Strong jump-traceability

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Approximating incomputable functions

How do we find the values of incomputable functions?

- Limit computation: $f = \lim_{s\to\infty} f_s$, with $\langle f_s \rangle$ uniformly computable.
- Domination: $f(n) \leqslant g(n)$, where g is computable.

Limit computations

Theorem (Shoenfield)

The following are equivalent for a function $f : \omega \to \omega$:

- *f is the discrete limit of a uniformly computable sequence of functions;*
- *f is Δ^0_2-definable in arithmetic;*
- *f is Turing-computable from the Halting set $\emptyset'$.*

Other notions involving limit computations: Ershov hierarchy (α-computably approximable functions), weak truth-table reducibility to $\emptyset'$.
Mixing with the Turing jump gives notions such as superlowness (every **a**-partial computable function has a computably bounded approximation), and its more mysterious analogue superhighness.

Domination

Theorem (Martin)

The following are equivalent for a Turing degree **a***:*

- **a** *contains a function f which dominates every computable function.* *i.e., for every computable function g, for all but finitely many $n \in \mathbb{N}$, $f(n) > g(n)$.*
- **a** *is high: The Turing jump of* **a** *computes $\emptyset''$.*

Other notions: hyperimmune-free degrees, array computable degrees.

Random reals are dominated by functions in the ground model (effectively: are $\emptyset'$-hyperimmune-free), Cohen generic reals escape domination.

The partial ordering ω^ω modulo domination gives rise to cardinal invariants of the continuum.

A finer notion

Idea similar to domination: specify a finite number of possible values.

Definition (Terwijn,Zambella;Ishmukhametov;following Raisonnier and Bartoszynski)

A trace is sequence of finite sets. A (possibly partial) function f is traced by a trace $\langle T(n) \rangle$ if for all $n \in \operatorname{dom} f$, $f(n) \in T(n)$.

Effectiveness considerations: a trace T is called computably enumerable if the sets $T(n)$ are uniformly c.e.

Bounds on traces

Rates of growth are measured by order functions.

Definition (Schnorr)

An order function is a non-decreasing and unbounded computable function from ω to $\omega \setminus \{0\}$.

Let h be an order function. An h-trace is a trace T such that for all n, $|T(n)| \leqslant h(n)$.

Compared to domination, this notion allows us to bound the number of possible values of a function, without having to bound their size.

A variety of traceability notions

Theorem (Terwijn,Zambella;Kjos-Hanssen,Nies,Stephan)

The following are equivalent for a Turing degree **a***:*

- **a** *is low for Schnorr randomness: every Schnorr random set is Schnorr random relative to* **a***.*
- *Every function in* **a** *has a computable* id*-trace.*

Theorem (Ishmukhametov)

The following are equivalent for a c.e. degree **a***:*

- **a** *has a strong minimal cover in the Turing degrees.*
- *Every function in* **a** *has a c.e.* id*-trace.*

Theorem (Downey,G)

The following are equivalent for a c.e. degree **a***:*

- **a** *bounds a critical triple in the c.e. degrees.*
- *Some function in* **a** *has no computably bounded c.e. trace.*

Strong jump-traceability

Definition (Nies)

Let h be an order function. A Turing degree $\mathbf{a}$ is h-jump-traceable if every $\mathbf{a}$-partial computable function has a c.e. h-trace.

Definition (Figueira,Nies,Stephan)

A Turing degree $\mathbf{a}$ is strongly jump-traceable if for every order function h, $\mathbf{a}$ is h-jump-traceable.

Existence and "uniqueness"

Theorem (Figueira,Nies,Stephan)

There is a non-computable SJT degree. Indeed, a c.e. SJT degree

Unlike all other traceability notions:

Theorem (Downey,G)

There are only countably many SJT degrees. Indeed, they are all Δ^0_2.

