FORCING AND CLASSES OF HYP-DOMINATING FUNCTIONS

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Abstract. This paper is aimed at showing separations between three subsets of ω^ω, namely HYP-SNE, HYP-SME, and HYP-DOM. These classes are natural computational analogues of cardinal characteristics from Cichon's diagram and are known to satisfy $\mathsf{HYP}\text{-}\mathsf{SNE} \subseteq \mathsf{HYP}\text{-}\mathsf{SME} \subseteq \mathsf{HYP}\text{-}\mathsf{DOM}$. To show that both of these inclusions are strict we introduce effectivizations of Laver and Hechler forcing, which we believe are of independent interest. Our techniques allow us to show similar results relative to any Turing ideal closed under $\leq_{\mathsf{HYP}}$.

1. Introduction

Given a Turing ideal $\mathcal{I} \subseteq \omega^\omega$, $\mathcal{I}$-DOM, $\mathcal{I}$-SME and $\mathcal{I}$-SNE are three of the so-called *non-lowness classes* of [6], relativized to $\mathcal{I}$. In [6] the authors explain that these classes are natural correspondents (with respect to $\mathcal{I}$) of the cardinal characteristics $\mathfrak{b}$, $\mathsf{add}(\mathcal{M})$, and $\mathsf{add}(\mathcal{N})$. The class $\mathcal{I}$-DOM consists of those reals which $\mathcal{I}$-compute a function dominating all functions in $\mathcal{I}$. The classes $\mathcal{I}$-SNE and $\mathcal{I}$-SME consist of those reals which $\mathcal{I}$-compute, respectively, a Schnorr null set S containing all the $\mathcal{I}$-coded Schnorr null sets, and a $\mathbf{\Sigma}^0_2$-name for a meagre set M containing all the $\mathcal{I}$-coded meagre sets. The computable theory of cardinal characteristics tells us that $\mathcal{I}\text{-}\mathsf{SNE} \subseteq \mathcal{I}\text{-}\mathsf{SME} \subseteq \mathcal{I}\text{-}\mathsf{DOM}$.

The analogy with set theory suggests that the inclusions above should be proper. In [6, Section 8], the authors asked for conditions on an ideal $\mathcal{I}$ corresponding to $\mathcal{I}\text{-}\mathsf{DOM} \neq \mathcal{I}\text{-}\mathsf{SME}$ and $\mathcal{I}\text{-}\mathsf{SME} \neq \mathcal{I}\text{-}\mathsf{SNE}$, noting that this is not the case when $\mathcal{I} = \mathsf{REC}$. Indeed, for any fixed $\alpha < \omega_1^{\mathrm{CK}}$, writing $\mathcal{I}_{\omega^\alpha}$ for the ideal obtained by closing REC under α-jump, we get $\mathcal{I}_{\omega^\alpha}\text{-}\mathsf{SNE} = \mathcal{I}_{\omega^\alpha}\text{-}\mathsf{DOM}$.

We introduce modifications of Laver and Hechler forcing inspired by the bushy tree forcing of Kumabe and Lewis [1] to show that $\mathcal{I}\text{-}\mathsf{SNE} \subsetneq \mathcal{I}\text{-}\mathsf{SME} \subsetneq \mathcal{I}\text{-}\mathsf{DOM}$ for every ideal $\mathcal{I}$ closed under $\leq_{\mathsf{HYP}}$. This provides a near-optimal solution to the problem raised in [6].

1.1. **Computable cardinal characteristics.**

We now give a very brief introduction to cardinal characteristics of the continuum and their computable analogues, for more see ([2], [1], [6] and [4]).

1.1.1. *Cardinal characteristics as problems.*

In the context of cardinal characteristics, a *problem* is a relation $R \subseteq \omega^\omega \times \omega^\omega$. Elements in the domain of R are *instances* of R, and if x is an instance of R, then a real $y \in \omega^\omega$ such that $(x, y) \in R$ is a *solution* for the instance x of R.

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[1]The paper where bushy tree forcing is first introduced is [8]. For a systematic exposition forcing with bushy trees see [7].

Remark 1.1. In the rest of the paper we also work with problems which, at first sight, appear to have instances and solutions which are not elements of ω^ω. For example, we work with problems which have Borel subsets of ω^ω as instances. In reality all our problems are defined on the codes of these objects. This is not only a notational matter, but it reflects an actual mathematical issue. For more on this, see for example [6, Remark 8.1].

Definition 1.2 (Complete solution set)**.** Let $R \subseteq \omega^\omega \times \omega^\omega$ be a problem and let $S \subseteq \omega^\omega$. We say that S is a *complete solution set* for R if for every instance $x \in \operatorname{dom}(R)$ there is a solution $y \in S$ to the instance x.

Definition 1.3 (Norm of a problem)**.** Let $R \subseteq \omega^\omega \times \omega^\omega$ be a problem. The *norm of* R, denoted $|R|$, is the cardinal $\min\{|S| : S \subseteq \omega^\omega \text{ is a complete solution set for } R\}$.

Example 1.4. Consider the domination preorder $\leq^* \subseteq \omega^\omega \times \omega^\omega$ defined as $f \leq^* g$ if and only if $\forall^\infty n\ f(n) \leq g(n)$. Viewing $\leq^*$ as a problem, we have that $|\leq^*|$ is the least size of a family $S \subseteq \omega^\omega$ such that, for every $g \in \omega^\omega$, there is $f \in S$ with $g \leq^* f$. The cardinal $|\leq^*|$ is better known as the *dominating number*, denoted $\mathfrak{d}$.

The cardinal $\mathfrak{d}$ is a cardinal characteristic of the continuum. Although there is no single definition of what a cardinal characteristic of the continuum is, all cardinal characteristics arise as norms of problems, and if A is a cardinal characteristic, then $\mathsf{ZFC} \vdash \omega_1 \leq A \leq \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum.

Since the continuum hypothesis is consistent with ZFC, if ZFC is consistent, then there are models in which all cardinal characteristics coincide and have value $\omega_1 = \mathfrak{c}$. Hence, set theorists have worked on showing statements of the form $\mathsf{ZFC} \vdash |A| \leq |B|$, or of the form "it is consistent with ZFC that $|A| < |B|$", where A and B are problems associated with cardinal characteristics.

Showing that ZFC proves an inequality $|A| \leq |B|$ is typically done by showing that A is "an easier problem" than B. This is highlighted in the setup of [2].

Definition 1.5. A *Tukey connection* from A to B is a pair of (possibly partial) functions (φ, ψ), each of type $\omega^\omega \to \omega^\omega$, such that for every $x \in \operatorname{dom}(A)$, $\varphi(x)$ is defined and belongs to $\operatorname{dom}(B)$ and, for every B-solution y to $\varphi(x)$, $y \in \operatorname{dom}(\psi)$ and $\psi(y)$ it is an A-solution of x.

Notice that, in the definition of Tukey connection, if we asked that φ and ψ were computable, then we would recover the definition of a strong Weihrauch reduction.

Lemma 1.6 (Folklore)**.** *Let A and B be problems, and let (φ, ψ) be a Tukey connection from A to B. Then $|A| \leq |B|$.*

Proof. Let S be a complete solution set for B. We show that $\psi[S]$ is a complete solution set for A. Indeed, let $x \in \operatorname{dom}(A)$. Then $x \in \operatorname{dom}(\varphi)$, $\varphi(x) \in \operatorname{dom}(B)$, and since S is a complete solution set for B, there is some $y \in S$ with $(\varphi(x), y) \in B$. By definition of Tukey connection we have $y \in \operatorname{dom}(\psi)$ and $(x, \psi(y)) \in A$, so x has an A-solution in $\psi[S]$. This shows that $\psi[S]$ is a complete solution set for B.

Now let $\hat{S}$ be a complete solution set for B of minimal cardinality. Since $\psi[\hat{S}]$ is a complete solution set for A and $|\psi[\hat{S}]| \leq |\hat{S}|$, we obtain $|A| \leq |B|$. $\square$

Featuring in this paper is *Cichon's diagram*, arguably a cornerstone of modern set theory. It summarizes the ZFC provable inequalities among ten cardinal characteristics of the continuum. Two of these are related to the domination preorder

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

FIGURE 1. Cichon's diagram. An arrow going from A to B denotes $\mathsf{ZFC} \vdash A \leq B$. No binary relation between the cardinals in the diagram, other than the ones shown, can be proved in ZFC.

($\leq^*$), while the remaining eight are related to the σ-ideals $\mathcal{M}$ and $\mathcal{N}$ of, respectively, meagre and null subsets of ω^ω. We will only define the cardinals (and the related problems) in Cichon's diagram which are relevant for this paper. Readers interested in more information should consult, e.g. [2].

1.1.2. *Effective morphisms.* In recent years researchers have noticed that the Tukey connections used to show ZFC provable inequalities between cardinal characteristics in Cichon's diagram, are, in some sense, computable (see [15], [4] and [6])). In particular, the authors of [6] identify a class of functions $\mathcal{F}$ such that:

- every inequality $|A| \leq |B|$ between cardinals in Cichon's diagram is witnessed by a Tukey connection (φ, ψ) with φ and ψ both in $\mathcal{F}$,
- for every $\varphi \in \mathcal{F}$ and every $x \in \mathrm{dom}(\varphi)$, $\varphi(x) \leq_{\mathrm{T}} x$
- if φ and ψ are in $\mathcal{F}$, then $\varphi \circ \psi$ is in $\mathcal{F}$.

They referred to Tukey connections in which both functions are in $\mathcal{F}$ as *effective morphisms*, and used the notation $A \leq_{\mathrm{W}} B$ to say "there is an effective morphism from A to B". It is routine to check that the relation $\leq_{\mathrm{W}}$ so defined is a preorder.

Theorem 1.7 (Folklore, see [15], [4] and [6]). *Let $|A|$ and $|B|$ be cardinal characteristics from Cichon's diagram. Then $A \leq_{\mathrm{W}} B$.*

The existence of an effective morphism from problem A to problem B is more informative than the mere existence of a Tukey connection, as it gives computational information about the problems.

Definition 1.8 (Non-lowness classes). Let A be a problem with $|A| \geq \omega_1$ and let $\mathcal{I}$ be a countable Turing ideal. The *non-lowness class* of A relative to $\mathcal{I}$ is defined as $\mathrm{NL}^{\mathcal{I}}(A) = \{x \in \omega^\omega : \exists y \in \mathcal{I}\ \exists z \leq_{\mathrm{T}} x \oplus y\ (z \in \mathrm{dom}(A) \wedge \forall w \in \mathcal{I}\ (z, w) \notin A)\}$.

Intuitively, the non-lowness class of A relative to $\mathcal{I}$ is given by those reals x which can compute (with the help of some oracle from $\mathcal{I}$) an instance of A which has no solution in $\mathcal{I}$. If $\omega_1 \geq |A|$, then $\mathrm{NL}^{\mathcal{I}}(A) \neq \emptyset$ because $\mathcal{I}$ is assumed countable, hence it cannot contain complete solution set for A.

Notation 1.9. In the rest of the paper, for every Turing ideal $\mathcal{I}$, we write $x \leq^{\mathcal{I}} y$ (or y $\mathcal{I}$-computes x) to mean that there is $z \in \mathcal{I}$ such that $x \leq_{\mathrm{T}} y \oplus z$.

Notice that $\mathcal{I}$ is a superscript in the notation $\leq^{\mathcal{I}}$. We chose this notation because we will also frequently have to write $\leq_{\mathsf{HYP}}$ to denote hyperarithmetical reducibility. It is well known and easy to see that $\leq^{\mathsf{HYP}} \neq \leq_{\mathsf{HYP}}$.

Lemma 1.10 (See [15], [4] and [6])**.** *Let $\mathcal{I}$ be a countable Turing ideal and let A and B be problems with $\omega_1 \leq \min\{|A|, |B|\}$. We have $\mathrm{NL}^{\mathcal{I}}(A) \subseteq \mathrm{NL}^{\mathcal{I}}(B)$.*

Proof. Let (φ, ψ) be an effective morphism from A to B, let $x \in \mathrm{NL}^{\mathcal{I}}(A)$ and let $y \leq^{\mathcal{I}} x$ be an instance of A which has no solutions in $\mathcal{I}$. If $\varphi(y)$ had a B-solution $z \in \mathcal{I}$, then by definition of effective morphism $\psi(z)$ would be an A-solution to y. Since $\psi(z) \leq_{\mathrm{T}} z$, we have $\psi(z) \in \mathcal{I}$, which is a contradiction. Hence $\varphi(y) \in \mathrm{NL}^{\mathcal{I}}(B)$. Since $\varphi(y) \leq_{\mathrm{T}} y \leq^{\mathcal{I}} x$, it follows that $x \in \mathrm{NL}^{\mathcal{I}}(B)$. $\square$

Lemma 1.10, together with Theorem 1.7 gives new perspective on some well-known inclusions between classes of computational strength (for more, see [6]). The contrapositive of Lemma 1.10 reads: if $\mathrm{NL}^{\mathcal{I}}(A) \not\subseteq \mathrm{NL}^{\mathcal{I}}(B)$ for some countable Turing ideal $\mathcal{I}$, then there cannot be an effective morphism from A to B.

In [6, Section 8], the authors point out that one of the key ingredients in the typical set theoretic proof of the consistency of $|A| < |B|$ is showing that, if V is a countable model of ZFC, then $\mathrm{NL}^{V}(B) \not\subseteq \mathrm{NL}^{V}(A)$. Indeed, one usually starts with a model of CH and iterates for ω_2 times a forcing notion $\mathbb{P}$ which adds some $x \in \mathrm{NL}^{V}(B) \setminus \mathrm{NL}^{V}(A)$, and the model so obtained satisfies $\omega_1 = |A| < |B| = \omega_2$.

Putting everything together, we get that if $|A|$ and $|B|$ are cardinal characteristics from Cichon's diagram, then $\mathsf{ZFC} \vdash |A| \leq |B|$ if and only if $A \leq_{\mathrm{W}} B$ if and only if, for every countable Turing ideal $\mathcal{I}$, $\mathrm{NL}^{\mathcal{I}}(A) \subseteq \mathrm{NL}^{\mathcal{I}}(B)$. This lends credence to the claim that the non-lowness classes are analogues of cardinal characteristics, while also having a computational flavor (and making sense regardless of the set theoretic universe one works in).

1.1.3. *Understanding exceptional collapses.* There are cardinal characteristics $|A|$ and $|B|$ in Cichon's diagram such that (i) ZFC proves $|A| \leq |B|$, (ii) it is consistent that $|A| < |B|$, but (iii) we have $\mathrm{NL}^{\mathsf{REC}}(A) = \mathrm{NL}^{\mathsf{REC}}(B)$. One example of this phenomenon, which we will not explore here, is given by the cardinals $\mathsf{cov}(\mathcal{M})$ and $\mathfrak{d}$. ZFC proves $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$, and it is consistent that the inequality is strict, but their corresponding non-lowness classes relative to the ideal of computable functions, namely the class of functions which can compute a weakly 1-generic set and the class of functions which can compute a hyperimmune set respectively, coincide (see [6, Section 8.3]). There is only one other phenomenon of this kind in the computable Cichon's diagram, and it involves the cardinals $\mathfrak{b}$, $\mathsf{add}(\mathcal{M})$, and $\mathsf{add}(\mathcal{N})$. This collapse is the starting point for this paper.

Definition 1.11. The *bounding number* $\mathfrak{b}$ is $|(\leq^*)^\perp|$, where $f(\leq^*)^\perp g$ if and only if $g \not\leq^* f$ (read: f does not dominate g, or g *escapes* f).

The *additivity of the meager ideal* $\mathsf{add}(\mathcal{M})$ is the least cardinality of a set $\mathcal{F}$ of $\mathbf{\Sigma}^0_2$-meager sets such that $\bigcup \mathcal{F}$ is not meager. The problem which gives rise to $\mathsf{add}(\mathcal{M})$ is $\mathsf{Spill}(\mathcal{M})$, defined as follows: an instance is a (code for a) $\mathbf{\Sigma}^0_2$-meager set $A \subseteq \omega^\omega$, and a solution to the instance A is a (code for a) $\mathbf{\Sigma}^0_2$-meager set $B \subseteq \omega^\omega$ with $B \not\subseteq A$.

The *additivity of the null ideal* $\mathsf{add}(\mathcal{N})$ is the least cardinality of a set $\mathcal{G}$ of $\mathbf{\Pi}^0_2$-null sets such that $\bigcup \mathcal{G}$ is not null. The problem which gives rise to $\mathsf{add}(\mathcal{N})$ is $\mathsf{Spill}(\mathcal{N})$, defined as follows: an instance is a (code for a) $\mathbf{\Pi}^0_2$-null set $A \subseteq \omega^\omega$, and a solution to the instance A is a (code for a) $\mathbf{\Pi}^0_2$-null set $B \subseteq \omega^\omega$ with $B \not\subseteq A$.

We have that ZFC proves $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M}) \leq \mathfrak{b}$, but these cardinals are consistently distinct. We now give abbreviations for the non-lowness classes associated to these cardinals (more precisely, to the problems which give rise to them).

Definition 1.12. For any countable Turing ideal $\mathcal{I}$, we define

- $\mathcal{I}$-DOM = $\mathrm{NL}^{\mathcal{I}}((\leq^*)^{\perp})$. These are the reals x which can $\mathcal{I}$-compute a function y which dominates every function in $\mathcal{I}$,
- $\mathcal{I}$-$\mathsf{SME} = \mathrm{NL}^{\mathcal{I}}(\mathsf{add}(\mathcal{M}))$: these are the reals x which can $\mathcal{I}$-compute a (code for a) $\mathbf{\Sigma}^0_2$-meager set A such that, if B is a $\mathbf{\Sigma}^0_2$-meager set which is coded in $\mathcal{I}$, then $B \subseteq A$. We call elements of $\mathcal{I}$-SME *strongly $\mathcal{I}$-meager engulfing*,
- $\mathcal{I}$-$\mathsf{SNE} = \mathrm{NL}^{\mathcal{I}}(\mathsf{add}(\mathcal{N}))$: these are the reals x which can $\mathcal{I}$-compute a (code for a) $\mathbf{\Pi}^0_2$-null set A such that, if B is a $\mathbf{\Pi}^0_2$-null set which is coded in $\mathcal{I}$, then $B \subseteq A$. We call elements of $\mathcal{I}$-SNE *strongly $\mathcal{I}$-null engulfing*.

In the definition of $\mathcal{I}$-SNE, we use Schnorr tests to code $\mathbf{\Pi}^0_2$-null sets.

By Theorem 1.7, we have that $\mathcal{I}$-$\mathsf{SNE} \subseteq \mathcal{I}$-$\mathsf{SME} \subseteq \mathcal{I}$-$\mathsf{DOM}$. In contrast with Cichon's diagram, we have $\mathcal{I}$-SNE = $\mathcal{I}$-SME = $\mathcal{I}$-DOM when $\mathcal{I} = \mathsf{REC}$ (more generally, whenever $\mathcal{I}$ has a top Turing degree), and also, for every $\alpha < \omega_1^{\mathrm{CK}}$, letting $\mathcal{I}_\alpha = \{x \in \omega^\omega : \exists \beta < \alpha \ (x \leq_{\mathrm{T}} 0^{(\beta)})\}$, $\mathcal{I}_\alpha$-$\mathsf{SNE} = \mathcal{I}_\alpha$-$\mathsf{SME} = \mathcal{I}_\alpha$-$\mathsf{DOM}$. This is due to the fact that, for all $\alpha < \omega_1^{\mathrm{CK}}$, $\mathcal{I}_\alpha$-$\mathsf{DOM} = \mathcal{I}_\alpha$-$\mathsf{WL}$, where the class $\mathcal{J}$-WL (for *weak listing* of $\mathcal{J}$) is defined as $\{x \in \omega^\omega : \exists y \leq^{\mathcal{J}} x \ \forall z \in \mathcal{J} \ \exists n \ (y^{[n]} = z)\}$, and for every countable Turing ideal $\mathcal{I}$, $\mathcal{I}$-$\mathsf{WL} \subseteq \mathcal{I}$-$\mathsf{SNE}$. Proofs of these facts can be found in [11].

As anticipated, the goal of this paper is to show that HYP-$\mathsf{SNE} \subsetneq \mathsf{HYP}$-$\mathsf{SME} \subsetneq \mathsf{HYP}$-$\mathsf{DOM}$, and more generally that $\mathcal{I}$-$\mathsf{SNE} \subsetneq \mathcal{I}$-$\mathsf{SME} \subsetneq \mathcal{I}$-$\mathsf{DOM}$ for every ideal $\mathcal{I}$ which is closed under $\leq_{\mathsf{HYP}}$. The discussion above shows that these results are in some sense optimal. This can be interpreted as saying that any ideal $\mathcal{I}$ closed under $\leq_{\mathsf{HYP}}$ (in particular, HYP) "knows enough set theory" to realize that $\mathsf{Spill}(\mathcal{N})$ is an easier problem than $\mathsf{Spill}(\mathcal{M})$, which is in turn an easier problem than $(\leq^*)^{\perp}$. In the analogy between cardinal characteristics and non-lowness classes, we could say that being closed under $\leq_{\mathsf{HYP}}$ is enough to obtain evidence that $\mathsf{add}(\mathcal{N}) \neq \mathsf{add}(\mathcal{M}) \neq \mathfrak{b}$.

The discussion above suggests another problem of interest: studying the circumstances in which $\mathcal{I}$-$\mathsf{WL} \neq \mathcal{I}$-$\mathsf{SNE}$. This, together with more results on listing of HYP (and listing of ideals of the form $\mathcal{I}_\alpha$ above), is in [11].

2. Preliminaries

Throughout the paper we assume the reader has some familiarity with the basic concepts of computability theory (in particular, with the relation $\leq_{\mathrm{T}}$, the concepts of Turing ideal and Turing jump), the theory of Π^1_1 and hyperarithmetical subsets of ω and of ω^ω, and several other basic notions of (effective) descriptive set theory. Standard references for these topics are, respectively, [14], [16] and [12]. We work with concepts coming from the topology and the measure of the real line (which we silently identify with either ω^ω or 2^ω, depending on the context, as is customary in set theory). For more on these topics, we refer to [1]. We also work with forcing, but we do not assume any familiarity with it and introduce all the notions necessary for (computability theoretic) forcing.

In the rest of the section, we first introduce some basic general notions from logic and topology, and then we move on to forcing and specifically forcing notions designed to add reals.

2.1. **Notation for trees, partial orders, and closure operators.** We denote the prefix relation on $\omega^{<\omega}$ as $\preceq$. We denote the concatenation of strings σ and τ as $\sigma^\frown\tau$. If $n \in \omega$, we write $\sigma^\frown n$ instead of $\sigma^\frown\langle n\rangle$. If $\sigma, \tau \in \omega^{\leq\omega}$, we write $\sigma \leq \tau$ when $\sigma(i) \leq \tau(i)$ for every i in their common domain, unless we specify otherwise. In particular this allows us to compare strings of different length. We consider the pair $(\omega^{<\omega}, \preceq)$ as a partial order to deal with upwards closures, defined as usual.

Definition 2.1. If $(P, \leq)$ is a partial order, given a set $A \subseteq P$, the *upwards closure* of A is the set $\uparrow^P A = \{y \in P : \exists x \in A\ (x \leq y)\}$. Normally the ambient partial order P will be clear from the context, and we will simply write $\uparrow A$. If $A = \{x\}$ is a singleton, we write $\uparrow x$ instead of $\uparrow\{x\}$. A set $A \subseteq P$ such that $\uparrow A = A$ is called *upwards closed*.

Most of this paper involves trees, which we now introduce formally to fix terminology that will be used throughout.

Definition 2.2. A *tree* on ω is a set $T \subseteq \omega^{<\omega}$ which is closed under initial segment, i.e. if $\sigma \in T$ and $\tau \preceq \sigma$, then $\tau \in T$.

The elements of these trees are finite sequences of elements of ω, sometimes called *strings* or *nodes*. Given a node σ in a tree T, the sequences of the form $\sigma^\frown n$, with $n \in \omega$, which belong to T are the *children* of σ in T. A node $\sigma \in T$ is called *splitting* if it has at least two children on T. The *stem* of T, denoted $\mathrm{stem}(T)$, is the $\preceq$-least splitting element in T.

