A Short Note on Constructing Decidable Graphs From Other Structures

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Abstract

We show that every structure—even in an infinite language—can be transformed into a graph which is bi-interpretable with the original structure, such that we can compute the full elementary diagram of one from the other.

1 Introduction

It has long been known that graphs are universal in the sense that any structure can be transformed into a graph with the same properties. Such a construction appeared in [1] where it was shown that for any structure, one can build a graph with the same computability-theoretic properties. Montalbán has introduced the notion of effective bi-interpretability [2]; in this language, every structure is effectively bi-interpretable with a graph. See Section 3.2 for the definition of effective bi-interpretation.

In this paper, we consider what happens if one wants to preserve the full elementary diagram of a structure. Given a decidable structure, one wants to produce a decidable graph which is effectively bi-interpretable with it. If the language is finite, then it is well-known that this can be done. The ideas required appear in [3], for example, where a model-theoretic interpretation is constructed. A model-theoretic bi-interpretation gives a way of translating elementary first-order formulas from one structure to the other (see Section 3.1). The new content of this paper is the case of an infinite language. Our structures will all be countably infinite with domain ω.

Theorem 1.1. *For every countable structure $\mathcal{A}$, there is a graph $G(\mathcal{A})$ that is effectively bi-interpretable with $\mathcal{A}$. The bi-interpretation is independent of the given structure. Moreover, from the elementary diagram of a copy of $\mathcal{A}$ we can compute the elementary diagram of the copy of $G(\mathcal{A})$ interpreted inside of it, and from the elementary diagram of a copy of $G(\mathcal{A})$ we can compute the elementary diagram of the copy of $\mathcal{A}$ interpreted inside of it, in a uniform way.*

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This theorem will be proved in Section 4.

As mentioned above the essential difficulty in the case of graphs is that of an infinite language. For a finite language, the bi-interpretations for graphs are model-theoretic bi-interpretations. However, for other universal structures, such as groups [1] or fields [4], the bi-interpretations are not model-theoretic even in the case of a finite language. In the case of groups for example, one needs tuples of arbitrary sizes to code words of arbitrary sizes.

Question 1.2. Does Theorem 1.1 hold for groups or fields in place of graphs?

2 Applications

This work came about because each of the authors wanted to translate to a finite language (or to a natural class, like that of graphs, groups, or fields) computability-theoretic examples which have interesting properties related to their full elementary diagrams. Similar kinds of constructions for particular kinds of structures were used in, for example, [5] and [6]. The construction appearing here is much more general and can turn any kind of decidable structure into a decidable graph. We now give several applications.

2.1 Decidable categoricity spectra

As one example, this solves an open problem of the first author related to complexity of isomorphisms between decidable structures. For a decidable structure $\mathcal{S}$, the *decidable categoricity spectrum* (or *autostability spectrum relative to strong constructivizations*) of $\mathcal{S}$ is the set of all Turing degrees capable of computing isomorphisms among arbitrary decidable copies of $\mathcal{S}$. In [7, Question 7.3] it was asked: Given a decidable structure $\mathcal{S}$ in an arbitrary computable language, can one always find a decidable structure $\mathcal{A}_{\mathcal{S}}$ from a familiar class (say, the class of graphs) such that the decidable categoricity spectra of $\mathcal{A}_{\mathcal{S}}$ and $\mathcal{S}$ coincide? Here we give the positive answer to this question:

Corollary 2.1. *Let $\mathcal{S}$ be a decidable structure. There is a decidable graph G such that the decidable categoricity spectra of $\mathcal{S}$ and G are the same.*

2.2 Index set of decidably presentable structures

The second application is to the complexity of the index set of decidably presentable structures. In [8], it was shown that the index set of the decidably presentable structures is Σ^1_1-complete. The language used was an infinite language. Using Theorem 1.1, we can transfer this result to the class of graphs:

Corollary 2.2. *The index set of the decidably presentable graphs is Σ^1_1-complete.*

2.3 Degree spectra

Slaman [9] and Wehner [10] constructed structures with a copy in all non-computable degrees, but with no computable copy. In [11], Hirschfeldt showed that there is a structure $\mathcal{A}$ which has no computable copy, but which has a D-decidable copy for every non-computable D. Applying Theorem 1.1, we get:

Corollary 2.3. *There is a graph that has no computable copy but has a D-decidable copy for every non-computable D.*

In general, we can transfer any such result about the decidable copies of a structure to get a graph with decidable copies in the same degrees.

2.4 Computable categoricity vs. relative computable categoricity

Goncharov [12] showed that there are structures which are computably categorical but not relatively computably categorical. If one adds more decidability, then these two notions do coincide. In particular, Goncharov [13] showed that for a 2-decidable structure, they are the same. Kudinov [14] then showed that there is a 1-decidable structure which is computably categorical but not relatively computably categorical.

Corollary 2.4. *There is a 1-decidable graph which is computably categorical but not relatively computably categorical.*

Proof. This follows from a slight extension of Theorem 1.1: from a 1-decidable copy of $\mathcal{A}$ we can compute a 1-decidable copy of the copy of $G(\mathcal{A})$ interpreted within it. To see this, we must modify the proof of Claim 4.9 to prove: Using the 1-diagram of $\mathcal{A}$ we can compute the 1-diagram of $H(\mathcal{A})$. We will sketch a proof of this modification, though the sketch will refer to material which appears later and the reader is adivsed to return to this sketch later.

