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Defining continuity of real functions of real variables

J F Harper
Victoria University of Wellington, New Zealand

Continuity of a real function of a real variable has been defined in various ways over almost 200 years. Contrary to popular belief, the definitions are not all equivalent, because their consequences for four somewhat pathological functions reveal five essentially different cases. The four defensible ones imply just two cases for continuity on an interval if that is defined by using pointwise continuity at each point. Some authors had trouble: two different textbooks each gave two arguably inconsistent definitions, three more changed their definitions in their second editions, two more claimed continuity at a point for functions not defined there, and one gave a definition implying it for a function with no limit there.

Introduction

When called upon to teach real analysis some years ago, I was amazed to find that the definition of continuity in our textbook (Bartle and Sherbert 1992)[1] was significantly different from the one I had learnt as a student from Siddons *et al.* (1952) and Hardy (1952), even though that definition is at the root of the entire subject. Bolzano (1817) gave the first formal definition, and Cauchy's (1821)) is equivalent to it. But at least four more have been given since that are equivalent neither to theirs nor to one another. This is proved by seeing what the definitions imply for the continuity at $x = 0$ of four functions $f_0, f_{\mathbb{Q}}, f_+, f_S$, which are defined below.

When I was a child reading *Through the looking glass*, I thought Lewis Carroll was joking when he wrote '"When *I* use a word," Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean—neither more nor less."' I now believe he was issuing a warning; after all he was a professional mathematician keenly interested in logic.

We shall see in the section 'Continuity of four functions' that it is not only undergraduates who find continuity difficult. Duhamel (1841) gave a definition allowing a function to be continuous at a point where it had no limit, Goursat (1904) had an error that Hedrick corrected in a footnote, and Hobson (1907) had an equivalent error. Courant (1937) had two definitions that appear to conflict with each other. According to Siddons *et al.* (1952), a function can be continuous in a closed interval but not at its endpoints. The definitions in all three of Jordan (1882), Whittaker (1902), and Hardy (1908) must have been found unsatisfactory, because they were changed in the second editions of those books: Jordan (1893), Whittaker and Watson (1915), and Hardy (1914).

[1]Page numbers for definitions of continuity are given in the third section 'Continuity of four functions', either in citations or in square brackets in quotations. In a quotation, a row of dots not in square brackets is in the original text, but [...] indicates an omission.

Textbooks and teaching

Cauchy's (1821) was the first of several textbooks on analysis by distinguished French mathematicians (Struik 1967, 178). A later book (Goursat 1902) was translated into English by E R Hedrick, Professor at the University of Missouri. On completing his PhD at Göttingen he then spent some months in Paris where he met Goursat (Ford 1943). In 1903 he became Professor at the University of Missouri (Ford 1943). His translator's preface in Goursat (1904, v) says inter alia 'The lack of standard texts on mathematical subjects in the English language is too well known to require insistence. I earnestly hope that this book will help to fill the need so generally felt throughout the American mathematical world'. He was not the only American trained in Europe who helped to fill that need: James Pierpont, PhD (Vienna) and a Yale professor (Ore 1939), wrote Pierpont (1905). The dates of Whittaker (1902), Hobson (1907) and Hardy (1908) suggest that the same need was felt at the same time on both sides of the Atlantic.

Weierstrass did not write a textbook but his contributions to analysis and his teaching were outstanding. Struik (1967, 158) said 'His lectures, always meticulously prepared, enjoyed increasing fame; it is mainly through these lectures that Weierstrass' ideas have become the common property of mathematicians'.

Cambridge University was 'Britain's premier institution for mathematical instruction', but it lagged behind Europe in the nineteenth century in terms of the importance that was placed on rigour (Rice and Wilson 2003, 174). Hardy (1940) said that he taught himself analysis from Jordan's book when Professor Love, primarily an applied mathematician, recommended it. There are two reasons to suspect that Hardy was using Jordan (1882), not Jordan (1893) although that was published before he entered Trinity College as an undergraduate in 1896. The 1882 edition is in that college's library but the 1893 edition is not, and Hardy's (1908) first definition of continuity is equivalent to the one in Jordan (1882) but not to the one in Jordan (1893). By the twentieth century analysis was taken seriously in Cambridge. The title pages show that Whittaker, Hobson, and Hardy all wrote their first editions there, and Watson also wrote his contribution to Whittaker and Watson (1915) there, but Whittaker had gone to Edinburgh by then.