Given a tree T and a node $\sigma \in T$, the *full subtree* of T generated by σ is the set

$$T_\sigma = \{\tau \in T : (\tau \preceq \sigma \vee \sigma \preceq \tau)\}$$

A node on a tree T which does not have any extension in T is called a *leaf*. A tree is *pruned* if it has no leaves, and it is *perfect* if every node has at least two incompatible extensions. A *path* in T is a sequence $f \in \omega^\omega$ such that $f\restriction n \in T$ for every n. We denote by $[T]$ the set of paths of T.

Notation 2.3. We call a set of strings $S \subseteq \omega^{<\omega}$ a *code for an open* when we are viewing S as encoding the open set $[S]^\prec = \{f \in \omega^\omega : \exists\sigma \in S\ (\sigma \prec f)\}$. For any $S \subseteq \omega^{<\omega}$ we can define a set $\mathrm{pf}(S) = \{\sigma \in S : \forall\sigma' \prec \sigma\ \sigma' \notin S\}$. The set $\mathrm{pf}(S)$ consists of the $\prec$-minimal elements of S, so in particular it is prefix-free, and it satisfies $\mathrm{pf}(S) \leq_\mathrm{T} S$ and $[\mathrm{pf}(S)]^\prec = [S]^\prec$.

The forcing notions we introduce to prove our main results rest on concepts of *bigness notions*. These turn out to be topological closure operators.

Definition 2.4. Let X be a set. A function $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a *topological closure operator* if it preserves the empty set, so $\mathrm{cl}(\emptyset) = \emptyset$, it is *extensive*, so $A \subseteq \mathrm{cl}(A)$ for every $A \subseteq X$, it is *idempotent*, so $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for every $A \subseteq X$, and it *commutes with binary union*, so $\mathrm{cl}(A\cup B) = \mathrm{cl}(A)\cup\mathrm{cl}(B)$ for every $A, B \subseteq X$.

Note that if a function cl is extensive, then cl is also idempotent if and only if $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$ for every $A \subseteq X$. It is well known (and an immediate consequence of the definition) that topological closure operators are monotone with respect to $\subseteq$. We state an easy fact about the definability of monotone operators.

Lemma 2.5. *Let $f, g\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be such that for every x, $f(x)$ and $g(x)$ are $\Pi^1_1(x)$, uniformly in x. Assume moreover that g is monotone with respect to $\subseteq$. Then, for every x, $g(f(x))$ is $\Pi^1_1(x)$, uniformly in x.*

Proof. Let φ and ψ be Π^1_1 formulae such that for every $n \in \omega$ and $x \subseteq \omega$, $\varphi(n,x) \leftrightarrow n \in f(x)$ and $\psi(n,x) \leftrightarrow n \in g(x)$.

By monotonicity of g, we obtain that for any $n \in \omega$ and $x \subseteq \omega$, $n \in g(x)$ if and only if $n \in g(y)$ for every $y \supseteq x$. We exploit this fact to prove our claim. We obtain, for every $n \in \omega$ and $x \subseteq \omega$:

$$n \in g(f(x)) \leftrightarrow \psi(n, f(x)) \leftrightarrow \forall y \subseteq \omega\ [(\forall m \in \omega\ \varphi(m,x) \to m \in y) \to \psi(n,y)].$$

Note that the last formula displayed says "for any $y \subseteq \omega$, if y is a superset of $f(x)$, then $n \in g(y)$". This is equivalent to

$$\forall y \subseteq \omega\ [(\exists m \in \omega\ \varphi(m,x) \wedge m \notin y) \vee \psi(n,y)]$$

and this formula is Π^1_1. $\square$

Our notation for computability theory is mostly standard. Here, we point out the following.

Notation 2.6. Let Φ be a Turing functional. We use the symbol Φ to denote two formally different objects: a computable (with respect to some oracle) partial monotone function on finite strings, and the unique partial computable (continuous) function $\omega^\omega \to \omega^\omega$ that this function induces.

Moreover, if Φ is a Turing functional and $x \in \omega^\omega$, we write $\Phi(x)\downarrow$ to mean that $x \in \mathrm{dom}(\Phi)$, viewing Φ as a partial function from ω^ω to itself. When we look at Φ as a function on strings, we adopt the further convention that elements in the range of Φ are defined on initial segments of ω, i.e. we write $\Phi(\sigma)(n)\downarrow$ if and only if $\Phi(\sigma)(m)\downarrow$ for every $m \leq n$.

2.2. **Forcing.**

The main results in this paper are proved by forcing over countable ideals. This technique is widespread in computability theory. Here we give a brief introduction to this incarnation of forcing.

2.2.1. *Basic definitions.*

Definition 2.7. A *forcing notion* is a triple $(\mathbb{P}, \leq, \mathbb{1})$, where $\mathbb{P}, \leq$ is a partial order on $\mathbb{P}$, and $\mathbb{1}$ is the top element of the partial order. Elements of forcing notions are called *conditions*. If $p, q \in \mathbb{P}$ and $p \leq q$, we say that p *extends* q. Two elements of $\mathbb{P}$ are called *compatible* if they have a common extension, *incompatible* otherwise.

If $\mathcal{I}$ is a Turing ideal, we say that $(\mathbb{P}, \leq, \mathbb{1})$ is a *forcing notion in* $\mathcal{I}$ if $\mathbb{P} \subseteq \mathcal{I}$.[2] Given any forcing notion $(\mathbb{P}, \leq, \mathbb{1})$ with $\mathbb{1} \in \mathcal{I}$, $(\mathbb{P} \cap \mathcal{I}, \leq, \mathbb{1})$ is a forcing in $\mathcal{I}$.

Definition 2.8. A *filter* G on a forcing notion $(\mathbb{P}, \leq, \mathbb{1})$ is a subset of $\mathbb{P}$ which is nonempty, upwards closed and *downward directed* (i.e. for every $p, q \in G$, p and q are compatible).

We say that a filter G on a partial order $\mathbb{P}$ *meets* a set $D \subseteq \mathbb{P}$ if $G \cap D \neq \emptyset$.

Definition 2.9. Let $(\mathbb{P}, \leq, \mathbb{1})$ be a forcing notion, let $p \in \mathbb{P}$ and $D \subseteq \mathbb{P}$. The set D is called *dense below* p if for every $q \leq p$ there is some $r \in D$ such that $r \leq q$. The set D is *dense* if it is dense below p for every $p \in \mathbb{P}$ (equivalently if it is dense below $\mathbb{1}$).

The most important result in forcing theory is the following.

[2] This presupposes some coding of the elements of $\mathbb{P}$ as elements of ω^ω.

Lemma 2.10 (Rasiowa-Sikorski). *Let $(\mathbb{P}, \leq, \mathbb{1})$ be a forcing notion, let $p \in \mathbb{P}$ and let $(D_n)_{n\in\omega}$ be a countable sequence of dense sets. There exists some filter $G \subseteq \mathbb{P}$ which meets each D_n.*

Every forcing notion $(\mathbb{P}, \leq, \mathbb{1})$ which we consider in this paper is designed to add a real. We call such forcing notions *real forcings*.[3]

Definition 2.11. A *real forcing* is a quadruple $(\mathbb{P}, \leq \mathbb{1}, \pi)$, where $(\mathbb{P}, \leq \mathbb{1})$ is a forcing notion and $\pi \colon \mathbb{P} \to \omega^{<\omega}$ is such that

- if $p \leq q$, then $\pi(q) \subseteq \pi(p)$,
- for every m, the set $L_m = \{p \in \mathbb{P} : |\pi(p)| \geq m\}$ is dense in $\mathbb{P}$.

Any filter G which meets all the L_m's thus determines a real $\pi(G) = \bigcup_{p\in G} \pi(p)$. A real forcing is in $\mathcal{I}$ if $(\mathbb{P}, \leq \mathbb{1})$ is in $\mathcal{I}$ and π is computable.

In the rest of the paper, we say that a filter $G \subseteq \mathbb{P}$ is in $\mathrm{dom}(\pi)$ to mean that G meets all the sets L_m in the definition above. Since this is a countable family of dense sets, any sufficiently generic G will be in $\mathrm{dom}(\pi)$. It is easy to see that, if $G \in \mathrm{dom}(\pi)$ and $p \in G$, then $\pi(G) \in [\pi(p)]$ (here, $[t]$ denotes the basic clopen subset of ω^ω determined by the finite string t).

2.2.2. *Forcing over countable ideals.*

Definition 2.12. Let $\mathcal{I}$ be a Turing ideal and let $(\mathbb{P}, \leq, \mathbb{1}, \pi)$ be a real forcing in $\mathcal{I}$. For any $G \in \mathrm{dom}(\pi)$, we define $\mathcal{I}(G) = \mathcal{I}(\pi(G)) = \{x \in \omega^\omega : \exists y \ (x \leq_{\mathrm{T}} y \oplus \pi(G))\}$.

Often we will introduce a special notation for the real $\pi(G)$, such as x_G or y_G. It is immediate from the definition that $\mathcal{I}(G)$ is a Turing ideal.

Remark 2.13. An alternative way to characterize $\mathcal{I}(G)$ is to say that it is the set of reals which are images of $\pi(G)$ under Turing functionals coded in $\mathcal{I}$. Thus, in the analogy with set theory, forcing over an ideal $\mathcal{I}$ allows us to construct another ideal $\mathcal{I}(G)$, whose elements are named by Turing functionals in $\mathcal{I}$.

As usual, we can use properties of $\mathbb{P}$ to infer the truth of formulas about $\mathcal{I}(G)$, for sufficiently generic filters G.

Definition 2.14. Let $\mathcal{I}$ be a Turing ideal, let$(\mathbb{P}, \leq, \mathbb{1}, \pi)$ be a real forcing in $\mathcal{I}$, and let $\varphi(x)$ be a formula in the language of second order arithmetic with one free second order variable x (we allow second order parameters in φ). We define the relation $p \Vdash \varphi(\pi(G))$ by induction on the construction of φ, as:

- if φ is Δ^0_0, then $p \Vdash \varphi(\pi(G))$ if and only if $\varphi(x)$ holds for all $x \in [\pi(p)]$,
- if $\varphi = \psi \wedge \chi$, then $p \Vdash \varphi$ if and only if $p \Vdash \psi$ and $p \Vdash \chi$,
- if $\varphi = \neg\psi$, then $p \Vdash \varphi$ if and only if there is no $q \leq p$ such that $q \Vdash \psi$
- if $\varphi = \exists n \psi$, then $p \Vdash \varphi$ if and only if $\{q \in \mathbb{P} : \exists m \in \omega (q \Vdash \psi(m))\}$ is dense below p,
- if $\varphi = \exists f \psi$, then $p \Vdash \varphi$ if and only if $\{q \in \mathbb{P} : \exists \Gamma \in \mathcal{I} \ (q \Vdash \Gamma(\pi(G)) \downarrow \wedge \psi(\Gamma(\pi(G), \pi(G))))\}$, where Γ is intended to be a Turing functional, is dense below p.

---

[3]It is not hard to see that one could also interpret each forcing notion in this paper as a partial order consisting of closed subsets of either ω^ω or 2^ω (or $2^\omega \times \omega^\omega$ for the forcing $\mathbb{I}$ of Section 6.2). For a general view on forcing with closed sets in a context similar to ours, see [5].

Now we could define, for any countable Turing ideal $\mathcal{I}$ and any real forcing in $\mathcal{I}$, a notion of $\mathcal{I}$-genericity in analogy to how it is done in set theory. We would say a filter $G \subseteq \mathbb{P}$ is *$\mathcal{I}$-generic* if it meets every every dense set $D \subseteq \mathbb{P}$ which is definable in $\mathcal{P}(\omega)$ by a formula of second order arithmetic with parameters from $\mathcal{I}$. Note that any $\mathcal{I}$-generic G belongs to $\mathrm{dom}(\pi)$ in the sense of Definition 2.11, as each $L_m \subseteq \mathbb{P}$ is easily definable in terms of π.

With this definition, we could prove by induction on the construction of formulas the following folklore result (sometimes called the *forcing theorem*).

Theorem 2.15. *Let $\mathcal{I}$ be a Turing ideal, let $(\mathbb{P}, \leq, \mathbb{1}, \pi)$ be a real forcing in $\mathcal{I}$, and let $\varphi(x)$ be a formula in the language of second order arithmetic. Then $p \Vdash \varphi(\pi(G))$ if and only if, for every $\mathcal{I}$-generic filter $G \subseteq \mathbb{P}$ which contains p, we have $\mathcal{I}(G) \Vdash \varphi(\pi(G))$.*

In fact, we will not need the notion of $\mathcal{I}$-genericity: every time we want to ensure that a real built via forcing has a given property, we specify precisely the dense sets that we are interested in meeting with our generic filter in order to ensure such property. These dense sets are always fewer than those necessary for $\mathcal{I}$-genericity. Throughout the paper, we frequently write sentences such as "... for every sufficiently generic filter G ...". These are to be interpreted as: "... there is a countable family of dense sets $(D_n)_{n\in\omega}$ such that, for every filter G which meets all the D_n's ...".

2.2.3. *Some notions of forcing.* Here we describe some (classes of) real forcing notions which feature in the paper.

Definition 2.16. Cohen forcing consists of the triple $(2^{<\omega}, \supseteq, \emptyset)$. The map π is simply the inclusion from $2^{<\omega}$ to $\omega^{<\omega}$. When referring to Cohen forcing, we denote the set $2^{<\omega}$ as $\mathbb{C}$.

Since the conditions of Cohen forcing are finite, we can assume that $\mathbb{C}$ is in $\mathcal{I}$ for any ideal $\mathcal{I}$. Others also call the triple $(\omega^{<\omega}, \supseteq, \emptyset)$ Cohen forcing. These two are known to be equivalent in a precise sense, which we will not explore here.

Definition 2.17. A real forcing $(\mathbb{P}, \leq, \mathbb{1}, \pi)$ is an *arboreal forcing* if elements of $\mathbb{P}$ are perfect subtrees of $\omega^{<\omega}$, $\leq$ is $\subseteq$, π is the map $T \mapsto \mathrm{stem}(T)$, and, for every $T \in \mathbb{P}$ and every $\tau \in T$, the tree T_τ is a condition of $\mathbb{P}$.

Now we can introduce Laver and Hechler forcing. These are arboreal forcing notions, and they will play a key role throughout the paper.

Definition 2.18. Laver forcing $(\mathbb{L}, \leq)$ is defined as follows: a condition is a pruned tree $T \subseteq \omega^{<\omega}$ such that, for every $\sigma \in T$, if $\sigma \succeq \mathrm{stem}(T)$, then, there are infinitely many n such that $\sigma^\frown n \in T$, and if $T, S \in \mathbb{L}$, then $T \leq S$ if and only if $T \subseteq S$. The map π which makes $\mathbb{L}$ a real forcing sends $T \in \mathbb{L}$ to $\mathrm{stem}(T)$.

Definition 2.19. Hechler forcing $(\mathbb{H}, \leq)$ is defined as follows: a condition is a pair $(\tau, f) \in \omega^{<\omega} \times \omega^\omega$ and $(\tau, f) \leq (\sigma, g)$ if $\tau \succeq \sigma$, $f \geq g$, and, for every $i \in \mathrm{dom}(\tau) \setminus \mathrm{dom}(\sigma)$, $\tau(i) \geq g(i)$.

It is not hard to see that Hechler forcing can be presented as an arboreal forcing: to see this, associate to any condition $(\tau, f) \in \mathbb{H}$, the tree given by $\{\sigma \preceq \tau\} \cup \{\sigma \succeq \tau : \forall |\tau| \leq i < |\sigma|\ (\sigma(i) \geq f(i))\}$.

As mentioned above, if $\mathcal{I}$ is any Turing ideal, one can consider $(\mathbb{L} \cap \mathcal{I}, \leq)$ and $(\mathbb{H} \cap \mathcal{I}, \leq)$, and obtain real forcings in $\mathcal{I}$.

It is immediate to see that, if $\mathcal{I}$ is any Turing ideal and G is a sufficiently generic filter on $(\mathbb{L} \cap \mathcal{I}, \leq)$ or $(\mathbb{H} \cap \mathcal{I}, \leq)$, then $\pi(G)$ is in $\mathcal{I}$-DOM. In [6], the authors mention that, if $\mathcal{I}$ is a countable ω-model of ATR_0, then Laver forcing and Hechler forcing can be used to show, respectively, that $\mathcal{I}$-$\mathsf{SME} \nsubseteq \mathcal{I}$-$\mathsf{DOM}$ and $\mathcal{I}$-$\mathsf{SNE} \nsubseteq \mathcal{I}$-$\mathsf{SME}$.

Despite this, Dan Turetsky showed that these forcings likely cannot be used to prove the separation relative to HYP-ideals. Indeed, any filter that is generic enough for either of these two forcing notions (relativized to the ideal HYP) will inevitably give rise to a real in HYP-WL. In particular, neither Laver nor Hechler forcing relativized to HYP can help us in separating the non-lowness classes of our interest. We provide proofs for these results in Appendix A.

2.3. **Cohen reals, random reals, and their relationships to SME and SNE.** In this section we briefly review some facts about Cohen and random reals, adapted to our setting.

2.3.1. *Represented spaces.* A *representation* of a set X is a surjective partial function $\delta \colon \omega^\omega \to X$. A pair (X, δ), where δ is a representation of X, is called a *represented space*. We study problems based on Borel subsets of ω^ω by means of their codes. We briefly recall the standard way to code Borel sets (we only need Borel sets of finite rank here).

Definition 2.20. Fix a bijection $b \colon \omega \to \omega^{<\omega}$. A function $f \in \omega^\omega$ is a *code* (or *name*) for the open set $A_f = \bigcup_{i\in\omega}[b(i)]$, where $[\sigma] = \{f \in \omega^\omega : \sigma \preceq f\}$. The function $f \mapsto A_f$, which we denote as $\delta_{\mathbf{\Sigma}^0_1}$, is a representation of the set of open subsets of ω^ω. We obtain a representation for closed sets by stipulating that, for every $f \in \omega^\omega$, $\delta_{\mathbf{\Pi}^0_1}(f) = \omega^\omega \setminus \delta_{\mathbf{\Sigma}^0_1}$. For any $n \in \omega$, we define representations

$$\delta_{\mathbf{\Sigma}^0_{n+1}}(f) = \bigcup_{i\in\omega} \delta_{\mathbf{\Pi}^0_n}(f^{[i]}), \text{ and } \delta_{\mathbf{\Pi}^0_{n+1}}(f) = \omega^\omega \setminus \delta_{\mathbf{\Sigma}^0_{n+1}}(f).^4$$

In this paper we will handle Borel sets of rank at most 2, and we will refer to Borel codes quite informally. In particular, we may refer to codes of open sets as subsets of $\omega^{<\omega}$ (like in Notation 2.3). Similarly, a code for a closed set C is any $T \subseteq \omega^{<\omega}$ with $[T] = C$. Finally, we may refer to $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$ codes as simply consisting of sequences of, respectively, codes for closed sets or codes for open sets.

If $A \subseteq \omega^\omega$ is a Borel set and $\mathcal{I}$ is a Turing ideal, we say that A is $\mathcal{I}$-*coded* if there is some Borel code for A in $\mathcal{I}$. Similarly, when we say that A is an $\mathcal{I}$-*coded* $\mathbf{\Sigma}^0_n$ set (resp. $\mathbf{\Pi}^0_n$), we mean that there is a $\mathbf{\Sigma}^0_n$ (resp. $\mathbf{\Pi}^0_n$) name for A in $\mathcal{I}$.

The cardinal characteristics in Cichon's diagram are related to the σ-ideals of meager and null subsets of ω^ω (equivalently, 2^ω). By definition, each meager set is contained in a $\mathbf{\Sigma}^0_2$-meager set, and each null set is contained in a $\mathbf{\Pi}^0_2$-null set. It is known that, to compute the cardinal characteristics related to these ideals ($\mathsf{add}, \mathsf{cov}, \mathsf{cof}$ and non), one can restrict attention to these sets.

2.3.2. *Cohen reals, random reals and their relationships with other non-lowness classes.*

[4]For every $n \in \omega$, using DeMorgan's Laws, we get $\delta_{\mathbf{\Pi}^0_{n+1}}(f) = \bigcap_{i\in\omega} \delta_{\mathbf{\Sigma}^0_n}(f^{[i]})$.

Definition 2.21. Let $\mathcal{I}$ be a Turing ideal. We say that $x \in \omega^\omega$ is *weakly $\mathcal{I}$-generic* if x belongs to no $\mathcal{I}$-coded $\mathbf{\Sigma}^0_2$-meagre set. Equivalently, x is weakly $\mathcal{I}$-generic if and only if it belongs to no $\mathcal{I}$-coded closed nowhere dense set.

It is known that weak $\mathcal{I}$-genericity is the relativization to $\mathcal{I}$ of Cohen genericity.

Lemma 2.22 (Folklore)**.** *A real x is weakly $\mathcal{I}$-generic if and only if $x = \bigcup_{\sigma \in G} \sigma$, where G is a sufficiently generic filter on $\mathbb{C}$.*

Remark 2.23. The class of reals which can compute a weakly $\mathcal{I}$-generic is the non-lowness class of the problem $\mathsf{Capture}(\mathcal{M})$, defined as follows: an instance is a real $x \in \omega^\omega$ and a solution to the instance x is a (code for a) $\mathbf{\Sigma}^0_2$-meagre set M such that $x \in M$.

The following Lemma is (the dual of) [6, Proposition 3.4].

Lemma 2.24. *There is an effective morphism from $\mathsf{Spill}(\mathcal{M})$ to $\mathsf{Capture}(\mathcal{M})$, hence every $x \in \mathcal{I}\text{-}\mathsf{SME}$ computes a weakly $\mathcal{I}$-generic.*

We will need one more result relating weakly $\mathcal{I}$-generics with strongly $\mathcal{I}$-meagre engulfing reals. This can also be seen as coming from an effective morphism, but an explicit definition of this morphism requires the introduction of the operator of *sequential composition* on problems. The interested reader should consult [6, Section 4].

Lemma 2.25. *Let $\mathcal{I}$ be any Turing ideal, let x be weakly $\mathcal{I}$-generic and let y be $\mathcal{I}(x)$-dominating. Then $y \in \mathcal{I}\text{-}\mathsf{SME}$.*

Lemma 2.25 follows from [6, Proposition 4.9]. The latter Proposition is an effective-morphism-based proof of the equalities in Cichon's diagram summarized in [6, Corollary 4.10], originally due to Truss, A. Miller, and Fremlin.

Now we go on to the definition of random reals which is relevant to us. Among many concepts of effective randomness which have been studied, the authors of [6] identified Schnorr randomness to be the correct one to develop the theory of effective cardinal characteristics (see [6, Remark 3.10]).

Definition 2.26. A *Schnorr test* is a sequence $(U_n, \lambda_n)_{n \in \omega}$ where, for every $n \in \omega$: U_n is the code for an open subset of 2^ω, $\lambda_n = \lambda(U_n)$, (λ is the Lebesgue measure), and $\lim_n \lambda_n = 0$.

The Schnorr test $(U_n, \lambda_n)_{n \in \omega}$ is thought of as a name for the $\mathbf{\Pi}^0_2$-null set $\bigcap_{n \in \omega} U_n$.

In the rest of the paper, given a Turing ideal $\mathcal{I}$, we say that a set $A \subseteq \omega^\omega$ is an $\mathcal{I}$-coded Schnorr null set if there is a Schnorr test $x \in \mathcal{I}$ which names A.

Remark 2.27. Note that a Schnorr test can be thought of as a name p for a $\mathbf{\Pi}^0_2$-set A as in Definition 2.20, together with a sequence of reals which give the measure of the open sets whose intersection constitutes A.

It is easy to see that if p is a $\mathbf{\Pi}^0_2$-name for a null set A, then p' (the Turing jump of p) computes a Schnorr test naming A. Therefore, if $\mathcal{I}$ is a jump ideal, the class of $\mathcal{I}$-coded $\mathbf{\Pi}^0_2$-null sets coincides with the class of $\mathcal{I}$-coded Schnorr null sets.

We regard problems in the effective Cichon's diagram related to null sets as being defined in terms of Schnorr null sets.

Definition 2.28. Let $\mathcal{I}$ be any Turing ideal. We say that $x \in \omega^\omega$ is *$\mathcal{I}$-Schnorr random* (also, just $\mathcal{I}$-random where there is no risk of confusion) if x belongs to no $\mathcal{I}$-coded Schnorr null set.