Given an existential formula φ and $\bar{g} \in H(\mathcal{A})$, we must decide effectively whether $H(\mathcal{A}) \models \varphi(\bar{g})$. The definitions of $\bar{g}$ from $h_{i_1}, \ldots, h_{i_k}$ are both existential and universal, so $\varphi(\bar{h})$ can be taken to be existential. Arguing as in Claim 4.9, we get that $H(\mathcal{A}) \models \varphi(\bar{h})$ if and only if $H_m^{**}(\mathcal{A}) \models \varphi(\bar{h})$. Applying the Reduction Theorem, Theorem 3.2, to an existential formula results in an existential formula ψ; this is because the model-theoretic interpretation uses both existential and universal formulas. Then

$$H_m^{**}(\mathcal{A}) \models \varphi(h_{i_1}, \ldots, h_{i_k}) \iff \mathcal{A} \models \psi(a_{i_1}, \ldots, a_{i_k}).$$

We can decide this effectively using the 1-diagram of $\mathcal{A}$.

Note that we do not need an existential form of Gaifman's Theorem; we just need to use the fact that φ is equivalent to a Boolean combination of r-local formulas and r-local sentences. $\square$

2.5 Further applications

Other possible applications of our approach include obtaining new results on theory spectra and Σ_n spectra for graphs [15, 16, 17]. Nevertheless, it seems that our main result does not have immediate consequences for these spectra. The issue is that, because our bi-interpretation is not model-theoretic, we do not immediately get an interpretation between the theories of $\mathcal{A}$ and $H(\mathcal{A})$. We leave open the question of whether our methods can be modified to obtain, e.g., that the class of graphs is universal with respect to Σ_n spectra.

3 Background

3.1 Model-Theoretic Interpretations

An interpretation of a structure $\mathcal{A}$ in a structure $\mathcal{B}$ is essentially a definition of a copy of $\mathcal{A}$ inside $\mathcal{B}$. In a model-theoretic interpretation, the domain of that copy is a definable subset of B^n for some n, and the definition uses elementary first-order formulas.

Definition 3.1. An n-dimensional *model-theoretic interpretation* of $\mathcal{A} = (A, P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \ldots)$ in $\mathcal{B}$ consists of:

- a definable subset $Dom_\Gamma \subseteq \mathcal{B}^n$,
- a definable equivalence relation $\sim$ on Dom_Γ,
- for each relation P_i of arity k, a definable set $R_i \subseteq Dom_\Gamma^k$ which respects $\sim$,
- a surjective map $f_\Gamma \colon Dom_\Gamma \to A$ which induces an isomorphism

$$f_\Gamma \colon (Dom_\Gamma/\sim; R_0/\sim, R_1/\sim, \ldots) \to \mathcal{A}.$$

If φ is a formula about $\mathcal{A}$, then we can translate all of the relations in $\mathcal{A}$ using their definitions in $\mathcal{B}$ to get a formula in the language of $\mathcal{B}$.

Theorem 3.2 (Reduction Theorem; Theorem 5.3.2 of [3])**.** *Let $\mathcal{A}$ be a σ-structure, $\mathcal{B}$ be a τ-structure, and Γ an n-dimensional interpretation of $\mathcal{A}$ in $\mathcal{B}$. For every σ-formula $\varphi(\bar{x})$ there is a τ-formula $\varphi_\Gamma(\bar{y})$ such that for all $\bar{b} \in Dom_\Gamma$,*

$$\mathcal{A} \models \varphi(f_\Gamma[\bar{b}]) \iff \mathcal{B} \models \varphi_\Gamma(\bar{b}).$$

Moreover, it follows from the proof that we can compute φ_Γ effectively.

There is a well-studied notion of model theoretic bi-interpretation, but we will not need it here.

3.2 Effective Interpretations

Effective interpretations were first introduced by Montalbán [18] but they are essentially the same as the parameterless version of the well-studied notion of Σ-reducibility which was introduced by Ershov [19]. The elementary first-order definitions of a model-theoretic interpretation are now replaced by effective $\Delta_1^{\mathtt{c}}$ definitions, and the interpretation is allowed to use tuples of arbitrary sizes.

Definition 3.3. An *effective interpretation* of $\mathcal{A} = (A, P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, \ldots)$ in $\mathcal{B}$ consists of:

- a $\Delta_1^{\mathtt{c}}$-definable subset $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq \mathcal{B}^{<\omega}$,
- a $\Delta_1^{\mathtt{c}}$-definable equivalence relation $\sim$ on $Dom_{\mathcal{A}}^{\mathcal{B}}$,
- a sequence of uniformly $\Delta_1^{\mathtt{c}}$-definable sets $R_i \subseteq (Dom_{\mathcal{A}}^{\mathcal{B}})^k$, where k is the arity of P_i, which respect $\sim$,
- a surjective map $f_{\mathcal{A}}^{\mathcal{B}} \colon Dom_{\mathcal{A}}^{\mathcal{B}} \to A$ which induces an isomorphism

$$f_{\mathcal{A}}^{\mathcal{B}} \colon (Dom_{\mathcal{A}}^{\mathcal{B}}/\sim; R_0/\sim, R_1/\sim, \ldots) \to \mathcal{A}.$$

Two structures $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable if they are each effectively interpretable in the other, and moreover, the composition of the interpretations—i.e., the isomorphisms which map $\mathcal{A}$ to the copy of $\mathcal{A}$ inside the copy of $\mathcal{B}$ inside $\mathcal{A}$, and $\mathcal{B}$ to the copy of $\mathcal{B}$ inside the copy of $\mathcal{A}$ inside $\mathcal{B}$—are definable.

