BOREL WADGE CLASSES AND SELIVANOV'S FINE HIERARCHY II: TURING DEGREES

NOAM GREENBERG, RENRUI QI, AND DAN TURETSKY

ABSTRACT. We answer the question: which levels of Selivanov's fine hierarchy contain new Turing degrees?

1. Introduction

Selivanov and Yamaleev [SY18a] asked a natural question: at what levels of the fine hierarchy do we get new Turing degrees? The fine hierarchy is an analogue of Wadge's hierarchy of Borel pointclasses, ordered by inclusion. For a survey see [Sel08]. In the paper [GQT] we showed how to extend the fine hierarchy to all levels of the hyperarithmetic hierarchy.

Cooper [Coo71] showed that there is a d.c.e. set that is not Turing equivalent to any c.e. set. This was extended to the finite levels of the difference hierarchy [JS84], [Sel85], answering the question for the first ω levels of the fine hierarchy. Selivanov and Yamaleev showed, however, that the $(\omega + 1)$-st level, while containing more *sets*, does not contain any new Turing degrees; however the $(\omega + 2)$-nd level does. In [SY18b], the authors extended their result from $\omega + 2$ to $\omega + n$ for all natural $n \geqslant 2$. In [MSY20], Melnikov, Selivanov and Yamaleev showed that the level $\omega^\omega + 2$ contains new Turing degrees, using a complicated $0^{(3)}$-priority argument.

In this paper we answer the main question, explicitly posed in [SY18b].

Theorem 1.1. *Let $\alpha < \omega_1^{\mathrm{ck}}$ be an ordinal which is not the successor of a limit, and let Γ be the class at level α of the extended fine hierarchy. There is a set $A \in \Gamma$ that is not Turing equivalent to any set in $\Delta(\Gamma)$.*

To prove the theorem, we use two main tools. Since we need to approximate sets that are arbitrarily high in the hyperarithmetic hierarchy, we need to use the method of iterated priority arguments, originally due to Ash [Ash86] and Harrington (unpublished, see [Kni90a, Kni90b]), made more dynamic in [Mon14] and somewhat simplified in [GT22] (a non-effective version was introduced independently in [DSR07]). To be able to consider all the complicated combinations that yield the levels of the fine hierarchy, we need a method for dynamically approximating sets in these classes. To do that, we use the descriptions of the classes in the extended hierarchy, that we introduced in [GQT], following work in [DGHTT] and [GT]. These are descriptions that generalise descriptions of Borel Wadge classes, given by Louveau [Lou83] and by Louveau and Saint Raymond [LSR88].

Throughout the paper, we use the notation and terminology that we introduced in [GQT]. Using these, Theorem 1.1 says that if Γ is finitely described and $\delta(\Gamma) > 1$

The authors were partially supported by the Marsden Fund of New Zealand.

is not the successor of a limit, then there is some $A \in \Gamma$ that is not Turing equivalent to any set in $\Delta(\Gamma)$.

Before we embark on the proof, let us first we explain the restriction on $\delta(\Gamma)$.

Proposition 1.2. *Suppose that $\delta(\Gamma) = \alpha + 1$, where α is a limit. Then every $A \in \Gamma$ is Turing equivalent to some set in $\Delta(\Gamma)$. Indeed, it is Turing equivalent to some set in Λ, where Λ is the predecessor of Γ (the class with $\delta(\Lambda) = \alpha$).*