Again the class of reals which compute an $\mathcal{I}$-Schnorr random is the non-lowness class of the problem $\mathsf{Capture}(\mathcal{N})$, which is defined as $\mathsf{Capture}(\mathcal{M})$, with "$\mathbf{\Sigma}^0_2$-meagre" replaced by "Schnorr null".

There is an analogue of Lemma 2.24 (see [6, Proposition 3.9]).

Lemma 2.29. *There is an effective morphism from* $\mathsf{Spill}(\mathcal{N})$ *to* $\mathsf{Capture}(\mathcal{N})$*, hence every* $x \in \mathcal{I}\text{-}\mathsf{SNE}$ *computes an* $\mathcal{I}$*-Schnorr random.*

Lastly, (the duals of) [6, Proposition 3.6 and Theorem 3.18] give:

Lemma 2.30. *There is an effective morphism from* $\mathsf{Spill}(\mathcal{N})$ *to* $\mathsf{Spill}(\mathcal{M})$ *and an effective morphism from* $\mathsf{Spill}(\mathcal{M})$ *to* Esc*, hence* $\mathcal{I}\text{-}\mathsf{SNE} \subseteq \mathcal{I}\text{-}\mathsf{SME} \subseteq \mathcal{I}\text{-}\mathsf{DOM}$*.*

Remark 2.31. It is well known (see for example [6, Subsection 3.2]) that, in the definition of cardinal characteristics related to the ideals of meagre and null sets, working with ω^ω, 2^ω, or the real line $\mathbb{R}$ does not make any difference. The same holds for the definition of non-lowness classes related to these problems, including the definitions of weak genericity and randomness. We will use this fact silently throughout the paper.

2.3.3. *The Laver property.* We review the definition of the Laver property, a well-known property of certain forcing notions (among which, of course, Laver forcing). A forcing notion with the Laver property does not add any Cohen reals. We give a definition adapted to our context of forcing over a countable ideal $\mathcal{I}$, and prove that a real forcing in $\mathcal{I}$ with our version of the Laver property adds no weakly $\mathcal{I}$-generic.

Definition 2.32. Let $h \in \omega^\omega$ and let Φ be a Turing functional. We say that Φ is *h-bounded* if for all $x \in \omega^\omega$ and all $m \in \omega$, if $\Phi(x)(m)\downarrow$, then $\Phi(x)(m) \leq h(m)$.

For any Turing ideal $\mathcal{I}$, we denote by $\mathrm{B}_\mathcal{I}$ the set of Turing functionals in $\mathcal{I}$ which are h-bounded for some $h \in \mathcal{I}$.

For any set $A \subseteq \omega^\omega$ and any Turing ideal $\mathcal{I}$, we write $\mathrm{B}_\mathcal{I}(A)$ to denote the set $\{g \in \omega^\omega : \exists \Phi \in \mathrm{B}_\mathcal{I}\, \exists f \in A\ (g = \Phi(f))\}$. Note that for all A and all $f \in A$, if there are Φ and h in $\mathcal{I}$ such that $\Phi(f) \in \omega^\omega$ and $\Phi(f) \leq h$, then there is $\Phi' \in \mathrm{B}_\mathcal{I}$ such that $\Phi'(f) = \Phi(f)$. An example of such Φ' is $\Phi'(x)(m) = \Phi(x)(m) \mod h(m)$. In other words $\mathrm{B}_\mathcal{I}(A)$ consists of those functions f which are not $\mathcal{I}$-escaping and are computable in some $x \oplus y$ where $x \in A$, $y \in \mathcal{I}$.

Lemma 2.33. *For every* $A \subseteq \omega^\omega$ *and every Turing ideal* $\mathcal{I}$, $\mathrm{B}_\mathcal{I}(A) = \mathrm{B}_\mathcal{I}(\mathrm{B}_\mathcal{I}(A))$.

Proof. One direction follows immediately from the fact that for all $\Phi, \Psi \in \mathrm{B}_\mathcal{I}$, $\Phi \circ \Psi \in \mathrm{B}_\mathcal{I}$. For the converse, notice that if $\Phi \in \mathrm{B}_\mathcal{I}$ and $h \in \mathcal{I}$ is a bound for Φ as in Definition 2.32, then $(\mathrm{id}_{\omega^\omega} \mod h) \circ \Phi = \Phi$. Now let $y = \Phi(x)$ for some $x \in A$ and $\Phi \in \mathrm{B}_\mathcal{I}$. Then $y = (\mathrm{id}_{\omega^\omega} \mod h) \circ \Phi(x)$. □

Definition 2.34. Given any Turing ideal $\mathcal{I}$, we say that a set $A \subseteq \omega^\omega$ has the *$\mathcal{I}$-Laver property* if for all $f \in A$ there is some $F \in ([\omega]^{<\omega})^\omega \cap \mathcal{I}$ such that, (i) for all n, $|F(n)| \leq 2^n$, and (ii) for all $n \in \omega$, $f(n) \in F(n)$.

A real forcing $(\mathbb{P}, \leq, \mathbb{1}, \pi)$ in $\mathcal{I}$ has the *$\mathcal{I}$-Laver property* if, for every sufficiently generic filter $G \subseteq \mathbb{P}$, $\mathrm{B}_\mathcal{I}(\pi(G))$ has the $\mathcal{I}$-Laver property.

A function $F \in ([\omega]^{<\omega})^\omega$ such that $|F(n)| \leq 2^n$ for every $n \in \omega$ is called a *trace.* If $g \in \omega^\omega$ is such that for every $n \in \omega$, $g(n) \in F(n)$, we say that F *traces* g. In these terms, A has the $\mathcal{I}$-Laver property if each of its element is traced by some function in $\mathcal{I}$.

Lemma 2.35. *Let $\mathcal{I}$ be any Turing ideal, let $A \subseteq \omega^\omega$ be such that $A = \mathrm{B}_\mathcal{I}(A)$, and assume that A satisfies the $\mathcal{I}$-Laver property. Then $\mathcal{I}(A)$ does not contain any weakly $\mathcal{I}$-generic reals.*

Proof. We show that for every $f \in \mathcal{I}(A)$ there is an $\mathcal{I}$-coded closed nowhere dense set containing f. Note that if $f \in \mathcal{I}(A) \cap \omega^\omega$ is weakly $\mathcal{I}$-generic, then f computes a set $\tilde{f} \in \mathcal{I}(A) \cap 2^\omega$ which is also weakly $\mathcal{I}$-generic. Therefore, we can assume without loss of generality that $f \in 2^\omega$. In particular, $f \in \mathrm{B}_\mathcal{I}(A) = A$.

For every $f \in A \cap 2^\omega$, define $\hat{f} \in A$ as $\hat{f}(n) = f \restriction 2^n$. By assumption there is a trace $F \in \mathcal{I}$ such that $\hat{f}(n) \in F(n)$ for all $n \in \omega$. So for every n, $F(n)$ is a set of $\leq 2^n$ (codes for) binary strings of length 2^n. Let $T = \{\rho \in 2^{<\omega} : \forall m \leq \log_2(|\rho|)\ (\rho \restriction 2^m \in F(m))\}$. Then $T \in \mathcal{I}$, T is a tree and $|T \cap 2^{(2^n)}| \leq |F(n)| \leq 2^n$ for every $n \in \omega$, so $[T]$ is nowhere dense. By construction, $f \in [T]$, so f is not weakly $\mathcal{I}$-generic. □

Corollary 2.36. *Let $\mathcal{I}$ be a Turing ideal and let $(\mathbb{P}, \leq, \mathbb{1}, \pi)$ be a real forcing in $\mathcal{I}$ with the Laver property. If $G \subseteq \mathbb{P}$ is sufficiently generic, then $\mathcal{I}(\pi(G))$ contains no weakly $\mathcal{I}$-generic.*

2.3.4. *σ-centeredness.*

Definition 2.37. A notion of forcing $(\mathbb{P}, \leq)$ is *σ-centered* if there is a countable partition $(A_n)_{n\in\omega}$ of $\mathbb{P}$ such that for any n, if $p, q \in A_n$, then p and q are compatible.

It is known that σ-centered forcing notions add no random real (see, for example, [3, Lemma 3.7]). In Sections 5 and 6, we will use forcing notions which are in some sense effectively σ-centered and hence add no Schnorr random.

3. Bigness on $\omega^{<\omega}$

In this section we define a notion of bigness for subsets of $\omega^{<\omega}$. This is the key notion for the effectivization of both Laver and Hechler forcing.

Definition 3.1 (ω-closure in T)**.** For any tree $T \subseteq \omega^{<\omega}$ and any $B \subseteq \omega^{<\omega}$, we define by recursion a sequence $(R^T_{B,\alpha})_{\alpha\in\omega_1}$ as $R^T_{B,0} = \uparrow B$ and, for $0 < \alpha < \omega_1$,

$$R^T_{B,\alpha} = \{\tau \in T : \exists^\infty n\ \exists\ \beta < \alpha\ (\tau^\frown n \in R_{B,\beta})\}.$$

We call $(R^T_{B,\alpha})_{\alpha\in\omega_1}$ *the B-T-ranking*.

We define the *ω-closure of B in T* as $\mathrm{cl}^T_\omega(B) = \bigcup_{\alpha\in\omega_1} R_{B,\alpha}$ and, for any $\sigma \in \mathrm{cl}^T_\omega(B)$, we write B-T-$r(\sigma) = \alpha$ if $\alpha = \min\{\beta : \sigma \in R^T_{B,\beta}\}$. If $\sigma \in \mathrm{cl}^T_\omega(B)$, we say that B is *ω-big in T above* σ. Otherwise, we say that B is ω-small in T above σ.

In case $T = \omega^{<\omega}$, we omit it from our notation.

We set out to show that, for any tree $T \subseteq \omega^{<\omega}$, the function cl^T_ω is a uniformly Π^1_1 topological closure operator.

Lemma 3.2. *For any tree T, the function $\mathrm{cl}^T_\omega \colon \mathcal{P}(\omega^{<\omega}) \to \mathcal{P}(\omega^{<\omega})$ is monotone with respect to $\subseteq$, extensive, and it satisfies $\mathrm{cl}^T_\omega(\emptyset) = \emptyset$.*

Proof. Immediate from the definition of cl^T_ω. □

Lemma 3.3. *Let $T \subseteq \omega^{<\omega}$ be a tree, $B \subseteq \omega^{<\omega}$, and let $\sigma \in \mathrm{cl}^T_\omega(B)$. Then B-T-$r(\sigma) < \omega_1^{T\oplus B}$.*

Proof. We assume that T and B are computable for notational convenience.

Consider the B-T ranking $(R^T_{B,\alpha})_{\alpha\in\omega_1}$. Notice that $R^T_{B,\alpha} \leq_{\mathrm{T}} 0^{(\alpha\cdot 2)}$ for all $\alpha < \omega_1^{\mathrm{CK}}$, and this is uniform in the notation for α. Consequently, the predicate $a \in \mathcal{O} \wedge \sigma \in R^T_{B,|a|}$ is Π^1_1. We claim that $R_{\omega_1^{\mathrm{CK}}} = \bigcup_{\alpha\in\omega_1^{\mathrm{CK}}} R^T_{B,\alpha}$, or equivalently, there are no strings of rank ω_1^{CK} or higher.

Indeed, assume $\sigma \in R_{\omega_1^{\mathrm{CK}}}$. By definition, this means that there are infinitely many $n \in \omega$ for which there is $\alpha < \omega_1^{\mathrm{CK}}$ with $\sigma^\frown n \in R^T_{B,\alpha}$. This means that the set $\{n \in \omega : \exists\alpha < \omega_1^{\mathrm{CK}}\ (\sigma^\frown n \in R^T_{B,\alpha})\}$ is an infinite Π^1_1 set. As such, it has an infinite Δ^1_1 subset, say A. It follows that the set $\{\alpha \in \omega_1^{\mathrm{CK}} : \exists n \in A\, (\sigma^\frown n \in R^T_{B,\alpha})\}$ is a Δ^1_1 set of ordinals, hence (by Σ^1_1 boundedness) it is bounded above by some $\beta < \omega_1^{\mathrm{CK}}$. By definition of the B-T-ranking, we have $\sigma \in R^T_{B,\beta}$. □

Corollary 3.4. *Let $T \subseteq \omega^{<\omega}$ be a tree, and let $B \subseteq \omega^{<\omega}$ and $X \subseteq \omega$ be such that B is $\Pi^1_1(X)$. Then $\mathrm{cl}^T_\omega(B)$ is $\Pi^1_1(X \oplus T)$.*

Proof. By Lemma 3.3, $\mathrm{cl}^T_\omega(B) = \{\sigma \in T : \exists\alpha < \omega_1^{T\oplus B}\ (B\text{-}T\text{-}r(\sigma) = \alpha)\}$, so cl^T_ω is a uniformly $\Pi^1_1(T)$ function. Hence, Lemma 2.5 and Lemma 3.2 imply our claim. □

The ω-closure in T of a set B can be characterized combinatorially using trees analogous to Laver conditions.

Definition 3.5 (ω-bushy tree)**.** Let $T \subseteq \omega^{<\omega}$ be a tree and let $\tau \in \omega^{<\omega}$. Then T is *ω-bushy above τ* if, for every $\sigma \in T$, if σ is not a leaf and $\sigma \succeq \tau$, then there are infinitely many n such that $\sigma^\frown n \in T$.

Lemma 3.6. *Let $T \subseteq \omega^{<\omega}$ be a tree, let $\tau \in T$ and let $B \subseteq \omega^{<\omega}$. B is ω-big in T above τ if and only if there is an ω-bushy tree $S \subseteq T$ which is $\Delta^1_1(B \oplus T)$, with* $\mathrm{stem}(S) = \tau$, *$[S] = \emptyset$ and such that all the leaves of S are contained in B. Moreover, if $\tau \notin B$, then we can take S to have all its leaves in* $\mathrm{pf}(B)$.

Proof. First suppose there is a tree $S \subseteq T$ as above and let $\sigma \in S$ with $\tau \preceq \sigma$. We show by induction on $r(\sigma)$, the tree rank of σ in S, that $\sigma \in \mathrm{cl}^T_\omega(B)$, and in particular $r(\sigma) \geq B\text{-}T\text{-}r(\sigma)$.[5]

If $r(\sigma) = 0$, then σ is a leaf in S, so it is in B. By definition σ is then in $\mathrm{cl}^T_\omega(B)$ and $B\text{-}T\text{-}r(\sigma) = 0$. If $r(\sigma) = \alpha > 0$ and $\tau \preceq \sigma$, then σ has infinitely many children in S because S is ω-bushy above τ, and inductively we can assume that these are all in $\mathrm{cl}^T_\omega(B)$ and have B-T rank less than or equal to their tree rank. This immediately implies that σ is also in $\mathrm{cl}^T_\omega(B)$ and $B\text{-}T\text{-}r(\sigma) \leq r(\sigma)$. This concludes the induction, and it shows that $\tau \in \mathrm{cl}^T_\omega(B)$.

Conversely, assume that $\tau \in \mathrm{cl}^T_\omega(B)$. We build a subtree $S \subseteq T$ as above. If $B\text{-}T\text{-}r(\tau) = 0$, then the tree $\{\sigma \preceq \tau\}$ suffices. Now assume $\alpha = B\text{-}T\text{-}r(\tau) < 0$ and consider the tree $S \subseteq \{\sigma \preceq \tau\} \cup \{\sigma \in R^T_{B,\alpha} : \sigma \succeq \tau\}$ whose nodes (extending τ) correspond to descending sequences of B-T-ranks. Formally: $\sigma \in S$ if and only if $\sigma \preceq \tau$ or $\sigma \succeq \tau$, $\{\rho : \tau \preceq \rho \preceq \sigma\} \subseteq R^T_{B,\alpha}$, and, for all $\sigma', \sigma'' \preceq \sigma$, if $\sigma' \prec \sigma''$, then $B\text{-}T\text{-}r(\sigma') > B\text{-}T\text{-}r(\sigma'')$. S is arithmetical in the sequence $(R^T_{B,\beta})_{\beta\leq\alpha}$, so it is $\Delta^1_1(T \oplus B)$. It is also clear that S is a tree with stem τ. The fact that S is wellfounded is immediate by definition: an infinite path in S would correspond to

[5] Recall that the tree rank of S is defined as $r(\sigma) = 0$ if σ is a leaf in S and, if σ is not a leaf in S, $r(\sigma) = \sup\{r(\sigma^\frown n) + 1 : \sigma^\frown n \in S\}$.

an infinite descending sequence of ordinals. So we only need to show that S is ω-bushy above τ, and all of its leaves are in $\mathrm{pf}(B)$.

To that end, let $\sigma \in S \subseteq \mathrm{cl}^T_\omega(B)$. If B-T-$r(\sigma) = 0$, then $\sigma \in \mathrm{pf}(B)$ and σ is a leaf in S. If B-T-$r(\sigma) > 0$, then there are infinitely many n such that $\sigma^\frown n \in T$ has strictly lower B-T-rank, and by definition these all belong to S. This shows that S is ω-bushy above τ and also that all the leaves of S are in $\mathrm{pf}(B)$. □

An immediate Corollary of Lemma 3.6 is that, when we are determining whether a set B is big in a tree T above some string τ, we can without loss of generality assume that B is prefix free. This will be tacitly used several times in the rest of the paper.

To show that the functions cl^T_ω are topological closure operator, we still need to show that they are idempotent and commute with binary union. We start with idempotency. The following proof exploits the fact that ω-bushy trees can be easily concatenated to obtain another ω-bushy tree. This concept will come back in the proof of idempotency of another closure operators (namely in Lemma 6.9 and Corollary 6.10), related to a different notions of bigness.

Lemma 3.7. *Let $T \subseteq \omega^{<\omega}$ be a tree and let $B \subseteq \omega^{<\omega}$. We have $\mathrm{cl}^T_\omega(\mathrm{cl}^T_\omega(B)) = \mathrm{cl}^T_\omega(B)$.*

Proof. We show that $\mathrm{cl}^T_\omega(\mathrm{cl}^T_\omega(B)) \subseteq \mathrm{cl}^T_\omega(B)$. Let $\tau \in \omega^{<\omega}$ and assume $\mathrm{cl}^T_\omega(B)$ is ω-big in T above τ. Let $S \subseteq T$ be a tree which witnesses this fact. By definition, all leaves of S are in $\mathrm{cl}^T_\omega(B)$. Let L be the set of leaves of S, and by Lemma 3.6, for every $\sigma \in L$, let $T_\sigma \subseteq T$ be a tree which witnesses that B is ω-big in T above σ. Then the tree $T' = S \cup \bigcup_{\sigma \in L} T_\sigma$ is a subtree of T which is ω-bushy above τ, has no infinite paths, and has all its leaves in B. Hence T' witnesses that B is ω-big in T above τ. The reverse inclusion follows from the fact that cl^T_ω is extensive. □

We now show that cl^T_ω always commutes with binary union.

Lemma 3.8. *Let $T \subseteq \omega^{<\omega}$ be a tree and let $A, B \subseteq \omega^{<\omega}$. Then $\mathrm{cl}^T_\omega(A) \cup \mathrm{cl}^T_\omega(B) = \mathrm{cl}^T_\omega(A \cup B)$.*

Proof. The inclusion $\mathrm{cl}^T_\omega(A) \cup \mathrm{cl}^T_\omega(B) \subseteq \mathrm{cl}^T_\omega(A \cup B)$ follows from monotonicity.

For the other inclusion, let $\tau \in \mathrm{cl}^T_\omega(A \cup B)$. By induction on $(A \cup B)$-T-$r(\tau)$ we show that $\tau \in \mathrm{cl}^T_\omega(A) \cup \mathrm{cl}^T_\omega(B)$.

If $(A \cup B)$-T-$r(\tau) = 0$, then either $\tau \in A$, in which case $\tau \in \mathrm{cl}^T_\omega(A)$, or $\tau \in B$, in which case $\tau \in \mathrm{cl}^T_\omega(B)$. If $(A \cup B)$-T-$r(\tau) = \alpha > 0$, then there are infinitely many $n \in \omega$ such that $\tau^\frown n \in \mathrm{cl}^T_\omega(A \cup B)$ and $\tau^\frown n$ has $(A \cup B)$-T rank lower than α. Inductively we can assume that for all such n, $\tau^\frown n \in \mathrm{cl}^T_\omega(A) \cup \mathrm{cl}^T_\omega(B)$, and either A-T-$r(\tau^\frown n) < \alpha$ or B-T-$r(\tau^\frown n) < \alpha$. By the pigeonhole principle this implies that τ has either A-T rank α or B-T rank α. In the first case we get $\tau \in \mathrm{cl}^T_\omega(A)$ and in the second case we get $\tau \in \mathrm{cl}^T_\omega(B)$. □

Lemmas 3.2, 3.7, and 3.8 together give:

Proposition 3.9. *For every $T \subseteq \omega^{<\omega}$, the function $\mathrm{cl}^T_\omega \colon \mathcal{P}(\omega^{<\omega}) \to \mathcal{P}(\omega^{<\omega})$ is a uniformly Π^1_1 closure operator.*

We state an easy result on how the notion of ω-big change when the tree under consideration changes.

Lemma 3.10. *Let $T, S \subseteq \omega^{<\omega}$ be trees with $T \subseteq S$, let $\tau \in T$ and let $B \subseteq \omega^{<\omega}$. Then $\mathrm{cl}^T_\omega(B) \subseteq \mathrm{cl}^S_\omega(B)$.*

We conclude the section noting that ω-bigness is Π^1_1-complete, both in the sense of m-reduction (for subsets of ω) and in the sense of Wadge reduction (for subsets of $2^\omega/\omega^\omega$). The proof of this fact, due to Marcone, is based on the the concept of *smooth trees* (present in [9] in the form of smooth barriers. See also [17, Exercises VI.1.8 and VI.1.9]).

Proposition 3.11 (Marcone)**.** *The set $O = \{B \in \mathcal{P}(\omega^{<\omega}) : \langle,\rangle \in \mathrm{cl}_\omega(B)\}$ is $\boldsymbol{\Pi}^1_1$ complete. Similarly, the set of (codes for) computable $B \subseteq \omega^{<\omega}$ with $\langle,\rangle \in \mathrm{cl}_\omega(B)$ is Π^1_1-complete.*

Proof. We only show the first statement, the second one being very similar.

We already know that O is $\boldsymbol{\Pi}^1_1$, so we only have to show $\boldsymbol{\Pi}^1_1$ hardness. We do so by exhibiting a Wadge reduction from the set of wellfounded trees to O. Given any tree $T \subseteq \omega^{<\omega}$, we define the tree $T^+ = \{\sigma \in \omega^{<\omega} : \exists \tau \in T\ (|\tau| = |\sigma| \wedge \tau \leq \sigma)\}$. It is not hard to see that if $f \in [T]$ and $f \leq g$, then $g \in [T^+]$. We define a continuous function by mapping the tree T to $\omega^{<\omega} \setminus T^+$.

Now if T is wellfounded, then so is T^+ (see, e.g. [17, Exercise VI.1.8]), hence $\omega^{<\omega} \setminus T^+$ is easily seen to be ω-big above $\langle,\rangle$. If on the other hand T is illfounded, then by our previous observation we have that, for any Laver tree L, $[L] \cap [T^+] \neq \emptyset$. By Lemma 3.6, this implies that $\omega^{<\omega} \setminus T^+$ is not ω-big above $\langle,\rangle$. $\square$

4. Separating dominating from strongly meagre engulfing

We introduce, for any countable ideal $\mathcal{I}$ which is closed under relative hyperarithmeticity, an effectivization of Laver forcing based on ω-bushy trees with ω-small bad sets. We then prove that this forcing adds $\mathcal{I}$-dominating reals, and that it has the $\mathcal{I}$-Laver property. Therefore, by Lemma 2.24 and Corollary 2.36, we can conclude that this forcing notion adds reals in $\mathcal{I}$-$\mathsf{DOM} \setminus \mathcal{I}$-$\mathsf{SME}$.

Notation 4.1. For the rest of the section, we fix a countable Turing ideal $\mathcal{I} \subseteq \omega^\omega$ which is $\leq_{\mathsf{HYP}}$-downward closed.