Definition 3.4. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if there are effective interpretations of $\mathcal{A}$ in $\mathcal{B}$ and of $\mathcal{B}$ in $\mathcal{A}$ such that the compositions

$$f^{\mathcal{A}}_{\mathcal{B}} \circ \tilde{f}^{\mathcal{B}}_{\mathcal{A}} \colon \mathrm{Dom}_{\mathcal{B}}^{(\mathrm{Dom}^{\mathcal{B}}_{\mathcal{A}})} \to \mathcal{B} \quad \text{and} \quad f^{\mathcal{B}}_{\mathcal{A}} \circ \tilde{f}^{\mathcal{A}}_{\mathcal{B}} \colon \mathrm{Dom}_{\mathcal{A}}^{(\mathrm{Dom}^{\mathcal{A}}_{\mathcal{B}})} \to \mathcal{A}$$

are $\Delta^{\mathtt{c}}_1$-definable in $\mathcal{B}$ and $\mathcal{A}$ respectively.

Two structures which are effective bi-interpretable share essentially all the same computability-theoretic properties (see [2, Lemma 5.3]).

3.3 Gaifman's Theorem

The proof of Theorem 1.1 will rely on the local nature of first-order logic, and in particular, Gaifman's result that every formula in the language of graphs can be replaced by one that only looks at subgraphs of a certain bounded radius.

The *Gaifman graph* $\mathcal{G}(\mathcal{A})$ of an $\mathcal{L}$-structure $\mathcal{A}$ is the undirected graph (A, E) where there is an edge between u and v if there is a tuple $\bar{w}$ containing u and v, and a relation $R \in \mathcal{L}$, such that $\bar{w} \in R$. If $\mathcal{A}$ is itself an undirected graph—or a graph together with unary relations—then $\mathcal{A} = \mathcal{G}(\mathcal{A})$. This will always be the case in this paper, so from now on we will just talk about a graph G.

The *distance* $d(a, b)$ is the length of the shortest path from a to b in G. Given a tuple $\bar{a} = (a_1, \ldots, a_\ell) \in G$, and $r \in \mathbb{N}$, the *r-sphere* centered at $\bar{a}$ in G is

$$S^G_r(\bar{a}) = \{b \in G \mid d(a_i, b) \leq r \text{ for some } i\}.$$

For each fixed r, there are first-order formulas that say "$d(x, y) > r$", "$d(x, y) = r$", and "$d(x, y) < r$". So the bounded quantifiers of the form $\forall x \in S_r(\bar{y})$ and $\exists x \in S_r(\bar{y})$ are expressible in first-order logic.

A formula $\varphi(\bar{x})$ is called *r-local* around $\bar{x}$ if every quantifier that appears in it is of the form $\forall y \in S_k(\bar{x})$ or $\exists y \in S_k(\bar{x})$ for $k \leq r$. A sentence ψ is called *basic r-local* if it is of the form

$$\exists x_1 \ldots \exists x_m \left(\bigwedge_{1 \leq i \leq m} \varphi(x_i) \quad \wedge \quad \bigwedge_{1 \leq i \leq j \leq m} d(x_i, x_j) > 2r \right)$$

where $\varphi(x)$ is an r-local formula around x.

Theorem 3.5 (Gaifmain [20])**.** *Every first-order sentence is equivalent to a Boolean combination of basic r-local sentences, and every first-order formula $\varphi(x_1, \ldots, x_n)$ is equivalent to a Boolean combination of r-local formulas around $x_1, \ldots, x_n$ and basic r-local sentences, for some r.*

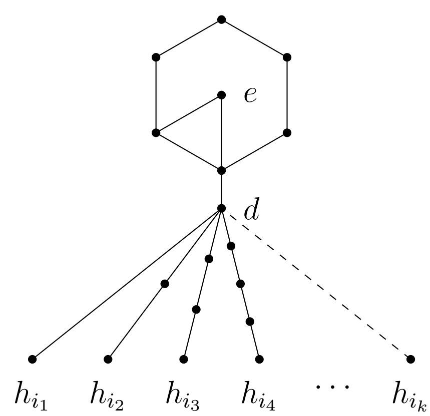
Gaifman gave bounds on r but we will not need them.

4 Proof of the Main Theorem

The construction, from a structure $\mathcal{A}$, of the graph $G(\mathcal{A})$ will be the same as Montalbán's construction in [21]; similar constructions appear in Appendix A of [1] and Theorem 5.5.1 of [3]. However, we must see what happens to the elementary diagram. The proof of Theorem 1.1 will be in two steps. First, we build from $\mathcal{A}$ a graph together with a unary relation. Now that we are in a finite language, we can use a model-theoretic interpretation to eliminate the unary relation.