The École Polytechnique was pre-eminent in France (Struik 1967, 144–145, 148). Cauchy, Duhamel, and Jordan all taught there, but Goursat was at the École Normale Supérieure.

Continuity of four functions

We use four real functions of one real variable, each of which is continuous at the origin by some definitions but not others. Let the three functions $f_0, f_+, f_{\mathbb{Q}}$ be zero where indicated, and undefined elsewhere, and let f_S be defined for all real x, as follows:

$f_0(x) = 0$ if $x = 0$ (defined only at a single point);
$f_+(x) = 0$ if $x \geq 0$ (defined on one half of the real line);
$f_{\mathbb{Q}}(x) = 0$ if $x \in \mathbb{Q}$ (defined on a dense set, the rationals, with a dense complement);
$f_S(x) = \left\{ \begin{array}{ll} 0 & \text{if } x = 0 \\ \sin 1/x & \text{if } x \neq 0 \end{array} \right\}$ (having no limit at one point but taking every value in its range nearby).

Testing for continuity of those functions at $x = 0$ by various authors' definitions gives the results indicated. They are listed in chronological order. I used published translations where they are available, but translated the relevant parts of Duhamel (1841) and Jordan (1882, 1893) myself. In those cases the original French is also quoted, to allow checking of the translation.

Each subsection heading specifies which (if any) of the four functions $f_0, f_{\mathbb{Q}}, f_+, f_S$ are continuous at $x = 0$ by the definition given there.

Bolzano (1817): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

Translation by Russ (2004, 256):

> According to a correct definition, the expression *that a function fx varies according to the law of continuity for all values of x inside or outside certain limits** means only that: *if* x is any such value, the difference $f(x + \omega) - fx$ can be made smaller than any given quantity provided ω can be taken as small as we please.
> * There are functions which are continuously variable for all values of their argument, e.g., $\alpha + \beta x$. But there are others which vary according to the law of continuity only for values of their argument inside or outside certain limits. Thus $x + \sqrt{(1-x)(2-x)}$ is continuous only for all values of $x \langle +1$ or $\rangle +2$ but not for the values between $+1$ and $+2$.

The definition is for continuity on an interval. The footnote implies that a function is not continuous on the boundary of its domain of definition. We shall see that some later authors agree but some do not.

Cauchy (1821): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

Translation by Bradley and Sandifer (2009, 26):

> Let $f(x)$ be a function of the variable x, and suppose that for each value of x between two given limits, the function always takes a unique finite value. If, beginning with a value of x contained between these limits, we give the variable x an infinitely small increment α, the function itself is incremented by the difference
>
> $$f(x+\alpha) - f(x)$$
>
> which depends both on the new variable α and on the value of x. Given this, the function $f(x)$ is a continuous function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference
>
> $$f(x+\alpha) - f(x)$$
>
> decreases indefinitely with the numerical value of a.[...]
> We also say that the function $f(x)$ is a continuous function of the variable x in a neighborhood of a particular value of the variable x whenever it is continuous between two limits of x that enclose that particular value, even if they are very close together.

Bradley and Sandifer (2009, 26) has a footnote ‘Cauchy defines continuity only on the interior of a bounded interval, and for the whole interval, not just at a single point’.

Duhamel (1841, 5): f_+, f_S

> *Continuité*. Une variable est continue lorsqu’elle ne peut passer d’une valeur quelconque à une autre sans passer par toutes les valeurs intermédiaires. Une fonction est dite *continue* lorsqu’en faisant varier d’une manière continue les quantités dont elle dépend, elle est constamment réelle et varie elle-même d’une manière continue, c’est-à-dire qu’elle ne peut passer d’une valeur à une autre sans passer par toutes les intermédiaires. Une fonction peut être continue tant que les variables dont elle dépend restent renfermées entre certaines limites, et cesser de l’être, ou devenir discontinue, en dehors de ces limites.

My translation:

> *Continuity*. A variable is continuous when it cannot pass from any value to another without passing through all the intermediate values. A function is called *continuous* when on making the quantities on which it depends vary in a continuous manner, it is always real and itself varies in a continuous manner, that is to say that it cannot pass from one value to another without passing through all the intermediate values. A function can be continuous so long as the variables on which it depends remain confined between certain limits, and ceases to be, or becomes discontinuous, outside these limits.