Definition 4.2. We define ω-bushy tree forcing (in $\mathcal{I}$) as the following partial order, which we denote as $(\mathbb{O}, \leq)$. Conditions in $\mathbb{O}$ are triples $\boldsymbol{p} = (T^{\boldsymbol{p}}, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ where:

- $T^{\boldsymbol{p}} \subseteq \omega^{<\omega}$ is a tree in $\mathcal{I}$ which is ω-bushy above $\tau^{\boldsymbol{p}}$,
- $\mathrm{stem}(T^{\boldsymbol{p}}) = \tau^{\boldsymbol{p}}$,
- $B^{\boldsymbol{p}} \subseteq \omega^{<\omega}$ is upwards closed, $\tau^{\boldsymbol{p}} \notin \mathrm{cl}^{T^{\boldsymbol{p}}}_\omega(B^{\boldsymbol{p}})$ and there exists $Z \in \mathcal{I}$ with $B^{\boldsymbol{p}} \in \Pi^1_1(Z)$,
- every leaf of $T^{\boldsymbol{p}}$ is in $B^{\boldsymbol{p}}$.

We define extension as $\boldsymbol{p} \leq \boldsymbol{q}$ if $T^{\boldsymbol{p}} \subseteq T^{\boldsymbol{q}}$, $B^{\boldsymbol{p}} \supseteq B^{\boldsymbol{q}}$, and $\tau^{\boldsymbol{p}} \succeq \tau^{\boldsymbol{q}}$. The condition $\mathbb{1}_{\mathbb{O}} = (\omega^{<\omega}, \emptyset, \langle\rangle)$ is the top element of $(\mathbb{O}, \leq)$.

By Lemma 2.5, the condition $B^{\boldsymbol{p}} \in \Pi^1_1(Z)$ implies that also $\mathrm{cl}^T_\omega(B^{\boldsymbol{p}}) \in \Pi^1_1(Z)$. Moreover, we always have $\tau^{\boldsymbol{p}} \notin B^{\boldsymbol{q}}$.

We say that $B^{\boldsymbol{p}}$ is the set of *bad strings* (or *bad set*) in $\boldsymbol{p}$. We will see later that $\boldsymbol{p}$ forces the generic real outside of $[B^{\boldsymbol{p}}]^{\prec}$. The connection between $\mathbb{O}$ and Laver forcing is explained in the following Lemma.

Lemma 4.3. *Let $\boldsymbol{p} \in \mathbb{O}$, then $S = T^{\boldsymbol{p}} \setminus \uparrow \mathrm{cl}^{T^{\boldsymbol{p}}}_\omega(B^{\boldsymbol{p}})$ is a Laver tree with stem $\tau^{\boldsymbol{p}}$.*

Proof. The set S is a tree because $T^{\boldsymbol{p}}$ is a tree and $\uparrow \mathrm{cl}_\omega^{T^{\boldsymbol{p}}}(B^{\boldsymbol{p}})$ is upwards closed. To show that S is a Laver tree with stem $\tau^{\boldsymbol{p}}$ it suffices to show that S is pruned and ω-bushy above $\tau^{\boldsymbol{p}}$, so let $\sigma \succeq \tau^{\boldsymbol{p}}$ with $\sigma \in S$. Since $\sigma \notin \uparrow \mathrm{cl}_\omega^{T^{\boldsymbol{p}}}(B^{\boldsymbol{p}})$, σ is not a leaf in $T^{\boldsymbol{p}}$ and hence it has infinitely many successors in $T^{\boldsymbol{p}}$. Moreover, $\sigma \notin \mathrm{cl}_\omega^{T^{\boldsymbol{p}}}(B^{\boldsymbol{p}})$, so only finitely many of these can be in $\mathrm{cl}_\omega^{T^{\boldsymbol{p}}}(B^{\boldsymbol{p}})$. Since σ has infinitely many successors in $T^{\boldsymbol{p}}$, it follows that σ still has infinitely many successors in S. This shows that S is both pruned and ω-bushy above $\tau^{\boldsymbol{p}}$, proving the claim. $\square$

The following lemma is straightforward, and it shows that $\mathbb{O}$ is a real forcing.

Lemma 4.4. *For any $m \in \omega$, the set $L_m = \{\boldsymbol{p} \in \mathbb{O} : |\tau^{\boldsymbol{p}}| \geq m\}$ is dense in $\mathbb{O}$.*

Proof. Let $m \in \omega$ and let $\boldsymbol{q} \in \mathbb{O}$. Since $S = T^{\boldsymbol{q}} \setminus \uparrow \mathrm{cl}_\omega^{T^{\boldsymbol{p}}}(B^{\boldsymbol{q}})$ is a Laver tree, let $\tau \in S$ be a string of length m. It is not hard to check that $(T^{\boldsymbol{q}}_\tau, B^{\boldsymbol{q}}, \tau) \in L_m$ and $(T^{\boldsymbol{q}}_\tau, B^{\boldsymbol{q}}, \tau) \leq (T^{\boldsymbol{q}}, B^{\boldsymbol{q}}, \tau^{\boldsymbol{q}})$. $\square$

So any sufficiently generic filter G on $\mathbb{O}$ adds a new real, namely $x_G = \bigcup_{\boldsymbol{p} \in G} \tau^{\boldsymbol{p}}$. Recall that we write that $G \in \mathrm{dom}(\pi)$ to mean that G meets all the L_m's. Since we know that these are countably many dense sets, every sufficiently generic G is in $\mathrm{dom}(\pi)$. We now that $\boldsymbol{p} \in \mathbb{O}$ forces the generic real to belong to $[T^{\boldsymbol{p}}] \setminus [B^{\boldsymbol{p}}]^{\prec}$.

Lemma 4.5. *Let $\boldsymbol{p} \in \mathbb{O}$ and let G be a filter on $\mathbb{O}$ with $G \in \mathrm{dom}(\pi)$. Then $f_G = \bigcup_{\boldsymbol{p} \in G} \tau^{\boldsymbol{p}}$ belongs to $[T^{\boldsymbol{p}}] \setminus [B^{\boldsymbol{p}}]^{\prec}$. In particular, $\boldsymbol{p} \Vdash f_G \in [T^{\boldsymbol{p}}] \setminus [B^{\boldsymbol{p}}]^{\prec}$.*

Proof. We first show that $f_G \in [T^{\boldsymbol{p}}]$: fix $k \in \omega$ and let $\boldsymbol{q} \in G$ be such that $|\tau^{\boldsymbol{q}}| \geq k$. Let $\boldsymbol{s} \in G$ be a common extension of $\boldsymbol{p}$ and $\boldsymbol{q}$. Then $f_G \restriction k \preceq \tau^{\boldsymbol{q}} \preceq \tau^{\boldsymbol{s}}$, $\tau^{\boldsymbol{s}} \in T^{\boldsymbol{s}} \subseteq T^{\boldsymbol{p}}$, so that $f_G \restriction k \in T^{\boldsymbol{p}}$. Since k was arbitrary, we obtain $f_G \in [T^{\boldsymbol{p}}]$.

We now show that $f_G \notin [B^{\boldsymbol{p}}]^{\prec}$. Towards a contradiction, suppose there is $k \in \omega$ are such that $f_G \restriction k \in B^{\boldsymbol{p}}$. Let $\boldsymbol{q} \in G$ be such that $|\tau^{\boldsymbol{q}}| \geq k$ and let $\boldsymbol{s} \in G$ be a common extension of $\boldsymbol{p}$ and $\boldsymbol{q}$. We have $f_G \restriction k \in B^{\boldsymbol{p}}$ and $f_G \restriction k \preceq \tau^{\boldsymbol{q}} \preceq \tau^{\boldsymbol{s}}$, so, since $B^{\boldsymbol{p}}$ is upwards closed, $\tau^{\boldsymbol{s}} \in B^{\boldsymbol{p}}$. Since $B^{\boldsymbol{s}} \supseteq B^{\boldsymbol{p}}$, it follows that $\tau^{\boldsymbol{s}} \in B^{\boldsymbol{s}}$, a contradiction. $\square$

We say that $\boldsymbol{p}$ *strongly forces* $x_G \in A$, where $A \subseteq \omega^\omega$, if $[T^{\boldsymbol{p}}] \setminus [B^{\boldsymbol{p}}]^{\prec} \subseteq A$. Correspondingly, $\boldsymbol{p}$ strongly forces $x_G \notin A$ if $\boldsymbol{p}$ strongly forces $x_G \in \omega^\omega \setminus A$. We now see an immediate consequence of Lemma 4.5 in terms of strong forcing.

Corollary 4.6. *Let $A \in \mathcal{I} \cap \omega^\omega$ be the code of an open set. The set of conditions $\boldsymbol{q} \in \mathbb{O}$ which either strongly force $x_G \in [A]^{\prec}$ or strongly force $x_G \notin [A]^{\prec}$ is dense.*

Proof. Let $\boldsymbol{p} \in \mathbb{O}$. If $\tau^{\boldsymbol{p}} \in \mathrm{cl}_\omega^T(A)$, then this is witnessed by some $S \subseteq T^{\boldsymbol{p}}$ with $S \leq_{\mathsf{HYP}} T^{\boldsymbol{p}} \oplus A$. Hence $(S, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) \in \mathbb{O}$ and, by definition, we have $[S] \subseteq [A]^{\prec}$.

If $\tau^{\boldsymbol{p}} \notin \mathrm{cl}_\omega^T(A)$, then, by Lemma 3.8, $\boldsymbol{q} = (T^{\boldsymbol{p}}, B^{\boldsymbol{p}} \cup \uparrow A, \tau^{\boldsymbol{p}}) \in \mathbb{O}$, $\boldsymbol{q}$ strongly forces $x_G \notin [A]^{\prec}$. $\square$

Like Laver forcing, $\mathbb{O}$ adds dominating reals.

Lemma 4.7. *For every $h \in \omega^\omega \cap \mathcal{I}$, the set $D_h = \{\boldsymbol{p} \in \mathbb{O} : \forall g \in T^{\boldsymbol{p}}\ (g \geq^* h)\}$ is dense.*

Proof. Let $\boldsymbol{p} \in \mathbb{O}$ and $h \in \omega^\omega \cap \mathcal{I}$. We want to show that there is $\boldsymbol{q} \leq \boldsymbol{p}$ in D_h. Let $T = \{\tau \in T^{\boldsymbol{p}} : \forall m \geq |\tau^{\boldsymbol{p}}|\ (\tau(m) > h(m))\}$. Note that for every $f \in [T]$, $f \geq^* h$. So, to show that that $(T, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) \in D_h$, it suffices to show that $(T, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ is a condition. This is obvious since C remains ω-small in T over $\tau^{\boldsymbol{p}}$ (cf. Lemma

3.10), and T is ω-bushy above $\tau^{\boldsymbol{p}}$ as any non-leaf node in $T^{\boldsymbol{p}}$ which extends $\tau^{\boldsymbol{p}}$ has infinitely many children, so infinitely many of these majorize h. $\square$

Corollary 4.8. *If G is a sufficiently generic filter for $\mathbb{O}$, then f_G dominates all functions in $\omega^\omega \cap \mathcal{I}$.*

We now show that the machinery of *fusion* can be used to build conditions of $\mathbb{O}$.

Definition 4.9. Let T be a tree on ω which is ω-bushy above its stem.

We define a family of subsets of T recursively as follows: $P_0(T) = \{\mathrm{stem}(T)\}$, and $P_{i+1}(T) = P_i(T) \cup \{\sigma^\frown n : \sigma \in P_i(T) \wedge n = \min\{m \in \omega : \sigma^\frown m \in T \setminus P_i(T)\}\}$.

Define the sequence of partial orders $(\leq_i)_{i\in\omega}$: as $T \leq_i T'$ if and only if $T \subseteq T'$ and $P_i(T) = P_i(T')$.

Note that for every tree T as above, we have $T = \{\sigma : \sigma \prec \mathrm{stem}(T)\} \cup \bigcup_{i\in\omega} P_i(T)$, and $|P_i(T)| \leq 2^i$. Moreover, if $\sigma \in P_i(T)$ and σ is not a leaf of T, then $P_{i+1}(T) \setminus P_i(T)$ contains a child of σ. It follows that if $j > i$, $\sigma \in P_i(T)$ and σ is not a leaf in T, then $P_j(T)$ contains at least $j - i$ children of σ.

Lemma 4.10 (Fusion for ω-bushy trees)**.** *Let $(T_i)_{i\in\omega}$ be a sequence of trees, each of them ω-bushy above its stem. Assume that $T_{i+1} \leq_i T_i$ for each $i \in \omega$ and let $T = \bigcap_{i\in\omega} T_i$. Then T is ω-bushy above its stem.*

Proof. First, notice that by induction we get $T_j \leq_i T_i$ for all $i \leq j$, so $P_i(T_i) \subseteq T$ and $P_i(T_i) = P_i(T)$ for all i. In particular, $\mathrm{stem}(T) = \sigma = \mathrm{stem}(T_i)$ for every $i \in \omega$.

Now let $\tau \in T$ such that $\sigma \preceq \tau$. Let $i = \min\{j \in \omega : \tau \in P_j(T)\}$. Note that if τ is not a leaf in T_{i+1}, then $P_{i+1}(T_{i+1})$ contains one child of τ, and $P_{i+1}(T_{i+1}) \subseteq T_k$ for every $k \geq i+1$, hence τ is not a leaf T_k for every $k \in \omega$. In this case, since each T_k is ω-bushy above σ and $\tau \in T_k$, it follows that τ has infinitely many children in T_k, and, for $k > i$, $k - i$ of these children are in $P_k(T_k) \subseteq T$. This shows that τ has infinitely many children in T, so that T is ω-bushy above σ. On the other hand, if τ is a leaf in T_{i+1}, then τ is also a leaf in T as $T \subseteq T_{i+1}$. $\square$

Remark 4.11. The proof of Lemma 4.10 also shows that a node $\tau \in T$ is a leaf in T if and only if there is some $j \in \omega$ such that τ is a leaf in T_j.

In the following we refer to sequences $(T_i)_{i\in\omega}$ which satisfy the hypothesis of Lemma 4.10 as *fusion sequences*, and we refer to the tree T constructed therein as the *fusion* of the sequence. Now we introduce some notation to make fusion arguments more readable.

Notation 4.12. Given a tree T which is ω-bushy above its stem σ and a number $i \in \omega$, define the finite sequence of trees: $(T^i(\tau))_{\tau\in P_i(T)}$ as

$$T^i(\tau) = T_\tau \setminus \bigcup_{\tau' \in P_i(T), \tau \prec \tau'} T_{\tau'},$$

where T_τ is the full subtree of T generated by τ as in Definition 2.2.

Notice that for every $i \in \omega$, the sequence $(T^i(\tau))_{\tau\in P_i(T)}$ is a finite partition of $T \setminus \{\sigma \preceq \mathrm{stem}(T)\}$. Moreover, for every τ, $T^i(\tau)$ is ω-bushy above $\mathrm{stem}(T^i(\tau)) = \tau$.

We use a fusion argument to show that a result analogous to Corollary 4.6 holds for $\boldsymbol{\Pi}^0_2$ sets.

Proposition 4.13. *Let $C \subseteq \omega^\omega$ be a $\mathbf{\Pi}^0_2$ set coded in $\mathcal{I}$, so that there is a sequence $(A_n)_{n\in\omega} \in \mathcal{I}$ such that $C = \bigcap_{n\in\omega}[A_n]^{\prec}$. The set of conditions $\boldsymbol{q}$ which either strongly force $x_G \in C$ or $x_G \notin C$ is dense in $\mathbb{O}$.*

Proof. Fix $\boldsymbol{p} \in \mathcal{I}$ and let $X \in \mathcal{I}$ be such that $\mathrm{cl}^{T^{\boldsymbol{p}}}_\omega(B^{\boldsymbol{p}})$ is $\Pi^1_1(X)$ and $T^{\boldsymbol{p}} \leq_{\mathrm{T}} X$.

If there is some $\boldsymbol{q} \leq \boldsymbol{p}$ and some $n \in \omega$ such that A_n is ω-small in $T^{\boldsymbol{q}}$ above $\tau^{\boldsymbol{q}}$ then, by Corollary 4.6, $\boldsymbol{q}' = (T^{\boldsymbol{q}}, B^{\boldsymbol{q}} \cup A_n, \tau^{\boldsymbol{q}})$ strongly forces $x_G \notin A_n$. Hence, $\boldsymbol{q}'$ strongly forces $x_G \notin C$.

If the conditions above are not satisfied, we build $\boldsymbol{q} \leq \boldsymbol{p}$ such that $\boldsymbol{q} = (T^{\boldsymbol{q}}, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$, where $T^{\boldsymbol{q}}$ is obtained from $T^{\boldsymbol{p}}$ by fusion. Note that our assumptions imply that for every $\sigma \in T^{\boldsymbol{p}} \setminus \mathrm{cl}^{T^{\boldsymbol{p}}}_\omega(B^{\boldsymbol{p}})$ and every tree $T \in \mathcal{I}$ which is ω-bushy above its stem and such that $T \subseteq T^{\boldsymbol{p}}$ and $\sigma \in T$, each A_n is ω-big in T above σ. We define a fusion sequence as follows: we start with $T_0 = T^{\boldsymbol{p}}$. At step i, assume we have a sequence of trees $(T_k)_{k\leq i}$ such that $T_k \leq_j T_j$ for every $j \leq k \leq i$ and $[T_i] \setminus [B^{\boldsymbol{p}}]^{\prec} \subseteq \bigcap_{j<i}[A_j]^{\prec}$.

Consider the partition of T_i given by $(T^i_i(\tau))_{\tau\in P_i(T_i)}$. For any $\tau \in P_i(T_i)$, we simultaneously follow the $\Pi^1_1(T^i_i(\tau) \oplus C)$ enumeration of $\mathrm{cl}^{T^i_i(\tau)}_\omega(A_i)$ and the $\Pi^1_1(T^i_i(\tau) \oplus X)$ enumeration of $\mathrm{cl}^{T^i_i(\tau)}_\omega(B^{\boldsymbol{p}})$ to check if τ appears. Our assumptions guarantee that one of these two cases will eventually realize.

If we find first that $\tau \in \mathrm{cl}^{T^i_i(\tau)}_\omega(B^{\boldsymbol{p}})$, we set $S_i(\tau) = W_B$, where W_B is the first $\Delta^1_1(X)$ witness of this fact that we find (cf. Lemma 3.6). If, on the other hand, we find first that $\tau \in \mathrm{cl}^{T^i_i(\tau)}_\omega(A_i)$, we set $S_i(\tau) = W_C$ where W_C is the $\Delta^1_1(T^i_i(\tau) \oplus C)$ witness of this fact that we find first. We obtain that for each $\tau \in P_i(T_i)$, $[S_i(\tau)] \setminus [B^{\boldsymbol{p}}]^{\prec} \subseteq [A_i]^{\prec}$, we can define $T_{i+1} = \bigcup_{\tau\in P_i(T_i)} S_i(\tau)$ and obtain that $[T_{i+1}] \setminus [B^{\boldsymbol{p}}]^{\prec} \subseteq \bigcap_{j\leq i}[A_j]^{\prec}$. By construction, we have $T_{i+1} \leq_i T_i$. Proceeding this way results in a fusion sequence $(T_i)_{i\in\omega} \in \mathcal{I}$. The fusion tree T (cf. Lemma 4.10) is by definition such that $T \leq_0 T_0$, the triple $\boldsymbol{q} = (T, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ is a condition in $\mathbb{O}$, and $\boldsymbol{q} \leq \boldsymbol{p}$. By construction $\boldsymbol{q}$ strongly forces $x_G \in C$. $\square$

Note that if $\Phi \in \mathcal{I}$ is a Turing functional, then the set $\mathrm{dom}(\Phi) = \{x \in \omega^\omega : \Phi(x) \in \omega^\omega\}$ is a $\mathbf{\Pi}^0_2$ set with code in $\mathcal{I}$. Hence Proposition 4.13 shows that in $\mathbb{O}$ one can always strongly force either divergence or convergence of Turing functionals.

We can now show that $\mathbb{O}$ enjoys the $\mathcal{I}$-Laver property.

Proposition 4.14. *Let $\boldsymbol{p} \in \mathbb{O}$ and let $\Phi \in \mathcal{I}$ be a Turing functional which is bounded by $h \in \mathcal{I}$. Assume that there is no $\boldsymbol{q} \leq \boldsymbol{p}$ which strongly forces the divergence of Φ. Then there is some $\boldsymbol{q} \leq \boldsymbol{p}$ and some trace $F \in \mathcal{I}$ such that $\boldsymbol{q}$ strongly forces that x_G is traced by $F(n)$. In particular, $\mathbb{O}$ has the Laver property.*

Proof. By Proposition 4.13 we can assume that $\boldsymbol{p}$ strongly forces that Φ converges. Fix a set $X \in \mathcal{I}$ such that $T^{\boldsymbol{p}} \leq_{\mathrm{T}} X$ and $B^{\boldsymbol{p}} \in \Pi^1_1(X)$. We build a fusion sequence and a trace F simultaneously, so that the fusion of our sequence gives rise to a condition $\boldsymbol{q}$ as above.

Consider the sequence $C_m = \{\tau \in T^{\boldsymbol{p}} : \Phi(\tau)(m)\downarrow\}$. In our assumptions on $T^{\boldsymbol{p}}$, for every $\tau \notin \mathrm{cl}^{T^{\boldsymbol{p}}}_\omega(B^{\boldsymbol{p}})$ and every $T \in \mathcal{I}$ which is ω-bushy above τ and such that $T \subseteq T^{\boldsymbol{p}}$, the set C_m is ω-big in T. Notice moreover that, since Φ is h-bounded, it follows that $C_m = \bigcup_{j\leq h(m)} V_{j,m}$, where $V_{j,m} = \{\tau \in T^{\boldsymbol{p}} : \Phi(\tau)(m) = j\}$. Thus, by Lemma 3.8, for any T and τ, if $\tau \in \mathrm{cl}^T_\omega(C_m)$, then $\tau \in \bigcup_{j\leq h(m)} \mathrm{cl}^T_\omega(V_{j,m})$.

Now we let $T_0 = T^{\boldsymbol{p}}$, and at step i we assume we have a sequence of trees $(T_k)_{k\leq i}$ such that $T_k \leq_j T_j$ for every $j \leq k \leq i$, $(T_k, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) \in \mathbb{O}$ for every $k \leq i$, and we have the initial segment of a trace $F_i\colon \{0,\dots,i\} \to [\omega]^{<\omega}$ (so $|F_i(k)| \le 2^k$ for all $k \in \mathrm{dom}(F_i)$) such that, for every $x \in [T_i] \setminus [B^{\boldsymbol{p}}]^{\prec}$ and every $k \le i$, $\Phi(x)(k) \in F_i(k)$. To build T_{i+1}, similarly to what we did in the proof of Proposition 4.13, for every $\tau \in P_i(T_i)$, we run the $\Pi^1_1(X \oplus \bigoplus_m C_m)$ enumerations of $\mathrm{cl}_\omega^{T_i^i(\tau)}(B^{\boldsymbol{p}})$ and $\mathrm{cl}_\omega^{T_i^i(\tau)}(V_{j,i+1})$ for every $j \le h(i+1)$. By our preceding discussions, we are sure that $\tau \in \mathrm{cl}_\omega^{T_i^i(\tau)}(B^{\boldsymbol{p}}) \cup \bigcup_{j \le h(i+1)} \mathrm{cl}_\omega^{T_i^i}(V_{j,i+1})$, so we can wait until we see a witness for one of these facts. Then we set $S_i(\tau) = W$ where W is the first $\Delta^1_1(X \oplus \bigoplus_m C_m)$ witness to whichever fact we find out first. If, for some $\tau \in P_i(T_i)$, the first of these facts that we find out is $\tau \in \mathrm{cl}_\omega^{T_i^i(\tau)}(V_{j,i+1})$ for some $j \le h(i+1)$, we let $j(\tau)$ be such value. We then define $F_{i+1} = F_i \cup \{i+1, \{j(\tau) : \tau \in P_i(T_i)\}\}$. The entire construction is hyperarithmetical in its inputs, so the resulting sequences $(T_i)_{i\in\omega}$ and $(F_i)_{i\in\omega}$ are in $\mathcal{I}$.

A verification analogous to that of Proposition 4.13 shows that $(T_i)_{i\in\omega} \in \mathcal{I}$ is a fusion sequence, $F = \bigcup_{i\in\omega} F_i \in \mathcal{I}$ is a trace, and, letting $T = \bigcap_{i\in\omega} T_i$, we have $(T, B^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) \in \mathbb{O}$. Moreover, by construction we have that if $x \in [T] \setminus [B^{\boldsymbol{p}}]^{\prec}$, then $\Phi(x)$ is traced by F. □

As anticipated, Proposition 4.14, combined with Corollary 2.36 yields:

Corollary 4.15. *Let G be a sufficiently generic filter on $\mathbb{O}$. $\mathcal{I}(f_G)$ does not contain any weakly $\mathcal{I}$-generic real.*

This establishes the main result of this section.

