A BUBBLE RISING IN VISCOUS FLUID: LAGRANGE'S EQUATIONS FOR MOTION AT A HIGH REYNOLDS NUMBER

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Abstract A gas bubble rising steadily in a pure liquid otherwise at rest at a moderate Weber number is, to a good approximation, of oblate spheroidal shape. Previous analytical calculations of that shape at high Reynolds numbers have ignored viscosity. This paper shows that if one includes viscosity by incorporating Rayleigh's dissipation integral in Lagrange's equations, then the speed of rise is that given by Moore, and the shape is that found for *inviscid* flow by El Sawi using the virial integral and by Benjamin using Hamiltonian theory.

Keywords: Lagrange's equations, spheroidal bubble, viscous flow, irrotational

1. INTRODUCTION

When a gas bubble rises steadily, at high Reynolds number Re and moderate Weber number W, in a fluid otherwise at rest, the flow has long been known to be irrotational to a good approximation, except in weak viscous boundary layers around the surface and down the wake (Moore, 1965). In this context "weak" means that the boundary layer is required only to bring the shear stress at the bubble surface to zero from its nonzero value of order $\eta U/a$ in the irrotational flow, where η is the dynamic viscosity of the liquid, U is the speed of rise, and a is the radius of the sphere with the same volume as the bubble. The Reynolds number Re and the Weber number W are defined by

$$Re = 2Ua/\nu, \qquad W = 2\rho U^2 a/\sigma, \tag{1.1}$$

where ν is the kinematic viscosity η/ρ, ρ is the density and σ is the surface tension. Because the boundary layer need not bring U to zero, the velocity in it is reduced by an amount of order $URe^{-1/2}$ instead of the $O(U)$ in a conventional strong layer, except for a small region of linear size $O(aRe^{-1/6})$ where the velocity reduction is $O(URe^{-1/6})$ (Moore, 1965). If Re is not sufficiently large and if W is sufficiently large, that reduction is not small and a recirculating eddy may appear in the rear stagnation region (Ryskin and Leal, 1984; Christov and Volkov, 1985; Dandy and Leal, 1986; Blanco and Magnaudet, 1995), but this paper ignores that possibility.

Lagrange's equations have occasionally been used in publications on bubbles in viscous irrotational flow (Voinov and Golovin, 1970; Ceschia and Nabergoj, 1978; Kok, 1993). Voinov and Golovin ignored distortion from spherical shape, and Ceschia and Nabergoj ignored gravity. A complication in both of those papers was allowing for change in volume of the bubble. That must be done if one wishes to study a pulsating bubble, but it may be neglected if, as here, one merely wishes to consider a bubble rising steadily under gravity in a liquid. Kok (1993) used Lagrange's equations for a pair of bubbles of constant size, but he assumed that his bubbles were spherical. In unpublished work, Wilson and Blake (personal communication, J. R. Blake) have studied a cloud of nearly spherical bubbles, by using multipole expansions and solving Lagrange's equations numerically.

There have been many more publications using Rayleigh's dissipation function with irrotational flow involving bubbles, a method pioneered by Levich (1949) and used to good effect by Moore (1963) for a spherical bubble with corrections due to boundary layers, by Moore (1965) with additional corrections due to distortion from spherical shape, and by Sangani and Didwania (1993) for a swarm of spherical bubbles, but none of these authors used Lagrange's equations. The purpose of this paper is to show that those equations can both simplify the work and extend the applicability of the results.

2. THEORY

If a Newtonian viscous fluid of constant density and viscosity, whose velocity tends to zero at infinity, contains a bubble of constant volume and is acted on by conservative and viscous forces, and the flow is irrotational, and is uniquely determined by n generalised coordinates q_i and their time derivatives $\dot{q}_i$, $i = 1 \ldots n$, and the total kinetic energy is $T(q_i, \dot{q}_i)$, the total potential energy is $V(q_i)$, and the total rate of viscous dissipation of energy is $D(q_i, \dot{q}_i)$, then (Rayleigh, 1873; Voinov and Golovin, 1970; Ceschia and Nabergoj, 1978) Lagrange's equations reduce to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{1}{2}\frac{\partial D}{\partial \dot{q}_i} = 0. \tag{1.2}$$