Darboux (1875, 62 and 111) showed that a function such as f_S obeys that definition. Because $\lim_{x\to 0}\sin(1/x)$ does not exist, no other definition mentioned in the present work makes f_S continuous at $x = 0$. However, Whittaker and Watson (1927, 42–43) still called a definition equivalent to Duhamel’s the popular definition of continuity, but did not mention him.

Weierstrass (1861): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

Weierstrass never published his work on the foundations of analysis but many of his students took notes of his lectures (Dugac 1973, 43), including this definition of pointwise continuity from the notes of H A Schwarz in 1861 (Dugac 1973, 119–120). K Bing translated it in Calinger (1982, 559–560):

> If $f(x)$ is a function of x and if x is a definite value, then the function will change to $f(x + h)$ if x changes to $x + h$; [...] If it is now possible to determine for h a bound δ such that for *all* values of h which in their absolute value are smaller than δ, $f(x + h) - f(x)$ becomes smaller than any magnitude ε, however small, then one says that infinitely small changes of the argument correspond to infinitely small changes of the function. [...] If now a function is such that to infinitely small changes of the argument there correspond infinitely small changes of the function, one then says that it is a *continuous function* of the argument, or that it changes continuously with this argument.

Heine (1872): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

Translation in Birkhoff (1973, 25):

> §B.2.1. *Definition*. A function $f(x)$ is called *continuous at an individual value* $x = X$ when, given any quantity ε no matter how small, there exists another positive number η_0 with the property that $f(X \pm \eta) - f(X)$ cannot exceed ε for any positive number η smaller than η_0.
> §B.2.2. *Definition*. A function $f(x)$ is called *continuous* from $x = a$ to $x = b$ when it is continuous for each individual value $x = \xi$ between a and b, including a and b.

Birkhoff (1973, 25) has a footnote 'Heine omits absolute value signs throughout'. On Heine's definitions $f_+(x)$ is continuous from $x = a$ to $x = b$ if $0 < a \leq b$, but not if $a = 0$.

Jordan (1882, 11): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

> On dit qu'une fonction $y = f(x)$ est *continue* pour la valeur $x = a$ si, quelque petite que soit la quantité ε, on peut toujours déterminer une seconde quantité γ_1 telle que l'on ait
>
> $$f(a+h) - f(a) < \varepsilon$$
>
> pour toutes les valeurs de h comprises entre $-\gamma_1$ et $+\gamma_1$.
> La fonction $f(x)$ sera continue de $x = a$ à $x = b$ si elle est continue pour toutes les valeurs de x comprises dans cet intervalle.

My translation:

> One says that a function $y = f(x)$ is *continuous* for the value $x = a$ if, however small be the quantity ε, one can always determine a second quantity γ_1 such that one has
>
> $$f(a+h) - f(a) < \varepsilon$$
>
> for all the values of h between $-\gamma_1$ and $+\gamma_1$.
> The function $f(x)$ will be continuous from $x = a$ to $x = b$ if it is continuous for all the values of x included in this interval.

Jordan (1893, 46): $f_0, f_+, f_{\mathbb{Q}}$

> Soit $f(x, y, \ldots)$ une fonction des n variables $x, y, \ldots$ définie dans une ensemble E.
> Soient $(a, b, \ldots)$ un point déterminé de E; $h, k, \ldots$ des quantités variables, assujetties à la seule condition que le point $(a + h, b + k, \ldots)$ appartienne aussi à E.

> Si, pour tout valeur de la quantité positive ε, on peut determiner une autre quantité positive δ, telle que l'on ait
>
> $$|f(a+h, b+k, \cdots) - f(a, b, \cdots)| < \varepsilon$$
>
> pour tous les systèmes de valeurs de h, k, … pour lesquels on a
>
> $$|h| < \delta, \qquad |k| < \delta, \qquad \cdots,$$
>
> on dira que la fonction $f(x, y, \ldots)$ est *continue au point*$(a, b, \ldots)$.

My translation:

> Let $f(x, y, \ldots)$ be a function of the n variables $x, y, \ldots$ defined in a set E.
> Let $(a, b, \ldots)$ be a fixed point of E; $h, k, \ldots$ variable quantities, subject to the one condition that the point $(a+h, b+k, \ldots)$ belongs also to E.
> If, for any value of the positive quantity ε, we can find another positive quantity δ, such that we have
>
> $$|f(a+h, b+k, \cdots) - f(a, b, \cdots)| < \varepsilon$$
>
> for all the systems of values of h, k, … for which we have
>
> $$|h| < \delta, \qquad |k| < \delta, \qquad \cdots,$$
>
> we shall say that the function $f(x, y, \ldots)$ is *continuous at the point* $(a, b, \ldots)$.

