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THE LEADING EDGE OF A SURFACE FILM
ON CONTAMINATED FLOWING WATER

by

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SUMMARY

Trace impurities often collect on the upstream side of obstacles in the surface of flowing water. The transition from apparently free surface to heavily polluted surface can be quite sharp, and changes in the surface level have been observed there. We treat theoretically the coupled problems of the viscous boundary layer and the convective mass transfer of surface-active material, and show how the experimentally measurable surface slope depends on the speed of the flow underneath.

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1. INTRODUCTION

Some years ago, Sellin (1) observed a small dip at the upstream edge of a region where the surface of flowing water was contaminated. Mockros and Krone (2) observed a ripple elevated slightly at the leading edge. In the hope of resolving the apparent conflict, and also of providing a method for measuring surface contamination dynamically, we analyze theoretically a model of the flow which is idealized enough for simple calculation but realistic enough to be interesting. Our theory is developed from that in Dixon's thesis (3).

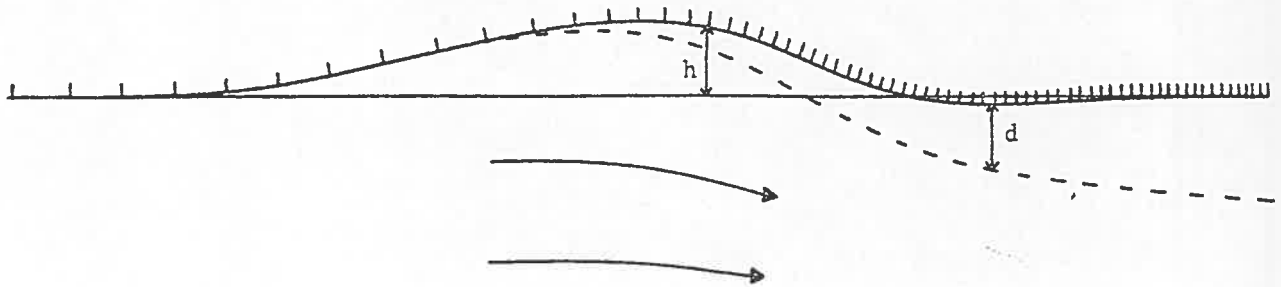


Figure 1. The flow configuration (vertically exaggerated). The degree of surface contamination is indicated by schematic "adsorbed molecules". Flow is from left to right. d = displacement thickness measured from the fluid surface, h = elevation of the surface above a horizontal reference plane.

Assuming that the flow indicated in Figure 1 is two-dimensional and steady, and consists of an unbounded uniform stream of speed U under a laminar viscous boundary layer at the surface, we may write its equations of motion and continuity as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (1.1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1.2)$$

where u , v are the velocity components in the direction of x along the surface and y perpendicularly into the fluid, ν is the kinematic viscosity, and we have ignored second-order effects due to surface curvature. Because vorticity diffuses much faster than surface contaminants, we may neglect diffusive mass transfer in a first approximation to the theory describing the transition region between (i) the upstream "free" surface (where $u \rightarrow U$, the displacement thickness $d \rightarrow 0$, and the surface pressure Π of the contaminant $\rightarrow \Pi_f$), and (ii) the downstream "clogged" surface where $u = o(U)$. If also the surfactant is thermodynamically ideal, so that its surface pressure is proportional to surface excess (4), the continuity equation for the surfactant is

$$\Pi u = \text{constant} = \Pi_f U \quad (1.3)$$

on the surface. Finally, the surface tension gradient must balance the surface shear stress in the liquid, and because the surface tension is $\sigma_p - \Pi$ where σ_p is the surface tension of uncontaminated liquid, we have

$$\frac{\partial \Pi}{\partial x} = \mu \frac{\partial u}{\partial y}, \quad (1.4)$$

to first order on $y = 0$, μ being the dynamic viscosity.

2. THE BOUNDARY LAYER

The problem described by equations (1.1) to (1.4) is made dimensionless by the use of variables u^* , v^* , x^* , y^* and a Reynolds number R , defined by

$$u = Uu^*, v = UR^{-1/2}v^*, x = \alpha x^*, y = \alpha R^{-1/2}y^*, \quad (2.1)$$

$$\text{where } \alpha = \nu R/U \text{ and } R = (\Pi_f/\mu U)^2 . \quad (2.2)$$

We also let the dimensionless displacement thickness d^* be $dR^{1/2}/\alpha$. Our boundary-layer analysis is valid if $R \gg 1$.

The field equations become

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 , \quad (2.3)$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}} , \quad (2.4)$$

with boundary conditions $u^* \rightarrow 1$, $v^* \rightarrow 0$ at $y^* \rightarrow \infty$, while $v^* = 0$ and

$$\frac{\partial u^*}{\partial x^*} = -u^{*2} \frac{\partial u^*}{\partial y^*} , \quad (2.5)$$

on $y^* = 0$. Equation 2.5 is found by eliminating Π between (1.3) and (1.4).

