for all m we have $(x,n,m) \in X_\ell^{\mathrm{c}}$. We may assume that (x,ℓ,n) is the tuple with $(x,n,m) \in X_\ell^{\mathrm{c}}$ for all m (so $x \in X_\ell$) but $x \notin X_\ell^s$ at any true stage s. Since $x \notin X_\ell^s$ for all true stages s, $(x,\ell,n) \notin G_s$ for all true stages s. For some true stage s, for all (x',ℓ',n') less than (x,ℓ,n) in our chosen enumeration, we will either have that x' does not appear to be labeled ℓ' as witnessed by n' at all true stages after s (and so (x',ℓ',n') can never be in G_t for any $t \geq s$) or that $x' \in X_{\ell'}$ (with least witness n') and $(x',\ell',n') \in G_s$ (so that $(x',\ell',n') \in G_t$ for all $t > s$). So x appears to be labeled ℓ as witnessed by n at all stages after s. Then at stage $s+1$, we have $G_{s+1} = G_s \cup \{(x,\ell,n)\}$ and $s+1$ is a true stage. So $x \in X_\ell^{s+1}$, a contradiction. $\square$

We will say that a node or T-element x from $\mathcal{D}[s]$ is *labeled ℓ (at stage s)* if $x \in X_\ell^s$.

5.3. $\exists\forall$ **Marker extensions.** Given a structure $\mathcal{A}$ together with a relation X on $\mathcal{A}$, we will describe how to make a certain kind of Marker extension of $(\mathcal{A},X)$. The Marker extension will be very similar to the labels of Section 3; in fact, the labels from that section were just the Marker extensions of unary relations. (We did not treat Marker extensions in general earlier to keep things simple and because it was more convenient to use different notation.)

We will define a three-sorted structure $M(\mathcal{A},X)$ whose first sort is a copy of the structure $\mathcal{A}$. Once again we have three different lemmas which say that the construction works.

Lemma 5.8. *X is definable in $M(\mathcal{A},X)$ by an $\exists\forall$ formula.*

Lemma 5.9. *If $\mathcal{A}$ is computable and X is Σ_2^0, then we can build a computable copy of $M(\mathcal{A},X)$ uniformly in $\mathcal{A}$ and X.*

Lemma 5.10. *If $(\mathcal{A},X)$ is decidable, then we can build a decidable copy of $M(\mathcal{A},X)$ uniformly in the elementary diagram of $(\mathcal{A},X)$.*

The first lemma says that if $M(\mathcal{A},X)$ is computable then $X \subseteq \mathcal{A}$ is a Σ^0_2, and if $M(\mathcal{A},X)$ is decidable then $(\mathcal{A},X)$ is decidable. The second lemma says that we can build a computable copy of $M(\mathcal{A},X)$ by building a computable copy of $\mathcal{A}$ with X being Σ^0_2. The final lemma says that we can build a decidable copy of $M(\mathcal{A},X)$ by building a decidable copy of $(\mathcal{A},X)$. Note that if $(\mathcal{A},X)$ is decidable, then, in particular, X is computable. Note that because we are now dealing with decidable structures, Lemma 5.10 is different from the corresponding Lemma 3.3 from Section 3.

To build a computable copy of $M(\mathcal{A},X)$, we can instead think of ourselves as building a copy of $\mathcal{A}$ with a Σ^0_2 copy of the relation X on $\mathcal{A}$. To build a decidable copy of $M(\mathcal{A},X)$, we think of ourselves as building a decidable copy of $(\mathcal{A},X)$. If we are diagonalizing against a decidable copy of $M(\mathcal{A},X)$, we can think of ourselves as diagonalizing against a decidable copy of $(\mathcal{A},X)$.

Definition of $M(\mathcal{A},X)$. Let n be the arity of X. We will refer to the sorts as $\mathcal{A}$, $\mathcal{S}_1$, and $\mathcal{S}_2$. The language of $M(\mathcal{A},X)$ will be the language of $\mathcal{A}$ augmented with functions $f{:}\,\mathcal{S}_1 \to \mathcal{A}^n$ and $g{:}\,\mathcal{S}_2 \to \mathcal{S}_1$ and a unary relation $R \subseteq \mathcal{S}_2$.

For each element $\bar{x} \in \mathcal{A}^n$, there will be infinitely many elements y of the second sort $\mathcal{S}_1$ with $f(y) = \bar{x}$. Each element of $\mathcal{S}_1$ will be the pre-image, under f, of some $\bar{x} \in \mathcal{A}^n$. For each element y of $\mathcal{S}_1$, there will be infinitely many elements $z \in \mathcal{S}_2$ with $g(z) = y$, and each element of $\mathcal{S}_2$ will be the pre-image, under g, of some $y \in \mathcal{S}_1$.