Proof. By [GQT, Prop. 5.19] (see [GQT, Exmp. 5.18]), $\Gamma \equiv \Lambda^+$ (or its dual; of course, up to Turing degree, this does not make a difference). Let $P \in \Gamma$; so there are disjoint c.e. sets C_1 and C_2, and sets $A \in \Lambda$ and $B \in \check{\Lambda}$, such that

$$P = (C_1 \cap A) \cup (C_2 \cap B)$$

(see the argument for the containment of $\Lambda^+ = \mathrm{SU}_{0,1}(\Lambda, \mathbf{0})$ in $\mathrm{BiSep}(\Sigma^0_1, \Lambda, \{\varnothing\})$ in the proof of [GQT, Lem. 4.15]; that step did not rely on the admissibility of Λ^+). By [GQT, Prop. 5.20], Λ and $\check{\Lambda}$ are closed under taking intersections with Δ^0_2 sets; so $C_1 \cap A \in \Lambda$ and $C_2 \cap B \in \check{\Lambda}$. Since Λ is closed under taking joins ([GQT, Prop. 2.7]), the join $(C_1 \cap A) \oplus (C_2 \cap B)$ is Turing equivalent to a set in Λ. And P is Turing equivalent to that join: in one direction, from the join of two sets we can easily compute their union. In the other direction, to tell if $x \in (C_1 \cap A)$, and if $x \in (C_2 \cap B)$, we ask if $x \in P$. If the answer is "no", then the answer to both questions is "no". If the answer is "yes", then the answer to one of these questions is "yes" and the other "no" (as C_1 and C_2 are disjoint). To tell which, we wait until x is enumerated into either C_1 or C_2. □

2. Acceptable descriptions

For calculating heights of sets in the extended fine hierarchy, we used *admissible* descriptions, ones in which non-default outcomes have to increase the ordinal level of the class. For the proof of Theorem 1.1, it will be convenient to use another kind of finite class descriptions, that will make our book-keeping simpler.

Definition 2.1. A finite class description Γ is *acceptable* if it is efficient, and for every internal node $\sigma \in T_\Gamma$, $\eta_\sigma = 1$.

That is, only one change is allowed, and we may assume that this one change is from the default outcome to some non-default outcome. Acceptable class descriptions were used in [DGHTT] for the proof, presented in that paper, of the Louveau and Saint Raymond separation theorem. These are close to the "type 2 descriptions" used in [LSR88].

Proposition 2.2. *Every finitely described class has an acceptable description.*

Proof. By [GQT, Lem. 4.5 and Prop. 4.11], it suffices to show that for every very admissible class description ([GQT, Def. 4.6]) there is an equivalent acceptable description. This is shown by induction on the complexity of the very admissible description. Thus, it suffices to give an acceptable description Γ equivalent to $\mathrm{SU}_{\xi,n}(\Theta, \Lambda)$, where $\Lambda \subseteq \Theta$ are acceptable, $o(\Lambda) \geqslant \xi$ and $o(\Theta) > \xi$. There are two cases. If $\Lambda = \Theta$, then we let Γ be the class description given in Fig. 1, where $\Theta_k = \Theta$ for even k, and $\Theta_k = \check{\Theta}$ for odd k.

If $\Lambda < \Theta$ then we let Γ be the class description given in Fig. 2, with the same definition for the Θ_k's.

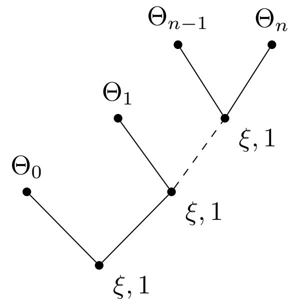

Figure 1. Acceptable description, case I

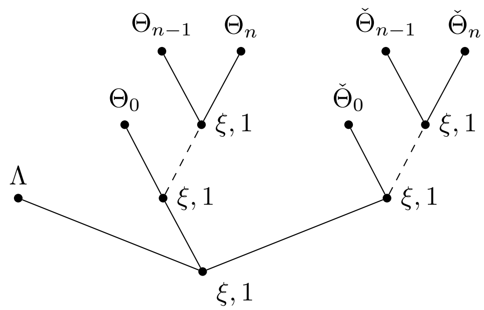

Figure 2. Acceptable description, case II: $\Lambda < \Theta$.