In our case we assume that the bubble is spheroidal (Moore, 1965; El Sawi, 1970; El Sawi, 1974; Benjamin, 1987), that q_1 is the height of the bubble's centre above a fixed level, so that the speed of rise $U = \dot{q}_1$, and that $q_2 = \chi$, the ratio of horizontal to vertical semi-axis of the bubble. Thus

$$T = \frac{2}{3}\pi a^3 U^2 K, \tag{1.3}$$

$$V = -\frac{4}{3}\pi \rho g a^3 q_1 + \sigma A, \tag{1.4}$$

$$D = 12\pi\eta U^2 a G + O(q_2^2), \tag{1.5}$$

and if $\zeta = (\chi^2 - 1)^{-1/2}$, so that $\cot^{-1}\zeta = \sec^{-1}\chi$,

$$K = \frac{1 - \zeta\cot^{-1}\zeta}{\zeta\cot^{-1}\zeta - \chi^{-2}}, \tag{1.6}$$

$$A = \text{ surface area } = 2\pi a^2 L, \tag{1.7}$$

$$L = (\zeta\cosh^{-1}\chi + \chi)\chi^{-1/3}, \tag{1.8}$$

$$G = \frac{(\zeta^2+1)\{(1-\zeta^2)\cot^{-1}\zeta + \zeta\}}{3\zeta\chi^{2/3}\{1 + (\zeta^2+1)(1-\zeta\cot^{-1}\zeta)\}^2}, \tag{1.9}$$

(Lamb, 1932; Moore, 1965; Harper, 1970; Harper, 1971). Strictly, T should also contain a term in $\dot{q_2}^2$, but the contribution of that term to equation (1.2) vanishes in steady flow. So does the term in that equation which accounts for the work done in changing the size of a bubble. Neither additional term is considered here. Lagrange's equations then give, for $i = 1, 2$ respectively,

$$U = \frac{ga^2}{9\nu G}, \tag{1.10}$$

$$W = 6\frac{\partial L/\partial\chi}{\partial K/\partial\chi}. \tag{1.11}$$

Equation (1.10) reproduces the first-order result obtained by simply equating D to the rate of loss of gravitational potential energy (Moore, 1965), and equation (1.11) reproduces the known inviscid result using virial theory (El Sawi, 1970), which is also deducible by Hamiltonian methods (Benjamin, 1987). One would expect equation (1.10) to emerge in the present theory, because the present assumptions are the same as Moore's if his boundary-layer correction to the irrotational flow is ignored. However it is at first sight surprising that equation (1.11) for a viscous liquid should be identical with the inviscid theory of El Sawi and Benjamin, but it happens that in steady flow there is no contribution to the $i = 2$ Lagrange equation from viscosity. That is because D contains terms in $\dot{q_1}^2$ and $\dot{q_2}^2$ but not in $\dot{q_1}\dot{q_2}$, because reversing one of $\dot{q_1}, \dot{q_2}$, but not both, must leave D unchanged.

3. CONCLUSIONS

Lagrangian theory confirms Moore's leading-order approximation for the speed of rise as a function of Reynolds number and axis ratio χ, and it shows for the first time that the El Sawi-Benjamin inviscid theory giving the Weber number W as a function of χ still holds to leading order for a bubble rising in a viscous liquid if Re is large.

The advantages of the present method are that it gives a simple route to the result of El Sawi and Benjamin, and it does so without using their assumption of inviscid flow. The disadvantage of the method is that it gives neither the structure of the viscous boundary layer at the surface, nor higher approximations to the pressure there. As a result (Moore, 1965) the theory needs corrections to the drag of order $Re^{-1/2}$ and to the shape of order Re^{-1}, and those corrections account for part of the discrepancy between the present theory and detailed computation (Ryskin and Leal, 1984; Christov and Volkov, 1985). The remainder of the discrepancy arises because irrotational theory cannot describe a standing eddy at the rear of the rising bubble. Moore showed that in the limit of large Re no such eddy appears, but because the velocity in the rear stagnation region is reduced by $O(Re^{-1/6})$ of its value, it would not be surprising if eddies appeared there at moderate Reynolds number. The computational work revealed them.

Acknowledgments

I am grateful for financial support to the Marsden Fund administered by the Royal Society of New Zealand, for hospitality during the preparation of this paper to the Isaac Newton Institute, Cambridge, and for useful discussions to Professor D. W. Moore, Imperial College, London, and Professor J. R. Blake, University of Birmingham.

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