For every $\bar{x} \in \mathcal{A}^n$, there will be infinitely many $y \in f^{-1}(\bar{x})$ such that there are infinitely many $z \in g^{-1}(y)$ with $R(z)$, and infinitely many $z \in g^{-1}(y)$ with $\neg R(z)$. If $\bar{x} \notin X$, this will be the case for all $y \in f^{-1}(\bar{x})$, but if $\bar{x} \in X$, then there will also be infinitely many $y \in f^{-1}(\bar{x})$ such that for all $z \in g^{-1}(y)$, $R(z)$.

Proof of Lemma 5.8. X is defined by the formula

$$\bar{x} \in X \iff (\exists y)\left[f(y) = \bar{x} \wedge (\forall z)\left[f(z) = y \to R(z)\right]\right]. \qquad \square$$

Proof of Lemma 5.9. Let X be defined by

$$\bar{x} \in X \iff (\exists y)(\forall z)\left[(\bar{x},y,z) \in X^{\mathsf{c}}\right]$$

where X^{c} is computable and, if $\bar{x} \in X$, then there are infinitely many y witnessing this (and, for all y, if there is z with $(\bar{x},y,z) \notin X^{\mathsf{c}}$, then there are infinitely many such z). We can find such a set X^{c} uniformly in a Σ^0_2 index for X.

The copy of $M(\mathcal{A},X)$ we build will have the decidable copy of $\mathcal{A}$ in the first sort, the second sort will contain elements $(\bar{x},s)$, and the third sort will contain the elements $(\bar{x},s,t)$. We will have $f(\bar{x},s) = \bar{x}$ and $g(\bar{x},s,t) = (\bar{x},s)$. It only remains to define the relation R. Given s and t, we will have $R(\bar{x},s,t)$ if and only if $(\bar{x},s,t) \in X^{\mathsf{c}}$. This defines a computable copy of $M(\mathcal{A},X)$. $\square$

Proof of Lemma 5.10. The copy of $M(\mathcal{A},X)$ we build will have the decidable copy of $\mathcal{A}$ in the first sort, the second sort will contain elements $(\bar{x},s)$, and the third sort will contain elements $(\bar{x},s,t)$. We will have $f(\bar{x},s) = \bar{x}$, and $f(\bar{x},s,t) = (\bar{x},s)$. Define $R(\bar{x},s,t)$ if t is odd, or if s and t are even and $\bar{x} \in X$.

We claim that this is decidable. Given a tuple $\bar{a} \in M(\mathcal{A},X)$ and a formula $\varphi(\bar{x})$, we want to decide whether $M(\mathcal{A},X) \models \varphi(\bar{a})$. First, we may rewrite φ in the language where we replace the language of $\mathcal{A}$ with the predicates

$$P^{\theta(y_1,\ldots,y_n)} = \{(a_1,\ldots,a_n) \in \mathcal{A}^n : \mathcal{A} \models \varphi(a_1,\ldots,a_n)\}$$

where θ is a formula, possibly involving quantifiers, in the language of $\mathcal{A}$.

We will show that $\varphi(\bar{x})$ is equivalent, in $M(\mathcal{A}, X)$, to a quantifier-free formula $\psi(\bar{x})$ in an expanded language with the predicate

$$Q = \{(\bar{x}, s) \in \mathcal{S}_1 : s \text{ is odd, } t \text{ is odd, or } \bar{x} \notin X\}$$

and the predicates $P^{\theta(y_1,\ldots,y_n)}$, where θ is now allowed to contain the predicate R. Note that the predicates Q and P^θ are computable in $M(\mathcal{A}, X)$, and so we can decide whether $M(\mathcal{A}, X) \vDash \psi(\bar{a})$, and hence whether $M(\mathcal{A}, X) \vDash \varphi(\bar{a})$.

Arguing by induction, it suffices to show that if $\varphi(x_1, \ldots, x_n)$ is a quantifier-free formula possibly involving Q and P^θ (where θ may involve R), $(\exists x_n)\varphi(x_1, \ldots, x_n)$ is equivalent in $M(\mathcal{A}, X)$ to a quantifier-free formula $\psi(x_1, \ldots, x_{n-1})$. The argument is essentially the same as Lemma 3.3, though $f(g(x_n))$ is now a tuple rather than a single element. □

6. Decidably presentable structures

In this section, we will add a guessing argument to the construction from the previous section to show that the index set of decidably presentable structures is Σ^1_1-complete (Theorem 1.1). The argument will also complete the proof of Theorem 1.2. See Section 6.4.