That Γ is equivalent to $\mathrm{SU}_{\xi,n}(\Theta,\Lambda)$ follows from [GQT, Prop. 3.12]; it is similar to the argument of [GQT, Exmp. 3.13]. In the first case, we use the equivalence with the description in [GQT, Fig. 5]. □

For the proof of Theorem 1.1, we will need not only Proposition 2.2; we will need its proof, i.e., we will need particular properties of the class descriptions shown in Figs. 1 and 2. For example, one such property is some amount of "predictability" of labels obtained by taking non-default outcomes. If Γ is a description as in Fig. 1, then every internal $\sigma \in S_\Gamma$ has exactly one non-default outcome; so if we instruct these nodes to choose the non-default outcome, we know what leaf of S_Γ we will obtain: the rightmost one. On the other hand, if Γ is as in Fig. 2, there will be two leaves of S_Γ obtained by only taking non-default outcomes; however, both of these leaves τ satisfy $\xi_\tau < \omega_1$; this follows from the assumption that $\Lambda < \Theta$ in that case.

3. The main argument

In this section we present the main argument for Theorem 1.1. The proof itself will be a mild elaboration of this argument. Here we prove the following. Recall that for any class Γ, we denoted by Γ^+ the successor class of Γ, the class of Σ-type with $\delta(\Gamma^+) = \delta(\Gamma) + 1$.

Theorem 3.1. *For any finitely described Γ, there is a set $X \in \Gamma^{++}$ which is not Turing equivalent to any set in Γ^+.*

To prove Theorem 3.1, we first observe that the theorem is already known for the classes $\Gamma^+ = D_n(\Sigma^0_1)$; for every n, there is an $(n+1)$-c.e. set not Turing equivalent to any n-c.e. set, in fact, to any set in $\Delta(D_n(\Sigma^0_1))$; this is a standard finite injury construction due to Cooper [Coo71]. We assume henceforth that $\delta(\Gamma) \geqslant \omega$.

3.1. **Preparation and discussion.** Let Γ be a finitely described class with $\delta(\Gamma) \geqslant \omega$. By Proposition 2.2, we fix an acceptable description of this class, that we also call Γ. Observe that Γ^+ and Γ^{++} are also acceptable. We note however, that as mentioned above, we will in fact need special properties of the class descriptions from Figs. 1 and 2, so we assume that Γ (and all sub-descriptions) are of this form; we will later specify the required properties.

We fix $(M_e)_{e\in\omega}$, an effective listing of Γ^+-names ([GQT, Prop. 2.8]). We will build a computable Γ^{++}-name N, and we must satisfy the following requirements:

$$R_{e,i,j}:\ \neg[\Phi_i^{M_e} = N \wedge \Phi_j^N = M_e]$$

(Here we are identifying N and M_e with the sets that they name.)

The construction must be computable, since the name N needs to be computable. It will be a finite injury construction, with additional complexity layered on top.

The salient class descriptions. The descriptions for Γ^+ and Γ^{++} are shown in Fig. 3. We will not be making use of the middle branch of Γ^{++}, so consider the pruned version shown in Fig. 4. [GQT, Prop. 3.12] implies that this is an equivalent description.

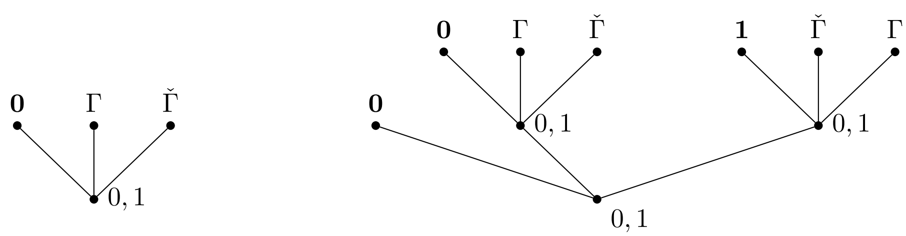

FIGURE 3. The class descriptions for Γ^+ and Γ^{++}.

