Continuous higher randomness

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Higher randomness and continuity

Long computations

Let $\langle L_e \rangle_{e<\omega}$ be an effective list of all computable linear orderings. Recall that

$$\mathrm{CWO} = \{e < \omega : L_e \text{ is a well-ordering }\}$$

is Π^1_1-complete. That is, a set $A \subseteq \omega$ is Π^1_1 if and only if it is m-reducible to CWO.

Informally, this shows that Π^1_1 sets are in some sense "generalised c.e.". If A is Π^1_1 and f is an m-reduction of A to CWO, then we say that $n \in A$ is enumerated into A at stage $\mathrm{otp}(f(n))$.

Long computations

For example, consider the reduction property. If A, B are c.e., and $A \cup B = \omega$, then there are c.e. $\hat{A} \subseteq A$ and $\hat{B} \subseteq B$ which partition ω: for each $n < \omega$, put n into $\hat{A}$ if n enters A before it (perhaps) enters B.

Similarly, suppose that A, B are Π^1_1, and $A \cup B = \omega$; let f, g be m-reductions of A, B to CWO. Let $\hat{A} = \{n \in A : \mathrm{otp}(f(n)) \leqslant \mathrm{otp}(g(n))\}$ and $\hat{B} = \{n \in B : \mathrm{otp}(g(n)) < \mathrm{otp}(f(n))\}$ (with the obvious interpretation of $\leqslant$ for ill-founded order-types).

Formalising the intuition

This intuition can be formalised using a set theoretic understanding of computability. Recall that a set $A \subseteq \omega$ is c.e. if and only if it is Σ_1-definable (with parameters) in the structure $(H_\omega, \in)$.

What makes the basic theorems of computability work is that H_ω is admissible: the image of every finite set by a computable function is bounded below ω.

The structure $L_{\omega_1^{\mathrm{ck}}}$ is admissible as well (this follows from Spector's Σ^1_1 bounding principle). Call a subset of $L_{\omega_1^{\mathrm{ck}}}$ c.e. if it is Σ_1-definable in that structure (with parameters). A set $A \subseteq \omega$ is c.e. if and only if it is Π^1_1.

Higher randomness

Computable enumerability is the most basic concept in recursion theory; everything else can be derived from it. For example, a set is computable if it is c.e. and co-c.e.

Similarly, a set $\mathcal{U} \subseteq 2^\omega$ is effectively open if it is generated by a c.e. set of strings.

Definition (Hjorth,Nies)

A ML test is a sequence $\langle \mathcal{U}_n \rangle$ of uniformly effectively open sets with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$. The test captures $\bigcap_n \mathcal{U}_n$. MLR is the set of reals not captured by any ML test.

Higher randomness

Theorem (Schnorr,Levin;Hjorth,Nies)

The following are equivalent for $X \in 2^\omega$:

1. $X \in \mathrm{MLR}$.
2. $K(X \upharpoonright_n) \geqslant^+ n$.

Some work is needed because it is not the case that every effectively open set is generated by a c.e. antichain. Some approximation is required.

Randomness and computability

For a Turing operator Φ and a string σ, let

$$\Phi^{-1}[\sigma] = \{Y \in 2^\omega \, : \, \Phi(Y) \succcurlyeq \sigma\}.$$

(This may contain reals on which Φ is not total).

Theorem (Levin,Zvonkin;Miller,Yu)

The following are equivalent for $X \in 2^\omega$:

1. $X \in \mathrm{MLR}$.
2. *For any Turing operator Φ, $\lambda\left(\Phi^{-1}[X\restriction_n]\right) \leqslant^\times 2^{-n}$.*
3. *For every A and Φ, if $\Phi(A) = X$ then the use $\varphi(A, n) \geqslant^+ n$.*

(2) is the continuous analogue of the discrete measure (prefix-free complexity) characterisation of randomness, and is essentially the same as the supermartingale characterisation of randomness.

Randomness and computability

The same holds for MLR, but we need to identify what operators we use. A Turing operator is a c.e. set of pairs (σ, τ) (the pair says that with an oracle extending σ outputs τ). We write $A \leqslant_{\mathrm{T}} B$ if there is a Turing functional Φ such that $\Phi(B) = A$.

However, note that $\lambda(\Phi^{-1}[\sigma])$ is a supermartingale if Φ is consistent. In countable world this is not an issue. In the higher setting, it is.

Interlude: think why we can’t always get consistency.

On consistency

Why would we consider inconsistent functionals?

First reason: "philosophical". Computable^B should mean c.e.^{B} and co-c.e.^{B}

If we require continuity, there is only one way to define c.e.^{B}:

Definition

A c.e. operator is a c.e. set of pairs $(\sigma, x) \in 2^{<\omega} \times \omega$. For $B \in 2^\omega$ and a c.e. operator Φ, we let

$$\Phi(B) = \{x : \exists \sigma \prec B.\ (\sigma, x) \in \Phi\}.$$

Sets of this form are called B-c.e.

More on consistency

Second reason: practical. Consider for example:

Theorem (Hirschfeldt,Miller)

If $X \in \mathsf{MLR} \setminus \mathsf{W2R}$ then there is some noncomputable c.e. set $A \leqslant_T X$.