6.1. **Overview of the construction.** As before, fix a Σ^1_1 set S and a computable listing $(\mathcal{D}_i)_{i\in\omega}$ of the (possibly partial) 2-diagrams of the 2-decidable structures. We will build a uniformly computable sequence of computable structures $(\mathcal{M}_n)_{n\in\omega}$, and a sequence of uniformly decidable structures $(\mathcal{M}^-_n)_{n\in\omega}$. We want that if $n \in S$ then $\mathcal{M}_n \cong \mathcal{M}^-_n$ (and so $\mathcal{M}_n$ is decidable), and if $n \notin S$ then $\mathcal{M}_n$ is not 2-presentable because it is not isomorphic to any $\mathcal{D}_i$. We could have taken the $\mathcal{D}_i$ to be decidable, but by taking them to be 2-decidable we will simultaneously prove the $n \geq 2$ case of Theorem 1.1. (See Section 6.4.)

Once again, the structures $\mathcal{M}_n$ will be the disjoint unions of infinitely many structures $(\mathcal{A}_{n,i})_{i\in\omega}$, each distinguished in $\mathcal{M}_n$ by some unary relation P_i. The languages of the $\mathcal{A}_i$ will be disjoint (but essentially the same, i.e., disjoint copies of the same language). Similarly, the $\mathcal{M}^-_n$ will be disjoint unions of decidable structures $(\mathcal{A}^-_{n,i})_{i\in\omega}$. We may assume that each of the structures $\mathcal{D}_i$ is a partial structure of this form. We want:

1. If $n \in S$, then for each i, $\mathcal{A}_{n,i} \cong \mathcal{A}^-_{n,i}$.
2. If $n \notin S$ and $\mathcal{D}_i$ is a 2-decidable structure, then $\mathcal{A}_{n,i}$ will not be isomorphic to the structure with domain P_i in the 2-decidable structure $\mathcal{D}_i$.

Thus, if $n \in S$, then $\mathcal{M}_n \cong \mathcal{M}^-_n$ and so $\mathcal{M}_n$ has a decidable presentation; $\mathcal{M}^-_n$ is decidable by Lemma 2.7 because the languages of the $\mathcal{A}^-_i$ are disjoint. On the other hand, if $n \notin S$, then $\mathcal{M}_n$ is not 2-presentable as it cannot be isomorphic to any 2-decidable structure $\mathcal{D}_i$.

The structure $\mathcal{A}_{n,i}$ will be a labeled Marker extension in the sense of Section 5. Once again, we make a non-standard definition.

Definition 6.1. Let $\mathcal{L}$ be a language and let $(\ell_k)_{k\in\omega}$ and L be additional unary predicates, and $\leq$ a binary relation. A presentation $\mathcal{A}$ of a structure in the language $\mathcal{L} \cup \{\leq, L\} \cup \{\ell_k \mid k \in \omega\}$ is:

- *computable* if $\mathcal{A} \mid \mathcal{L}$ is computable in the usual sense and the relations L, ℓ_k, and $\leq$ are uniformly Σ^0_2.
- *2-decidable* if $\mathcal{A} \mid \mathcal{L}$ is 2-decidable in the usual sense, the relations L and ℓ_k are uniformly Σ^0_2, and the definable subsets of $\mathcal{A}$ defined by existential formulas in which $\leq$ occurs positively are computable.
- *decidable* if $\mathcal{A} \mid \mathcal{L} \cup \{\leq\}$ is decidable in the usual sense and the relations L and ℓ_k are uniformly Σ^0_2.

Note that the requirements of the relation $\leq$ and the labels are different.

For each n and i, we will construct a structure $\mathcal{A}$ as a computable structure in this sense, and $\mathcal{A}^-$ as a decidable structure in this sense, while diagonalizing against a partial 2-decidable structure $\mathcal{D}$. $\mathcal{A}_{n,i}$ will be the computable structure (in the normal sense) $M(\mathcal{A} \mid \mathcal{L}, \leq)^{\{L,\ell_k | k \in \omega\}}$; it is computable by Lemmas 5.2 and 5.9. Similarly, $\mathcal{A}^-_{n,i}$ will be the decidable structure (in the normal sense) $M(\mathcal{A} \mid \mathcal{L}, \leq)^{\{L,\ell_k | k \in \omega\}}$; it is decidable by Lemmas 5.3 and 5.10.

The 2-decidable structure $\mathcal{D}$ against which we diagonalize will be obtained by Lemmas 5.8 and 5.1 from the structure with domain P_i in $\mathcal{D}_i$. The labels will be Σ^0_2, so to guess at them, we will use the approximations from Section 5.2. An existential formula in which $\leq$ occurs positively is an $\exists\forall$ formula, as $\leq$ has an $\exists\forall$ definition in $\mathcal{D}_i$, and so can be decided using the 2-diagram of $\mathcal{D}_i$. Recall from that section that for each label ℓ, we can find a sequence of computable sets X^s_ℓ such that, if $s_0 < s_1 < s_2 < \cdots$ are the true stages, the elements labeled ℓ are exactly $\bigcup_{i \in \omega} X^{s_i}_\ell$. Recall also that we say that a node or T-element x from $\mathcal{D}[s]$ is *labeled ℓ (at stage s)* if $x \in X^s_\ell$. Thus, the labels which hold at any true stage are actual labels of elements of $\mathcal{D}$.