Theorem 4.16. *There are reals $x \in \mathcal{I}$-DOM which do not compute any weakly $\mathcal{I}$-generic, and in particular do not belong to $\mathcal{I}$-SME.*

Proof. Let G be a sufficiently generic filter on $\mathbb{O}$. By Corollary 4.8 and Corollary 4.15, x_G is an example of such a real. □

5. Interlude: adding a dominating real without adding randoms.

In the context of set theory, one builds a real in $\mathcal{V}$-$\mathsf{SME}\setminus\mathcal{V}$-$\mathsf{SNE}$ by forcing with the two step iteration $\mathbb{C}\star\mathbb{H}$. A generic for such forcing yields a pair of reals (x, y) such that x is $\mathcal{V}$-Cohen generic and y is $\mathcal{V}(x)$-dominating. By Lemma 2.25, y is $\mathcal{V}$-SME, and, since $\mathbb{C}\star\mathbb{H}$ is σ-centered, there are no $\mathcal{V}$-random reals in $\mathcal{V}(x, y)$. In Section 6 we will obtain an effectivization of $\mathbb{C}\star\mathbb{H}$, and we will use it to show that, if $\mathcal{I}$ is closed under relative hyperarithmeticity, then there is some real in $\mathcal{I}$-$\mathsf{SME}\setminus\mathcal{I}$-$\mathsf{SNE}$. Here we show an intermediate result which is helpful to understand the notion of forcing which we will introduce in Section 6.

Notation 5.1. For the rest of the section, we fix a countable Turing ideal $\mathcal{I} \subseteq \omega^\omega$ which is $\le_{\mathsf{HYP}}$-downward closed.

We shall now see that ω-small sets are naturally linked to Hechler forcing, and use this connection to obtain a forcing notion in $\mathcal{I}$ which is sufficiently effectively σ-centered, so that it adds no $\mathcal{I}$-randoms.

Definition 5.2. We define Hechler forcing with bad sets (in $\mathcal{I}$) $\mathbb{H}_{\mathsf{B}}$ as the following partial order: conditions are triples $(\tau^{\boldsymbol{p}}, f^{\boldsymbol{p}}, B^{\boldsymbol{p}})$ such that

- $\tau^{\boldsymbol{p}} \in \omega^{<\omega}$,
- $f^{\boldsymbol{p}} \in \omega^\omega \cap \mathcal{I}$,

- $B^{\boldsymbol{p}} \subseteq \omega^{<\omega}$ is upwards closed, $\tau^{\boldsymbol{p}} \notin \mathrm{cl}_\omega(B^{\boldsymbol{p}})$ and there is some $X \in \mathcal{I}$ such that $B^{\boldsymbol{p}}$ is $\Pi^1_1(X)$.

A condition $\boldsymbol{p}$ extends another condition $\boldsymbol{q}$ ($\boldsymbol{p} \leq \boldsymbol{q}$) if $B^{\boldsymbol{p}} \supseteq B^{\boldsymbol{q}}$ and $(\tau^{\boldsymbol{p}}, f^{\boldsymbol{p}})$ extends $(\tau^{\boldsymbol{q}}, f^{\boldsymbol{q}})$ in Hechler forcing. A real x *satisfies* a condition $\boldsymbol{p}$ if $\tau^{\boldsymbol{p}} \prec x$, $x(i) \geq f^{\boldsymbol{p}}(i)$ for every $i \geq |\tau^{\boldsymbol{p}}|$, and there is no $m \in \omega$ such that $x{\restriction}m \in B^{\boldsymbol{p}}$. We denote by $[\boldsymbol{p}]$ the collection of reals which satisfy $\boldsymbol{p}$.

As one would expect, $\mathbb{H}_{\mathsf{B}}$ is σ-centered.

Lemma 5.3. *For any $\sigma \in \omega^{<\omega}$, let $A_\sigma = \{\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}} : \tau^{\boldsymbol{p}} = \sigma\}$. Any two conditions $\boldsymbol{p}, \boldsymbol{q} \in A_\sigma$ are compatible, hence $\mathbb{H}_{\mathsf{B}}$ is σ-centered.*

Proof. Given $\boldsymbol{p}$ and $\boldsymbol{q}$ such $\tau^{\boldsymbol{p}} = \tau^{\boldsymbol{q}}$, let $f = \max\{f^{\boldsymbol{p}}, f^{\boldsymbol{q}}\}$. Then $\boldsymbol{r} = (\tau^{\boldsymbol{p}}, f, B^{\boldsymbol{p}} \cup B^{\boldsymbol{q}})$ is a condition by Lemma 3.8, and $\boldsymbol{r}$ is a common extension of both $\boldsymbol{p}$ and $\boldsymbol{q}$. □

It is not hard to see that the map $\pi\colon \mathbb{H}_{\mathsf{B}} \to \omega^{<\omega}$ given by $\boldsymbol{p} \mapsto \tau^{\boldsymbol{p}}$ makes $\mathbb{H}_{\mathsf{B}}$ into a real forcing in $\mathcal{I}$. A sufficiently generic filter G on $\mathbb{H}_{\mathsf{B}}$ adds an $\mathcal{I}$-dominating real $x_G = \bigcup_{\boldsymbol{p} \in G} \tau^{\boldsymbol{p}} \in \omega^\omega$. The proof of these facts follows the blueprint of Lemmas 4.4, 4.5, 4.7 and Corollary 4.8, and the proofs are essentially identical, so we do not repeat them.

Lemma 5.4. *For any $m \in \omega$, the set $L_m = \{\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}} : |\tau^{\boldsymbol{p}}| \geq m\}$ is dense.*

Lemma 5.5. *Let $G \subseteq \mathbb{H}_{\mathsf{B}}$ be a filter in $\mathrm{dom}(\pi)$. For all $\boldsymbol{p} \in G$, we have $x_G \in [\boldsymbol{p}]$, in particular, there is no $k \in \omega$ such that $x{\restriction}k \in B^{\boldsymbol{p}}$.*

Lemma 5.6. *For every function $h \in \omega^\omega \cap \mathcal{I}$, the set $D_h = \{\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}} : \forall g \in [\boldsymbol{p}]\ (g \geq^* h)\}$ is dense.*

Corollary 5.7. *If G is a sufficiently generic filter on $\mathbb{H}_{\mathsf{B}}$, then $f_G \in \omega^\omega$ dominates all functions in $\mathcal{I}$.*

Similarly to what we did for $\mathbb{O}$, we say that a condition $\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}}$ *strongly forces* $x_G \in A$ (where $A \subseteq \omega^\omega$) if $[p] \subseteq A$.

We now show that the use of bad sets allow us to recover a weaker version of Corollary 4.6 for $\mathbb{H}_{\mathsf{B}}$.

Lemma 5.8. *Let $A \subseteq \omega^{<\omega} \cap \mathcal{I}$ be upwards closed and let $\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}}$. If $\tau^{\boldsymbol{p}} \in \mathrm{cl}_\omega(A)$, then there is $\boldsymbol{q} \leq \boldsymbol{p}$ with $[\boldsymbol{q}] \subseteq [A]^\prec$, and if $\tau^{\boldsymbol{p}} \notin \mathrm{cl}_\omega(A)$, then $\boldsymbol{p'} = (\tau^{\boldsymbol{p}}, f^{\boldsymbol{p}}, B^{\boldsymbol{p}} \cup {\uparrow}A) \in \mathbb{H}_{\mathsf{B}}$, and $[\boldsymbol{p'}] \cap [B]^\prec = \emptyset$.*

Proof. If $\tau^{\boldsymbol{p}} \notin \mathrm{cl}_\omega(A)$, then Lemma 3.8 implies that $\boldsymbol{p'} = (\tau^{\boldsymbol{p}}, f^{\boldsymbol{p}}, B^{\boldsymbol{p}} \cup A) \in \mathbb{H}_{\mathsf{B}}$, and by Lemma 5.5 $[\boldsymbol{p'}] \cap [B]^\prec = \emptyset$.

On the other hand, if $\tau^{\boldsymbol{p}} \in \mathrm{cl}_\omega(A)$, then this is witnessed by a wellfounded ω-bushy tree T with stem $\tau^{\boldsymbol{p}}$ and all leaves contained in A. We can build by recursion a partial path f in T such that, if $i \geq |\tau^{\boldsymbol{p}}|$, $f(i) \geq f^{\boldsymbol{p}}(i)$. Since T is wellfounded, there must be some k such that $f{\restriction}k$ is a leaf in T, so it is in A. The triple $\boldsymbol{q} = (f{\restriction}k, f^{\boldsymbol{p}}, B^{\boldsymbol{p}})$ is a condition of $\mathbb{H}_{\mathsf{B}}$, $\boldsymbol{q} \leq \boldsymbol{p}$, and $[\boldsymbol{q}] \subseteq [A]^\prec$. □

Definition 5.9. Let $\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}}$ be any condition. Define the set of possible finite extensions of $\boldsymbol{p}$ as $E_{\boldsymbol{p}} = \{\sigma : (\sigma, f^{\boldsymbol{p}}, B^{\boldsymbol{p}}) \in \mathbb{H}_{\mathsf{B}} \wedge (\sigma, f^{\boldsymbol{p}}, B^{\boldsymbol{p}}) \leq \boldsymbol{p}\}$.

It is immediate to see that if $\boldsymbol{q} \leq \boldsymbol{p}$ is any extension, the stem $\tau^{\boldsymbol{q}}$ belongs to $E_{\boldsymbol{p}}$.

Corollary 5.10. *Let $A \subseteq \omega^{<\omega} \cap \mathcal{I}$ be an upwards closed set and let $\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}}$. Then $\boldsymbol{p} \Vdash x_G \in [A]^\prec$ if and only if $E_{\boldsymbol{p}} \subseteq \mathrm{cl}_\omega(A)$.*

Proof. If there is $\tau \in E_{\boldsymbol{p}} \setminus \mathrm{cl}_\omega(A)$, by Lemma 5.8 the condition $(\tau, f^{\boldsymbol{p}}, B^{\boldsymbol{p}} \cup A) \leq \boldsymbol{p}$ forces (actually strongly forces) $x_G \notin [A]^{\prec}$.

Otherwise, we claim that the set of conditions $\boldsymbol{q}$ such that $\tau^{\boldsymbol{q}} \in A$ is dense below $\boldsymbol{p}$. Indeed, let $\boldsymbol{r} \leq \boldsymbol{p}$. The stem $\tau^{\boldsymbol{r}}$ is in $E_{\boldsymbol{p}}$, hence $\tau^{\boldsymbol{r}} \in \mathrm{cl}_\omega(A)$. By Lemma 5.8, it follows that there is some $\boldsymbol{q} \leq \boldsymbol{r}$ with $\tau^{\boldsymbol{q}} \in A$. □

Note the difference between Corollary 5.10 and Corollary 4.6: the latter can be stated purely in terms of strong forcing, whereas the former cannot. This is the result of a combinatorial difference between Laver and Hechler forcing. In the terminology of this paper, the reason why Corollary 4.6 works is that a (-n upwards closed) set $A \subseteq \omega^{<\omega}$ is either ω-big or ω-small. The combinatorial analogue of this in the context of Hechler forcing would be the following: for any upwards closed set $A \subseteq \omega^{<\omega}$, either A or $\omega^{<\omega} \setminus A$ are ω-small. This statement is readily seen to be false. This and related matters are explored in [10].

Lemma 5.11. *Let $\Phi \in \mathcal{I}$ be a Turing functional and let $\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}}$. If there is no $\boldsymbol{q} \leq \boldsymbol{p}$ which strongly forces $\Phi(x_G)\uparrow$, then $\boldsymbol{p} \Vdash \Phi(x_G)\downarrow$.*

Proof. For any $m \in \omega$, let $C(m) = \{\rho \in \omega^{<\omega} : \Phi(\rho)(m)\downarrow\}$. If there is some $m \in \omega$ and some $\sigma \in E_{\boldsymbol{p}}$ with $\sigma \notin \mathrm{cl}_\omega(C_m)$, then the triple $\boldsymbol{q} = (\sigma, f^{\boldsymbol{p}}, B^{\boldsymbol{p}} \cup C_m)$ is a condition and $[\boldsymbol{q}] \cap C_m = \emptyset$. Hence $\boldsymbol{q}$ strongly forces $\Phi(x_G)\uparrow$.

If $E_{\boldsymbol{p}} \subseteq \mathrm{cl}_\omega(C_m)$ for all $m \in \omega$, then Corollary 5.10 implies that $\boldsymbol{p} \Vdash x_G \in \bigcap_{m\in\omega}[C_m]^{\prec}$, hence $\boldsymbol{p} \Vdash \Phi(x_G)\downarrow$. □

Now we show that if G is a sufficiently generic filter for $\mathbb{H}_{\mathsf{B}}$, then $\mathcal{I}(x_G)$ does not contain any $\mathcal{I}$-random reals.

Proposition 5.12. *Let Φ be an $\mathcal{I}$-coded $\{0,1\}$-valued Turing functional. The set*

$$C_\Phi = \{\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}} : \boldsymbol{p} \Vdash \Phi(x_G)\uparrow \vee \exists S \in \mathcal{I}\ (S \text{ Schnorr null } \wedge \boldsymbol{p} \Vdash \Phi(x_G) \in S)\}.$$

is dense. In particular, if G is a sufficiently generic filter for $\mathbb{H}_{\mathsf{B}}$, then $\mathcal{I}(x_G)$ does not contain any $\mathcal{I}$-randoms.

Proof. Let $\boldsymbol{p} \in \mathbb{H}_{\mathsf{B}}$: we show that it $\boldsymbol{p}$ has an extension which belongs to C_Φ. We define the sets $C_m \subseteq \omega^{<\omega}$ consisting of strings on which Φ converges on the first m bits as in the proof of Lemma 5.11.

Again, if there are $m \in \omega$ and $\sigma \in E_{\boldsymbol{p}} \setminus \mathrm{cl}_\omega(C_m)$, then $\boldsymbol{q} = (\sigma, f^{\boldsymbol{p}}, B^{\boldsymbol{p}} \cup C_m)$ strongly forces Φ to diverge, hence $\boldsymbol{q} \in C_\Phi$.

Now assume that $E_{\boldsymbol{p}} \subseteq \bigcap_{m\in\omega} \mathrm{cl}_\omega(C_m)$ and let $X \in 2^\omega$ be such that $B^{\boldsymbol{p}}$ is $\Pi^1_1(X)$ and $f^{\boldsymbol{p}} \leq_{\mathrm{T}} X$. We define $F = \{\sigma \succeq \tau^{\boldsymbol{p}} : \forall i\ (|\tau^{\boldsymbol{p}}| \leq i < |\sigma| \to \sigma(i) \geq f^{\boldsymbol{p}}(i))\}$. The set F is the set of strings σ which respect $f^{\boldsymbol{p}}$, i.e. the set of strings σ such that $(\sigma, f^{\boldsymbol{p}})$ extends $(\tau^{\boldsymbol{p}}, f^{\boldsymbol{p}})$ in the ordering of Hechler forcing. Note that F is arithmetical in $f^{\boldsymbol{p}}$ (unlike $E_{\boldsymbol{p}}$).

We build a $\Delta^1_1(\Phi \oplus X)$ sequence of infinite binary trees $(R_\sigma)_{\sigma\in F}$, as follows. We simultaneously run the $\Pi^1_1(\Phi \oplus X)$ enumerations of $\mathrm{cl}_\omega(B^{\boldsymbol{p}})$ and $\mathrm{cl}_\omega(V_\nu)$, where $V_\nu = \{\rho \in \omega^{<\omega} : \Phi(\rho)\restriction|\nu|\downarrow = |\nu|\}$. If we ever find out that $\sigma \in F \cap \mathrm{cl}_\omega(B^{\boldsymbol{p}})$, we set $P_\sigma = 2^{<\omega}$. As long as we do not find this out, we put $\nu \in 2^{<\omega}$ inside the set P_σ just in case we find $\sigma \in \mathrm{cl}_\omega(V_\nu)$

Our assumption guarantees that if $\sigma \in F \setminus \mathrm{cl}_\omega(B^{\boldsymbol{p}})$, then for every $m \in \omega$, $\sigma \in \mathrm{cl}_\omega(C_m)$, so by Lemma 3.8 $\sigma \in \mathrm{cl}_\omega(V_{\nu,\sigma})$ for some $\nu \in 2^m$. Hence, if $\sigma \notin \mathrm{cl}_\omega(B^{\boldsymbol{p}})$, then P_σ is an infinite $\Pi^1_1(\Phi \oplus X)$ tree. On the other hand, if $\sigma \in \mathrm{cl}_\omega(B^{\boldsymbol{p}})$, then we find this out at some stage $\alpha < \omega_1^{\Phi\oplus X}$, and then let $R_\sigma = 2^{<\omega}$.

In either case, we can find an infinite $\Delta^1_1(\Phi \oplus X)$ set $Q_\sigma \subseteq P_\sigma$, uniformly in $\sigma \in F$: for each $\sigma \in F$, we wait for the least ordinal α such that $(P_\sigma)_\alpha$ is infinite. The sequence $(Q_\sigma)_{\sigma \in F}$ is hence $\Delta^1_1(\Phi \oplus X)$, and closing each Q_σ under prefixes gives a $\Delta^1_1(\Phi \oplus X)$ sequence of infinite binary trees $(R_\sigma)_{\sigma \in F}$.

Now let $(g_\sigma)_{\sigma \in F}$ be a sequence of elements of 2^ω such that $(g_\sigma)_{\sigma \in F} \leq_{\mathrm{T}} ((R_\sigma)_{\sigma \in F})'$ and, for every $\sigma \in F$, $g_\sigma \in [R_\sigma]$. We use this sequence to obtain some $h \colon \omega \to 2^{<\omega}$ such that $h \in \mathcal{I}$, $h(n) \in 2^n$ for every n, and, for every $\sigma \in F$, there are infinitely many $n \in \omega$ such that $g_\sigma \restriction n = h(n)$. The function h allows us to define an $\mathcal{I}$-coded Schnorr null set $S = \{x \in 2^\omega : \exists^\infty n \ (x \restriction n = h(n))\}$. The fact that S is a null set follows as $S = \bigcap_{n \in \omega} \bigcup_{m \geq n} [h(m)]$, and $\lambda([h(m)]) = 2^{-m-1}$. The fact that S is coded as a Schnorr null set in $\mathcal{I}$ follows from Remark 2.27 as $\mathcal{I}$ is a jump ideal.

We claim that $\boldsymbol{p} \Vdash x_G \in S$. To see this, let $\boldsymbol{q} \leq \boldsymbol{p}$ and let $n \in \omega$. We show that there is some $m \geq n$ and some $\boldsymbol{r} \leq \boldsymbol{q}$ such that $\boldsymbol{r}$ strongly forces $x_G \in [h(m)]$. Indeed, by construction we have that the tree $R_{\tau^{\boldsymbol{q}}}$ consists of strings ν such that $\tau^{\boldsymbol{q}} \in \mathrm{cl}_\omega(V_\nu)$. Now consider $g_{\tau^{\boldsymbol{q}}}$ and let $m \geq n$ be least such that $h(m) = g_{\tau^{\boldsymbol{q}}} \restriction m$. Since $\tau^{\boldsymbol{q}} \in \mathrm{cl}_\omega(V_{h(m)})$, reasoning as in the proof of Lemma 5.8, there is some $\rho \in E_{\boldsymbol{q}} \cap V_{h(m)}$. The condition $\boldsymbol{r} = (\rho, f^{\boldsymbol{q}}, B^{\boldsymbol{q}})$ strongly forces $x_G \in [h(m)]$. □

Theorem 5.13. *There are reals $x \in \mathcal{I}$-DOM which do not compute any $\mathcal{I}$-random, and in particular do not belong to $\mathcal{I}$-SNE.*

Proof. Let G be a sufficiently generic filter on $\mathbb{H}_{\mathsf{B}}$. By Corollary 5.7 and Proposition 5.12, x_G is an example of such a real. □

6. Separating strongly meagre engulfing from strongly null engulfing

As anticipated, we now separate the notions of $\mathcal{I}$-SME and $\mathcal{I}$-SNE for countable ideals $\mathcal{I}$ closed under relative hyperarithmeticity. This is achieved by obtaining the correct effectivization of the iteration $\mathbb{C} \star \mathbb{H}$. As in Section 5, we exploit the fact that our forcing notion is σ-centered to obtain the following stronger result: for any ideal $\mathcal{I}$ as above, there is some $x \in \mathcal{I}$-SME which computes no $\mathcal{I}$-random.

To get a suitable forcing notion, we want to retrieve results analogous to Lemmas 5.8 and 5.11. We do so using bad sets once more. This requires the introduction of a bigness notion (i.e. a closure operator) on $\mathcal{P}(2^{<\omega} \times \omega^{<\omega})$ which behaves well with respect to forcing (in analogy with the relationship between cl_ω and forcing with $\mathbb{H}_{\mathsf{B}}$ which is visible in Section 5). That is the content of the next subsection.

6.1. **Bigness on $2^{<\omega} \times \omega^{<\omega}$.** We introduce a uniformly Π^1_1 closure operator on $\mathcal{P}(2^{<\omega} \times \omega^{<\omega})$ which reflects the properties that we will need to handle the iteration $\mathbb{C} \star \mathbb{H}$ over ideals closed under $\leq_{\mathsf{HYP}}$.

Definition 6.1. Let $B \subseteq 2^{<\omega} \times \omega^{<\omega}$ and let $(\sigma, \tau) \in 2^{<\omega} \times \omega^{<\omega}$. We say that (σ, τ) has B-rank 0 (in symbols, B-$r(\sigma, \tau) = 0$) if $(\sigma, \tau) \in B$. For any $\alpha > 0$, we say that (σ, τ) has B-rank α (B-$r(\sigma, \tau) = \alpha$) if

$$\alpha = \min\{\beta : \exists \sigma' \succ \sigma \ \forall \sigma'' \succeq \sigma' \ \exists \sigma''' \succeq \sigma'' \ \exists^\infty n \ (B\text{-}r(\sigma''', \tau^\frown n) < \alpha)\}.$$

We say that B is *big* above (σ, τ) if (σ, τ) has B-rank α for some ordinal α. Otherwise, we say that B is *small* above (σ, τ).

In case B is big above (σ, τ), we write $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(B)$ (the p is for *pairs*). This defines a function $\mathrm{cl}_{\mathrm{P}} \colon \mathcal{P}(2^{<\omega} \times \omega^{<\omega}) \to \mathcal{P}(2^{<\omega} \times \omega^{<\omega})$.

We now show that $\mathrm{cl_P}$ is a uniformly Π^1_1 closure operator, in analogy with the functions cl^T_ω defined in Section 3.

Lemma 6.2. *The* $\mathrm{cl_P}\colon \mathcal{P}(2^{<\omega}\times\omega^{<\omega})\to\mathcal{P}(2^{<\omega}\times\omega^{<\omega})$ *is monotone with respect to* $\subseteq$*, extensive (i.e.* $B\subseteq \mathrm{cl_P}(B)$ *for every* B*), and* $\mathrm{cl_P}(\emptyset)=\emptyset$*.*

Proof. Immediate from the definition of $\mathrm{cl_P}$. $\square$

We show that, for any $B\subseteq 2^{<\omega}\times\omega^{<\omega}$ and $(\sigma,\tau)\in 2^{<\omega}\times\omega^{<\omega}$, if (σ,τ) has a B-rank, then its B-rank is strictly smaller than ω_1^B. This implies that $\mathrm{cl_P}$ is a uniformly Π^1_1 function.

Lemma 6.3. *Let* $B\subseteq 2^{<\omega}\times\omega^{<\omega}$*, let* $(\sigma,\tau)\in\mathrm{cl_P}(B)$*. Then* $B\text{-}r(\sigma,\tau)<\omega_1^B$*.*

Proof. We assume that B is computable, and we show that $B\text{-}r(\sigma,\tau)<\omega_1^{\mathrm{CK}}$. The full statement follows as usual by relativization. We want to show that there is no (σ,τ) of rank ω_1^{CK}. For every α, let R_α be the set of pairs with B-rank less than or equal to α.