Jordan now explicitly excluded points where a function is not defined. Other changes are that he now dealt with functions of several variables from the start, and included the absolute value signs omitted from his 1882 definition.

Whittaker (1902, 41): either $f_0, f_+, f_{\mathbb{Q}}$ or none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

> Let $f(z)$ be a quantity which, for all values of z lying within given limits, depends on z.
> Let z_1 be a point situated within those limits. Then $f(z)$ is said to be *continuous* at the point z_1 if, corresponding to any positive quantity ε, however small, a finite positive quantity η can be found, such that the inequality
>
> $$|f(z) - f(z_1)| < \varepsilon$$
>
> is satisfied so long as $|z - z_1|$ is less than η.

Whittaker did not define 'lying within given limits'; a possible interpretation is 'in the domain of definition of $f(z)$', but his 'within' may have been meant to exclude boundary points of the domain, because on page 43 he said that $(1 - z^2)^{-1/2}$ is continuously dependent on z along the straight line joining the origin ($z = 0$) to a point $z = Z$, where Z is real. He did not exclude the possibility of $Z = \pm 1$ although $(1 - z^2)^{-1/2} \to$

∞ as $z \to \pm 1$. If boundary points are excluded none of $f_0, f_+, f_{\mathbb{Q}}, f_S$ is continuous at the origin; if they are included, all except f_S are.

Goursat (1904): f_+

Note: 'the interval (a, b)' means the closed interval for Goursat, because after supposing that x 'can assume all values between two given numbers a and b $(a < b)$' he said

> [2] Let y be another variable, such that to each value of x between a and b, and also for the values a and b themselves, there corresponds one definitely determined value of y. Then y is called a function of x, defined in the interval (a, b); and this dependence is indicated by writing the equation $y = f(x)$.
> [...]
> Let $y = f(x)$ be a function defined in a certain interval (a, b), and let x_0 and $x_0 + h$ be two values of x in that interval. If the difference $f(x_0 + h) - f(x_0)$ approaches zero as the absolute value of h approaches zero, the function $f(x)$ is said to be *continuous for the value* x_0.
> [6] The function $y = x \sin 1/x$, for example, is a perfectly continuous function of x, for $x = 0$,* and y approaches zero as x approaches zero. [...]
> Finally, the function
>
> $$y = \frac{xe^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}$$
>
> is continuous at $x = 0$,* but the ratio y/x approaches two different limits according as x is always positive or always negative while it is approaching zero.
>
> * After the value zero has been assigned to y for $x = 0$.—TRANSLATOR.

The footnote correcting Goursat's oversight was added by Professor Hedrick, who did more than just translate. His preface (Goursat 1904, v) says inter alia

> Few alterations have been made from the French text. Slight changes of notation have been introduced occasionally for convenience, and several changes and additions have been made at the suggestion of Professor Goursat[...]. To him is due all the additional matter not to be found in the French text, except the footnotes which are signed, and even these, though not of his initiative, were always edited by him.

Pierpont (1905): $f_+, f_{\mathbb{Q}}$

Pierpont's $\mathfrak{R}$ is the set $\mathbb{R}$ of all real numbers, and $\mathfrak{R}_m$ is $\mathbb{R}^m$, real m-dimensional space.

> [153] A point x for which Dist(a, x) is small, is said to be *near* a. What is to be considered as *small*, depends on the problem in hand.