For the remainder of the construction, we can fix n and i as long as we work uniformly in n and i. We must construct $\mathcal{A}$ and $\mathcal{A}^-$ computable and decidable respectively such that:

(1) If $n \in S$, then for each i, $\mathcal{A} \cong \mathcal{A}^-$.
(2) If $n \notin S$ and $\mathcal{D}$ is a 2-decidable structure, then $\mathcal{A}$ will not be isomorphic to $\mathcal{D}$.

6.2. **Acting for a guess.** We build $\mathcal{A}$ as a computable structure by stages. The language of $\mathcal{A}$ will be the same as it was before, except that now we have a different non-standard definition of what it means to be a computable structure (Definition 6.1) than we did before (Definition 4.1). As before, we build $\mathcal{A}^-$ in the same way as $\mathcal{A}$, but without building the ρ-component.

The relation $\leq$ will be the relation defining the linear orders W_ν. Because $\leq$ and the labels just need to be Σ^0_2 in the end, at each stage, we have approximations $\leq_s$ and labels ℓ^s_k and L^s. It will not necessarily be true that if x is labeled ℓ at stage s, then it will be labeled ℓ at stage t, or that if $x \leq_s y$, then $x \leq_t y$. If, in fact, $s \leq_1 t$, then $\leq_t$ will extend $\leq_s$, and anything labeled ℓ at stage s will still be labeled ℓ at stage t. Note that by Lemma 5.4, if $s < t < u$, $s,t \leq_1 u$, then $s \leq_1 t$. Thus, this last requirement need only be checked at stage u for the greatest $t < u$ with $t \leq_1 u$.

Let $s_0 < s_1 < s_2 < \cdots$ be the true path of the approximation of the labels of $\mathcal{D}$. For each label ℓ, the set of elements of $\mathcal{A}$ which are labeled ℓ are exactly those which are labeled ℓ at some true stage. Similarly, we will have $x \leq y$ if and only if there is a true stage s at which $x \leq_s y$. To see that these are in fact Σ^0_2 sets, note that the set of true stages is a Π^0_1 subset of ω: s is a true stage if and only if, for all $t > s$, $s \leq_1 t$. Then, for example, $x \leq y$ if and only if there is a true stage s such that $x \leq_s y$. This is Σ^0_2, and so $\mathcal{A}$ is a computable structure in the sense of Definition 6.1.

At each stage s, we can revert $\leq_s$ and the labels back to the way they were at the previous stage s^* with which s agrees, i.e., the largest $s^* \leq_1 s$, and then act in the same way that we did in the previous construction. Recall that in the previous construction, the only actions we took were adding labels, adding elements to W_ρ, and adding new components $\tau^\square_{i,s}$. Thus we can essentially revert everything that we did, and the construction along the true path is almost exactly the same as it was before.

The one difference has to do with the nodes $\tau^\square_{i,s}$, and the elements of W_ρ. Before, we said that we would add one new element to W_ρ at each stage. Now, if we are at a stage s, and s^* is the previous stage which agrees with s, then we may have added elements to W_ρ at stages in between, and we cannot now remove those elements. So now, what we do instead is begin with W_ρ having infinitely many elements, but with $\leq$ not defined on any of them. When we add a new element to the linear order, what we mean is just that we take one of these elements and define $\leq$ on it. In the end, along the true path, every element of W_ρ ends up being in the domain of $\leq$.

Similarly, we may add nodes $\tau_{i,s}^{\square}$ which we later want to remove. But there are only two different isomorphism types of $\tau_{i,s}^{\square}$ once we remove the labels, depending on the value of $\square$. So we can begin with a reserve of infinitely many of these. When we add a new $\tau_{i,s}^{\square}$-component, what we mean is that we take it from the reserve, and we can return them to the reserve if they were added at a stage which is not on the true path. If we add these components from the reserve in an appropriate way, we end up adding all of them along the true path.

Stage 0. Begin at stage 0 with $\mathcal{A}[0]$ as follows. In $\mathcal{A}[0]$, there will be nodes ρ and $\sigma_i^{\square}$ for $\square \in \{\leftarrow\!\shortmid, \mapsto, \leftrightarrow\}$. We have $T_\rho = \mathcal{C}_n$ and $T_{\sigma_i^{\square}} = \mathcal{C}_\infty$. We put a infinitely many elements in W_ρ, but only one of them in the linear order $\preceq_0$. In $W_{\sigma_i^{\square}}$ we put a linear order $\preceq$ isomorphic to either ω or ω^*, depending on whether $\square$ is $\mapsto$ or $\leftarrow\!\shortmid$.

