Cupping and jump classes in the c.e. degrees

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23rd March 2011
The c.e. Turing degrees form an upper semilattice: every two c.e. degrees $a$ and $b$ have a least upper bound $a \lor b$.

**Definition**
A c.e. degree $a$ is **cuppable** if there is some c.e. degree $b < 0'$ such that $a \lor b = 0'$.

Such a degree $b$ is called a **cupping partner** for $a$. It contains the “missing information” that helps $a$ decide the halting problem.
Existence of cuppable degrees

**Theorem (Sacks)**

*There is an incomplete cuppable degree. In other words, \(0'\) is join-reducible.*

In fact, every c.e. degree is join-reducible.
Noncuppable degrees

Theorem (Cooper; Yates)

Not every c.e. degree is cuppable.

The noncuppable c.e. degrees form a definable ideal.
Noncuppability is special to the c.e. degrees

Theorem (Posner, Robinson)
Every $\Delta^0_2$ degree is cuppable to $0'$ by some incomplete $\Delta^0_2$ degree.
Jump classes

The jump operator is used to classify strata of c.e. degrees: those that are closer to being computable, compared to those that are closer to being complete.

**Definition**
- A c.e. degree $a$ is low if $a' = 0'$. It is high if $a' = 0''$.
- A c.e. degree $a$ is $\text{low}_2$ if $a'' = 0''$. It is $\text{high}_2$ if $a'' = 0'''$.
- etc.
Cuppability and jump classes

**Theorem (Sacks)**
There is a low cuppable degree; in fact a low cuppable degree which has a low cupping partner.

**Theorem (Harrington)**
There is a high noncuppable degree.
A notion dual to cupping is that of **capping**.

**Definition**
A c.e. degree $a$ is **cappable** if there is some c.e. degree $b > 0$ such that $a \land b = 0$.

In other words, $a$ and $b$ have no “common information”.

A major difference between cupping and capping is that the c.e. degrees do **not** form a lower semilattice: some pairs of degrees do not have greatest lower bounds.
Cupping and capping

There is some interaction between cupping and capping.

**Theorem (Harrington)**

*Every c.e. degree is either cappable or cuppable. Some c.e. degrees are both.*

But:

**Theorem (Lachlan)**

*Let \(a\) be a cappable and cuppable c.e. degree. No degree \(b\) can be both a cupping and a capping partner for \(a\).*
Cupping, capping and lowness

The full picture of the interaction between cupping and capping:

**Theorem (Ambos-Spies, Jockusch, Shore, Soare)**

A c.e. degree is noncappable if and only if it has a low cupping partner.

We say that it is *low cuppable*.

The main concept enabling this characterisation is that of **prompt permission**; noncappable c.e. sets resemble $\emptyset'$ in that they get enumerated quickly. The theorem gives a definable decomposition of the c.e. degrees into a filter and an ideal.
Cupping and lowness

A corollary of Harrington’s and the decomposition theorems is the following:

**Corollary**

*Not every cuppable degree is low cuppable.*

Compare:

**Theorem (Shore)**

*Every cappable degree is high cappable.*
High cupping

**Theorem**
*There is a cuppable degree which only has high cupping partners.*

So non-high cuppability is distinct from cuppability.
The proof is a modification of the construction of a noncuppable degree. In that construction, we enumerate a set $A$, and need to:

- make it nonrecursive – this requires enumerating numbers into $A$; and
- ensure that if $W$ is another c.e. set, and $A \oplus W \geq_T \emptyset'$, then $\emptyset' \leq_T W$.

To overcome the tension between these requirements, before enumerating numbers into $A$, we agitate $W$ to change by enumerating a small number into $\emptyset'$, and freezing $A$. This gives us clearance to redefine the reduction of $\emptyset'$ to $W$ and so to enumerate a number into $A$ without ruining that reduction.
In the current construction, we need to make $A$ cuppable – which means enumerating more numbers into $A$.

Now if $A \oplus W$ is complete, then we need to merely ensure that $A$ is high, or “complete in the limit”: we construct a $W$-computable sequence $\langle X_s \rangle$ such that $\lim_s X_s = \emptyset'$. Every subrequirement is now allowed to be injured finitely often, which enables the extra enumerations into $A$. 
The continuity of cupping

**Theorem (Ambos-Spies, Lachlan, Soare)**

*No cuppable degree has a minimal cupping partner.*
Being array recursive is a strong form of being low$^2$, yet incomparable with lowness. In the c.e. degrees, it coincides with being c.e. traceable.

**Theorem**

Let $b$ be a low cupping partner for $a$. Then there is some $c \leq b$, a cupping partner for $a$, which is both low and array recursive.
Theorem (Downey, G, J. Miller, Weber)

Every low cuppable degree is AR cuppable, but some AR cuppable degree is not low cuppable.

Hence not only cuppability in general is distinct from low cuppability, but in fact low\textsubscript{2} cuppability is distinct from low cuppability.
A certain collapse of the hierarchy

**Theorem**
Every low$_2$ cuppable degree is AR cuppable.

**Question**
Do some levels of the hierarchy of the low$_n$ and non-high$_n$ cuppable degrees collapse?
Joining to degrees other than 0′

Some degrees are badly not low cuppable:

**Theorem (Cholak,Groszek,Slaman)**

*There is a degree $a$ such that for every low degree $b$, $a \lor b$ is low.*

This gives another ideal of the c.e. degrees. However, this result cannot be extended to other jump classes:

**Theorem (Jockusch,Li,Yang)**

*Every c.e. degree can be joined to a high degree by a low$_2$ degree.*
Refining the jump classes using strong reducibilities

Definition (Bickford, Mills, Mohrherr)
A set \( A \) is superlow if \( A' \leq_{wtt} \emptyset' \).
This notion is invariant in the Turing degrees.
Superlowness and cupping

**Theorem (Diamondstone)**
Not every low cuppable degree is superlow cuppable.

**Theorem (Ng)**
There is a c.e. degree $a$ such that for every superlow c.e. degree $b$, $a \lor b$ is superlow.
A common specialisation

**Theorem (G,Nies)**

There is a low cuppable degree $a$ such that for every superlow c.e. degree $b$, $a \lor b$ is superlow.

In fact, every strongly jump-traceable c.e. degree has this property of being “almost superdeep”.
12th Asian Logic Conference

- 15th-20th December 2011, in Wellington.
- Additional workshop for students on the 14th.
- There is some NSF funding for US-based people to attend. Talk to Antonio or write to Rod.