The points x, such that

$$\mathrm{Dist}(a,x) \leq \rho, \qquad \rho > 0.$$

form an aggregate [...] denoted by

$$D_\rho(a), \qquad D(a), \qquad D_\rho.$$

[157] Let A be a point aggregate in $\mathfrak{R}_m$. Any point p of $\mathfrak{R}_m$ is a limiting point of A, if however small $\rho > 0$ is taken, $D_\rho(p)$ contains an infinity of points of A.
If every domain of p *contains at least one other point,* p *is a limiting point of* A.[...]
[158] If p is a limiting point of A and p itself lies in A, it is called a proper limiting point.
[171] Let $f(x)$ be a one-valued function defined over a domain D. Let

$$A = a_1, a_2, a_3 \cdots \qquad (1$$

be *any* sequence of points in D such that

$$\lim a_n = a; \qquad a \text{ finite or infinite}; \qquad a_n \neq a.$$

If the sequence

$$f(a_1), f(a_2), f(a_3) \cdots \qquad (2$$

has a limit η, finite or infinite, always the same, however the sequence A be chosen, we say η *is the limit of* $f(x)$ *for* $x = a$ and write

$$\eta = \lim_{x=a} f(x),$$

[208] Let $f(x_1 \cdots x_m)$ be defined over a domain D. Let $a = (a_1 \cdots a_m)$ be a proper limiting point of D. If

$$\lim_{x=a} f(x_1 \cdots x_m) = f(a_1 \cdots a_m),$$

the function f is continuous at a.

Hobson (1907): $f_+, f_{\mathbb{Q}}$

[61] *If a linear set of points not finite in number (denoted by* G*) is in the interval (*a, b*), then a point* P*, in whose arbitrarily small neighbourhood there exists at least one point of* G *not identical with* P*, is called a limiting point of the set* G*, whether* P *belongs to* G *or not.*
[221] Let the domain of the independent variable x be continuous, and either bounded or unbounded; and denote the function y at the point x by $f(x)$.

> *The function $f(x)$ is said to be continuous at the point α of the domain of x, if, corresponding to any arbitrarily chosen positive number ε whatever, a positive number δ dependent on ε can be found, such that $|f(\alpha + \eta) - f(\alpha)| < \varepsilon$, for all positive or negative values of η which are numerically less than δ, and which are such that $\alpha + \eta$ is in the domain of x. At an end-point of a limited domain, the values of η will have one sign only.*
> [222] The domain of the independent variable has hitherto been considered continuous; it is however clear from a consideration of the definition of continuity, either in Cauchy's or in Heine's form, that the [223] definition is applicable in case the domain of the independent variable is not continuous, but consists of any set of points which contains limiting points that belong to the set. It is, of course, only at such a limiting point that the question of continuity arises; [...] the notion of continuity of a function is applicable whatever be the domain of the independent variable, except when it consists of an isolated set of points.

The trap that caught Goursat (see section 'Goursat (1904): f+') also caught Hobson:

> [236] Let $f(x) = (x - a)\sin 1/(x - a)$; then $f(a + 0) = 0, f(a - 0) = 0$. This function is continuous at $x = a$, and makes an infinite number of oscillations in any neighbourhood of that point.

Hardy (1908): none of $f_0, f_{\mathbb{Q}}, f_+, f_S$

> [171] To be able to define continuity *for all values of x* we must first [172] define continuity *for any particular value of x.* Let us therefore fix on some particular value of x, say the value $x = \xi$ [...]. What are the characteristic properties of $\phi(x)$ associated with this value of x?
> In the first place $\phi(x)$ *is defined for* $x = \xi$.[...]
> Secondly $\phi(x)$ *is defined for all values of x near* $x = \xi$; i.e. we can find an interval, including $x = \xi$ in its interior, for all points of which $\phi(x)$ is defined.
> Thirdly *if x approaches the value ξ from either side $\phi(x)$ approaches the limit $\phi(\xi)$.*
> [...]Definition. *The function $\phi(x)$ is said to be continuous for $x = \xi$ if it tends to a limit as x tends to ξ from either side, and each of those limits is equal to $\phi(\xi)$.*
> [...] our definition is equivalent to '*$\phi(x)$ is continuous for $x = \xi$ if, given ε, we can choose η so that $|\phi(x) - \phi(\xi)| < \varepsilon$ if $0 \leq |x - \xi| \leq \eta$).*

Hardy (1914, 175–176): f_+

The only significant change to the definition of continuity in Hardy (1908, 172) is the following addition:

> We have often to consider functions defined only in an interval (a, b). [...] We shall say that $\phi(x)$ is continuous for $x = a$ if $\phi(a + 0)$ exists and is equal to $\phi(a)$, and for $x = b$ if $\phi(b - 0)$ exists and is equal to $\phi(b)$.

Here (a, b) is the closed interval $\{x: a \leq x \leq b\}$. Hardy did not explicitly consider the possibility $a = b$ but his definitions for continuity at a point and on a closed interval are consistent with each other even in that case: $\phi(x)$ not continuous at $x = a$.