Unlike before, we will not immediately add infinitely many nodes $\tau_{i,0}^{\square}$, but rather will "schedule" two such nodes (one for each of $\square = \mapsto$ and $\square = \leftarrow\!\shortmid$) to be added at each stage. We do, at stage 0, create an infinite *reserve* of nodes which will, at some later stage, become one of the $\tau_{i,s}^{\square}$. To each of these nodes ν in the reserve, we have T_ν be a copy of $\mathcal{C}_n$, and W_ν a linear order isomorphic to ω for half of the nodes, and ω^* for the other half.

To each node or T-element x associated to a node ρ or $\sigma_i^{\square}$, we choose two unique labels ℓ_1 and ℓ_2, as primary and secondary labels, and label x with them.

Set $\text{scope}(0) = 0$.

Action at stage s. Let $s_1, \dots, s_n < s$ be the previous stages with $s_i \leq_1 s$. We say that these stages are the *s-true stages*, and if they were expansionary stages, then we say that they are *s-true expansionary stages*. Let s^* be the last s-true expansionary stage.

At stage s, so far we have built $\mathcal{A}[s-1]$, $\preceq_{s-1}$ and certain labels ℓ^{s-1} on $\mathcal{A}[s-1]$. The first thing we need to do is to undo anything that we have done since the stage s_n. So we begin stage s with the order $\preceq_{s_n}$ and only the labels which held at stage s_n; any changes to $\preceq$ or the labels after stage s_n and up to, and including, stage $s-1$ are discarded. Also, return all of the nodes $\tau_{i,s'}^{\square}$, for $s_n < s' < s$, to the reserve.

Now we need to decide whether the stage s is expansionary. Stage s is expansionary if there are:

(1) nodes $\nu_0, \dots, \nu_r$ of $\mathcal{A}[s_n - 1]$, containing among them the first $\text{scope}(s^*)$ nodes of $\mathcal{A}$;
(2) T-elements $\bar{a}_0 \in T_{\nu_0}, \dots, \bar{a}_r \in T_{\nu_r}$, containing among them the first $\text{scope}(s^*)$ elements of each of these components;
(3) nodes $\mu_0, \dots, \mu_r$ of $\mathcal{A}[s]$, containing among them the first $\text{scope}(s^*)$ nodes of $\mathcal{D}$; and
(4) T-elements $\bar{d}_0 \in T_{\mu_0}, \dots, \bar{d}_r \in T_{\mu_r}$, containing among them the first $\text{scope}(s^*)$ elements of each of these components

such that:

- the atomic types of $\nu_0, \dots, \nu_r; \bar{a}_0, \dots, \bar{a}_r$ in $\mathcal{A}[s_n - 1]$ and $\mu_0, \dots, \mu_r; \bar{d}_0, \dots, \bar{d}_r$ in $\mathcal{D}[s_n]$ are the same, and
- each of the elements from $\nu_0, \dots, \nu_r; \bar{a}_0, \dots, \bar{a}_r$ has the same labels in $\mathcal{A}[s_n - 1]$ as the corresponding elements from $\mu_0, \dots, \mu_r; \bar{d}_0, \dots, \bar{d}_r$ have in $\mathcal{D}[s_n]$.

Otherwise, stage s is not expansionary. If stage s is expansionary, let $\text{scope}(s) \geq \text{scope}(s^*) + 1$ be large enough that $\nu_0, \dots, \nu_r$ are among the first $\text{scope}(s)$ nodes of $\mathcal{A}$, $\bar{a}_0, \dots, \bar{a}_r$ are among the first $\text{scope}(s)$ elements of their components, $\mu_0, \dots, \mu_r$ are among the first $\text{scope}(s)$ nodes of $\mathcal{D}$, and $\bar{d}_0, \dots, \bar{d}_r$ are among the first $\text{scope}(s)$ elements of their components.

If stage s is expansionary, then continue by *updating the target* followed by *renewing labels* as described below. If the stage s is not expansionary, the target and direction are the same as they were at the last s-true expansionary stage.

At all stages, expansionary or not, we finish by adding a new element to the linear order $\leq$ in W_ρ. In $\mathcal{A}[s_n]$, finitely many of the elements of W_ρ are bear some relation $\leq$, and these are linearly ordered. If direction(s) = right, pick the least element x of W_ρ which does not bear any such relation, and put this new element to the right of the linear order we have built so far. Otherwise, if direction(s) = left, do the same but add the new element to the left. This defines $\leq_s$.

Let s^* be the last s-true expansionary stage. Take two nodes, which we call $\tau^{\leftarrow}_{s,s^*}$ and $\tau^{\mapsto}_{s,s^*}$, from the reserve (with W_η isomorphic to ω^* and ω respectively). Label these with the same labels as ρ.

Updating the target. In $\mathcal{D}[s]$, find the least node, if one exists, which is labeled exactly by the labels of ρ (and not by L). Set target(s) to be this node. (If no such element exists, target(s) is undefined and direction(s) = right.)