Structure theorems

Theorem (Cholak,Downey,G;Diamondstone,G,Turetsky)
The SJT degrees form an ideal of the Turing degrees.

Most results are first proved for c.e. degrees and then extended to all SJT degrees using:

Theorem (Diamondstone,G,Turetsky)
The ideal of SJT degrees is generated by its c.e. elements.

Theorem (Ng)
No single level of the jump-traceability hierarchy coincides with SJT.

Alternative definitions

Theorem (Figueira,Nies,Stephan;DGT)

The following are equivalent for a Turing degree $\mathbf{a}$*:*

- $\mathbf{a}$ *is SJT (recall what this means: every* $\mathbf{a}$*-partial computable function has arbitrarily good traces);*
- *every* $\mathbf{a}$*-partial computable function has arbitrarily good approximations.*

Kučera's programme

Theorem (Kučera)

Every Δ_2^0 Martin-Löf random set computes an incomputable c.e. set.

Q: what kind of random sets compute what kind of c.e. sets?

Theorem (Hirschfeldt,Nies,Stephan)

Every c.e. set computable from an incomplete ML-random set is K-trivial.

The converse is open.

The covering problem for SJT

Theorem (Kučera,Nies;G,Turetsky)

A c.e. degree is SJT if and only if it is computable from a Demuth random set.

The difference between Martin-Löf randomness and Demuth randomness is that when specifying components of the null sets we exclude, we can change our minds a computably bounded number of times.

What *many* random sets can compute

Theorem (G,Hirschfeldt,Nies;DGT)

The following are equivalent for a Turing degree **a***:*

- **a** *is computable from every ω-computably-approximable ML-random set;*
- **a** *is computable from every superlow ML-random set;*
- **a** *is SJT.*

Theorem (G,Hirschfeldt,Nies)

The following are equivalent for a c.e. degree **a***:*

- **a** *is computable from every superhigh ML-random set.*
- **a** *is SJT.*

Application: superlow cupping

Theorem (G,Nies;DGT)

Every SJT degree $\mathbf{a}$ *is superlow preserving: for every superlow degree* $\mathbf{b}$, $\mathbf{a} \vee \mathbf{b}$ *is also superlow.*

Corollary (Diamondstone)

The notions of low cupping and superlow cupping differ in the c.e. degrees.

Box-promotion

Say A is c.e. and has SJT degree.

We test possible initial segments σ of A by specifying that $g^A(x) = \sigma$ with A-use σ. The test is successful if σ shows up in the trace for g.

- If the test is not successful there is no loss and no gain.
- If the test is successful but σ is incorrect, then the "box" x was promoted.

We make progress by aggregating boxes into "metaboxes". One successful but incorrect test ensures promotion of many boxes, which can then be used for more than one test.

This allows us to limit the number of errors in exponentially many tests by a linear bound.

Box-promotion: the join theorem

Say A and B are c.e. sets of SJT degree. We want to trace $\Psi^{A,B}(n)$.

Test the two halves of the computation separately, on "A-boxes" and on "B-boxes".

What if A is correct and B is not? Then: we lost some A-boxes but promoted many B-boxes.

Now shift priorities.

Cost functions

Cost functions give an analytic yardstick for how good computable approximations are.

Roughly, a cost function $c \colon \mathbb{N} \to \mathbb{R}^+$ specifies the "price" (in the limit) of modifying the value on x of a computable approximation of a Δ^0_2 set.

In turn, we can measure the simplicity of a cost function itself by how easy it is to approximate its values. For example, if we have a computable bound on how many times the cost passes a rational threshold, we call the cost function benign.

Theorem (G,Nies;DGT)

A set A has SJT degree if and only if for every benign cost function c, A has a computable approximation whose c-cost is finite.

Cost function constructions

The general theme: classes of degrees are characterised by the variety of constructions that they allow. A cost function characterisation says that all the degrees in the class are produced by a corresponding cost function construction.

