Anti-randomness and near global hegemony

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COMPUTABILITY

We work with idealised computers. They input and output finite binary strings, and give rise to (partial) computable functions.

A set is computably enumerable if it is the range of a computable function.

Reals

There is an (almost) bijection between the set of reals in $[0, 1]$ and the set of infinite binary strings.

This allows us, for example, to use reals as *oracles* for computers.

Open sets

Recall that a basis for a topology for $\mathbb{R}$ consists of open intervals $B(q, r)$ with rational centre and rational length.

Definition

An open set $O \subset \mathbb{R}$ is computably enumerable if the set

$$\{(q, r) \,:\, B(q, r) \subseteq O\}$$

is computably enumerable.

Relativisation

Using a real as an oracle, computers may be able to enumerate more sets. This gives rise to more open sets: if **a** is a real, then an open set $O \subset \mathbb{R}$ is computably enumerable in **a** if

$$\{(q, r) : B(q, r) \subseteq O\}$$

is c.e. in **a**.

Fact

Every open set can be enumerated by some real.

UNIFORMITY

A sequence of c.e. open $\langle O_n \rangle$ sets is uniformly enumerable if, effectively in n, we can enumerate the basic intervals contained in O_n.
That is, if

$$\{(n, q, r) \,:\, B(q, r) \subseteq O_n\}$$

is computably enumerable.

Effectively G_δ sets

Definition

A set $A \subset \mathbb{R}$ is **effectively G_δ** if it is the intersection of a sequence of uniformly c.e. open sets.

Complements of effectively G_δ sets are effectively F_σ.

Strongly random reals

Definition

A real **a** is strongly random if it is not an element of any effectively G_δ set of measure 0.

MARTIN-LÖF RANDOMNESS

Let $A = \bigcap_n W_n$ be an effectively G_δ set. Then the measure of A is zero iff $\mu(W_n) \to 0$.

DEFINITION

A Martin-Löf test is an effectively G_δ set A of measure 0 such that there is a uniformly c.e. sequence $\langle W_n \rangle$ such that $A = \bigcap W_n$ and $\mu(W_n) \to 0$ *effectively*.

A real is Martin-Löf random if it is not the element of any Martin-Löf test.

KOLMOGOROV COMPLEXITY

Let f be a computable function, and suppose that σ and τ are strings and that $f(\sigma) = \tau$. Then we call σ a description of τ.

For a definite notion of complexity, we use a universal function, one that simulates all others.

We then let $K(\tau)$ be the length of a shortest description of τ.

A string τ of length n is called incompressible if $K(\tau) \geqslant n$.

THEOREM (LEVIN, CHAITIN)

A real $\mathbf{a}$ *is Martin-Löf random iff every initial segment of* $\mathbf{a}$ *is incompressible.*

Anti-randomness

A string τ of length n is called very compressible if $K(\tau) \leqslant K(n)$.

Definition

A real **a** is K-trivial if every initial segment of **a** is very compressible.

Lowness for randomness

Relativising effecive open sets gives relativisation of all other notions, including randomness.

Definition

A real **a** is low for Martin-Löf randomness if every real which is ML-random is also ML-random relative to **a**.

Similarly we can define lowness for strong randomness.

Lowness for complexity

We can also relativise Kolmogorov complexity.

Definition

A real $\mathbf{a}$ is low for K if $K \leqslant^{+} K^{\mathbf{a}}$.

THEOREM (DOWNEY, HIRSCHFELDT, NIES)

The following are equivalent for any real **a***:*

- **a** *is* K*-trivial.*
- **a** *is low for ML-randomness.*
- **a** *is low for* K*.*

Domination

Let $f, g\colon \mathbb{N} \to \mathbb{N}$. We say that f dominates g if for all but finitely many n we have $g(n) < f(n)$.

for any $f\colon \mathbb{N} \to \mathbb{N}$, let $D(f)$ be the collection of all functions that are dominated by f.

A.E. DOMINATION

For any real **a**, let $C(\mathbf{a})$ denote the collection of all function $g\colon \mathbb{N} \to \mathbb{N}$ that are computable from **a**.

Definition

We say that a real **a** dominates a real **b** if every function computable from **b** is dominated by some function computable from **a**. That is,

$$C(\mathbf{b}) \subset \bigcup_{f \in C(\mathbf{a})} D(f).$$

A real **a** is a.e. dominating if the collection of reals **b** which are dominated by **a** has full measure.

Uniform domination

We say that a function f dominates a real $\mathbf{b}$ if every function computable from $\mathbf{b}$ is dominated by f. That is, if

$$C(\mathbf{b}) \subset D(f).$$

A function f is uniformly a.e. dominating if the collection of reals which are dominated by f has full measure.

A *real* is uniformly a.e. dominating if it computes some function that is uniformly a.e. dominating.

Motivation

Lebesgue measure is regular: for every measurable set A,

- $\mu(A)$ is the infimum of the measures of open sets containing A; and
- $\mu(A)$ is the supremum of the measures of closed sets contained in A.

It follows that there is an F_σ set B and a G_δ set C such that $B \subseteq A \subseteq C$ and such that $\mu(A) = \mu(B) = \mu(C)$.

Theorem (Dobrinen and Simpson)

A real **a** *is uniformly a.e. dominating iff for every effectively G_δ set C, there is some set B which is effectively F_σ relative to* **a** *such that $B \subset C$ and $\mu(B) = \mu(C)$.*

Existence

Theorem (Kurtz)
The halting problem is uniformly a.e. dominating.

A real **a** is called incomplete if it does not compute the halting problem.

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THEOREM (CHOLAK, GREENBERG, AND MILLER)

There is a c.e., incomplete, uniformly a.e. dominating real.