Now, look at the linear order $W_{\text{target}(s)}$. If it has a greatest element, set direction(s) = right. Otherwise, set direction(s) = left. We can recognize whether an element x of $W_{\text{target}(s)}$ is the greatest element by asking whether for all y, $x \not\leq y$ (where $x \not\leq y$ is definable by a $\forall\exists$ formula by Lemma 5.8). This is a $\forall\exists$ fact, and so we can ask the 2-diagram of $\mathcal{D}$.

Renewing labels. Recall that s^* was the previous expansionary stage. First, apply the label L to each node $\tau^{\square}_{i,s^*}$. Second, each of the nodes ρ and $\sigma^{\square}_i$ and their T-elements have two labels which only of themselves and which are not in the bag. Add each of the primary labels to the bag. The secondary labels become the primary labels. Then, label each of these elements with each label from the bag along with a new unique secondary label.

6.3. **Verification.** Let $s_0 < s_1 < s_2 < \cdots$ be the true path of the approximation of the labels of $\mathcal{D}$. If, at any true stage s, we though there was a label on an element of $\mathcal{D}$, then that element does actually have a label. Similarly:

(1) An element $x \in \mathcal{A}$ labeled ℓ in $\mathcal{A}$ if and only if it was labeled ℓ at some true stage s.
(2) A pair of elements $x, y \in \mathcal{A}$ have $x \leq y$ if and only if $x \leq_s y$ at some true stage s.

The same is true of $\mathcal{A}^-$.

The proofs of the following lemmas end up being almost exactly the same as proofs of the corresponding lemmas in the 1-decidable case, except that we talk only about true stages. We will repeat the statements of the lemmas, with the modifications to refer only to true stages. We will repeat a few of the proofs modified to talk about true stages, but the proofs are essentially the same.

Lemma 6.2. *W_ρ is isomorphic to either ω, ω^*, or $\omega^*+\omega$. These three cases correspond, respectively, to having* direction(s) = right *for all but finitely many true stages s,* direction(s) = left *for all but finitely many true stages s, and* direction(s) = right *and* direction(s) = left *for infinitely many true stages s each.*

Proof. At each true stage s we add a single element to the linear order $\leq$ in W_ρ on either the left or right hand side, depending on the direction. There are now infinitely many elements in W_ρ at stage 0, but since we always add the least one to the domain of $\leq$, every element of W_ρ ends up being in the linear order $\leq$. $\square$

We begin with the case $n \in S$.

Lemma 6.3. *If there are infinitely many true expansionary stages, then every node ρ or $\sigma^{\square}_i$ and their T-elements have exactly the same labels as each other. Each $\tau^{\square}_{i,s}$ is labeled L.*

Proof sketch. At each true expansionary stage, we renew the labels, and the labels that we place at such stages are never removed. Labels that we place at any other stage are later removed. Also, for every s which is not a true stage, and components $\tau^{\square}_{i,s}$ which we add are later returned to the reserve, so the only $\tau_{i,s}^{\square}$ which remain in the construction are those added at true stages; these are all labeled L. □

Lemma 6.4. *If $n \in S$, then $\mathcal{A}$ is isomorphic to $\mathcal{A}^-$.*

Proof sketch. We can argue as before that if there are infinitely many true expansionary stages, then the special ρ-component is isomorphic to some $\sigma_i^{\square}$-component, and there are infinitely many of these. On the other hand, if s is the last true expansionary stage, then we freeze the labels in the sense that no new labels are added at true stages (and so the labels on $\mathcal{A}$ are exactly those at stage s, because any other labels which are added are later removed). Then the special ρ-component is isomorphic to some $\tau_{i,s}^{\square}$-component, and there are infinitely many of these because we add one from the reserve at each true stage $t > s$. □

Lemma 6.5. *$\mathcal{A}^-$ is decidable.*

Proof sketch. Before, we had to argue that $\mathcal{A}^-$ was 1-decidable. The argument now is essentially the same, as $\mathcal{C}_n$, $\mathcal{C}_\infty$, ω, ω^*, and $\omega^* + \omega$ are all decidable. □

The next case is when $n \notin S$. If there are are finitely many expansionary stages, then $\mathcal{A}$ and $\mathcal{D}$ are not isomorphic. We will give the argument for the next lemma in full to show that it is essentially the same as before. We omit the proofs of the other lemmas, which can be modified in a similar way.