The construction says: only change $A(n)$ at stage s (to meet some Friedberg requirement P_e for example) if the cost $c_s(n)$ of such an action at stage s is smaller than the quota allotted for P_e to spend. Judiciously allot quotas so that total expenditure is finite.

SJT and K-triviality

K trivials:

1. Form a Σ^0_3 ideal, c.e.-generated.
2. Characterised by the standard cost function $c_{\mathcal{K}}$.
3. Computable from incomplete ML-random sets?
4. Bases for ML-randomness.

SJTs:

1. Form a Π^0_4 ideal, c.e.-generated.
2. Characterised by the benign cost functions.
3. Computable from Demuth random sets, and more.
4. Bases for $\text{Demuth}_{\text{BLR}}$ randomness.

SJT and K-triviality

Theorem (Cholak,Downey,G;Downey,G)

The SJT degrees are strictly contained in the K-trivial degrees.

Open problem: where do the K-trivial degrees lie in the jump-traceability hierarchy? In particular: is every K-trivial degree $\log(n)$-jump-traceable?

- (Hölzl,Kräling,Merkle) Every K-trivial degree is $M\log(n)$-jump-traceable for some constant M.
- (Cholak,Downey,G;Turetsky) There is a K-trivial degree which is not $o(\log(n))$-jump-traceable.

Relativising SJT

Lowness notions can often be partially relativised to obtain “weak reducibilities”. For example, K-triviality leads to $\leqslant_{\mathrm{LR}}$, a relation which measures how well an oracle derandomises ML-random sets.

Definition (Nies)

Let $A, B \in 2^\omega$. Then $A \leqslant_{\mathrm{SJT}} B$ if for every order function h, every A-partial computable function has a B-c.e. h-trace.

Question

Does $\leqslant_{\mathrm{SJT}}$ *imply* $\leqslant_{\mathrm{LR}}$*?*

SJT-hard degrees

Definition

- A set A is LR-hard if $\emptyset' \leqslant_{\mathrm{LR}} A$.
- A set A is SJT-hard if $\emptyset' \leqslant_{\mathrm{SJT}} A$.

Theorem (Kjos-Hanssen,Miller,Solomon)

A Turing degree is LR*-hard if and only if it is almost everywhere dominating.*

Question (Nies,Shore,...)

In the c.e. degrees, is there a minimal pair of LR*-hard degrees?*

Pseudojump operators

There are direct constructions of incomplete LR-hard and SJT-hard c.e. degrees. An indirect approach uses pseudojump inversion.

Definition (Jockusch,Shore)

A pseudojump operator is a function $J\colon 2^\omega \to 2^\omega$ such that for all $A \in 2^\omega$, $J(A)$ is uniformly c.e. in A and uniformly computes A. A pseudojump operator is increasing if for all A, $J(A) >_T A$.

Theorem (Jockusch,Shore)

For any pseudojump operator J there is a c.e. set A such that $J(A) \equiv_T \emptyset'$.

Question (Downey,Jockusch,LaForte)

Can this be combined with upper-cone avoidance? Can one always invert to minimal pairs?

Restrictions on pseudojump inversion

Theorem (Downey,G)

There is no minimal pair of SJT*-hard c.e. degrees. In fact, there is an incomputable c.e. set which is computable in every* SJT*-hard c.e. set.*

Corollary

There is a natural, increasing pseudojump operator J_{SJT} *which cannot be inverted to a minimal pair, or while avoiding upper cones.*

SJTH$^\spadesuit$

The ideal SJTH$^\spadesuit$ of all c.e. degrees which are reducible to all SJT-hard c.e. degrees is a new ideal in the c.e. degrees.
The extent of this ideal measures how restricted the construction of an incomplete SJT-hard c.e. set is.

If every K-trivial degree is $\frac{1}{10}\log(n)$-jump traceable, then there is no minimal pair of LR-hard degrees either.

Thank you