Let $(\sigma,\tau)\in R_{\omega_1^{\mathrm{CK}}}$. By definition we know that there is some $\sigma'\succ\sigma$ such that for every $\sigma''\succeq\sigma'$ there is some $\sigma'''\succeq\sigma''$ and infinitely many $n\in\omega$ such that $B\text{-}r(\sigma''',\tau^\frown n)<\omega_1^{\mathrm{CK}}$. For every $\sigma'''\succeq\sigma'$ such that there are infinitely many n with $B\text{-}r(\sigma''',\tau^\frown n)<\omega_1^{\mathrm{CK}}$, by an argument like that of Lemma 3.6 (essentially, an application of Σ^1_1 boundedness), we know that there is some $\beta_{\sigma'''}<\omega_1^{\mathrm{CK}}$ such that for infinitely many $n\in\omega$, we have $B\text{-}r(\sigma''',\tau^\frown n)<\beta_{\sigma'''}$. We now define, for every $\sigma''\succeq\sigma'$, the ordinal $\gamma_{\sigma''}=\min_{\sigma'''\succeq\sigma''}\beta_{\sigma'''}$. We have already shown that $\gamma_{\sigma''}$ is defined and $\gamma_{\sigma''}<\omega_1^{\mathrm{CK}}$ for every $\sigma''\succeq\sigma'$. It is also easy to see that the function $\sigma''\mapsto\gamma_{\sigma''}$ is Π^1_1, and since it has a computable domain, it is actually Δ^1_1. Therefore, its range is bounded by some $\alpha<\omega_1^{\mathrm{CK}}$ hence, by definition of rank, we have $(\sigma,\tau)\in R_\alpha$. $\square$

Corollary 6.4. *Let* $B\subseteq 2^{<\omega}\times\omega^{<\omega}$*, and let* $X\subseteq\omega$ *be such that* B *is* $\Pi^1_1(X)$*. Then* $\mathrm{cl_P}(B)$ *is* $\Pi^1_1(X)$*.*

Proof. By Lemma 6.3, $\mathrm{cl_P}(B)=\{(\sigma,\tau)\in 2^{<\omega}\times\omega^{<\omega} : \exists\alpha<\omega_1^B\ (B\text{-}r(\sigma,\tau)=\alpha)\}$, and the statement $B\text{-}r(\sigma,\tau)=\alpha$ is $\Pi^1_1(B)$. This shows that $\mathrm{cl_P}$ is a uniformly Π^1_1 function. Hence, Lemma 2.5 and Lemma 6.2 imply our claim. $\square$

When working with the closure operators cl^T_ω, we know that $\sigma\in\mathrm{cl}^T_\omega(B)$ is witnessed by some wellfounded tree $S\subseteq T$ which is ω-bushy above σ and has all leaves in B. We introduce a combinatorial object which witnesses statements of the form $(\sigma',\tau')\in\mathrm{cl_P}(B')$.

Definition 6.5. A *bigness witness* is a pair (T,f) where $T\subseteq 2^{<\omega}\times\omega^{<\omega}$ and $f\colon T\to[T]^{<\omega}$ is such that:

- there is some pair $(\sigma,\tau)\in T$, called *the stem* of T, such that $(\sigma',\tau')\succeq(\sigma,\tau)$ for all $(\sigma',\tau')\in T$,
- for every $(\hat\sigma,\hat\tau)\in\bigcup\mathrm{ran}(f)$, there is some $\sigma'\succ\hat\sigma$ such that for every $\sigma''\succeq\sigma'$ there is some $\sigma'''\succeq\sigma''$ and infinitely many $n\in\omega$ such that $(\sigma''',\tau^\frown n)\in T$ and $(\hat\sigma,\hat\tau)\in f(\sigma''',\tau^\frown n)$,
- $f(\sigma,\tau)=\emptyset$ and for every $(\hat\sigma,\hat\tau)\in T$, if $(\tilde\sigma,\tilde\tau)\in f(\hat\sigma,\hat\tau)$, then $\tilde\sigma\prec\hat\sigma$ and there is some $n\in\omega$ such that $\tilde\tau=\hat\tau^\frown n$,

- for every $(\hat{\sigma}, \hat{\tau}) \in T$, letting $d = |\hat{\tau}| - |\tau|$, there is a sequence $(\sigma_i, \tau_i)_{i \leq d}$ such that $(\sigma_0, \tau_0) = (\sigma, \tau)$, $(\sigma_d, \tau_d) = (\hat{\sigma}, \hat{\tau})$ and, for all $i < d$, $(\sigma_i, \tau_i) \in f(\sigma_{i+1}, \tau_{i+1})$.

For every $(\hat{\sigma}, \hat{\tau}) \in T$, we say that any $(\tilde{\sigma}, \tilde{\tau}) \in f(\hat{\sigma}, \hat{\tau})$ is a *justification for* $(\hat{\sigma}, \hat{\tau})$. We say that $(\hat{\sigma}, \hat{\tau}) \in T$ is a *leaf* of T if $(\hat{\sigma}, \hat{\tau}) \notin \bigcup \mathrm{ran}(f)$, i.e. if it is not a justification for any other pair. We say that $(x, y) \in 2^\omega \times \omega^\omega$ is a *path* of (T, f), in symbols $(x, y) \in [T, f]$, if there is a sequence $(\sigma_i, \tau_i)_{i \in \omega}$ such that $(\sigma_0, \tau_0) = (\sigma, \tau)$, and, for every $i \in \omega$, $(\sigma_i, \tau_i) \prec (x, y)$ and (σ_i, τ_i) is a justification for $(\sigma_{i+1}, \tau_{i+1})$.

Remark 6.6. For any bigness witness (T, f) and any $s, t \in T$, define $s \lhd t$ if $s \in f(t)$, so if s is a justification of t.

A leaf of (T, f) is a maximal element for $\lhd$, and a path for (T, f) is an infinite $\lhd$-increasing sequence, so (T, f) has no paths if and only if $\rhd$ is wellfounded.

Note that $\lhd$ is a strict refinement of the prefix relation on pairs of strings, so (T, f) may have leaves which have proper $\prec$-extensions on T, and not all pairs of reals with arbitrarily long prefixes on T correspond to paths in (T, f).

As the name suggests, a set B is big above a pair (σ, τ) if and only if there is a bigness witness certifying this.

Lemma 6.7. *Let $B \subseteq 2^{<\omega} \times \omega^{<\omega}$ and let $(\sigma, \tau) \in 2^{<\omega} \times \omega^{<\omega}$. We have $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(B)$ if and only if there is a bigness witness (T, f) with stem (σ, τ), no paths, and all leaves in B. Moreover, if a bigness witness (T, f) as above exists, then there is one which is $\Delta^1_1(B)$.*

Proof. The proof is analogous to the proof of Lemma 3.6.

First, suppose that there is a bigness witness (T, f) as above. We define a relation $\lhd$ on T as in Remark 6.6: for every pair $s, t \in T$, we let $s \lhd t$ if and only if $s \in f(t)$. We claim that for every element $s \in T$, the $\lhd$-rank of s, defined as $r(s) = 0$ if s is a leaf and $r(s) = \sup\{r(t) + 1 : s \lhd t\}$, exists and it is greater or equal to the B-rank of s. In particular, one can assign a B-rank to (σ, τ), so by definition $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(B)$.

Every element of T has a $\lhd$-rank because $\rhd$ is wellfounded. We show that $r(s) \geq B\text{-}r(s)$ for every $s \in T$ by induction.

Given any $(\nu, \rho) \in T$, if $r(\nu, \rho) = 0$, then (ν, ρ) is a leaf of (T, f), so $(\nu, \rho) \in B$ and $B\text{-}r(\nu, \rho) = 0$. If $r(\nu, \rho) = \beta > 0$ then $(\nu, \rho) \in \bigcup \mathrm{ran}(f)$, so by definition of bigness witness there is some $\nu' \succ \nu$ such that for every $\nu'' \succeq \nu'$ there is some $\nu''' \succeq \nu''$ and infinitely many n such that $(\nu''', \rho^\frown n) \in T$ and $(\nu, \rho) \lhd (\nu''', \rho^\frown n)$, so that $r(\nu, \rho) > r(\nu''', \rho^\frown n) \geq B\text{-}r(\nu''', \rho^\frown n)$ for all such ν''' and n. By definition of B-ranking, this immediately implies that (ν, ρ) is B-ranked. and $r(\nu, \rho) \geq B\text{-}r(\nu, \rho)$.

For the converse we assume that $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(B)$ and we build a bigness witness (T, f) as above using the B-ranking function. We again assume that B is computable for notational simplicity. The general statement follows by relativization.

By Lemma 6.3, we know that $B\text{-}r(\sigma, \tau) = \alpha < \omega_1^{\mathrm{CK}}$. By definition of B-rank, one can see that for every $\beta < \omega_1^{\mathrm{CK}}$, the set $R_\beta = \{(\sigma', \tau') : B\text{-}r(\sigma', \tau') = \beta\}$ is computable in $\emptyset^{(\beta \cdot 5)}$, uniformly in (a notation for) $\beta \cdot 5$. So, the sequence $(R_\beta)_{\beta < \alpha}$ is Δ^1_1.

We start from $T_0 = \{(\sigma, \tau)\}$, and $f_0 = \{(\sigma, \tau), \emptyset\}$. Given $i \in \omega$, assume we have a sequence $(T_j, f_j)_{j \leq i}$ such that (i) for every $j \leq i$, $T_j \subseteq \mathrm{cl}_{\mathrm{P}}(B) \cap (2^{<\omega} \times \omega^{\leq |\tau| + j})$ is a bigness witness with stem (σ, τ), (ii) for every $j < i$, $T_j \subseteq T_{j+1}$ and $f_j \subseteq f_{j+1}$, and (iii) for every $j < i$, every $(\sigma', \tau') \in T_{j+1} \setminus T_j$, and every $(\hat{\sigma}, \hat{\tau}) \in f(\sigma', \tau')$, we have $B\text{-}r(\hat\sigma,\hat\tau) > B\text{-}r(\sigma',\tau')$. Note that, by definition of bigness witness, we have that for every $j < i$, every $(\sigma',\tau') \in T_{j+1} \setminus T_j$ is a leaf in (T_{j+1}, f_{j+1}).

We build T_{i+1} by extending every leaf of (T_i, f_i) which is not contained in B, as follows. Let (σ',τ') be such a leaf, and call $\beta = B\text{-}r(\sigma',\tau') > 0$. Look for the least $\gamma \le \beta$ such that there is some $\sigma'' \succ \sigma'$ such that for every $\sigma''' \succeq \sigma''$ there is some $\hat\sigma \succeq \sigma'''$ and infinitely many $n \in \omega$ such that $B\text{-}r(\hat\sigma, (\tau')^\frown n) < \gamma$. Once this γ is fixed, look for the least $\sigma'' \succ \sigma'$ as above. Now let $S_{(\sigma',\tau')}$ consist of all sequences $\hat\sigma \succeq \sigma''$ such that for infinitely many n we have $B\text{-}r(\hat\sigma, \tau'^\frown n) < \gamma$, and let $N_{(\sigma',\tau')}$ be given by the pairs $(\hat\sigma, \tau'^\frown n)$ such that $\hat\sigma \in S_{(\sigma',\tau')}$ and $B\text{-}r(\hat\sigma, (\tau')^\frown n) < \gamma$. We then define T_{i+1} as T_i together with the union of all the $N_{(\sigma',\tau')}$ as (σ',τ') ranges over the leaves of T_i which are not in B. On T_{i+1}, we extend f_i to f_{i+1} so that, for $(\hat\sigma,\hat\tau) \in T_{i+1} \setminus T_i$, $f_{i+1}(\hat\sigma,\hat\tau) = \{(\sigma',\tau') : (\hat\sigma,\hat\tau) \in N_{(\sigma',\tau')}\}$.

We let $T = \bigcup_{i\in\omega} T_i$ and $f = \bigcup_{i\in\omega} f_i$. It is immediate to see that $(T, f) \le_{\mathsf{HYP}} (R_\beta)_{\beta<\alpha}$, so in particular (T, f) is hyperarithmetical. Moreover, (T, f) is a bigness witness with stem (σ,τ) as it is the union of an increasing sequence of bigness witnesses with stem (σ,τ), and, by construction, (T, f) has no leaves outside B.

Now if $(x, y) \in 2^\omega \times \omega^\omega$ were a path in (T, f), then, given $(\sigma_i,\tau_i)_{i\in\omega}$ a sequence witnessing this, we would have $(\sigma_{i+1},\tau_{i+1}) \in N_{(\sigma_i,\tau_i)}$, so that $B\text{-}r(\sigma_i,\tau_i) > B\text{-}r(\sigma_{i+1},\tau_{i+1})$, for every $i \in \omega$. This is a contradiction. $\square$

Remark 6.8. Lemma 6.7 provides another proof of Corollary 6.4, as we can now say that $(\sigma,\tau) \in \mathrm{cl}_{\mathrm{P}}(B)$ if and only if this is certified by an hyperarithmetical, wellfounded bigness witness.

We now show that the operator $B \mapsto \mathrm{cl}_{\mathrm{P}}(B)$ is idempotent, using bigness witnesses for convenience. First, we note that bigness witnesses can be concatenated.

Lemma 6.9. *Let (T, f) be a bigness witness with stem (σ,τ) and no paths, and for every leaf $\ell = (\sigma^\ell,\tau^\ell) \in T$ let (T_ℓ, f_ℓ) be a bigness witness with stem (σ^ℓ,τ^ℓ) and no paths.*

The pair (T', f') given by $T' = T \cup \bigcup_\ell T_\ell$ and $f' = f \cup \bigcup f_\ell$ is a bigness witness with stem (σ,τ) and no paths. Moreover, if (σ',τ') is a leaf of (T', f'), then it is a leaf of some (T_ℓ, f_ℓ).

Proof. Immediate by definition. $\square$

Corollary 6.10. *Let $B \subseteq 2^{<\omega} \times \omega^{<\omega}$. We have* $\mathrm{cl}_{\mathrm{P}}(\mathrm{cl}_{\mathrm{P}}(B)) = \mathrm{cl}_{\mathrm{P}}(B)$

Proof. We show that $\mathrm{cl}_{\mathrm{P}}(\mathrm{cl}_{\mathrm{P}}(B)) \subseteq \mathrm{cl}_{\mathrm{P}}(B)$, as this is enough by Lemma 6.2. Let $(\sigma,\tau) \in \mathrm{cl}_{\mathrm{P}}(\mathrm{cl}_{\mathrm{P}}(B))$, and fix some bigness witness (T, f) with stem (σ,τ), no path, and all leaves in $\mathrm{cl}_{\mathrm{P}}(B)$. For any leaf (σ^ℓ,τ^ℓ) of (T, f), by definition of $\mathrm{cl}_{\mathrm{P}}(B)$, we have a bigness witness (T_ℓ, f_ℓ) with stem (σ^ℓ,τ^ℓ), no path, and all leaves in B. Concatenating (T, f) with the (T_ℓ, f_ℓ)'s as in Lemma 6.9 we obtain a bigness witness (T', f') with stem (σ,τ), no paths, and all leaves in B. By Lemma 6.7, we get $(\sigma,\tau) \in \mathrm{cl}_{\mathrm{P}}(B)$. $\square$

Lemma 6.11. *Let $A, B \subseteq 2^{<\omega} \times \omega^{<\omega}$. Then* $\mathrm{cl}_{\mathrm{P}}(A \cup B) = \mathrm{cl}_{\mathrm{P}}(A) \cup \mathrm{cl}_{\mathrm{P}}(B)$.

Proof. Since cl_{P} is monotone, we only need to show the left-to-right inclusion.

Let $(\sigma,\tau) \in \mathrm{cl}_{\mathrm{P}}(A\cup B)$. By induction on $\alpha = (A\cup B)\text{-}r(\sigma,\tau)$ we show that either $A\text{-}r(\sigma,\tau)$ is defined and less than or equal to α or $B\text{-}r(\sigma,\tau)$ is defined and less than or equal to α.

If $\alpha = 0$, then $(\sigma, \tau) \in A \cup B \subseteq \mathrm{cl_P}(A) \cup \mathrm{cl_P}(B)$.

Suppose now that $\alpha > 0$. By definition, there is some $\sigma' \succ \sigma$ such that for every $\sigma'' \succeq \sigma'$ there is some $\sigma''' \succeq \sigma''$ and infinitely many $n \in \omega$ such that $(A \cup B)$-$r(\sigma''', \tau^\frown n) < \alpha$.

For every $\sigma''' \succeq \sigma'$ and every $n \in \omega$, assign the pair $(\sigma''', \tau^\frown n)$ the label $(0, a)$ if A-$r(\sigma''', \tau^\frown n) < \alpha$, and the label $(0, b)$ if B-$r(\sigma''', \tau^\frown n) < \alpha$ (we allow a pair $(\sigma''', \tau^\frown n)$ to have two labels).

Now fix any $\sigma'' \succeq \sigma'$, and let $\sigma''' \succeq \sigma''$ be any extension such that for infinitely many $n \in \omega$, the pair $(\sigma''', \tau^\frown n)$ has $(A \cup B)$-rank strictly below α. By inductive assumption, any such pair $(\sigma''', \tau^\frown n)$ is labeled either $(0, a)$ or $(0, b)$. By the pigeonhole principle, there are either infinitely many n such that $(\sigma''', \tau^\frown n)$ has label $(0, a)$ or infinitely many n such that $(\sigma''', \tau^\frown n)$ has label $(0, b)$. In the first case we assign σ'' the label $(1, a)$, and in the second case we assign σ'' the label $(1, b)$. Note that this allows us to assign a label (potentially, both labels) of the form $(1, \cdot)$ to every string $\sigma'' \succeq \sigma'$.

Now there are two possible cases: either there is $\sigma'' \succeq \sigma'$ such that every $\hat{\sigma} \succeq \sigma''$ is labeled $(1, a)$, or we have that every $\sigma'' \succeq \sigma'$ has an extension $\sigma''' \succeq \sigma''$ which is labeled $(1, b)$. Spelling everything out, in the first case, we have some $\sigma'' \succeq \sigma' \succ \sigma$ such that every $\hat{\sigma} \succeq \sigma''$ has an extension σ''' such that there are infinitely many $n \in \omega$ with A-$r(\sigma''', \tau^\frown n) < \alpha$, so A-$r(\sigma, \tau) \leq \alpha$. In the other case, given any $\sigma'' \succeq \sigma'$, there is an extension σ''' which is labeled $(1, b)$. This means that σ'' itself is labeled $(1, b)$, so reasoning as above we can conclude that B-$r(\sigma, \tau) \leq \alpha$. □

Proposition 6.12. *The function* $\mathrm{cl_P}$ *is a uniformly* Π^1_1 *closure operator.*

Proof. This is the content of Lemmas 6.2, 6.4, 6.10, and 6.11. □

6.2. **Separating $\mathcal{I}$-SNE and $\mathcal{I}$-SME for HYP-ideals $\mathcal{I}$.** We again fix a countable Turing ideal $\mathcal{I}$ which is closed under $\leq_{\mathsf{HYP}}$. We introduce the forcing $\mathbb{I}$, which is version of $\mathbb{C} \star \mathbb{H}$, effectivized via bad sets. Forcing with $\mathbb{I}$ provides a real in $\mathcal{I}$-SME $\setminus$ $\mathcal{I}$-SNE, witnessing the separation of the two $\mathcal{I}$-non-lowness classes.

Definition 6.13. We define a forcing notion $\mathbb{I}$ as follows.

Elements of $\mathbb{I}$ are quadruples $\boldsymbol{p} = (\tau^{\boldsymbol{p}}, f^{\boldsymbol{p}}, \sigma^{\boldsymbol{p}}, B^{\boldsymbol{p}})$ such that

1. $\tau^{\boldsymbol{p}} \in \omega^{<\omega}$ and $\tau^{\boldsymbol{p}} = f^{\boldsymbol{p}}(\sigma^{\boldsymbol{p}})$,
2. $f^{\boldsymbol{p}} \in \mathcal{I}$ and $f^{\boldsymbol{p}} \colon 2^{<\omega} \to \omega^{<\omega}$ is monotone function such that $|f^{\boldsymbol{p}}(\sigma)| \leq |\sigma|$ and for every $m \in \omega$ and $\sigma' \succeq \sigma^{\boldsymbol{p}}$ there is $\sigma'' \succeq \sigma'$ such that $|f^{\boldsymbol{p}}(\sigma'')| \geq m$,
3. there is some $X \in \mathcal{I}$ such that $B^{\boldsymbol{p}}$ is $\Pi^1_1(X)$, and
4. $B^{\boldsymbol{p}}$ is upwards closed and $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) \notin \mathrm{cl_P}(B^{\boldsymbol{p}})$.

A condition $\boldsymbol{p}$ extends another $\boldsymbol{q}$ ($\boldsymbol{p} \leq \boldsymbol{q}$) if

1. $B^{\boldsymbol{p}} \supseteq B^{\boldsymbol{q}}$,
2. $\sigma^{\boldsymbol{p}} \succeq \sigma^{\boldsymbol{q}}$,
3. $|f^{\boldsymbol{p}}(\sigma)| \leq |f^{\boldsymbol{q}}(\sigma)|$ and $f^{\boldsymbol{p}}(\sigma) \geq f^{\boldsymbol{q}}(\sigma)$ for every $\sigma \succeq \sigma^{\boldsymbol{q}}$ in $\mathrm{dom}(f^{\boldsymbol{p}}) \cap \mathrm{dom}(f^{\boldsymbol{q}})$,
4. $\tau^{\boldsymbol{p}} \succeq \tau^{\boldsymbol{q}}$.

The component $f^{\boldsymbol{p}}$ of a condition $\boldsymbol{p} \in \mathbb{I}$ is meant to be a code for a Turing functional in $\mathcal{I}$. As usual, we will abuse notation and use the symbol $f^{\boldsymbol{p}}$ also for the partial function $2^\omega \to \omega^\omega$ which is induced by $f^{\boldsymbol{p}}$.

Intuitively, one can view a condition $\boldsymbol{p}$ as consisting of the following components: $\sigma^{\boldsymbol{p}}$ is "the Cohen part", $\tau^{\boldsymbol{p}}$ is the finite part of a Hechler condition, $f^{\boldsymbol{p}}$ is a "name" for a function which will give us the infinitary part of our Hechler condition, and $B^{\boldsymbol{p}}$ is a Cohen name for a set of bad strings which we want to avoid, as in Section 5. We will see that the condition $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) \notin \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}})$ is the appropriate notion of smallness to impose on our bad sets.

We now provide the basic properties of $\mathbb{I}$. First, it is immediate to see that $\mathbb{I}$ is also σ-centered.

Lemma 6.14. *The forcing notion $\mathbb{I}$ is σ-centered, indeed, if $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{I}$ and $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) = (\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}})$, then $\boldsymbol{p}$ and $\boldsymbol{q}$ are compatible.*

Proof. Assume $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) = (\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}})$. By Lemma 6.11 $B^{\boldsymbol{p}} \cup B^{\boldsymbol{q}}$ is small above $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$. Now let f be defined as $f(\sigma)(i) = \max\{f^{\boldsymbol{p}}(\sigma)(i), f^{\boldsymbol{q}}(\sigma)(i)\}$ for every σ and every $i < \min\{|f^{\boldsymbol{p}}(\sigma)|, |f^{\boldsymbol{p}}(\sigma)|\}$ (so $|f(\sigma)| = \min\{|f^{\boldsymbol{p}}(\sigma)|, |f^{\boldsymbol{q}}(\sigma)|\}$. We get that $\boldsymbol{s} = (\tau^{\boldsymbol{p}}, f, \sigma^{\boldsymbol{p}}, B^{\boldsymbol{p}} \cup B^{\boldsymbol{q}})$ is a condition extending both $\boldsymbol{p}$ and $\boldsymbol{q}$. □

We introduce, for every $\boldsymbol{p} \in \mathbb{I}$, a set $D_{\boldsymbol{p}} \subseteq 2^{<\omega} \times \omega^{<\omega}$. This intuitively corresponds to the set of pairs $(\sigma, \tau) \succeq (\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ such that $(\sigma, f^{\boldsymbol{p}}, \tau, \emptyset) \notin \mathbb{I}$, as the entries of τ above $|\tau^{\boldsymbol{p}}|$ are too small compared to the entries of $f^{\boldsymbol{p}}(\sigma)$ (or the entries of its possible extensions).

Definition 6.15. For any $\boldsymbol{p} \in \mathbb{I}$, we define $D_{\boldsymbol{p}} = \{(\sigma, \tau) \succeq (\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) : \forall \sigma' \succeq \sigma \; (|f^{\boldsymbol{p}}(\sigma')| \geq |\tau| \to f^{\boldsymbol{p}}(\sigma') \nleq \tau)\}$.