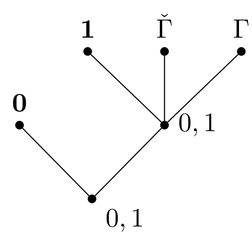

FIGURE 4. The pruned class description for Γ^{++}.

The basic diagonalisation strategy. First, we describe the strategy for meeting $R_{e,i,j}$ in isolation. The interaction between different requirements (and as we will see, between different followers for the same requirement) will force us to modify this basic strategy. The very main idea is that us, playing on Γ^{++}, have a "one-step" advantage over the opponent, who is playing on Γ^+. We both start with a default outcome **0**; but we can move to **1**, and in response the opponent must choose one of Γ or $\check{\Gamma}$: we can respond to this choice at a later step. Here is the one-requirement module.

(1) Choose a witness x; direct $N(x)$ to take the default outcome at the root (that is, we let $f^N_{\langle\rangle}(x,s)$ be the default child of the root $\langle\rangle$). Since this is labelled **0**, this means setting $N(x,s)=0$ for now. Search for strings τ_0 and ρ_0 with $\tau_0(x)=0$, $\Phi_i^{\rho_0}(x)=0$, and $\Phi_j^{\tau_0}\succcurlyeq\rho_0$. When found, restrain N to agree with τ_0.
(2) For every y with $\rho_0(y)=1$, wait until $M_e(y)$ has moved off the default outcome, i.e., wait until we see that $f^{M_e}_{\langle\rangle}(y,s)$ is a non-default outcome of the root of T_{Γ^+}.
(3) Direct $N(x)$ to take the non-default outcome at the root, and the default outcome above that. Recall that this is on the pruned description, so this means setting $N(x,s)=1$. Search for strings τ_1 and ρ_1 with $\tau_1(x)=1$, $\Phi_i^{\rho_1}(x)=1$, $\Phi_j^{\tau_1}\succcurlyeq\rho_1$, and $\tau_1(y)=\tau_0(y)$ for every $y<|\tau_0|$ other than x. When found, restrain N to agree with τ_1. Fix z least such that $\rho_0(z)\neq\rho_1(z)$.
(4) Wait until $M_e(z)$ moves off the default outcome at the root of T_{Γ^+}.
(5) $M_e(z)$ moving off the default outcome amounts to a choice our opponent is making for $M_e(z)$: it will be evaluated using the Γ-name $(M_e)_1$, or the $\check{\Gamma}$-name $(M_e)_2$ (where 1 and 2 here name the two non-default children of the root on T_{Γ^+}). We respond by choosing a non-default outcome for $N(x)$ at the non-default child of the root of $T_{\Gamma^+{+}}$ (call it σ), which similarly chooses between Γ and $\check{\Gamma}$. The choice of which depends on $\rho_0(z)$:
   - If $\rho_0(z)=0$, then we let define $f^N_\sigma(x,s)$ to point to the description which is the dual of the description $f^{M_e}_{\langle\rangle}(z,s)$ is pointing to (so N chooses Γ if M_e has chosen $\check{\Gamma}$, and vice versa). Have N match M_e on the interior nodes of this description.
   - If $\rho_0(z)=1$, then define $f^N_\sigma(x,s)$ to point to the same description as $f^{M_e}_{\langle\rangle}(z,s)$ is pointing to. Have N match M_e on the interior nodes of this description.

Note that τ_1 is an extension of τ_0, except that they disagree on x: $|\tau_1|\geqslant|\tau_0|$ and τ_0 and τ_1 agree on each $y<|\tau_0|$ other than x.