4.1 First Step: Constructing a Graph with a Unary Relation

Definition 4.1. Let $\mathcal{A} = (A, P_0, P_1, \ldots)$ be a structure. We define a structure $H(\mathcal{A})$ which is a graph together with a unary relation U as follows. Let $a_0, a_1, \ldots$ be an enumeration of the elements of A. $H(\mathcal{A})$ will have corresponding elements $h_0, h_1, h_2, h_3, \ldots$. The unary relation U picks out these elements $h_0, h_1, \ldots$ which correspond to the domain of $\mathcal{A}$. For each tuple $a_{i_1}, \ldots, a_{i_k}$ satisfying the k-ary relation P_n, we attach the following configuration to $h_{i_1}, \ldots, h_{i_k}$ where the top loop has a length of $2n+4$. We call such a configuration an *m-tag*, where m is the length of the top loop. The configuration shown is a 6-tag.



For each tuple $a_{i_1}, \ldots, a_{i_k}$ not satisfying P_n, we attach a $(2n+5)$-tag to $h_{i_1}, \ldots, h_{i_k}$. We call the node labeled d the *central node* of the tag.

Standard arguments (see [21] or Theorem 5.5.1 of [3]) show that $\mathcal{A}$ and $H(\mathcal{A})$ are effectively bi-interpretable. We will not give the argument in full detail here, but the following is a sketch.

Claim 4.2. *$\mathcal{A}$ and $H(\mathcal{A})$ are effectively bi-interpretable.*

Proof sketch. The interpretation of $\mathcal{A}$ in $H(\mathcal{A})$ is as follows. The domain of the interpretation is U. The equivalence relation is just equality. Given $a_1, \ldots, a_k \in U$ and a relation R_m of arity k, $a_1, \ldots, a_m$ satisfy R_m if there is a $(2m+4)$-tag attached to them, and they do not satisfy R_m if there is a $(2m+5)$-tag attached to them. Note that this interpretation only uses 1-tuples and the relations of $\mathcal{A}$ are definable using both an existential and a universal first-order formula.

The interpretation of $H(\mathcal{A})$ in $\mathcal{A}$ is more complicated and uses tuples of arbitrary sizes. The domain of $H(\mathcal{A})$ consists of the elements $(a_i)_{i\in\omega}$ together with the elements which make up the tags; using the coding of ω in tuples from $\mathcal{A}$ of arbitrary size, for the tag attached to $h_{i_1}, \ldots, h_{i_k}$ for the sake of P_j, we can use tuples $(j, n, a_{i_1}, \ldots, a_{i_k})$ with $n \in \omega$.

To see that this is a bi-interpretation, we note that the copy of $\mathcal{A}$ in $H(\mathcal{A})$ in $\mathcal{A}$ is just $\mathcal{A}$ itself. The copy of $H(\mathcal{A})$ in $\mathcal{A}$ in $H(\mathcal{A})$ is more complicated, but it is not hard to write down a relatively intrinsically computably enumerable (r.i.c.e.) isomorphism between the two. It is for this that we use the node labeled e above, so that each element of each tag is uniquely defined by an existential formula. See [21] or Theorem 5.5.1 of [3] for details. $\square$

If the language of $\mathcal{A}$ is infinite, then $\mathcal{A}$ and $H(\mathcal{A})$ are not model-theoretically bi-intepretable because a first-order formula defining the edge relation of $H(\mathcal{A})$ can use only finitely many symbols of the language of $\mathcal{A}$. If the language is finite, then $\mathcal{A}$ and $H(\mathcal{A})$ are model-theoretically bi-interpretable.

We will now show that we can compute the elementary diagrams of $H(\mathcal{A})$ and $\mathcal{A}$ from each other. We may assume that the language of $\mathcal{A}$ includes infinitely many relations of each arity; if it does not, we can add them in (interpreting them as the empty set). We can also assume that the arity of each relation in $\mathcal{A}$ is even, by replacing each relation of odd arity by a relation of arity one more which does not depend at all on the last entry. Let $\mathcal{A}\restriction_m$ be the reduct of $\mathcal{A}$ to the first m relation symbols.

One direction is easy:

Claim 4.3. *Using the elementary diagram of $H(\mathcal{A})$, we can compute the elementary diagram of $\mathcal{A}$.*

Proof. Fix a tuple $\bar{a} = (a_{i_1}, \ldots, a_{i_k}) \in \mathcal{A}$ and an elementary first-order formula φ. We want to decide whether $\mathcal{A} \models \varphi(\bar{a})$. Let m be such that φ uses only the relation $P_0, \ldots, P_m$. Then $\mathcal{A} \models \varphi(\bar{a})$ if and only if $\mathcal{A}\restriction_m \models \varphi(\bar{a})$. The interpretation of $\mathcal{A}$ in $H(\mathcal{A})$ induces a model-theoretic interpretation of $\mathcal{A}\restriction_m$ in $H(\mathcal{A})$. Then, by the Reduction Theorem, there is an elementary first-order formula ψ such that

$$\mathcal{A}\restriction_m \models \varphi(a_{i_1}, \ldots, a_{i_k}) \iff H(\mathcal{A}) \models \psi(h_{i_1}, \ldots, h_{i_k}).$$

We can decide this using the elementary diagram of $H(\mathcal{A})$. $\square$

The other direction is more complicated. For $m \in \omega$, we will define two new operators $H^*_m(\mathcal{A})$ and $H^{**}_m(\mathcal{A})$ which depend only on $\mathcal{A}\restriction_m$.