Lemma 6.6. *If $\mathcal{A}$ is isomorphic to $\mathcal{D}$, then there are infinitely many true expansionary stages.*

Proof. Suppose to the contrary that there is a last true expansionary stage s^*, and that $\mathcal{A}$ is isomorphic to $\mathcal{D}$ at the end of the construction, say by an isomorphism f. Then at every true stage after stage s^*, we never add any more nodes into $\mathcal{A}$ other than those $\tau_{i,s^*}^{\square}$ which were already scheduled to be added from the reserve, and we never add any new labels to any elements. Let $\mu_0, \dots, \mu_r$ be the first scope(s^*) nodes of $\mathcal{A}$ together with the inverse images, under f, of the first scope(s^*) nodes of $\mathcal{D}$. Let $\bar{a}_0 \in T_{\mu_0}, \dots, \bar{a}_r \in T_{\mu_r}$ be the first scope(s^*) elements of these components, together with the inverse images, under f, of the first scope(s^*) elements of $T_{f(\mu_0)}, \dots, T_{f(\mu_r)}$. Then, for sufficiently large true stages s, $\mu_0, \dots, \mu_r; \bar{a}_0, \dots, \bar{a}_r$ and $f(\mu_0), \dots, f(\mu_r); f(\bar{a}_0), \dots, f(\bar{a}_r)$ have the same labels in $\mathcal{A}[s']$ and $\mathcal{D}[s]$ respectively (where s' is the previous true stage before s). Such a stage s is expansionary, and hence a true expansionary stage. □

Lemma 6.7. *Let s_n be a true expansionary stage and suppose that $a \in \mathcal{A}$ and $d \in \mathcal{D}$ are nodes which are among the first* scope(s_n) *nodes of $\mathcal{A}$ and $\mathcal{D}$ respectively (or T-elements which are among the first* scope(s_n) *elements of their components, and associated to nodes which are among the first* scope(s_n) *nodes), and so that a has the same labels in $\mathcal{A}[s_{n-1}]$ as d does in $\mathcal{D}[s_n]$. Then for any true expansionary stage $s_m \geq s_n$, either a and d have the same labels in $\mathcal{A}[s_{m-1}]$ and $\mathcal{D}[s_m]$ respectively, or one of them is labeled L.*

Proof sketch. If a and d look the same at stage s_n, then at the next true expansionary stage, we do the usual thing with the labels where we put each primary label in the bag etc. (and we do not put any new labels on elements at any true stage in between). The same argument as before shows that a and d still have the same labels, unless one of them gets labeled L, and we argue inductive as before. □

Lemma 6.8. *Suppose that $\mathcal{A}$ and $\mathcal{D}$ are in fact isomorphic. Let s_n be a true expansionary stage, and let ν and μ be nodes of $\mathcal{A}$ and $\mathcal{D}$ respectively, which are among the first* scope(s_n) *nodes of those structures, and assume that neither are ever labeled L at a true stage. If, at stage s, ν and μ are labeled in the same way in $\mathcal{A}[s_{n-1}]$ and $\mathcal{D}[s_n]$ respectively, then $T_\nu \subseteq \mathcal{A}$ and $T_\mu \subseteq \mathcal{D}$ are isomorphic.*

Proof sketch. We build an isomorphism f between T_ν and T_μ, but now f is only defined at true expansionary stages instead of at true stages. As before, f maps an element of T_ν to the unique element of T_μ which is labeled in the same way. The proof is otherwise the same. □

At each true expansionary stage s, the target is a node ν from $\mathcal{D}$ which is labeled in the same way at stage s as ρ. So this lemma says that either the target ν later gets killed (at a true stage) by being labeled L, or ν is never killed and T_ν and T_ρ are isomorphic.

Finally, we have to prove that the construction does what we want in the case $n \notin S$.

Lemma 6.9. *If $n \notin S$, then $\mathcal{A}$ is not isomorphic to $\mathcal{D}$.*

Proof sketch. Suppose that $\mathcal{A}$ was isomorphic to $\mathcal{D}$ via f. There are infinitely many true expansionary stages. An argument as before shows that for sufficiently large true expansionary stages, the target is $f(\rho)$. Note that any action we take at any stage which is not true is undone, so we are only building the linear order W_ρ at true stages. So, as before, we build it to be not isomorphic to $W_{f(\rho)}$. □

6.4. **The $n \geq 2$ case of Theorem 1.2.** Note that we built $\mathcal{A}$ while diagonalizing against a 2-decidable structure $\mathcal{D}$. So in fact we have shown that

$$(\Sigma^1_1, \Pi^1_1) \leq_m (I_{d\text{-}pres}, \overline{I_{2\text{-}pres}}).$$

That is, for any Σ^1_1 set S, there is a computable function f such that

$$n \in S \Longrightarrow \text{the } f(n)\text{th computable structure has a decidable presentation}$$

and

$$n \notin S \Longrightarrow \text{the } f(n)\text{th computable structure has no 2-decidable presentation.}$$

This proves the $n \geq 2$ case of Theorem 1.2.

Question 6.10. Is it true that $(\Sigma^1_1, \Pi^1_1) \leq_m (I_{d\text{-}pres}, \overline{I_{1\text{-}pres}})$?

References

[AKMS89] C. J. Ash, J. F. Knight, M. Manasse, and T. A. Slaman. Generic copies of countable structures. *Ann. Pure Appl. Logic*, 42(3):195–205, 1989.