We also introduce, for every condition $\boldsymbol{p} \in \mathbb{I}$, a set of strings $C_{\boldsymbol{p}}$, which corresponds to the pairs (σ, τ) such that σ is "long enough" for $f^{\boldsymbol{p}}$ to converge on sufficiently many bits.

Definition 6.16. For any $\boldsymbol{p} \in \mathbb{I}$, we define $C_{\boldsymbol{p}} = \{(\sigma, \tau) \succeq (\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) : |f^{\boldsymbol{p}}(\sigma)| \geq |\tau|\}$.

It is immediate from the definition that, for any condition $\boldsymbol{p} \in \mathbb{I}$, the set $\mathrm{cl}_{\mathrm{P}}(C_{\boldsymbol{p}})$ consists of the entire set of pairs $2^{<\omega} \times \omega^{<\omega}$. The sets $C_{\boldsymbol{p}}$ also enjoy another useful property with respect to the operator cl_{P}, which we state now for future reference.

Lemma 6.17. *Let $\boldsymbol{p} \in \mathbb{I}$ and let $A \subseteq 2^{<\omega} \times \omega^{<\omega}$ be an upward closed set. We have $\mathrm{cl}_{\mathrm{P}}(A) = \mathrm{cl}_{\mathrm{P}}(A \cap C_{\boldsymbol{p}})$.*

Proof. Let $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(A)$ and let (T, f) be a bigness witness certifying this. It is trivial to extend (T, f) to obtain another bigness witness (T', f') all of whose leaves are in $C_{\boldsymbol{p}}$: given any leaf $(\hat{\sigma}, \hat{\tau})$ of (T, f), find the least extension $\sigma' \succ \hat{\sigma}$ such that $|f^{\boldsymbol{p}}(\sigma')| \geq |\hat{\tau}| + 1$, and add the pairs $(\sigma'', \tau^\frown n)$, for every $\sigma'' \succeq \sigma'$ and every $n \in \omega$, to T as "children" of (σ, τ). Since A is upward closed, all the leaves of (T', f') are then in $A \cap C_{\boldsymbol{p}}$, proving our claim. □

We now set out to show that for every $\boldsymbol{p} \in \mathbb{I}$ and any (σ, τ), there is $\boldsymbol{q} \leq \boldsymbol{p}$ with $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) = (\sigma, \tau)$ if and only if $(\sigma, \tau) \in C_{\boldsymbol{p}} \setminus (\mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}}) \cup D_{\boldsymbol{p}})$. Note that if $(\sigma, \tau) \in C_{\boldsymbol{p}}$, then we have $f^{\boldsymbol{p}}(\sigma) \leq \tau$ if and only if $(\sigma, \tau) \notin D_{\boldsymbol{p}}$.

Lemma 6.18. *For any $\boldsymbol{p} \in \mathbb{I}$, if $(\sigma, \tau) \in C_{\boldsymbol{p}} \setminus D_{\boldsymbol{p}}$, then $(\sigma, \tau) \notin \mathrm{cl}_{\mathrm{P}}(D_{\boldsymbol{p}})$. In other words, $D_{\boldsymbol{p}} \cap C_{\boldsymbol{p}} = \mathrm{cl}_{\mathrm{P}}(D_{\boldsymbol{p}}) \cap C_{\boldsymbol{p}}$.*

Proof. In our assumptions, let (T, f) be a bigness witness with stem (σ, τ) and no infinite branches. We show that (T, f) must have some leaves outside of $D_{\boldsymbol{p}}$, so that $(\sigma, \tau) \notin \mathrm{cl}_{\mathrm{P}}(D_{\boldsymbol{p}})$ by Lemma 6.7.

To that end, we start from (σ, τ) and we try define a branch in (T, f) as follows. Let $(\sigma_0, \tau_0) = (\sigma, \tau)$. Now inductively assume that, for some $n \in \omega$, we have defined a sequence $((\sigma_i,\tau_i) : i \le n)$ where, for every i, we have the following conditions: (i) $(\sigma_i,\tau_i) \in T$, (ii) $(\sigma_i,\tau_i) \in C_{\boldsymbol{p}}$ and $f^{\boldsymbol{p}}(\sigma_i) \le \tau_i$, (iii) $(\sigma_i,\tau_i) \in f(\sigma_{i+1},\tau_{i+1})$. If (σ_n,τ_n) is a leaf in T, then condition (ii) applied to n means exactly that $(\sigma_n,\tau_n) \notin D_{\boldsymbol{p}}$, and we are done. If (σ_n,τ_n) is not a leaf in T, we lengthen the sequence as follows.

Since $(\sigma_n,\tau_n) \in T$, let $\sigma' \succ \sigma_n$ be least such that for every $\sigma'' \succeq \sigma'$ there is some $\sigma''' \succeq \sigma''$ and infinitely many $m \in \omega$ such that $(\sigma''', \tau_n^\frown m) \in T$. Since $\sigma' \succeq \sigma^{\boldsymbol{p}}$, let $\sigma'' \succeq \sigma'$ be least such that $|f^{\boldsymbol{p}}(\sigma'')| \ge |f^{\boldsymbol{p}}(\sigma_n)| + 1 \ge |\tau_n| + 1$. Then let $\sigma''' \succeq \sigma''$ be least such that for infinitely many $m \in \omega$, we have $(\sigma''', \tau_n^\frown m) \in T$. Let $\sigma_{n+1} = \sigma'''$ and let $m_n \in \omega$ be the least such that $(\sigma''', \tau_n^\frown m_n) \in T$ and $m_n \ge f^{\boldsymbol{p}}(\sigma''')(|\tau_n|+1) = f^{\boldsymbol{p}}(\sigma'')(|\tau_n|+1)$.

We obtain that the sequence $((\sigma_i,\tau_i) : i \le n+1)$ satisfies conditions (i), (ii) and (iii). Therefore the sequence can be lengthened indefinitely, as long as we do not run into a leaf. Since $[T] = \emptyset$, it follows that this sequence must end up with some leaf (σ_m,τ_m), and by construction this leaf will not be in $D_{\boldsymbol{p}}$. □

Definition 6.19. For every $\boldsymbol{p} \in \mathbb{I}$, define $E_{\boldsymbol{p}} = \{(\sigma,\tau) : \exists \boldsymbol{q} \le \boldsymbol{p} : (\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) = (\sigma,\tau)\}$.

Proposition 6.20. *For every $\boldsymbol{p} \in \mathbb{I}$, we have $E_{\boldsymbol{p}} = C_{\boldsymbol{p}} \setminus (\mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}}) \cup D_{\boldsymbol{p}})$.*

Proof. Let $\boldsymbol{q} \le \boldsymbol{p}$. By definition of extension we immediately obtain $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) \succeq (\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$, $|f^{\boldsymbol{q}}(\sigma^{\boldsymbol{q}})| \le |f^{\boldsymbol{p}}(\sigma^{\boldsymbol{q}})|$, and $f^{\boldsymbol{q}}(\sigma^{\boldsymbol{q}}) = \tau^{\boldsymbol{q}} \ge f^{\boldsymbol{p}}(\sigma^{\boldsymbol{q}})$, so $|f^{\boldsymbol{p}}(\sigma^{\boldsymbol{q}})| \ge |\tau^{\boldsymbol{q}}|$ and $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) \in C_{\boldsymbol{p}} \setminus D_{\boldsymbol{p}}$. Moreover, since $\boldsymbol{q} \le \boldsymbol{p}$, $B^{\boldsymbol{q}} \supseteq B^{\boldsymbol{p}}$ and $B^{\boldsymbol{q}}$ is small above $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}})$, $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) \notin \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}})$.

Conversely assume that $(\sigma,\tau) \in C_{\boldsymbol{p}} \setminus (\mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}}) \cup D_{\boldsymbol{p}})$. So, we have $f^{\boldsymbol{p}}(\sigma) \le \tau$ and we can easily construct a monotone function $f' \in \mathcal{I}$ which satisfies the following requirements: (i) $f'(\sigma) = \tau$ (ii) for every $\hat\sigma \succeq \sigma$ and every i with $|\tau| \le i < |f^{\boldsymbol{p}}(\hat\sigma)|$, $f'(\hat\sigma)(i) = f^{\boldsymbol{p}}(\hat\sigma)(i)$.

The fact that $(\sigma,\tau) \notin D_{\boldsymbol{p}}$ implies that, for every $\sigma' \succeq \sigma^{\boldsymbol{p}}$, $|f'(\sigma')| \le |f^{\boldsymbol{p}}(\sigma')|$ and $f'(\sigma') \ge f^{\boldsymbol{p}}(\sigma')$. Since $B^{\boldsymbol{p}}$ is small above (σ,τ), and $f'(\sigma) = \tau$ by construction, it follows that $\boldsymbol{q} = (\tau, f', \sigma, B^{\boldsymbol{p}}) \in \mathbb{I}$ and $\boldsymbol{q} \le \boldsymbol{p}$. □

We now show that $E_{\boldsymbol{p}}$ contains pairs of arbitrarily long strings.

Corollary 6.21. *For any $m \in \omega$ and any $\boldsymbol{p} \in \mathbb{I}$, the set $E_{\boldsymbol{p}} \cap \{(\sigma,\tau) \succeq (\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) : |\sigma| \ge |\tau| \ge m\}$ is big above $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$.*

Proof. Fix $\boldsymbol{p} \in \mathbb{I}$ and $m \in \omega$. We claim that the set $S_m = \{(\sigma,\tau) \succeq (\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) : |f^{\boldsymbol{p}}(\sigma)| \ge |\tau| \ge m\}$ is big above $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$. To see this, we build a bigness witness with stem $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ and all leaves in S_m. If $m \le |\tau^{\boldsymbol{p}}|$, then $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}}) \in S_m$ and there is nothing to show. So, assume $m < |\tau^{\boldsymbol{p}}|$.

We start from $T_0 = \{(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})\}$ and $f_0 = \emptyset$. For $i < d = m - |\tau^{\boldsymbol{p}}|$, assume we have built a sequence of bigness witnesses $((T_j, f_j))_{j \le i}$ with stem $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ and such that every leaf (σ^ℓ, τ^ℓ) of T_i satisfies $|f^{\boldsymbol{p}}(\sigma^\ell)| \ge |\tau^\ell|$ and $f^{\boldsymbol{p}}(\sigma^\ell) \le \tau^\ell$. We build T_{i+1} starting from T_i as follows. For any leaf $(\sigma^\ell, \tau^\ell) \in T_i$, we take the least $\sigma' \succeq \sigma^\ell$ such that $|f^{\boldsymbol{p}}(\sigma')| \ge |f^{\boldsymbol{p}}(\sigma^\ell)| + 1$, and we add the set $\{(\sigma'', (\tau^\ell)^\frown n) : \sigma'' \succeq \sigma' \wedge n \ge f^{\boldsymbol{p}}(\sigma')(|\tau^\ell|)\}$ to T_i. We then extend f_i to f_{i+1} in the obvious way.

We obtain that for every $i \le d$, the pair (T_i, f_i) is a bigness witness with stem $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ and no infinite path, and every leaf (σ^ℓ, τ^ℓ) of T_i satisfies (i) $|f^{\boldsymbol{p}}(\sigma^\ell)| \ge |\tau^\ell|$, (ii) $f^{\boldsymbol{p}}(\sigma^\ell) \le \tau^\ell$, and (iii) $|\sigma^\ell| \ge |\tau^\ell| \ge i + |\tau^{\boldsymbol{p}}|$. So T_d witnesses that S_m is big above $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$. By Proposition 6.20, $E_{\boldsymbol{p}} \cap S_m = S_m \setminus (\mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}}) \cup D_{\boldsymbol{p}})$, and the latter is big above $(\sigma^{\boldsymbol{p}}, \tau^{\boldsymbol{p}})$ by Lemmas 6.11 and 6.18. □

As usual, if a filter $G \subseteq \mathbb{I}$ is in $\mathrm{dom}(\pi)$, we denote by (x_G, y_G) the pair of reals in $2^\omega \times \omega^\omega$ that are added by G.

We now show that, intuitively, if G is a sufficiently generic filter on $\mathbb{I}$, then x_G is "Cohen generic enough".

Lemma 6.22. *Let $D \subseteq \mathbb{C}$ be a dense set, and let $\pi\colon \mathbb{I} \to \mathbb{C}$ be the map $\boldsymbol{p} \mapsto \sigma^{\boldsymbol{p}}$. The set $\pi^{-1}[D]$ is dense in $\mathbb{I}$.*

Proof. Let $\boldsymbol{p} \in \mathbb{I}$ and let $D \subseteq \mathbb{C}$ be a dense set. We show that there is an extension $\boldsymbol{q}$ of $\boldsymbol{p}$ such that $\sigma^{\boldsymbol{q}} \in D$. By density of D in $\mathbb{C}$, let $\hat{\sigma} \succeq \sigma^{\boldsymbol{p}}$ be in D. Consider the function $f\colon 2^{<\omega} \to \omega^{<\omega}$ defined as

$$f(\sigma) = \begin{cases} f^{\boldsymbol{p}}(\sigma) & \text{if } \sigma \perp \sigma^{\boldsymbol{p}} \vee \sigma \preceq \sigma^{\boldsymbol{p}} \\ \tau^{\boldsymbol{p}} & \text{if } \sigma^{\boldsymbol{p}} \preceq \sigma \preceq \hat{\sigma} \\ f^{\boldsymbol{p}}(\sigma) & \text{if } \sigma \succeq \hat{\sigma} \end{cases}$$

The function f is essentially the same as $f^{\boldsymbol{p}}$, the only difference between the two being the fact that the convergence of f is slowed down so that its output is constant between $\sigma^{\boldsymbol{p}}$ and $\hat{\sigma}$. Moreover, by definition of cl_{P}, we have $(\hat{\sigma}, \tau^{\boldsymbol{p}}) \notin \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}})$. Hence $(\tau^{\boldsymbol{p}}, f, \hat{\sigma}, B^{\boldsymbol{p}})$ is a condition in $\mathbb{I}$ which extends $\boldsymbol{p}$. □

Corollary 6.23. *If $G \subseteq \mathbb{I}$ is a sufficiently generic filter, the real x_G is weakly $\mathcal{I}$-generic.*

We now prove that each condition $\boldsymbol{p}$ forces the pair of generic reals out of the set $[D_{\boldsymbol{p}} \cup B^{\boldsymbol{p}}]^{\prec}$.

Lemma 6.24. *For every $\boldsymbol{p} \in \mathbb{I}$, $\boldsymbol{p} \Vdash (x_G, y_G) \notin [D_{\boldsymbol{p}} \cup B^{\boldsymbol{p}}]^{\prec}$.*

Proof. Fix a condition $\boldsymbol{p}$ and a sufficiently generic filter G.

The proof that $(x_G, y_G) \notin [B^{\boldsymbol{p}}]^{\prec}$ is the same as the proof of the analogous statement in Lemma 4.5 and Lemma 5.5.

We explicitly show that $(x_G, y_G) \notin [D_{\boldsymbol{p}}]^{\prec}$. By contradiction assume that there are $\sigma \prec x_G$ and $\tau \prec y_G$ such with $(\sigma, \tau) \in D_{\boldsymbol{p}}$. Let $\boldsymbol{q} \in G$ be such that $|\sigma^{\boldsymbol{q}}| \geq |\tau^{\boldsymbol{q}}| \geq \max\{|\sigma|, |\tau|\}$, so $(\sigma, \tau) \preceq (\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}})$, and let $\boldsymbol{s} \in G$ be a common extension of $\boldsymbol{p}$ and $\boldsymbol{q}$. We obtain that $D_{\boldsymbol{s}} \supseteq D_{\boldsymbol{p}}$, and, since $(\sigma, \tau) \in D_{\boldsymbol{p}}$, $(\sigma^{\boldsymbol{s}}, \tau^{\boldsymbol{s}}) \succeq (\sigma, \tau)$, and $D_{\boldsymbol{p}}$ is upward closed, it follows that $(\sigma^{\boldsymbol{s}}, \tau^{\boldsymbol{s}}) \in D_{\boldsymbol{s}}$. This contradicts Proposition 6.20. □

The fact that each condition $\boldsymbol{p}$ forces $(x_G, y_G) \notin [D_{\boldsymbol{p}}]^{\prec}$ implies that, for any sufficiently generic G containing $\boldsymbol{p}$, $y_G \geq^* f^{\boldsymbol{p}}(x_G)$. We now show that from this fact we can derive that, in these assumptions, y_G is dominating for $\mathcal{I}(x_G)$.

Lemma 6.25. *For every $\Phi \in \mathcal{I}$, the set $A_\Phi = \{\boldsymbol{q} \in \mathbb{I} : \boldsymbol{q} \Vdash \Phi(x_G)\uparrow \vee\ \boldsymbol{q} \Vdash y_G \geq^* \Phi(x_G)\}$ is dense in $\mathcal{I}$.*

Proof. Let $\Phi \in \mathcal{I}$ be a Turing functional, and let $\boldsymbol{p} \in \mathbb{I}$. Suppose there is no $\boldsymbol{q} \leq \boldsymbol{p}$ such that $\boldsymbol{q} \Vdash \Phi(x_G)\uparrow$. We show that the set of conditions $S_\Phi = \{\boldsymbol{r} : \forall \hat{\sigma} \succeq \sigma^{\boldsymbol{r}}\, \forall i \geq |\tau^{\boldsymbol{r}}|\ (f^{\boldsymbol{r}}(\hat{\sigma})(i) \geq \Phi(\hat{\sigma})(i))\}$ is dense below $\boldsymbol{p}$. Note that if $\boldsymbol{r} \in S_\Phi$ and $(x, y) \notin [D_{\boldsymbol{r}}]^{\prec}$, we have that $y \geq f^{\boldsymbol{r}}(x)$, and if $x \in \mathrm{dom}(\Phi)$, then $f^{\boldsymbol{r}}(x) \geq^* \Phi(x)$. This implies that $\boldsymbol{r} \Vdash y_G \geq^* \Phi(x_G)$.

To show the density of S_Φ, let $\boldsymbol{q} \leq \boldsymbol{p}$ and let $f\colon 2^{<\omega} \to \omega^{<\omega}$ be defined as follows: for $\sigma \preceq \sigma^{\boldsymbol{q}}$, $f(\sigma) = f^{\boldsymbol{q}}(\sigma)$, and for $\sigma \not\preceq \sigma^{\boldsymbol{q}}$, $f(\sigma)(i) = \max\{f^{\boldsymbol{q}}(\sigma)(i), \Phi(\sigma)(i)\}$ for every $i < \min\{|\Phi(\sigma)|, |f^{\boldsymbol{q}}(\sigma)|\}$. Then the condition $\boldsymbol{r} = (\tau^{\boldsymbol{q}}, f, \sigma^{\boldsymbol{q}}, B^{\boldsymbol{q}})$ is in D_Φ and $\boldsymbol{r} \leq \boldsymbol{q}$. □

Corollary 6.26. *Let G be a filter on $\mathbb{I}$ which meets all the dense sets $(D_\Phi)_{\Phi\in\mathcal{I}}$ and let (x_G, y_G) be the pair of reals added by G. Then x_G is weakly $\mathcal{I}$-generic and y_G is $\mathcal{I}(x_G)$-dominating, hence y_G is $\mathcal{I}$-strongly meagre engulfing.*

Proof. Corollary 6.23 ensures that x_G is weakly $\mathcal{I}$-generic. To see that the real y_G is $\mathcal{I}(x_G)$-dominating, note that for every $\Phi \in \mathcal{I}$, if $\Phi(x_G)\downarrow$, then (in the notation of Lemma 6.25) $G \cap D_\Phi \neq \emptyset$ and hence $y_G \geq^* \Phi(x_G)$. This shows that y_G is dominating for $\mathcal{I}(x_G)$. By Lemma 2.25, y_G is $\mathcal{I}$-strongly meagre engulfing. $\square$

What is left to show now is that if G is sufficiently generic, then (x_G, y_G) does not $\mathcal{I}$-compute any $\mathcal{I}$-random. We adapt the construction of Proposition 5.12 to the forcing $\mathbb{I}$.

Theorem 6.27. *Let $\Phi \in \mathcal{I}$ be a $\{0,1\}$-valued Turing functional. The set of conditions $C_\Phi = \{\boldsymbol{p} \in \mathbb{I} : (\boldsymbol{p} \Vdash \Phi(x_G, y_G)\uparrow) \vee \exists S \in \mathcal{I}\ (S \text{ Schnorr null} \wedge \boldsymbol{p} \Vdash \Phi(x_G, y_G) \in S)\}$ is dense in $\mathcal{I}$.*

Proof. We use the proof of Proposition 5.12 as a blueprint, because the idea behind the construction is the same. Here, we spell out the details of how to adapt that proof to the context of forcing with $\mathbb{I}$.

We fix $\boldsymbol{p} = (\tau^{\boldsymbol{p}}, f^{\boldsymbol{p}}, \sigma^{\boldsymbol{p}}, B^{\boldsymbol{p}}) \in \mathbb{I}$ and we build an extension of $\boldsymbol{p}$ in C_Φ, using the family of sets $A_m = \{(\sigma, \tau) : \Phi(\sigma, \tau)(m)\downarrow\}$. To that end, fix $X \in \mathcal{I}$ such that $f^{\boldsymbol{p}} \leq_{\mathrm{T}} X$ and $B^{\boldsymbol{p}}$ is $\Pi^1_1(X)$.

If there are $m \in \omega$ and $(\sigma, \tau) \in E_{\boldsymbol{p}} \setminus \mathrm{cl}_{\mathrm{P}}(A_m)$, then by Proposition 6.20, we can consider $\boldsymbol{q} \leq \boldsymbol{p}$ with $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) = (\sigma, \tau)$ and let $\boldsymbol{q'} = (\tau^{\boldsymbol{q}}, f^{\boldsymbol{q}}, \sigma^{\boldsymbol{q}}, B^{\boldsymbol{q}} \cup A_m)$. Lemma 6.11 guarantees that $\boldsymbol{q'}$ is a condition of $\mathbb{I}$ and trivially $\boldsymbol{q'} \leq \boldsymbol{q} \leq \boldsymbol{p}$. Moreover, by Lemma 6.24, $\boldsymbol{q'} \Vdash (x_G, y_G) \notin [A_m]^{\prec}$, so $\boldsymbol{q'} \Vdash \Phi(x_G, y_G)\uparrow$ and $\boldsymbol{q'} \in C_\Phi$.

Otherwise, assume $E_{\boldsymbol{p}} \subseteq \bigcap_{m\in\omega} \mathrm{cl}_{\mathrm{P}}(A_m)$. We build a Schnorr null set $S \in \mathcal{I}$ such that $\boldsymbol{p} \Vdash \Phi(x_G, y_G) \in S$. The construction of the set S is as in the proof of Proposition 5.12: given any $(\sigma, \tau) \in C_{\boldsymbol{p}} \setminus D_{\boldsymbol{p}}$, we build an infinite set of "values" $Q_{\sigma,\tau}$ with the following property ($\star$): if $\boldsymbol{q} \leq \boldsymbol{p}$ is such that $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) = (\sigma, \tau)$, then for each $\nu \in Q_{\sigma,\tau}$ we can find an extension $\boldsymbol{s}$ of $\boldsymbol{q}$ such that $\Phi(\sigma^{\boldsymbol{s}}, \tau^{\boldsymbol{s}}) \succeq \nu$. We will build the Schnorr null set S using these sets $Q_{\sigma,\tau}$, and property ($\star$) is what ultimately allows us to argue that $\boldsymbol{p} \Vdash \Phi(x_G, y_G) \in S$.

We can build these sets exactly like we did in the proof of Proposition 5.12. We proceed as follows: simultaneously, we go along the $\Pi^1_1(X)$ enumeration of the set $\mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}})$ as well as those of the sets $\mathrm{cl}_{\mathrm{P}}(V_\nu)$, where $V_\nu = \{(\hat\sigma, \hat\tau) : \Phi(\hat\sigma, \hat\tau) \succeq \nu\}$, for every $\nu \in 2^{<\omega}$. If we ever find out that $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}})$, we stop the process and set $R_{\sigma,\tau} = 2^{<\omega}$. As long as we do not find a witness for $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}})$, we enumerate strings ν inside $R_{\sigma,\tau}$ as soon as we see that $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(V_\nu)$. Note that for every $m \in \omega$, if $(\sigma, \tau) \in C_{\boldsymbol{p}} \setminus (D_{\boldsymbol{p}} \cup \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}}))$, then $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(A_m)$, and since $\mathrm{cl}_{\mathrm{P}}(A_m) = \bigcup_{\nu\in 2^m} V_\nu$, we are guaranteed to enumerate some string of length m inside $R_{\sigma,\tau}$.