Definition 4.4. $H^*_m(\mathcal{A})$ consists of the elements $h_0, h_1, \ldots$ labeled with the unary predicate U together with the following tags:

1. tags for $P_0, \ldots, P_m$ just as in $H(\mathcal{A})$,
2. for each tuple $h_{i_1}, \ldots, h_{i_k}$, $k \leq 2m$, we add infinitely many $(2m+6)$-tags,
3. for each tuple $h_{i_1}, \ldots, h_{i_{2m}}$, and each $\ell \in \omega$, we add infinitely many $(2m+6)$-tags each of which has ℓ loops of length $4m$ attached to the central node (labeled d in the picture above) of the tag.

For each element h_i, we also add infinitely many loops of length $4m$ attached to h_i.

Definition 4.5. $H_m^{**}(\mathcal{A})$ is defined in the same way as $H_m^*(\mathcal{A})$, except that in (3) we restrict to $\ell \leq m$.

The idea is that we will have that $(H_m^*(\mathcal{A}), \{h_i\}_{i\in\omega}) \equiv_m (H_m^{**}(\mathcal{A}), \{h_i\}_{i\in\omega})$ (Claim 4.6), i.e., the h_i satisfy the same formulas of quantifier rank m in $H_m^*(\mathcal{A})$ as they do in $H_m^{**}(\mathcal{A})$. Then we will show that $H_m^{**}(\mathcal{A})$ is model-theoretically interpretable in $\mathcal{A}$ (Claim 4.7) and that $H_m^*(\mathcal{A})$ is locally equivalent to $H(\mathcal{A})$ (Claim 4.8).

Claim 4.6. $(H_m^*(\mathcal{A}), \{h_i\}_{i\in\omega}) \equiv_m (H_m^{**}(\mathcal{A}), \{h_i\}_{i\in\omega})$.

Proof. First note that a $(2m+6)$-tag with m loops of length $4m$ is $\equiv_m$-equivalent to a $(2m+6)$-tag with more than m loops of length $4m$. One sees this by playing m rounds of an Ehrenfeucht-Fraïssé game. Then, one can put together the strategies for each of these games into a strategy for $(H_m^*(\mathcal{A}), \{h_i\}_{i\in\omega})$ and $(H_m^{**}(\mathcal{A}), \{h_i\}_{i\in\omega})$. $\square$

Claim 4.7. *$H_m^{**}(\mathcal{A})$ is model-theoretically interpretable in $\mathcal{A}$.*

Proof. First, given a bound ℓ, we will describe how to code numbers $i \leq \ell$ into $(\ell+2)$-tuples from $\mathcal{A} \upharpoonright_\ell$. We can identify a number $i \leq \ell$ with the equivalence class of $(\ell+2)$-tuples of the form $(\underbrace{a, \ldots, a}_{i+1 \text{ times}}, b, \ldots)$ where $b \neq a$, and we can identify such a tuple with the corresponding natural number i using first-order formulas.

Let r be an upper bound on the arity of the relations $P_0, \ldots, P_m$. In the interpretation, the largest natural numbers we will need to encode are 8, r, and $4m+1$. Then the domain of the interpretation consists of the $5 \cdot \max(8, r, 4m+1)$-tuples of the following forms:

- $(0, a, \ldots)$;
- $(1, n, a_1, \ldots, a_k, i, j, \ldots)$ where $0 \leq n \leq m$, k is the arity of P_n, $1 \leq i \leq k$, and $0 \leq j < i-1$;
- $(2, n, a_1, \ldots, a_k, i, \ldots)$ where $0 \leq n \leq m$, k is the arity of P_n, and $i \leq 2n+5$ (if P_n holds of $a_1, \ldots, a_k$) or $i \leq 2n+6$ (if P_n does not hold of $a_1, \ldots, a_k$);
- $(3, k, b, a_1, \ldots, a_k, i, j, \ldots)$ where $k \leq 2m$, $b \in A$, $1 \leq i \leq k$, and $0 \leq j < i-1$;
- $(4, k, b, a_1, \ldots, a_k, i, \ldots)$ where $k \leq 2m$, $b \in A$, $i \leq 2m+7$;
- $(5, \ell, b, a_1, \ldots, a_{2m}, i, \ldots)$ where $\ell \leq m$, $b \in A$, $1 \leq i \leq k$, and $0 \leq j < i-1$;
- $(6, \ell, b, a_1, \ldots, a_{2m}, i, \ldots)$ where $\ell \leq m$, $b \in A$, $i \leq 2m+7$;
- $(7, \ell, j, b, a_1, \ldots, a_{2m}, i, \ldots)$ where $\ell \leq m$, $j < \ell$, $b \in A$, and $1 \leq i < 4m$;
- $(8, b, a, i, \ldots)$ where $a, b \in A$, $1 \leq i < 4m$.