Far upstream, u^* is close to unity for all y^* , and so we may linearize to an Oseen flow, in which we find

$$u^* \sim 1 - \epsilon \exp(x^* - y^*) , \quad (2.6)$$

$$v^* \sim \epsilon \exp(x^* - y^*) , \quad (2.7)$$

$$d^* \sim \epsilon \exp x^* , \quad (2.8)$$

and, from (1.3),

$$\Pi \sim \Pi_f(1 + \epsilon \exp x^*) , \quad (2.9)$$

where ϵ is an arbitrary constant and $\epsilon \exp x^*$ must be small for the linearization to be valid. Far downstream, $u^* \rightarrow 0$, equation (2.5) becomes nugatory, and the flow tends to Blasius's well-known flow past a flat plate, for which

$$d^* \sim 1.7208 (x^* + k)^{1/2} , \quad (2.10)$$

where k is another arbitrary constant.

To obtain some idea of the transition region, one can use a type of von Kármán-Pohlhausen approximation

$$u^* = 1 - A(x^*) \exp\{-y^*/\delta(x^*)\} , \quad (2.11)$$

determining A and δ from (2.5) which gives

$$\frac{dA}{dx^*} = (1-A)^2 \frac{A}{\delta} , \quad (2.12)$$

and from von Kármán's equation (5)

$$\frac{\partial u^*}{\partial y^*} = \frac{\partial}{\partial x^*} \int_0^\infty (1-u^*)u^* dy^* ,$$

which gives

$$\frac{A}{\delta} = \frac{d}{dx^*} (A\delta - \frac{1}{2}A^2\delta) . \quad (2.13)$$

It can now be shown from (2.12) and (2.13) that x^* and d^* ($=A\delta$) are the following functions of $B = 1-A$:

$$x^* = (1/B^2) + \ln \{(1/B^2) - 1\} , \quad (2.14)$$

$$d^* = 2(1-B)/\{B(1+B)\} . \quad (2.15)$$

In the upstream free region, $B \rightarrow 1$ and $d^* \sim \frac{1}{2}\exp(x^*-1)$ from these equations, in agreement with (2.8)