Hausdorff (1914): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

Translation by Pier (2001, 35):

> As is well known a real function $f(x)$ of a real variable is called continuous at the position a if to every preassigned $\sigma > 0$ one may choose $\rho > 0$ such that $|x - a| < \rho$ always implies $|f(x) - f(a)| < \sigma$. These two inequalities define the neighourhood U_a of the point a with radius ρ and part of the neighbourhood V_b of the point $b = f(a)$ with radius σ; hence the continuity condition expresses that for the given neighbourhood V_b one can choose a neighbourhood U_a such that the images of all points of U_a are in V_b. We adopt this also as the general definition of continuity so that right by the way it will suffice to consider ***A, B*** to be just topological spaces in which the neighbourhood axioms hold.
> Definition. The function $y = f(x)$ is called continuous at the point a if to every neighbourhood V_b of the point $b = f(a)$ there exists a neighbourhood U_a of the point a the image of which lies in V_b: $f(U_a) \subseteq V_b$.

Which functions are continuous in topological spaces depends on the topology; here Hausdorff was using the Euclidean topology in $\mathbb{R}$, but with some topological spaces ***A*** and ***B*** every function f: $\boldsymbol{A} \to \boldsymbol{B}$ is continuous, and with others only constant functions are (Gaal 1964, 181).

Whittaker and Watson (1915, 42–44): f_+

> [42] Let $f(x)$ be a function of x defined when $a \le x \le b$.
> Let x_1 be such that $a \le x_1 \le b$. If there exists a number l such that, corresponding to the arbitrary positive number ε, we can find a positive number η such that
>
> $$|f(x) - l| < \varepsilon,$$
>
> whenever $|x - x_1| < \eta$, $x \neq x_1$, and $a \le x \le b$, then l is called the limit of $f(x)$ as $x \to x_1$.
> It may happen that we can find a number l_+ (even when l does not exist) such that $|f(x) - l_+| < \varepsilon$ when $x_1 < x < x_1 + \eta$. We call l_+ the limit of $f(x)$ when x approaches x_1 from the right and denote it by $f(x_1 + 0)$; in a similar manner we define $f(x_1 - 0)$ if it exists.
> If $f(x_1 + 0)$, $f(x_1)$ and $f(x_1 - 0)$ all exist and are equal, we say that $f(x)$ is *continuous* at x_1; [...]

That definition, which is used on page 43 and requires $a \neq b$ because $x \neq x_1$, does not apply to $f_0, f_+, f_{\mathbb{Q}}$ or f_S, but page 44 has a definition applying to some functions that page 42 does not.

> [43] Let x and y be two functions of a real variable t which are continuous for every value of t such that $a \le t \le b$. We denote the dependence of x and y on t by writing
>
> $$x = x(t), \qquad y = y(t). \qquad\qquad (a \le t \le b)$$

The functions x and y are supposed to be such that they do not assume the same pair of values for any two different values of t in the range $a < t < b$. Then the set of points with coordinates (x, y) corresponding to these values of t is called a simple curve. [...]

[44] A simple curve is sometimes called *a closed one-dimensional region*; a simple curve with its end-points omitted is then called an *open one-dimensional region.*

3.32 *Continuous functions of complex variables.*

Let $f(z)$ be a function of z defined at all points of a closed region (one- or two-dimensional) in the Argand diagram, and let z_1 be a point of the region. Then $f(z)$ is said to be continuous at z_1 if, given any positive number ε, we can find a corresponding positive number η such that $|f(z) - f(z_1)| < \varepsilon$, whenever $|z - z_1| < \eta$ and z is a point of the region.

Although the section heading and 'Argand diagram' refer to complex variables, the real line $\mathbb{R}$ is part of the complex plane, and a one-dimensional closed region is in $\mathbb{R}$ if $y = 0$. The definition on page 44 makes $f_+(x)$ continuous at $x = 0$.

Courant (1937): $f_0, f_+, f_{\mathbb{Q}}$

[50] A function $f(x)$ is said to be *continuous* at the point ξ if it possesses the following property: at the point ξ the value of the function $f(x)$ is approximated to within an arbitrary pre-assigned degree of accuracy ε by all functional values $f(x)$ for which x is near enough to ξ. In other words, $f(x)$ is continuous at ξ if, for every positive number ε, no matter how small, there can be determined another positive number $\delta = \delta(\varepsilon)$ such that $|f(x) - f(\xi)| < \varepsilon$ for all points x for which $|x - \xi| < \delta$.