The relation $\sim$ holds between different tuples of the same form. The relation U holds of those tuples $(0, a, \ldots)$; $(0, a_i, \ldots)$ represents the element h_i. The elements with first entry 1 or 2 are used for the tags which code the relation P_n; we show the edge relations between these elements in Figure 1. The elements with first entry 3 or 4 are used for the $(2m+6)$-tags in a similar way (the element b is just used to get infinitely many of these, one for each $b \in A$). The elements with first entry 5, 6, or 7 are used for the $(2m+6)$-tags which have loops of length $4m$ (the elements beginning with 7 are used for the loops). Finally, the elements with first entry 8 are used for the loops of length $4m$ attached to each element h_a representing a (with b again ensuring that there are infinitely many loops for each h_a). This defines a copy of $H_m^{**}(\mathcal{A})$ inside of $\mathcal{A}$, and the edge relations are all definable by elementary first-order formulas. $\square$

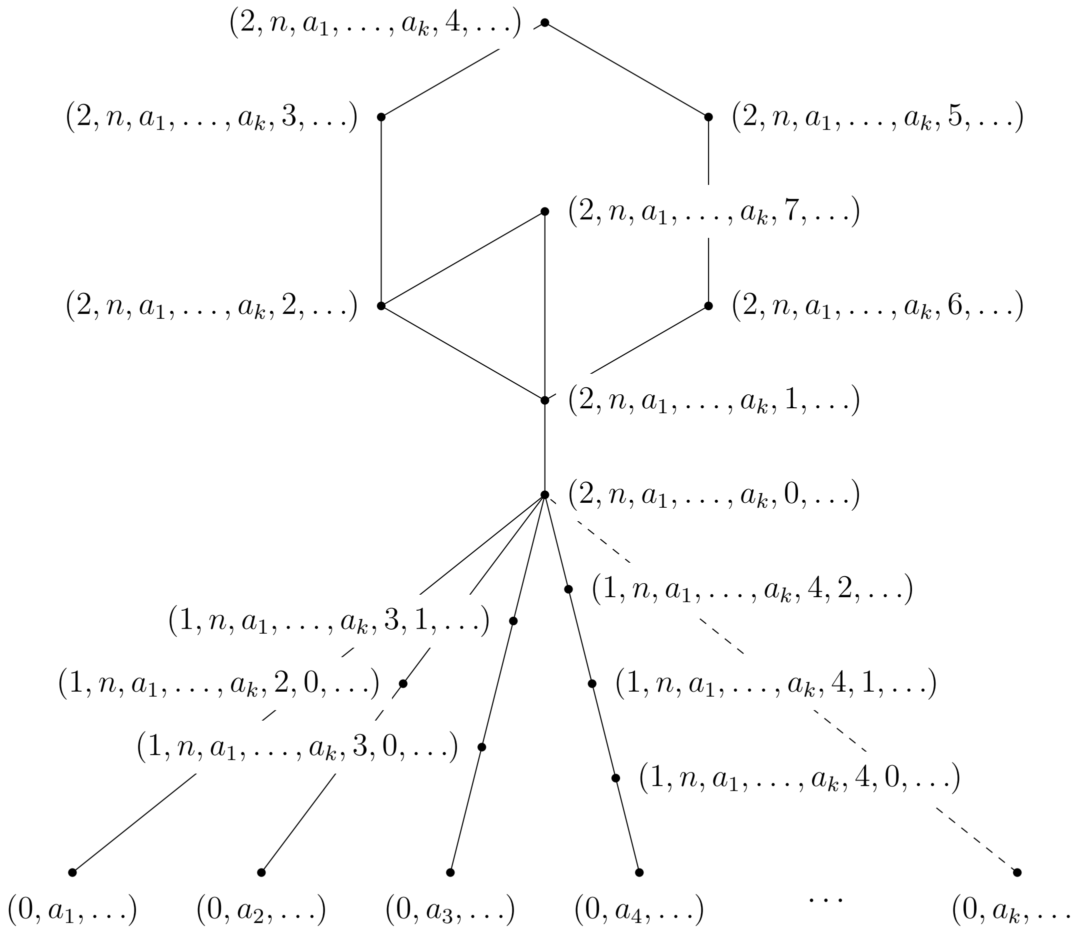


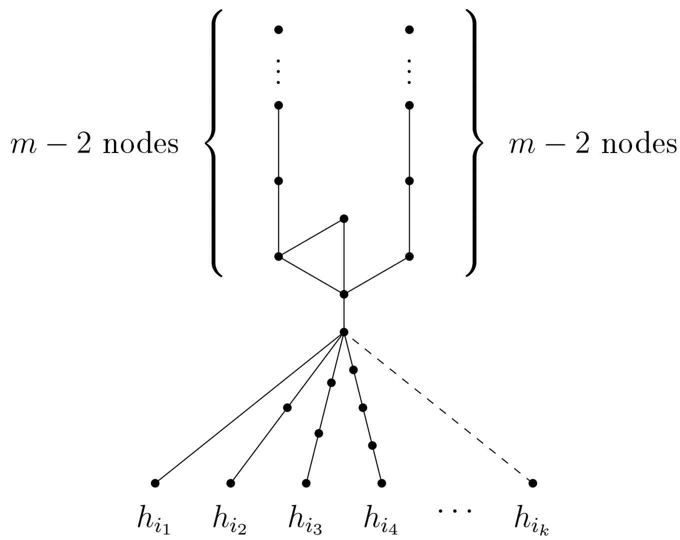
Figure 1: Interpretation of a tag which codes $P_n(a_1, \dots, a_k)$, $n \leq m$.