The proof does not give a consistent functional. Or:

Theorem (Franklin,Ng;Yu)

The following are equivalent for $X \in \mathsf{MLR}$:

- *X fails a difference test: there is a sequence $\langle \mathcal{U}_n \rangle$ of uniformly effectively open sets and an effectively closed set $\mathcal{P}$ such that $\lambda(\mathcal{P} \cap \mathcal{U}_n) \leqslant 2^{-n}$ and $X \in \mathcal{P} \cap \bigcap_n \mathcal{U}_n$.*
- *$X \geqslant_T O$.*

(some more work is needed).

Randomness and continuity

Theorem (van Lambalgen)

$A \oplus B \in \mathsf{MLR}$ if and only if $A \in \mathsf{MLR}$ and $B \in \mathsf{MLR}^A$.

A corollary of Levin-Zvonkin:

Theorem (Miller,Yu)

Suppose that $\mathcal{C}$ is a test-based, reasonable randomness notion stronger than MLR. *Then $\mathcal{C}$ is downward-closed in the $\leqslant_T$-degrees of* MLR*.*

(inconsistency must be dealt with).

Stronger notions of randomness

We mentioned in passing weak 2 randomness.

Theorem (Yu,Chong)

A left-c.e. random real is not weak 2 random.

They used the Lebesgue density theorem.

Simple proof.

Let $A = A_{\omega_1^{\mathrm{ck}}} = \lim_{s<\omega_1^{\mathrm{ck}}} A_s$, a monotone approximation. The set

$$\mathcal{D} = \left\{A_s \,:\, s \leqslant \omega_1^{\mathrm{ck}}\right\}$$

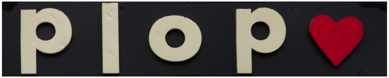
is closed. Let

$$\mathcal{U}_n = \bigcup_{s<\omega_1^{\mathrm{ck}}} [A_s \restriction n].$$

Then $\bigcap_n \mathcal{U}_n = \mathcal{D}$. It is countable, and so null. □

Corollary

Every weak 2 random real is difference random.



The converse of Hirschfeldt-Miller is not known.

Theorem (folklore?)

A real X is weak 2 random if and only if it is not captured by a test $\langle \mathcal{U}_n \rangle$ with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$ and $\mathcal{U}_n = [W_{f(n)}]$ for some $f \leqslant_T \varnothing'$.

That is, $\mathrm{W2R} = \mathrm{MLR}\langle \varnothing' \rangle$. The higher analogue fails, because there is an O-computable $X \in$ W2R (Kleene basis theorem). Again, think of the time trick required.

A new characterisation of W2R

Theorem

The following are equivalent for $X \in$ MLR:

- *$X \notin \mathrm{MLR}\langle O \rangle$.*
- *X is captured by a 'long test': a test $\langle \mathcal{U}_\alpha : \alpha < \omega_1^{\mathrm{ck}} \rangle$, uniformly c.e., with null intersection.*
- *X computes a noncomputable c.e. subset of ω_1^{ck}.*
- *There is some O-computable non-c.e. set $A \subseteq \omega$ which is c.e.^{X}*

Π^1_1 randomness

Theorem (Kechris)

There is a greatest null Π^1_1 set.

Its complement is the collection of Π^1_1-random reals. Since Π^1_1 is closed under number quantification, Π^1_1 randomness implies weak 2 randomness. Again because of the basis theorem, it does not imply MLR$\langle O \rangle$.

Question (Nies,Yu)

- *Is* Π^1_1-R $=$ W2R*?*
- *What is the Borel rank of* Π^1_1-R*?*

Remark (Chong,Nies,Yu)

For $X \in$ MLR, $X \in \Pi^1_1$-R *if and only if* $\omega_1^X = \omega_1^{\mathrm{ck}}$*.*

Steel showed that the Borel rank of reals which preserve ω_1^{ck} is $\omega_1^{\mathrm{ck}} + 2$.

The Borel rank of Π^1_1-R

Theorem
Π^1_1-R is $\underset{\sim}{\Pi}^0_3$ (and not $\underset{\sim}{\Sigma}^0_3$).

Why? Regularity behaves:

Lemma
Let $\mathcal{G}$ be Π^1_1. For every $\epsilon > 0$ there is an effectively closed set $\mathcal{P} \subseteq \mathcal{G}$ with $\lambda(\mathcal{G} - \mathcal{P}) < \epsilon$.

Lemma
Let $X \in$ MLR. Then $X \in \Pi^1_1$-R if and only if for any effectively G_δ set $\mathcal{G}$ there is some effectively closed $\mathcal{P} \subseteq \mathcal{G}$ of positive measure containing X.

A new inclusion

In fact, O can tell if an effectively closed set is null, and if it is contained in a given effectively G_δ set. Hence:

Theorem
MLR$\langle O \rangle \subset \Pi^1_1$-R.

We still do not know if Π^1_1-R $=$ W2R. A positive answer will be a strong converse to Hirschfeldt-Miller.

Lowness for randomness

K-triviality

Hjorth and Nies constructed noncomputable, K-trivial sets, but showed that lowness for Π^1_1-MLRand lowness for K are identical with being computable.

However, they did not use continuous relativisation.

Theorem

The following are equivalent for $A \in 2^\omega$:

1. *A is K-trivial.*
2. *A is low for K (but we need to say what this means!)*
3. *A is low for MLR.*
4. *A is a base for MLR.*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