Repeating the argument of Proposition 5.12, we can see that we are able to build a $\Delta^1_1(X)$ sequence of subtrees of the $T_{\sigma,\tau} \subseteq R_{\sigma,\tau}$, and hence a $\Delta^1_1(X)$ sequence $(g_{\sigma,\tau})_{(\sigma,\tau)\in C_{\boldsymbol{p}}\setminus D_{\boldsymbol{p}}}$ such that $g_{\sigma,\tau} \in [T_{\sigma,\tau}]$ for every (σ, τ). Using the sequence of the $g_{\sigma,\tau}$, we build a Schnorr null set $S \in \mathcal{I}$ again as in the proof of Proposition 5.12. We now claim that for every $(\sigma, \tau) \in C_{\boldsymbol{p}} \setminus D_{\boldsymbol{p}}$, the sets $Q_{\sigma,\tau} = \{g_{\sigma,\tau}\restriction n : n \in \omega\}$ satisfy property ($\star$).

Indeed, let $\nu \in Q_{\sigma,\tau}$ and let $\boldsymbol{q} \leq \boldsymbol{p}$ be such that $(\sigma^{\boldsymbol{q}}, \tau^{\boldsymbol{q}}) = (\sigma, \tau)$. In particular, $(\sigma, \tau) \notin \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{p}})$, so the fact that $\nu \in Q_{\sigma,\tau}$ implies that $(\sigma, \tau) \in \mathrm{cl}_{\mathrm{P}}(V_\nu)$. We want to claim that $(\sigma,\tau) \in \mathrm{cl}_{\mathrm{P}}(E_{\boldsymbol{q}} \cap V_\nu)$: indeed, by Proposition 6.20 $E_{\boldsymbol{q}} \cap V_\nu = (C_{\boldsymbol{q}} \cap V_\nu) \setminus (D_{\boldsymbol{q}} \cup \mathrm{cl}_{\mathrm{P}}(B^{\boldsymbol{q}}))$, and by Lemma 6.17 we have that $(\sigma,\tau) \in \mathrm{cl}_{\mathrm{P}}(C_{\boldsymbol{q}} \cap V_\nu)$, so by Lemma 6.11, since $(\sigma,\tau) \notin \mathrm{cl}_{\mathrm{P}}(D_{\boldsymbol{q}} \cup B^{\boldsymbol{q}})$, we get $(\sigma,\tau) \in \mathrm{cl}_{\mathrm{P}}(E_{\boldsymbol{q}} \cap V_\nu)$. Hence we can conclude that $E_{\boldsymbol{q}} \cap V_\nu \neq \emptyset$, so there is some $\boldsymbol{s} \leq \boldsymbol{q}$ such that $(\sigma^{\boldsymbol{s}}, \tau^{\boldsymbol{s}}) \in V_\nu$.

Again following the proof of Proposition 5.12, using property $(\star)$ allows us to show that $\boldsymbol{p} \Vdash \Phi(x_G, y_G) \in S$, so that $\boldsymbol{p} \in C_\Phi$. □

Corollary 6.28. *Let G be a sufficiently generic filter on $\mathbb{I}$. The set $\mathcal{I}(x_G, y_G)$ contains no $\mathcal{I}$-random*

Proof. By Theorem 6.27, it suffices to make sure that G meets the dense sets C_Φ for every $\{0,1\}$-valued Turing functional $\Phi \in \mathcal{I}$. □

Theorem 6.29. *There are reals $x \in \mathcal{I}\text{-}\mathsf{SME}$ which do not compute any $\mathcal{I}$-random, and in particular do not belong to $\mathcal{I}\text{-}\mathsf{SNE}$.*

Proof. Let G be a sufficiently generic filter on $\mathbb{I}$ and let $(x_G, y_G) \in 2^\omega \times \omega^\omega$ be the pair of reals added by G. By Corollary 6.26, $y_G \in \mathcal{I}\text{-}\mathsf{SME}$, and by Corollary 6.28, there are no $\mathcal{I}$-randoms in $\mathcal{I}(x_G, y_G)$. Hence y_G is an example of such a real. □

Appendix A. On Laver and Hechler forcing.

We prove some facts about using Laver and Hechler forcing in the context of this paper. First, we sketch a proof of the claim in [6, Section 8] that, if $\mathcal{I}$ is a countable ω-model of ATR_0,[6] then separations between $\mathcal{I}\text{-}\mathsf{DOM}$, $\mathcal{I}\text{-}\mathsf{SME}$ and $\mathcal{I}\text{-}\mathsf{SNE}$ can be achieved with Laver and Hechler forcing. More precisely, we show that ATR_0 suffices to handle Laver forcing and Hechler forcing similarly to how we handled $\mathbb{O}$ and $\mathbb{H}_{\mathsf{B}}$ in Sections 4 and 5. Hence, it is not hard to adapt the proofs of Sections 4 and 6 to obtain a proof of the claim above. Indeed, proofs in this context are simpler as ATR_0 allows us to do away with the bad sets.

We also report an argument due to Dan Turetsky which shows that, on the other hand, Laver and Hechler forcing cannot be used to show the separations $\mathsf{HYP}\text{-}\mathsf{SNE} \subsetneq \mathsf{HYP}\text{-}\mathsf{SME} \subsetneq \mathsf{HYP}\text{-}\mathsf{DOM}$.

Throughout the section we use a more flexible variant of Hechler forcing, sometimes called *Hechler tree forcing*. This alternative version is often used in the set theoretic literature and, although it is not equivalent (in the sense of forcing equivalence) to $\mathbb{H}$,[7] it is equivalent for our purposes (i.e. because it adds dominating reals, and is σ-centered).

Definition A.1. Let $(\mathbb{H}_{\mathsf{T}}, \leq)$ be the arboreal forcing notion whose conditions are trees $T \subseteq \omega^{<\omega}$ such that, for every $\tau \succeq \mathrm{stem}(T)$, $\forall^\infty n\ \tau^\frown n \in T$.

We use $\mathbb{H}_{\mathsf{T}}$ here as this version of Hechler forcing is the one which is more clearly connected to $\mathbb{L}$, as can be seen by their definition. One example of this connection is the following:

Lemma A.2. *Let $T \in \mathbb{H}_{\mathsf{T}}$ and let $S \in \mathbb{L}$. Then $[T] \cap [S] \neq \emptyset$.*

[6] ATR_0 is one of the big-five subsystems of second order arithmetic. Its defining axiom is *arithmetical transfinite recursion*, which allows for iterations of the Turing jump along arbitrary wellorders in the model. In this section we use some well-known features of ATR_0. For their proofs, see [17].

[7] For a comparison of the two, see [13].

Proof. A common branch can be built recursively, starting from $\langle , \rangle$. At stage n, we have a string $\sigma \in \omega^n \cap T \cap S$. Since σ has infinitely many children in S, and all but finitely many of the immediate successors of σ are in T, there are infinitely many k that $\sigma^\frown k \in S \cap T$. We can take the least at each stage. Proceeding like this yields a branch $f \in [S] \cap [T]$. $\square$

The fundamental result about these notions of forcing, which requires ATR_0, is the following.

Proposition A.3. *Let $\mathcal{I} \subseteq \omega^\omega$ be an ω-model of ATR_0. Let $B \subseteq \omega^{<\omega}$, $B \in \mathcal{I}$ be a code for an open set. There is either a Laver tree T with $\mathrm{stem}(T) = \langle , \rangle$ and $\mathcal{I} \vDash [T] \subseteq [B]^{\prec}$, or an Hechler tree S with $\mathrm{stem}(S) = \langle , \rangle$ and $\mathcal{I} \vDash [S] \cap [B]^{\prec} = \emptyset$.*

We stress that the statements $[T] \subseteq [B]^{\prec}$ and $[S] \cap [B]^{\prec} = \emptyset$ are Π^1_1, so not necessarily absolute between $\mathcal{I}$ and $\mathcal{P}(\omega)$.

Proof. For any $L \in \mathcal{I}$ such that $\mathcal{I} \vDash \mathrm{WO}(L)$, we can define a B-ranking over L similarly to how we defined the B-ranking in the proof of Lemma 3.3. If there is some L as above and some $m \in Ł$ such that $B\text{-}r(\langle , \rangle) = m$ (where this rank is computed with respect to L), then, following the proof of Lemma 3.6, we obtain an ω-bushy tree T' with root $\langle , \rangle$, all leaves in B, and such that $\mathcal{I} \vDash [T'] = \emptyset$. Extending every leaf of T' indefinitely leads to a Laver tree $T \in \mathcal{I}$ with $\mathcal{I} \vDash [T] \subseteq [B]^{\prec}$.

On the other hand, if, for every L with $\mathcal{I} \vDash \mathrm{WO}(L)$, the sequence $\langle , \rangle$ is unranked in the B-ranking associated to L, then, by Σ^1_1-boundedness in $\mathcal{I}$, there is some linear order $K \in \mathcal{I}$ with a minimal element, such that $\mathcal{I} \vDash \neg\mathrm{WO}(K)$ and there is a B-ranking $(R^B_m)_{m \in K}$ along K such that $\langle , \rangle \notin \bigcup_{m \in K} R^B_m$. Let $f \colon \omega \to K$ be a descending sequence in K with $f \in \mathcal{I}$. We use f to build a Hechler tree S as required. Note that, for every $\sigma \in \omega^{<\omega}$, we have $\sigma \notin R^B_{f(n)} \to \forall^\infty k \ (\tau^\frown k \notin R^B_{f(n+1)})$. We define $S_0 = \{\langle , \rangle\}$. At step n, we assume we have a tree S_n such that (i) all leaves of S_n are in $\omega^n \setminus R^B_{f(n)}$, (ii) for every $\tau \in S_n$, $|\tau| < n \to \forall^\infty k \ (\tau^\frown k \in S_n)$. We define $S_{n+1} = S_n \cup \{\tau^\frown k : \tau \in S_n \cap \omega^n \wedge \tau^\frown k \notin R^B_{f(n+1)}\}$. Proceeding like this, we let $S = \bigcup_{n \in \omega} S_n$. By construction S is a Hechler tree. Moreover, it follows by definition that no node of S is ranked 0, hence $\mathcal{I} \vDash [S] \cap [B]^{\prec} = \emptyset$. $\square$

Note that the tree S that we build is such that $\mathcal{P}(\omega) \vDash [S] \cap [B]^{\prec} = \emptyset$.

Remark A.4. There are many natural extensions of Proposition A.3. Since all of them are proved the same way, we do not repeat the proofs and just mention them here.

First, note that we could have swapped the Hechler tree and the Laver tree in the statement of Proposition A.3 to obtain that $[S] \subseteq [B]^{\prec}$ or $[T] \cap [B]^{\prec} = \emptyset$. This is achievable by modifying the arithmetical operator which we iterate to define our rankings (i.e. changing a $\exists^\infty$ to a $\forall^\infty$ in the operator of Definition 3.1).

Secondly, note that Proposition A.3 is concerned, essentially, with ω-bigness *in* $\omega^{<\omega}$. That is only to simplify notation, indeed, given any Laver tree $T \in \mathcal{I}$, there is a natural bijection between the portion of T above $\mathrm{stem}(L)$ and $\omega^{<\omega}$ which respects the relation $\preceq$. Using this bijection, we conclude the following: for every Laver tree $T \in \mathcal{I}$ and every $B \in \mathcal{I} \cap \mathcal{P}(\omega^{<\omega})$, there is either a Laver tree $T' \subseteq L$ with $\mathrm{stem}(T') = \mathrm{stem}(T)$ and $\mathcal{I} \vDash [T'] \subseteq [B]^{\prec}$, or a tree $S \subseteq T$ with (i) $\mathrm{stem}(S) = \mathrm{stem}(T)$, (ii) for every $\tau \in S$, if $\tau \succeq \mathrm{stem}(S)$, then all but finitely many of the children of τ in T are also in S, (iii) $[S] \cap [B]^{\prec} = \emptyset$. If T is actually a Hechler tree, then so is S.

The second author and Marcone have shown in [10] that Proposition A.3 is actually equivalent to ATR_0 (this equivalence is not restricted to ω-models).

As an aside, we recall the notion of β-model: a set $S \subseteq \mathcal{P}(\omega)$ is a β-model if it $\mathbf{\Sigma}^1_1$ statements are absolute between S and $\mathcal{P}(\omega)$. It is known that β-models are models of ATR_0 (see, e.g. [17, Theorem VII.2.7]), so Proposition A.3 holds for these. If $\mathcal{I}$ is a β-model, the statements $[T] \subseteq [B]^{\prec}$ and $[S] \cap [B]^{\prec}$ in the statement of Lemma A.3 can be interpreted equivalently in $\mathcal{I}$ or in $\mathcal{P}(\omega)$.

From now on, we work with a fixed ideal $\mathcal{I}$ which models ATR_0. Accordingly, we work with $\mathbb{L}$ and $\mathbb{H}_\mathsf{T}$ in $\mathcal{I}$.

Lemma A.5. *Let $T \in \mathcal{I}$ be a Laver tree and let $B \in \mathcal{I} \cap \mathcal{P}(\omega^{<\omega})$. There is some set $T' \in \mathcal{I}$ such that $T' \leq_0 T$ and either $T' \Vdash x_G \in [B]^{\prec}$ or $T' \Vdash x_G \notin [B]^{\prec}$.*

Proof. By Proposition A.3, there are two cases. There is either $T' \leq_0 T$ with $\mathcal{I} \vDash [T'] \subseteq [B]^{\prec}$, or $T' \leq_0 T$ with $[T'] \cap [B]^{\prec} = \emptyset$. In the first case, we claim that $T' \Vdash x_G \in [B]^{\prec}$. Indeed, if $S \leq T'$, then $\mathcal{I} \vDash [S] \subseteq [B]^{\prec}$, therefore there must be some $\sigma \in \uparrow B$ with $\sigma \succeq \operatorname{stem}(S)$. Clearly $S_\sigma \Vdash x_G \in [B]^{\prec}$, so $T' \Vdash x_G \in [B]^{\prec}$.

In the second case, T' strongly forces $x_G \notin [B]^{\prec}$. $\square$

We can follow the proof of Proposition 4.13, augmented as in the proof of Proposition 4.14 to obtain the following result.

Proposition A.6. *Laver forcing $\mathbb{L}$ in $\mathcal{I}$ has the Laver property.*

Proof. Fix a Turing functional $\Phi \in \mathcal{I}$ and a condition $T \in \mathbb{L} \cap \mathcal{I}$.

First, we can either find an extension $S \leq T$ which strongly forces Φ to diverge, or an extension $S \leq_0 T$ which forces Φ to be total. The first case boils down to showing that the code B for some open set is ω-small (from the point of view of $\mathcal{I}$) in T above some $\tau \in T$. If this is the case, then by Lemma A.5 we can find an $S \leq T$ with $\operatorname{stem}(S) = \tau$ which strongly forces $x_G \notin [B]^{\prec}$.

If this is not possible, we can follow a fusion argument to obtain that a tree $S \leq_0 T$ such that $\mathcal{I} \Vdash [S] \subseteq C$, where $C = \{x \in \omega^\omega : \Phi(x)\downarrow\}$. Again this is not enough to show strong forcing, but, reasoning as in Lemma A.5, we obtain that $S \Vdash \Phi(x_G)\downarrow$.

In case the functional Φ is h-bounded for some $h \in \mathcal{I}$, and again assuming we cannot force Φ to diverge below T, we can run the "improved" fusion argument of the proof of Proposition 4.14 to also build a trace $F \in \mathcal{I}$ alongside our condition $S \leq_0 T$. Reasoning as above we obtain that $S \Vdash \forall n\, (\Phi(x_G)(n) \in F(n))$. $\square$

Note that, in the proof of Lemma A.5, we do not have strong forcing inside open sets. This is due to the fact that it is possible to have $\mathcal{I} \vDash [T'] \subseteq [B]^{\prec}$ without having $[T'] \subseteq [B]^{\prec}$. Obviously, the same holds for the $\mathbf{\Pi}^0_2$ sets of Proposition A.6. We recover strong forcing (which we always had for $\mathbb{O}$ in Section 4) if $\mathcal{I}$ is a β-model.

Corollary A.7. *If G is a sufficiently generic filter on $\mathbb{L}$, then $\mathcal{I}(x_G)$ does not contain any weakly $\mathcal{I}$-generic. Hence $x_G \in \mathcal{I}\text{-}\mathsf{DOM} \setminus \mathcal{I}\text{-}\mathsf{SME}$.*

We now turn to Hechler forcing $\mathbb{H}_\mathsf{T}$. The fundamental result about forcing inside or outside open sets, which is parallel to Corollary 5.10, is the following.

Lemma A.8. *Let $T \in \mathbb{H}_{\mathsf{T}} \cap \mathcal{I}$ and let $B \in \mathcal{I}$, $B \subseteq \omega^{<\omega}$ be the code for an open. We have $p \Vdash x_G \in [B]^{\prec}$ if and only if $T \subseteq \mathrm{cl}_\omega(B)$ (here, cl_ω is intended inside $\mathcal{I}$).*

Proof. Let $\tau \in T \setminus \mathrm{cl}_\omega(B)$. By Proposition A.3, we can find an extension of T_τ, with stem τ, which strongly forces $x_G \notin [B]^{\prec}$.

Conversely suppose $T \subseteq \mathrm{cl}_\omega(B)$ and let $S \leq T$. Let $U \in \mathcal{I}$ be the witness for $\mathrm{stem}(S) \in \mathrm{cl}_\omega(B)$. With an abuse of notation, we write U also for the Laver tree obtained by prolonging U indefinitely above its leaves. By Lemma A.2 in $\mathcal{I}$, there is some $x \in \mathcal{I}$ with $x \in [S] \cap [U]$, and since $\mathcal{I} \Vdash [S] \subseteq [B]^{\prec}$ there must be some n such that $x{\restriction}n \in B$. This shows that S can be extended to strongly force $x_G \in [B]^{\prec}$, and hence $S \Vdash x_G \in [B]^{\prec}$. □

We obtain:

Lemma A.9. *Let $\Phi \in \mathcal{I}$ be a Turing functional and $T \in \mathbb{H}_{\mathsf{T}} \cap \mathcal{I}$. If there is no $S \leq T$ which strongly forces $\Phi(x_G)\uparrow$, then $T \Vdash \Phi(x_G)\downarrow$.*

Proof. The proof is essentially the same as that of Lemma 5.11. □

Proposition A.10. *If G is a sufficiently generic filter on $\mathbb{H}_{\mathsf{T}}$, then $\mathcal{I}(x_G)$ does not contain any $\mathcal{I}$-randoms. Hence $x_G \in \mathcal{I}\text{-}\mathsf{DOM} \setminus \mathcal{I}\text{-}\mathsf{SNE}$.*

Proof. This is proved similarly to Proposition 5.12, we give a brief sketch here. We fix a $\{0,1\}$-valued Turing functional $\Phi \in \mathcal{I}$ and a condition $T \in \mathbb{H}_{\mathsf{T}} \cap \mathcal{I}$. If we can force divergence of Φ below T, then we do so. Otherwise, thanks to Lemma A.9, we know that $T \Vdash \Phi(x_G)\downarrow$. We can now build a sequence of trees $(R_\sigma)_{\sigma \in T} \in \mathcal{I}$ as we did in the proof of Proposition 5.12. Our job here is actually easier, as we do not need to worry about the bad set and we know that for any $\sigma \in T$, there are infinitely many strings $\nu \in 2^{<\omega}$ such that $\{\tau \in T : \Phi(\tau){\restriction}|\nu| \downarrow = \nu\}$ is ω-big above σ. We can then do exactly as in the proof mentioned above to extract a sequence (again indexed by T) of branches through these trees, and use these to define a Schnorr null set $S \in \mathcal{I}$ with $T \Vdash \Phi(x_G) \in S$. □

With the notion of bigness of Section 6.1 we can obtain results on $\mathbb{C} \star \mathbb{H}_{\mathsf{T}}$ analogous to those we obtained for $\mathbb{I}$, reasoning as above. These allow us to conclude that if G is sufficiently generic for $\mathbb{C} \star \mathbb{H}_{\mathsf{T}}$, then $x_G \in \mathcal{I}\text{-}\mathsf{SME} \setminus \mathcal{I}\text{-}\mathsf{SNE}$.

We conclude with Turetsky's argument showing that Laver and Hechler forcing cannot be used to show the separations $\mathsf{HYP\text{-}SNE} \subsetneq \mathsf{HYP\text{-}SME} \subsetneq \mathsf{HYP\text{-}DOM}$.

Proposition A.11 (Turetsky)**.** *Let $(\mathbb{P}, \leq, \mathbb{1}, \pi)$ be an arboreal forcing notion in HYP, and assume that, for every $h \in \mathsf{HYP}$, the set $D_h = \{\boldsymbol{p} \in \mathbb{P} : \forall g \in [\boldsymbol{p}]\ \forall i \geq |\mathrm{stem}(\boldsymbol{p})| (g(i) \geq h(i))\}$ is dense in $\mathbb{P}$. Then there is a countable family of dense sets $(C_n)_{n\in\omega}$ such that any filter $G \subseteq \mathbb{P}$ in $\mathrm{dom}(\pi)$ which meets all the D_h's and all the C_n's yields a real $\pi(G) \in \mathsf{HYP\text{-}WL}$.*

We will rely on the following fact, proved in [11].

Lemma A.12. *Let $x \in \omega^\omega$ be HYP-dominating and such that $\mathcal{O}$ is $\Sigma^0_2(x)$. Then $x \in \mathsf{HYP\text{-}WL}$.*

Proof of Proposition A.11. We show that, for every computable tree $T \subseteq \omega^{<\omega}$, T has a hyperarithmetical branch if and only if it has a branch f with $f \leq^* \pi(G)$. Recall that a computable tree has a branch majorized by an hyperarithmetical function if and only if it has an hyperarithmetical branch (this is, essentially, because

Kőnig's Lemma for bounded trees is computable in the jump of its two inputs, the tree and the bound). Since the set of (indices for) computable trees with hyperarithmetical branches is Π^1_1-complete (by the Spector-Gandy Theorem, see [16, Theorem III.3.5]), and the predicate "T has a branch f with $f \leq^* \pi(G)$" is $\Sigma^0_2(\pi(G))$, this shows that $\mathcal{O}$ is $\Sigma^0_2(\pi(G))$, which, by Lemma A.12 immediately implies that $\pi(G)$ is HYP-WL.

Since $\pi(G)$ dominates all hyperarithmetical functions, one direction is clear. Conversely, assume that T does not have any branch which is majorized by a hyperarithmetical function, and let $\boldsymbol{p}$ be any condition. We show that the set of conditions $B_T \subseteq \mathbb{P}$ which force the statement "there is no branch in T which is majorized by $\pi(G)$" is dense below $\boldsymbol{p}$. The family of sets of the form B_T, where T ranges over all computable trees, is the family $(C_n)_{n\in\omega}$ required.

To see that B_T is dense below $\boldsymbol{p}$, note that the leftmost path $h \in [\boldsymbol{p}]$ is computable in $\boldsymbol{p}$ (because $\boldsymbol{p}$ is pruned), so it is hyperarithmetical. In particular, there is no $f \in [T]$ which is majorized by h. By Kőnig's Lemma, this means that the set $\{\sigma \in T : \sigma \leq h{\restriction}|\sigma|\}$ is a finite tree, so there is some n such that, for all $\sigma \in \omega^n$, if $\sigma \leq h{\restriction}n$, then $\sigma \notin T$. Hence, the condition $S_{h\restriction n} \in \mathbb{P}$ is an extension of $\boldsymbol{p}$ which forces the statement "there is no branch in T which is majorized by $\pi(G)$". □

Indeed, it is shown in [11] that if x is HYP-dominating and $\mathcal{O}$ is $\Sigma^0_2(x)$, then x is HYP-*listing*, i.e. x computes a sequence $(y_n)_{n\in\omega}$ such that $\mathsf{HYP} = \{y_n : n \in \omega\}$. Turetsky's argument, accordingly, implies reals that are sufficiently generic for arboreal forcing notions which add HYP-dominating reals are HYP-listing.

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