Let us explain why this strategy works:

- If we search forever at step (1), then we will have $N(x)=0$; in this case, either $\Phi_j^N\neq M_e$ or $\Phi_i^{M_e}(x)\neq 0=N(x)$.
- If we wait forever at step (2), then as **0** is the label of the default child of the root on Γ^+, there is some y with $M_e(y)=0$ and $\Phi_j^N(y)=\rho_0(y)=1$.
- If we search forever at step (3), then either $\Phi_j^N\neq M_e$ or $\Phi_i^{M_e}(x)\neq 1=N(x)$.

- If we wait forever at step (4), then $M_e(z) = 0$. By the fact that we did not wait at step (2), we have $\rho_0(z) = 0$, and so $\rho_1(z) = 1$. Then $\Phi_j^N(z) = \rho_1(z) = 1 \neq M_e(z)$.
- Suppose otherwise. Since τ_1 and τ_0 agree on all common entries except for x, we have $N \succ \tau_0$ iff $N(x) = 0$, and $N \succ \tau_1$ iff $N(x) = 1$. Thus $\Phi_j^N(z) = \rho_0(z)$ iff $N(x) = 0$.
  - If $\rho_0(z) = 0$, then $N(x) = 1 - M_e(z)$ by construction, so $\Phi_j^N(z) = 0$ iff $M_e(z) = 1$, and thus $\Phi_j^N(z) \neq M_e(z)$.
  - If $\rho_0(z) = 1$, then $N(x) = M_e(z)$ by construction, so $\Phi_j^N(z) = 1$ iff $M_e(z) = 0$, and thus $\Phi_j^N(z) \neq M_e(z)$.

In all cases, we guarantee our requirement.

Finding the τs.

The main complexity in this proof comes from locating τ_0 and τ_1. If N and M_e named c.e. sets, then we could simply let τ_i be initial segments of a stage s approximation of these sets, whenever we see enough convergence of Φ_i and Φ_j. However, sets in Γ will be a lot more complicated. In particular, since Γ will contain internal nodes σ with $\xi_\sigma > 0$, we cannot computably see values of N. A reasonable approximation to N is only obtained on the true stages, which of course cannot be identified computably. However, the above strategy for $R_{e,i,j}$ must be carried out computably (in order to construct a valid name). Thus, we will need to find τ_0 and τ_1 that may seem unrelated to the current version of N. We need, however, to be able to at times restrain N so that it extends τ_0 and τ_1, so not all strings will be suitable. How then can we know what are good values for these τ?

Fix some $y < |\tau_i|$ (for some candidate for τ_i); we need to decide if it is within our power to restrain the value of N on y. There are two cases: those y which are witnesses of some strategy, and those y which are not (the second case includes y which are chosen as witnesses and then discarded because of an injury).

Let us begin by discussing the second case. We will always choose our witnesses large. Thus, we know that y will never be a witness if it is not a witness when some larger number is considered in the construction, and also that if y is a witness and then discarded by injury, it will never again be a witness. When we are certain that y is not a witness, we will be able to decide any value we need for $N(y)$. Let us explain how.

Recall that we are using a description Γ which is of one of the forms given in Figs. 1 and 2, and further, than $\delta(\Gamma) \geqslant \omega$. This implies that either $0 < o(\Gamma) < \omega_1$; or that $o(\Gamma) = 0$, and the form is as in Fig. 2 where $o(\Theta) > 0$. [The other possibility, when $o(\Gamma) = \omega_1$, or Γ is as in Fig. 1 and $o(\Gamma) = 0$, gives us the classes with $\delta(\Gamma) < \omega$, namely, $\{\varnothing\}$, or $D_n(\Sigma_1^0)$, or their duals.]

In particular, this implies that if τ is any leaf of T_Γ that does not extend the default outcome at the root, then there is some $\sigma \prec \tau$ with $\xi_\sigma > 0$.