Claim 4.8. *Fix $m = 2r + 10 \in \omega$. Then:*

1. *$H(\mathcal{A})$ and $H^*_m(\mathcal{A})$ satisfy the same r-local sentences.*
2. *Each tuple $\bar{h} = (h_1, \dots, h_k)$ satisfies the same r-local formulas in $H(\mathcal{A})$ and $H^*_m(\mathcal{A})$.*

Proof. Recall that $S_m^{H(\mathcal{A})}(\{h_i\}_{i\in\omega})$ and $S_m^{H^*_m(\mathcal{A})}(\{h_i\}_{i\in\omega})$ are the m-spheres of the set $\{h_i\}_{i\in\omega}$ in $H(\mathcal{A})$ and $H^*_m(\mathcal{A})$ respectively, treated as substructures of $H(\mathcal{A})$ and $H^*_m(\mathcal{A})$ respectively. These m-spheres are isomorphic via an isomorphism which is the identity on $\{h_i\}_{i\in\omega}$. We build such an isomorphism as follows:

- Each i-tag in $H(\mathcal{A})$, $i \leq 2m + 5$, gets mapped to the corresponding i-tag in $H^*_m(\mathcal{A})$.
- Each i-tag attached to $h_{i_1}, \dots, h_{i_k}$ in $H(\mathcal{A})$, $k \leq 2m$ and $i \geq 2m + 6$, has only the following portion within distance m of the set $\{h_i\}_{i\in\omega}$:



So we can map this portion of any i-tag, $i \geq 2m+6$, to the corresponding portion of a $(2m+6)$-tag attached to $h_{i_1}, \ldots, h_{i_k}$ in $H^*_m(\mathcal{A})$, and vice versa. (Note that we use here the fact that $\mathcal{A}$ has infinitely many relations of each arity in its signature.)

- For an i-tag attached to $h_{i_1}, \ldots, h_{i_k}$ in $H(\mathcal{A})$, with $k > 2m$ and $i \geq 2m+6$, let d be the central node of the tag. For each j, there is a chain of length j between h_{i_j} and d. But if $j > 2m$, part of this chain is not within the m-sphere around $\{h_i\}_{i\in\omega}$. All that is left is a chain of length m from h_{i_j} and a chain of length m from d for each of $2m+1, \ldots, k$. We assumed above that the arity of each relation is even, and so k is even, and there are an even number of such chains from d.

  In $H^*_m(\mathcal{A})$, we have a $(2m+6)$-tag attached to $h_{i_1}, \ldots, h_{i_{2m}}$ with $(k-2m)/2$ loops of length $4m$ attached to the node d, as well as infinitely many loops of length $4m$ attached to $h_{i_{2m+1}}, \ldots, h_{i_k}$. For $j > 2m$, we can map the chain starting at h_{i_j} in $H(\mathcal{A})$ to one side of one of the loops attached to h_{i_j} in $H^*_m(\mathcal{A})$, and we can map the chains from d in $H(\mathcal{A})$ to either side of the loops attached to d in $H^*_m(\mathcal{A})$. We map the rest of the tags to each other as in the previous case.

It is easy to see from this that each tuple $\bar{h} = (h_1, \ldots, h_k)$ satisfies the same r-local formulas in $H(\mathcal{A})$ and $H^*_m(\mathcal{A})$, as the r-spheres $S_r^{H(\mathcal{A})}(\bar{h})$ and $S_r^{H^*_m(\mathcal{A})}(\bar{h})$ centered at $\bar{h}$ are isomorphic.

An r-local sentence is of the form

$$\exists x_1 \ldots \exists x_\ell \left(\bigwedge_{1\leq i\leq \ell} \varphi(x_i) \quad \wedge \quad \bigwedge_{1\leq i\leq j\leq \ell} d(x_i, x_j) > 2r = m - 10 \right),$$

where $\varphi(x)$ is an r-local formula around x. Note that in both $H(\mathcal{A})$ and $H^*_m(\mathcal{A})$, an element is either within distance r of the set $\{h_i\}_{i\in\omega}$, or its r-sphere is isomorphic to a chain of length $2r$. If φ is satisfied by an element whose r-sphere is isomorphic to a chain of length $2r$, then this r-local sentence is satisfied by both $H(\mathcal{A})$ and $H^*_m(\mathcal{A})$ as they both have infinitely many such elements at distance greater than $m = 2r$ from each other (we can pick such points on distinct i-tags, $i \geq 2m+6$). On the other hand, if φ is not satisfied by such an element, then any witnesses to $c_1, \ldots, c_m$ must be within distance $r+5$ of the set $\{h_i\}_{i\in\omega}$; the r-sphere of such elements is contained within the m-sphere of $\{h_i\}_{i\in\omega}$, and m-spheres of $\{h_i\}_{i\in\omega}$ in $H(\mathcal{A})$ and $H_m^*(\mathcal{A})$ are isomorphic. $\square$

Now we will show that given the elementary diagram of $\mathcal{A}$, we can compute the elementary diagram of $H(\mathcal{A})$.

Claim 4.9. *Using the elementary diagram of $\mathcal{A}$ we can compute the elementary diagram of $H(\mathcal{A})$.*

Proof. Given φ and $\bar{g} \in H(\mathcal{A})$, we must decide effectively whether $H(\mathcal{A}) \models \varphi(\bar{g})$. Since each element of $\bar{g}$ is definable from some tuple $h_{i_1},\dots,h_{i_k}$ in a first-order way, we may replace $\bar{g}$ with $\bar{h} = (h_{i_1},\dots,h_{i_k})$, and φ by some other formula (which we will also denote φ). We can do this effectively.