Highness

The halting set can be relativised. The halting set relative to a real $\mathbf{a}$ is denoted by $\mathbf{a}'$.

A real $\mathbf{a}$ is called high if $\mathbf{a}'$ computes $\mathbf{0}''$.

Theorem (Martin)

A real $\mathbf{a}$ is high iff there is some function f, computable from $\mathbf{a}$, which dominates all computable functions.

It follows that every uniformly a.e. dominating real is high.

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It follows that every uniformly a.e. dominating real is high.

Theorem (Greenberg and Miller; Binns, Kjos-Hanssen, Lerman, and Solomon)

There is a high real that is not uniformly a.e. dominating.

Almost complete reals

A real $\mathbf{a}$ is called almost complete if $\mathbf{0}'$ is low for ML-randomness relative to $\mathbf{a}$.

Remark (Miller)

For any two reals $\mathbf{a}$ *and* $\mathbf{b}$*, the following are equivalent:*

- $\mathbf{a}$ *is low for ML-randomness relative to* $\mathbf{b}$ *(that is, every random relative to* $\mathbf{b}$ *is random relative to* $\mathbf{a}$*;)*
- $\mathbf{a}$ *is low for* K *relative to* $\mathbf{b}$ *(that is,* $K^{\mathbf{b}} \leqslant^{+} K^{\mathbf{a}}$*.)*

Lowness

Lowness for randomness implies “traditional” lowness.

Definition

A real $\mathbf{a}$ is called low if $\mathbf{0}'$ computes $\mathbf{a}'$.

Theorem (Nies)

Every K-trivial real is low.

Corollary

Every almost complete real is high.

Positive domination

Definition

A real **a** is positively dominating if for every effectively continuous function Φ from $\mathbb{R}$ to $\mathbb{N}^{\mathbb{N}}$ whose domain has positive measure, the collection of reals $\mathbf{b} \in \operatorname{dom} \Phi$ such that $\Phi(\mathbf{b})$ is dominated by some function computable in **a** has positive measure too.

Lemma

A real **a** *is positively dominating iff for every effectively G_δ set C of positive measure, there is some set B which is effectively F_σ relative to* **a** *such that $B \subset C$ and $\mu(B) > 0$.*

Positive domination is implied by a.e. domination.

POSITIVE DOMINATION AND ALMOST COMPLETENESS

THEOREM (HIRSCHFELDT AND KJOS-HANSSEN)

The following are equivalent for any real $\mathbf{a}$*:*

- $\mathbf{a}$ *is positively dominating.*
- $\mathbf{a}$ *is almost complete.*

Strong almost completeness

Strong randomness yields its own lowness and almost completeness notions:

Definition

A real **a** is low for strong randomness if every strongly random real is also strongly random relative to **a**.

Definition

A real **a** is strongly almost complete if $\mathbf{0}'$ is low for strong randomness relative to **a**, that is, if every real which is strongly random relative to **a** is also strongly random relative to the halting problem.

Strong almost completeness implies almost completeness

Theorem (Downey, Nies and Weber)

Every real which is low for strong randomness is also low for ML-randomness.

Corollary

Every real which is strongly almost complete is almost complete.

Strong almost completeness and uniform domination

Theorem

The following are equivalent for a real **a***:*

- **a** *is strongly almost complete.*
- **a** *is uniformly a.e. dominating.*

THE CONCLUSION

THEOREM (MILLER)

A real is low for ML-randomness iff it is low for strong randomness.

COROLLARY

A real is almost complete iff it is strongly almost complete.

COROLLARY

The following are equivalent for a real **a***:*

- **a** *is positively dominating.*
- **a** *is a.e. dominating.*
- **a** *is uniformly a.e. dominating.*

What's left? 1. Reverse Mathematics

Recall that $\mathrm{ACA}_0 \vdash \mathrm{WKL}_0 \vdash \mathrm{WWKL}_0 \vdash \mathrm{DNR}_0 \vdash \mathrm{RCA}_0$.
ACA_0 implies $G_\delta - \mathrm{REG}$, and WKL_0 does not.

Theorem (Cholak, Greenberg and Miller)

- $\mathrm{RCA}_0 + G_\delta - \mathrm{REG}$ *doesn't imply* DNR_0.
- $\mathrm{WWKL}_0 + G_\delta - \mathrm{REG}$ *doesn't imply* WKL_0.
- $\mathrm{WKL}_0 + G_\delta - \mathrm{REG}$ *doesn't imply* ACA_0.

Question

Does $G_\delta - \mathrm{REG} + \mathrm{DNR}_0$ *imply* WWKL_0*?*

What's left? 2. ML Cupping

A real **a** ML-cups if there is some incomplete random **r** such that **a** and **r** together compute $\mathbf{0}'$.

Theorem (Nies)

If **a** *doesn't ML-cup then* **a** *is K-trivial.*

The converse is unknown.

It is known that if a K-trivial **a** does ML-cup via some random **r**, then **r** is almost complete.

There are K-trivial reals that are computable from every almost complete random real, and so do not ML-cup.

Question

Is every K-trivial computable from every almost complete random real?

What's left? 3. Strong jump traceability

Figueira, Nies, and Stephan define strongly jump traceable reals. This is a combinatorial notion.

Theorem (Downey and Greenberg)

- *Every c.e. strongly jump-traceable real is K-trivial, indeed, does not ML-cup.*
- *There is a K-trivial real that is not a strongly jump-traceable real.*
- *Every strongly jump-traceable real is computable from $\mathbf{0}'$ (in particular, there are only countably many.)*

Question

Is every strongly jump-traceable real K-trivial?

What's left? 4. And finally —

Question

Is there a direct proof that a.e. domination implies uniform a.e. domination?