Thus, when we see that y is not a witness for any requirement, we do the following:

- For any internal node σ of Γ^{++} with $\xi_\sigma = 0$, we move $f_\sigma^N(y)$ off the default outcome of σ (if they are not already off the default outcome). Here σ includes not only possible nodes in Γ or $\check{\Gamma}$, but also the two nodes below the copy of Γ. Thus, we ensure that $\ell^N(y)$ passes through some internal node σ with $\xi_\sigma > 0$.[1]

- If $\xi_\sigma > 0$, then σ has access to some positive number of jumps, so it can tell when injuries will occur and can locate the highest priority uninjured strategy which chooses some τ with $y < |\tau|$. When it finds such τ, it can define its outcome to point to some leaf labelled with $\tau(y)$ (since Γ is nontrivial, both **0** and **1** occur amongst its leaves).

Thus, the $R_{e,i,j}$ strategy, when searching for its τ, can consider values of both 0 and 1 at any y which is not a higher priority witness. If it finds a τ, any lower priority witnesses within the domain will be injured, and the resulting y will set themselves to be the correct value (under the assumption that the $R_{e,i,j}$ strategy is not itself injured).

[The reader may ask, if we have the power to set non-witnesses to any value, why don't we just declare in advance that we set them all to 0? But this gets the order of events wrong. The requirement $R_{e,i,j}$ must examine, as candidates for τ_i, all strings that are possibly consistent with N. Weaker witnesses will be initialised only when τ_i are founded. Until then, we do not know that weaker witnesses will in fact be later determined to be 0.]

Now, let us consider the first case: y is a witness of some stronger requirement. To illustrate this, we will consider the action of $R_0 = R_{e_0,i_0,j_0}$ and $R_1 = R_{e_1,i_1,j_1}$. R_0 chooses a witness x_0 and acts according to the previously described strategy. Every time it advances a step along that strategy, R_1 is injured. Thus R_1 may assume it knows which step R_0 settles at. If R_0 settles at any step other than 7, then $N(x_0)$ is determined by the step, and so R_1 will know the value of $N(x_0)$. Thus R_1 will be searching for τ which have $\tau(x_0) = N(x_0)$.

However, if R_0 reaches step (7), then $N(x_0)$ is determined by the behaviour of M_{e_0} on some z, in a manner that can involve one or more non-computable limits. To address this, we will have R_1 *guess* the value of $N(x_0)$. In fact, for ease of implementation, we will have R_1 guess the particular leaf $\ell^N(x_0)$ which is reached, which then determines $N(x_0)$. Thus R_1 will require many witnesses, apparently one for each possible value of $\ell^N(x_0)$. In fact, we will require many more, as we will shortly see.

This creates additional complexity, however. Suppose R_1 creates a witness x_1 guessing some value of $\ell^N(x_0)$. We then want to create x_2 guessing some other value of $\ell^N(x_0)$. However, since x_1 is an earlier witness, our search for τ's on behalf of x_2 must take into account $N(x_1)$. Every time x_1 advances a step of the strategy, we will injure x_2, so x_2 may assume it knows what step x_1 settles at (for this reason we will no longer speak of requirements being injured, but instead of individual witnesses being injured). If x_1 reaches step (7), then we will again have to guess the value of $\ell^N(x_1)$, which will require multiple witnesses for the different values.

It might seem that this will create an infinite regression, with each witness needing to guess the values of previous witnesses, such that we will never be ready to

[1]Also, our definition of N will be reasonable, in that we never decrease the ordinal $\beta^N_\sigma(y)$ before we move off the default outcome. That is, we never lock ourselves into the default outcome. On the other hand, earlier, when we believed that y was a witness, we may have already moved $f^N_\sigma(y)$ off the default outcome. Once that happens, we cannot change our mind again. So in the instruction above, we would not be able to say "once we discover y is not a witness, move $f^N_\sigma(y)$ to the default outcome."

place the witness guessing the third value of $\ell^N(x_0)$, let alone find witnesses for the next requirement R_2. We escape this trap by making the various x's promise to behave themselves, as we will now describe.