By Gaifman's Theorem, Theorem 3.5, $\varphi(x_1,\dots,x_k)$ is equivalent to a Boolean combination of r-local formulas around $x_1,\dots,x_k$ and basic r-local sentences for some r; we can search for such a Boolean combination. Fix m larger than $2r+10$ and the quantifier rank of φ. By the Claim 4.8, $H(\mathcal{A}) \models \varphi(\bar{h})$ if and only if $H_m^*(\mathcal{A}) \models \varphi(\bar{h})$. By Claim 4.6, $H_m^*(\mathcal{A}) \models \varphi(\bar{h})$ if and only if $H_m^{**}(\mathcal{A}) \models \varphi(\bar{h})$. By Claim 4.7, $H_m^{**}(\mathcal{A})$ is model-theoretically interpretable in $\mathcal{A}$, and so by the Reduction Theorem, Theorem 3.2, there is a formula ψ such that,

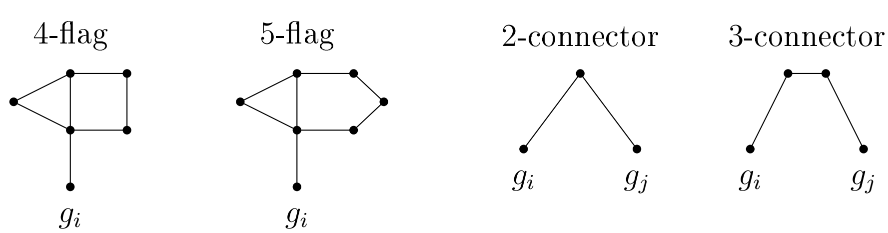
$$H_m^{**}(\mathcal{A}) \models \varphi(h_{i_1},\dots,h_{i_k}) \Longleftrightarrow \mathcal{A} \models \psi(a_{i_1},\dots,a_{i_k}).$$

We can decide this effectively using the elementary diagram of $\mathcal{A}$. $\square$

4.2 Second Step: Eliminating the Unary Relation

Given a graph H and a unary relation U, we will define a graph $G(H,U)$ which is effectively bi-interpretable with (H,U). Moreover, the interpretations will be model-theoretic and so by the Reduction Theorem (Theorem 3.2), we can compute the elementary diagram of $G(H,U)$ from that of (H,U) and vice versa.

Definition 4.10. Suppose that H is a symmetric irreflexive graph and U is a unary relation on H. A graph $G(H,U)$ is defined as follows. Let $h_0,h_1,h_2,\dots$ be an enumeration of the elements of H. The graph $G(H,U)$ will have corresponding elements $g_0,g_1,g_2,\dots$. If $h_i \in U$, then we attach a 4-flag to g_i (see the picture below). Otherwise, attach a 5-flag to g_i. If there is an edge from h_i to h_j in H, then connect g_i and g_j with a 2-connector. If there is no edge from h_i to h_j, then we connect g_i and g_j with a 3-connector.



Lemma 4.11. *The structures (H,U) and $G(H,U)$ are effectively bi-interpretable. Moreover, the interpretations are model-theoretic.*

Proof. Notice that any element x from $G(H,U)$ satisfies exactly one of the following three conditions:

1. x is attached to a flag (i.e. $x = g_i$ for some i);
2. x is a part of a flag;
3. x is connected to an element which is, in turn, attached to a flag (i.e. x is a part of a connector).

Each of the conditions is definable by an existential formula.

Therefore, the interpretation of (H,U) in $G(H,U)$ is defined as follows. The domain of the interpretation is the set of all x which are attached to a flag. The equivalence relation is equality. An element g_i satisfies U iff it is attached to a 4-flag. There is an edge from g_i to g_j in the interpretation iff they are connected by a 2-connector in $G(H,U)$. It is not difficult to show that this interpretation is definable by both an existential and a universal formula.

Now we describe a model-theoretic interpretation of $G(H,U)$ in (H,U). As in Claim 4.7, we can use an encoding of natural numbers into tuples from (H,U). The domain of the interpretation consists of the tuples of the following forms:

- $(0, a, \dots)$;
- $(1, a, i)$, where $i \leq 4$ (if a satisfies U) or $i \leq 5$ (if a does not satisfy U);
- $(2, a, b, i)$, where $i = 0$ (if there is an edge from a to b in H) or $i \leq 1$ (otherwise).

Note that the domain can be encoded using 13-tuples from (H,U): the first four coordinates encode a natural number $k \leq 2$, the fifth and the sixth entries are arbitrary a and b from H, and the last seven coordinates encode a natural number $i \leq 5$.

The equivalence relation $\sim$ holds between different tuples of the same form. The tuples $(0, a, \dots)$, $a \in H$, represent the elements g_k, $k \in \omega$, from $G(H,U)$. The tuples $(1, a, i)$ are used for the flags: for a fixed $a \in H$, the elements $(1, a, i)$ form the flag which is attached to g_k that is represented by $(0, a, \dots)$. The tuples $(2, a, b, i)$ are used for the connectors in a natural way. This defines a copy of $G(H,U)$ inside (H,U), and all required relations are definable by first-order formulas.

Notice that a copy of (H,U) in $G(H,U)$ in (H,U) is essentially just (H,U). Moreover, it is straightforward to show that there is a r.i.c.e isomorphism between $G(H,U)$ and a copy of $G(H,U)$ in (H,U) in $G(H,U)$. Hence (H,U) and $G(H,U)$ are effectively bi-interpretable. $\square$

This concludes the proof of Theorem 1.1.

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