In the first sentence, 'All functional values $f(x)$' excludes values of x such that $f(x)$ is undefined. In the second sentence, 'all points x for which $|x - \xi| < \delta$' does not exclude those values of x, and 'In other words' does not make it clear which definition was intended. Fortunately, the definition for functions of two variables shows that Courant really intended the first of his two pointwise definitions. because of the words 'More precisely'.

[463] Here again the concept of continuity is given by the following definition: *a function* u = (x,y)*, defined in a region* R*, is continuous at the point* (ξ, η) *of* R *if for all points* (x,y) *near* (ξ, η) *the value of the function* f(x,y) *differs but little from* f(ξ, η)*, the difference being arbitrarily small if only* (x,y) *is near enough to* (ξ, η)*.*

More precisely, *the function* f(x,y)*, defined in the region* R*, is continuous at the point* (ξ, η) *of* R*, provided that for every positive number* ε *it is possible to find a positive distance* δ = δ(ε) *(in general depending on* ε *and tending to* 0 *with* ε*) such that for all points of the region whose distance from* (ξ, η) *is less than* δ *(that is, for which the inequality*

$$(x - \xi)^2 + (y - \eta)^2 \leq \delta^2$$

holds) the relation

$$|f(x,y) - f(\xi,\eta)| < \varepsilon$$

is satisfied.

Siddons et al. (1952): none of $f_0, f_+, f_\mathbb{Q}, f_S$

[3.244] The following abbreviations will be used:

$x \to a+0$ $\quad x \to a$ through values greater than a.
$x \to a-0$ $\quad x \to a$ through values less than a.
(a, b) The *closed* interval $a \leq x \leq b$. ($a < x < b$ is called an *open* interval.)

[3.247] **Definition.** $f(x)$ is continuous at $x = c$ if $f(c)$ exists and

$$\lim_{x \to c-0} f(x) = \lim_{x \to c+0} f(x) = f(c).$$

[...]
$f(x)$ is continuous in (a, b) if it is continuous in $a < x < b$ and also $\lim_{x \to a+0} f(x) = f(a)$, $\lim_{x \to b-0} f(x) = f(b)$.

That implies that a function defined on only one side of a given point $x = a$ can be continuous on a closed interval with a as an endpoint, but not at $x = a$ itself. I failed to notice that oddity when I was a student using that book. I was also using Hardy (1952, 186), which has the same wording as Hardy (1914, 175–176), and avoids the problem. (See 'Hardy (1914, 175–176): f+' above.) Continuity on a closed interval of zero length is explicitly excluded here.

Rudin (1953, 65): $f_0, f_+, f_\mathbb{Q}$

Let f be defined on E. Then f is said to be continuous at a point x of E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for all points t of E for which $|t - x| < \delta$. If f is continuous at every point of E, then f is said to be continuous on E.
It should be noted that f has to be defined at the point x in order to be continuous at x. [...]
If x is an isolated point of E, then our definition implies that every function f which has E as its domain of definition is continuous at x.

Bartle and Sherbert (1992, 140): $f_0, f_+, f_\mathbb{Q}$

Let $A \subseteq \boldsymbol{R}$, let $f : A \to \boldsymbol{R}$, and let c be in A. We say that f is continuous at c if, given any neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there exists a neighborhood $V_\delta(c)$ of c such that if x is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\varepsilon(f(c))$.

Neighbourhoods had been defined on page 41:

$$V_\varepsilon(a) = \{x \in \boldsymbol{R} : |x - a| < \varepsilon\}.$$

Carlson (2006, 190): f_+

Suppose that $I \subseteq R$ is an interval. A function f: $I \to R$ is *continuous at* $x_0 \in I$ if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If x_0 is the left or right endpoint of the interval I, this limit is taken to be the limit from above or below, as appropriate.

Gowers et al. (2008, 32): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$

We say that f is *continuous at a* if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad (|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

Kudryavtsev (2011): $f_0, f_+, f_{\mathbb{Q}}$

Let f be a real-valued function defined on a subset E of the real numbers $\mathbb{R}$, that is, $f : E \to \mathbb{R}$. Then f is said to be continuous at a point $x_0 \in E$ (or, in more detail, continuous at x_0 with respect to E) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ with $|x - x_0| < \delta$ the inequality

$$|f(x) - f(x_0)| < \varepsilon$$

is valid.