Each witness x is based on guesses about the behaviour of previous witnesses. The witness x promises that if any of those guesses is wrong, then it will behave in a way which can be partly predicted from how the guess is wrong. Let us consider again x_0 and x_1. Suppose x_1's guess about $\ell^N(x_0)$ is wrong, and let σ be the meet of the true value of $\ell^N(x_0)$ and x_1's guess for $\ell^N(x_0)$ (so σ is the first node along $\ell^N(x_0)$ which x_1 has guessed the wrong outcome for). Let $\alpha_1 = \xi_\sigma$. Then x_1 promises that in $N(x_1)$, every internal node π with $\xi_\pi \geqslant \alpha_1$ will behave in a way which can be predicted from the knowledge of x_1's mistake.

Since x_2 makes a different guess for $\ell^N(x_0)$ than x_1 does, x_2 is guessing that x_1 has made a specific mistake about $\ell^N(x_0)$. Thus x_2's guess for $\ell^N(x_0)$ has already partly determined $\ell^N(x_1)$. As such, x_2 only needs to guess $\ell^N(x_1)$'s path on nodes π with $\xi_\pi < \alpha_1$. The witness x_2 will in turn make the same promises about its guesses for $\ell^N(x_0)$ and $\ell^N(x_1)$, and what it will do if those guesses are wrong. Then x_3 will make the same guess for $\ell^N(x_0)$ as x_2 made, but a different guess for $\ell^N(x_1)$. In particular, x_3's guess for $\ell^N(x_1)$ will differ from x_2's guess at some π with $\xi_\pi = \alpha_2 < \alpha_1$. So x_3 will also need to guess $\ell^N(x_2)$, but it will only need to make this guess on nodes with ordinal smaller than α_2. The ordinals keep decreasing in this fashion, so we will eventually reach an n such that x_{n+1} does not need to guess $\ell^N(x_n)$ at all: $\ell^N(x_n)$ will be entirely determined by x_n's guesses for previous witnesses. This is how we avoid needing an infinite stream of witnesses.

We will give the full details shortly.

Notice that because of injury and witnesses being chosen large, x_2 will be beyond any τ which x_1 chooses. This is why x_1 does not need to guess x_2's behaviour. For the same reason, x_0 does not need to guess x_1's behaviour.

The collection of witnesses.

Here we define the collection of witnesses which will be required for each requirement, as well as what guesses each witness will be making about previous witnesses. In the previous discussion, we had said that the finite injury nature of the construction allows a witness to know $N(y)$ for earlier witnesses y which settle before step (7). Since we are developing a complex machinery to guess the outcomes for witnesses which reach step (7), it will simplify the presentation to instead just use the same machinery to guess the outcomes at previous steps as well. So we will not be using the finite injury for that. Injury will still be important to ensure that any witness is beyond the domain of any τ chosen by a higher priority witness.

Definition 3.2. Let $\alpha_0 > \alpha_1 > \cdots > \alpha_{n-1} = 0$ list the ordinals occurring in the description of Γ^{++}. For $i < n$, let

$$S_i = \{\langle\rangle\} \cup \{\sigma \in T_{\Gamma^{++}} : \xi_{\sigma^-} \leqslant \alpha_i\}.$$

We also define $S_n = \{\langle\rangle\}$.

Note the similarity with S_Γ from the leaf selection game. Indeed, $S_{n-1} = S_{\Gamma^{++}}$, while S_0 is the full tree $T_{\Gamma^{++}}$.

Fix an effective listing $R_0, R_1, \ldots$ of all the requirements, with order-type ω. The witnesses required for R_k will depend on the witnesses chosen for stronger requirements, i.e., requirements $R_{k'}$ for $k' < k$. For requirement, we will construct a finite tree $Q = Q_k$, and for each $\rho \in Q$, a function $g_\rho = g_{\rho,k}$ and a set $Y_\rho = Y_{\rho,k}$.