[And14] U. Andrews. Decidable models of ω-stable theories. *J. Symb. Log.*, 79(1):186–192, 2014.

[Ash91] C. J. Ash. A construction for recursive linear orderings. *J. Symbolic Logic*, 56(2):673–683, 1991.

[BHT] Nikolay Bazhenov and Matthew Harrison-Trainor. On the orbits of computably enumerable sets. Preprint.

[CDH08] P. A. Cholak, R. G. Downey, and L. A. Harrington. On the orbits of computably enumerable sets. *J. Amer. Math. Soc.*, 21(4):1105–1135, 2008.

[CFG+07] W. Calvert, E. Fokina, S. S. Goncharov, J. F. F. Knight, O. Kudinov, A. S. Morozov, and V. Puzarenko. Index sets for classes of high rank structures. *J. Symbolic Logic*, 72(4):1418–1432, 2007.

[Chi90] J. Chisholm. Effective model theory vs. recursive model theory. *J. Symbolic Logic*, 55(3):1168–1191, 1990.

[CLLS00] P. A. Cholak, S. Lempp, M. Lerman, and R. A. Shore, editors. *Computability theory and its applications*, volume 257 of *Contemporary Mathematics.* American Mathematical Society, Providence, RI, 2000. Current trends and open problems.

[CM98] J. Chisholm and M. Moses. An undecidable linear order that is n-decidable for all n. *Notre Dame J. Formal Logic*, 39(4):519–526, 1998.

[DKL+15] R. G. Downey, A. M. Kach, S. Lempp, A. E. M. Lewis-Pye, A. Montalbán, and D. D. Turetsky. The complexity of computable categoricity. *Adv. Math.*, 268:423–466, 2015.

[DM08] R. G. Downey and A. Montalbán. The isomorphism problem for torsion-free abelian groups is analytic complete. *J. Algebra*, 320(6):2291–2300, 2008.

[FGK+15] E. B. Fokina, S. S. Goncharov, V. Kharizanova, O. V. Kudinov, and D. Turetski. Index sets of n-decidable structures that are categorical with respect to m-decidable representations. *Algebra Logika*, 54(4):520–528, 544–545, 547–548, 2015.

[Fok07] E. B. Fokina. Index sets of decidable models. *Sibirsk. Mat. Zh.*, 48(5):1167–1179, 2007.

[FS89] H. Friedman and L. Stanley. A Borel reducibility theory for classes of countable structures. *J. Symbolic Logic*, 54(3):894–914, 1989.

[GBM15a] S. S. Goncharov, N. A. Bazhenov, and M. I. Marchuk. The index set of Boolean algebras that are autostable relative to strong constructivizations. *Sibirsk. Mat. Zh.*, 56(3):498–512, 2015.

[GBM15b] S. S. Goncharov, N. A. Bazhenov, and M. I. Marchuk. Index sets of constructive models of natural classes that are autostable with respect to strong constructivizations. *Dokl. Akad. Nauk*, 464(1):12–14, 2015.

[GN02a] S. S. Goncharov and Dzh. Naĭt. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.

[GN02b] S. S. Goncharov and Dzh. Naĭt. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.

[Gon77] S. S. Gončarov. The number of nonautoequivalent constructivizations. *Algebra i Logika*, 16(3):257–282, 377, 1977.

[Gon80] S. S. Gončarov. The problem of the number of nonautoequivalent constructivizations. *Algebra i Logika*, 19(6):621–639, 745, 1980.

[Har68] J. Harrison. Recursive pseudo-well-orderings. *Trans. Amer. Math. Soc.*, 131:526–543, 1968.

[HKSS02] D. R. Hirschfeldt, B. Khoussainov, R. A. Shore, and A. M. Slinko. Degree spectra and computable dimensions in algebraic structures. *Ann. Pure Appl. Logic*, 115(1-3):71–113, 2002.

[LS07] S. Lempp and T. A. Slaman. The complexity of the index sets of $\aleph_0$-categorical theories and of Ehrenfeucht theories. In *Advances in logic*, volume 425 of *Contemp. Math.*, pages 43–47. Amer. Math. Soc., Providence, RI, 2007.

[Mar89] D. Marker. Non Σ_n axiomatizable almost strongly minimal theories. *J. Symbolic Logic*, 54(3):921–927, 1989.

[MPSS] R. Miller, B. Poonen, H. Schoutens, and A. Shlapentokh. A computable functor from graphs to fields. To appear.

[Ros82] J. G. Rosenstein. *Linear orderings*, volume 98 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982.

[Sel76] V. L. Selivanov. Enumerations of families of general recursive functions. *Algebra and Logic*, 15(2):128–141, 1976.

Department of Pure Mathematics, University of Waterloo, ON, Canada N2L 3G1
Email address: maharris@uwaterloo.ca
URL: http://www.math.uwaterloo.ca/~maharris/