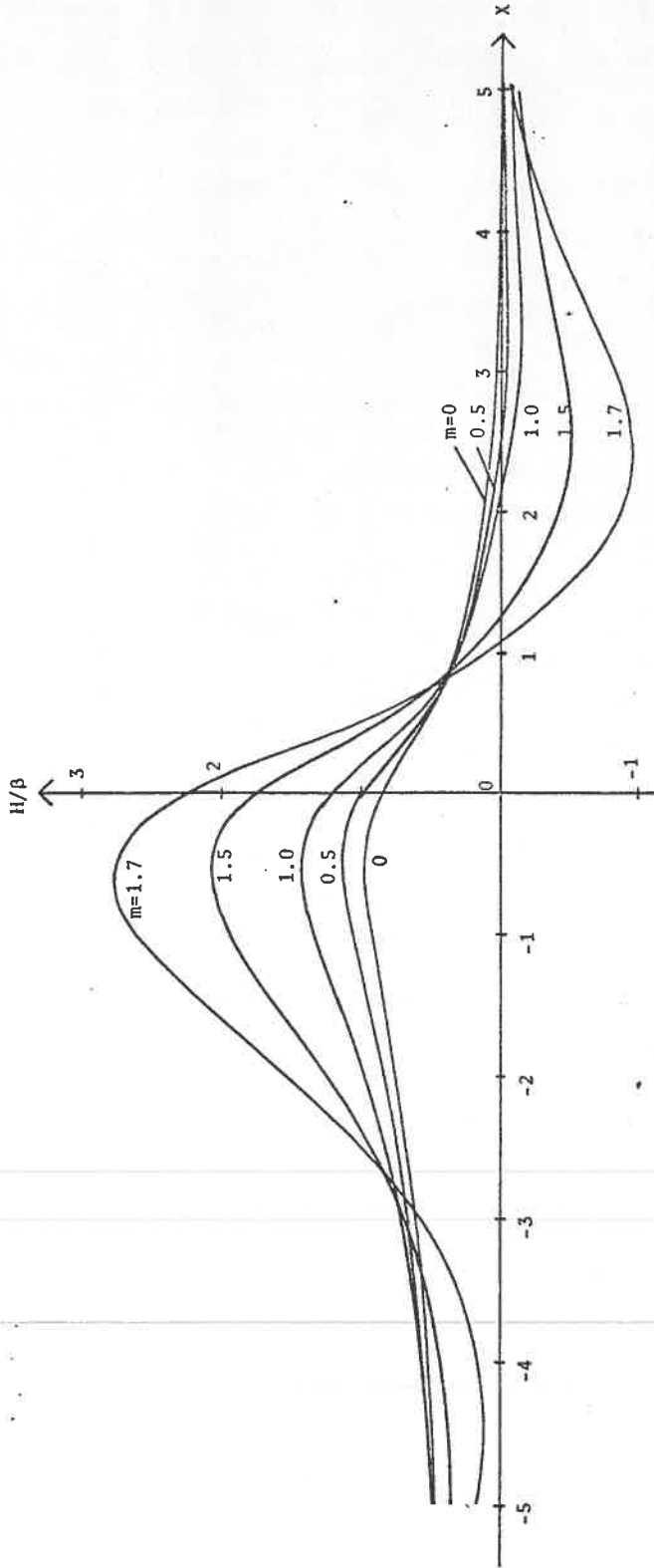


Figure 2. Graphs of dimensionless surface elevation $H(X)/\beta$ as a function of X , for several values of m .

in the downstream clogged region $B \rightarrow 0$ and $d^* \sim 2x^{*\frac{1}{2}}$, in rough agreement with (2.10). The transition occurs over a dimensionless distance of order unity, i.e. a physical distance of order α (see equation 2.1).

3. THE SURFACE ELEVATION

On our hypothesis of a high Reynolds number R and small surface slopes, we may assume that the pressure is constant across the surface boundary layer (apart from the hydrostatic term), and use Bernoulli's equation and thin-aerofoil theory to find its variation along the layer. Then the pressure p_i just under the fluid surface obeys

$$p_i/p + \frac{1}{2}q^2 + gh = \text{constant} \quad , \quad (3.1)$$

where g is the acceleration due to gravity, p is the fluid density, and q is the speed of the irrotational flow past the displacement surface shown dotted in Figure 1. Hence (6)

$$q^2 \doteq U^2 \left(1 + \frac{2}{\pi} \oint_{-\infty}^{\infty} \frac{d'(t) - h'(t)}{x-t} dt \right) \quad , \quad (3.2)$$

where the symbol \oint denotes the Cauchy principal value of the integral. To the first order, the Young-Laplace equation for the pressure difference across the surface is

$$p_i(x) = p_0 - \sigma h''(x) \quad , \quad (3.3)$$

where p_0 is the atmospheric pressure and σ the surface tension (variations in which are insignificant in the neighbourhood of the surface transition). With the boundary conditions that $p_i \rightarrow p_0$, $q \rightarrow U$, $h \rightarrow 0$ far upstream, we obtain from (3.1), (3.2), and (3.3) a dimensionless linear integro-differential equation for h in the form

$$\frac{d^2H}{dx^2} - H + \frac{m}{\pi} \int_{-\infty}^{\infty} \frac{H'(T)dT}{X-T} = \frac{m}{\pi} \int_{-\infty}^{\infty} \frac{D'(T)dT}{X-T} \quad , \quad (3.4)$$

where $h = \gamma H$, $x = \gamma X$, $t = \gamma T$, $d = \gamma D$, γ being the capillary length scale $(\sigma/\rho g)^{\frac{1}{2}}$, and the parameter m is given by

$$m = (\rho U^4/\sigma g)^{\frac{1}{2}} \quad . \quad (3.5)$$

Thus m is proportional to the square of the speed, and $m \geq 2$, or $U \geq 0.23 \text{ m s}^{-1}$, is the condition that stationary waves be possible on the surface, if $\rho = 1 \text{ Mg m}^{-3}$, $\sigma = 72 \text{ mN m}^{-1}$, $g = 9.81 \text{ m s}^{-2}$.

It transpires that the capillary length scale γ must have been much greater than the viscous length scale α in Sellin's experiments, and so we need only use the asymptotic forms (2.8) and (2.10) for the displacement thickness, instead of better approximations such as the implicit equations (2.14) and (2.15). The right-hand side of (3.4) then takes the simple form of $-\beta(-X)^{-\frac{1}{2}}$ for $X < 0$, 0 for $X > 0$, where

$$\beta = 0.8604 m(v/U\gamma)^{\frac{1}{2}} \quad . \quad (3.6)$$

Equation (3.4) is readily solved by Fourier integration. If we put

$$H(X) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} V(w) \exp(iwX) dw \quad , \quad (3.7)$$

and use Titchmarsh's (7) results for the Fourier transform of a Hilbert transform, we find that

$$V(w) = \frac{\frac{1}{2}\beta\{1 + i \operatorname{sgn}(w)\}}{|w|^{\frac{1}{2}}\{1 - mw \operatorname{sgn}(w) + w^2\}} \quad , \quad (3.8)$$

and hence

$$H(X) = \frac{2\beta}{\pi^{\frac{1}{2}}} \int_0^{\infty} \frac{\cos(t^2X + \frac{1}{2}\pi)}{1 - mt^2 + t^4} dt \quad , \quad (3.9)$$

provided that there are no eigensolutions of (3.4) bounded both as $X \rightarrow \infty$ and as $X \rightarrow -\infty$. If $0 < m < 2$ there are none, and if $m > 2$ there are just two eigensolutions, which correspond to the two surface waves of velocity U on still water, i.e. the waves at rest on our stream of velocity U . Sellin (1) ensured that $m < 2$ in his experiments. In that case the integral (3.9) can be computed numerically; graphs of $H(X)/\beta$ are shown in Figure 2 for several values of m . In every case the surface rises gradually to a maximum height a short distance upstream from the transition at