Conclusions

Bolzano (1817), Cauchy (1821), and Duhamel (1841) all defined continuity on an interval, but Darboux (1875) showed that Duhamel's definition was inadequate. Weierstrass (1861) began with continuity at a point. In modern notation his definition is equivalent to

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad (|x - a| < \delta) \Rightarrow \big((x \in D_f) \wedge (|f(x) - f(a)| < \varepsilon)\big), \tag{1}$$

where D_f is the domain of definition of the function f. His '*all* values of h' (see section 'Weierstrass (1861): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$') gives the requirement ($x \in D_f$), so (1) makes a function continuous at a point a only if it is defined throughout a neighbourhood of a (hence at a itself and at some points on each side of a).

Authors who follow Weierstrass usually omit ($x \in D_f$) because it is superfluous. The definition is then

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad (|x - a| < \delta) \Rightarrow (|f(x) - f(a)| < \varepsilon). \tag{2}$$

If $f(x)$ is defined throughout the neighbourhood $V_\delta(a) = \{x: |x - a| < \delta\}$, then (1), (2) are both true or both false. If not, there is at least one point x in $V_\delta(a)$ where $|f(x) - f(a)| < \varepsilon$ is meaningless, so (1) is false, (2) is meaningless, and f is not continuous at a by either (1) or (2).

The effect of Jordan (1893) was to replace (1) by

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad ((|x - a| < \delta) \wedge (x \in D_f)) \Rightarrow \Big(|f(x) - f(a)| < \varepsilon\Big), \tag{3}$$

i.e. to avoid trying to evaluate $f(x)$ at points x outside D_f but inside $|x - a| < \delta$. That allows functions like our $f_0, f_{\mathbb{Q}}, f_+$ to be continuous at the origin. Some people thought he had gone too far, but were happy with some extensions beyond Weierstrass's view implying that all three functions are discontinuous at the origin. It appears that continuity at one end of an interval was most acceptable, then continuity on a dense subset of $\mathbb{R}$, and continuity at an isolated point in the domain of definition was least acceptable.

That least acceptable case has now become the most acceptable. Hardy (1952, 186) repeated the definition in Hardy (1914, 171–172) following Goursat (1904, 2): effectively (3) but requiring that $a \in I \subseteq D_f$, where I is an interval of nonzero length. Of the fifty-five post-1952 books on my university library's Analysis shelves that defined continuity for functions $\mathbb{R} \to \mathbb{R}$, thirty-two used (3), eighteen used (2), and five followed Goursat (1904). So Jordan's (3) is now the most popular definition of pointwise continuity, Weierstrass's (2) ranks second, and Goursat's is third. All three are still in use in the present century: see sections 'Carlson (2006, 190): f_+', 'Gowers *et al.* (2008, 32): none of $f_0, f_+, f_{\mathbb{Q}}, f_S$' and 'Kudryavtsev (2011): $f_0, f_+, f_{\mathbb{Q}}$'.

The conflict between definitions of pointwise continuity seems not to be well known. That is probably because pointwise continuity of real functions of real variables is useful only for defining continuity on larger sets, often intervals. The post-1841 definitions all lead to one or other of two cases for continuity on a closed interval in $\mathbb{R}$, the difference being whether a function like f_+, which is defined only at and on one side of an endpoint, is deemed to be continuous at that endpoint or not. The extreme value theorem and the intermediate value theorem hold for a function f if its domain of definition includes a closed interval $[a, b]$ with $a < b$, and f is continuous on the open interval (a, b) and is continuous from the right at a and from the left at b. But people who wish to prove those theorems and who use (1) or (2) to define pointwise continuity are likely to obtain proofs valid only in $[a', b']$, where $a < a' < b' < b$ unless they deal separately with the endpoints a, b.

That may well be why Jordan and Hardy changed their definitions. Whittaker changed his too, but in his case improving the clarity may have been another reason. Bolzano (1817) had already considered continuity at endpoints; he denied it.

A significant matter of notation emerges from the quoted definitions: some of the authors used (a, b) for a closed interval and $a < x < b$ for an open one. Nowadays (a, b) usually means an open interval and $[a, b]$ a closed one.

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