$X = 0$, and then falls more steeply. Presumably Sellin (who measured surface slopes) did not detect the rise preceding the more obvious dip. He did find, in agreement with Figure 2, that the curvature was sharper at the upper end of the dip than at the lower. As $m \rightarrow 2$ the amplitude becomes large and the surface shape tends towards sinusoidal, because of the approach to resonance with a standing surface wave. It is easy to see from (3.9) that at $X = 0$, $H = \beta\pi^{1/2}/(4-2m)^{1/2}$.

4. ESTIMATION OF U AND Π_f

In the above theory, the surface elevation depends only on fluid properties (ρ, g, ν, σ) and U . Sellin used tap water ($\nu = 10^{-6} \text{ m}^2\text{s}^{-1}$ presumably), and obtained a mean dip in surface level of 0.175 mm and a mean maximum slope of 0.041. He did not report values of U , but the best fit with our theory is obtained if $U = 0.157 \text{ m s}^{-1}$, so that $m = 0.93$, $\beta = 0.388$, the theoretical dip (from the highest point on the fluid surface to the lowest) is 0.157 mm, and the maximum slope is 0.046.

To estimate Π_f we recall that the flux of surfactant into the clogged surface is $\Pi_f U$, and in a steady state this must balance the flux out. The available data do not allow us to calculate Π_f with any precision, but one can make an order-of-magnitude estimate if turbulence is negligible and if the surfactant solution is ideal everywhere, so that $\Pi = R_g T \Gamma = R_g T \lambda c$, where R_g is the gas constant, T the absolute temperature, Γ the surface excess, c the surfactant concentration in the adjacent solution, and λ a constant length, the adsorption depth (3, 4, 8).

If we define Π_s to be $R_g T \lambda$ times the concentration in the bulk solution, if the surface pressure on the clogged part of the surface is of order Π_c , and if the clogged and free parts have lengths l_c , l_s and diffusion boundary-layer thicknesses δ_c , δ_s respectively, then

$$\delta_s^2 \sim l_s \kappa / U, \quad \delta_c^3 \sim l_c^2 \kappa / U, \quad (4.1)$$

where κ is the diffusivity of the surfactant (8, 9), and the flux condition is

$$\lambda \Pi_f U \sim \kappa \Pi_c l_c / \delta_c. \quad (4.2)$$

Similarly we find that

$$\lambda \Pi_f U \sim \kappa \Pi_s l_s / \delta_s \quad (4.3)$$

if l_s is too short (or λ too large) for the free part of the surface to have approached thermodynamic equilibrium, while $\Pi_f = \Pi_s$ if equilibrium has been reached. Also, equation (1.4) and flat-plate boundary-layer theory show that

$$\Pi_c \sim \rho (\nu U^3 l_c)^{1/2}. \quad (4.4)$$

In Sellin's experiments l_s and l_c were both of order 0.45 m. Let us also take $\kappa = 4 \times 10^{-10} \text{ m}^2\text{s}^{-1}$ a typical value for organic surfactants. Then $\Pi_c \sim 40 \text{ mN m}^{-1}$ (which strains the assumption of ideality, but is a feasible surface pressure), $\Pi_f \lambda \sim 60 \mu\text{N}$ from (4.2), and $\Pi_f = \Pi_s$ if $\lambda \ll 34 \mu\text{m}$, whereas (4.3) holds and $\Pi_s \sim 2 \text{ mN m}^{-1}$ if $\lambda \gg 34 \mu\text{m}$. A necessary condition for validity of (4.1) is that $\lambda \gg 3 \mu\text{m}$: otherwise Π_f is too large for there to be a sharp transition between free and clogged surface. Many surfactants exist, of course, with $\lambda \gg 3 \mu\text{m}$, and the more insoluble ones have $\lambda \gg 34 \mu\text{m}$.

5. CONCLUSIONS

Sellin's observations of the surface slope at the upstream end of a contaminated region seem to have been accounted for, but they do not give much information about surface contamination. More useful for that purpose would be measurements of the lengths l_s and l_c of the free and clogged parts of the surface over a range of speeds of flow.

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