We will always have $\mathrm{dom}(g_\rho) \subseteq Y_\rho$, and in fact ρ will be a leaf of Q precisely when $\mathrm{dom}(g_\rho) = Y_\rho$. The requirement R_k will require one witness for each leaf of Q.

Definition 3.3. Fix k. We construct the tree $Q = Q_k$ recursively, depth first, left-to-right. We begin by adding a root $\langle\rangle$ to Q. We define $g_{\langle\rangle}$ to be the empty function. We let $Y_{\langle\rangle}$ be the set of all leaves of all trees $Q_{k'}$ for $k' < k$.

Suppose we have added a node ρ to Q of height $i \leqslant n$, also defining g_ρ and Y_ρ. If $Y_\rho = \mathrm{dom}(g_\rho)$, then ρ is a leaf of Q. Otherwise, consider extensions of g_ρ to Y_ρ such that all new values are leaves of S_i. For every such extension, we add a child $\rho\hat{\ }m$ and define $g_{\rho\hat{\ }m}$ to be that extension. We then let

$$Y_{\rho\hat{\ }m} = Y_\rho \cup \{\mu : \mu \text{ is a leaf of } Q \text{ and } \rho\hat{\ }l \leqslant \mu \text{ for some } l < m\}.$$

Note that this means that before defining $Y_{\rho\hat{\ }m}$, we must define the entire tree above $\rho\hat{\ }m$'s siblings to its left. The children of $\rho\hat{\ }m$ depend on the leaves extending these siblings.

Observe that for $k = 0$ we have $Y_{\langle\rangle} = \varnothing$ so $\langle\rangle$ is a leaf (and the unique node) of Q_0. Indeed, the strongest witness does not need to make any guesses, which is why the strongest requirement only needs one witness.

Claim 3.3.1. Each tree Q_k is well-defined, and finite. Further,

(a) Q_k has height $\leqslant n + 1$.
(b) For every $\rho \in Q_k$, Y_ρ consists of the leaves of trees $Q_{k'}$ for $k' < k$, as well as the leaves of Q_k that are lexicographically to the left of ρ.

Proof. The claim is proved by induction on k, and within step k, by induction on the creation of the tree.

First, we observe that if $\rho \in Q_k$ and is not a leaf, then the leftmost child of ρ is a leaf, whereas other children (if they exist) are not.

If $\rho \in Q_k$ has height n, and is not a leaf of Q_k, then as S_n is a singleton, ρ will have one child on Q_k, and that child will be a leaf. Thus, the instructions given suffice to proceed with the definition of Q_k in all cases; and Q_k has height at most $n + 1$.

To show that Q_k is finite, then, it suffices to show that if $\rho \in Q_k$ is not a leaf, then it has only finitely many children. This is ensured since by induction, Y_ρ is finite. □

Let L be the set of all leaves of all the trees Q_k. Recall that we will have a witness x_ν for every $\nu \in L$ (whose values may change by injury). For $\mu, \nu \in L$, we let $\mu < \nu$ if $\mu \in Y_\nu$, i.e., if μ occurs on an earlier tree than ν, or if μ is lexicographically left of ν on the same tree. We will let μ have priority over ν. For $\mu < \nu$, the value $g_\nu(\mu)$ will be x_ν's guess as to the behaviour of x_μ.

If μ occurs on an earlier tree, then μ is in the $Y_{\langle\rangle}$ of the tree containing ν; this means that $g_\nu(\mu)$ will be a leaf of S_0, i.e., of $T_{\Gamma^{++}}$. If μ and ν belong to the same tree, let ρ be the meet of these two leaves; so $\mu \geqslant \rho\hat{\ }l$ and $\nu \geqslant \rho\hat{\ }m$ for some $l < m$. So $\mu \in Y_{\rho\hat{\ }m}\backslash Y_\rho$, and $g_\nu(\mu)$ will be a leaf of $S_{|\rho|+1}$. Note though that a leaf of S_j can also be a leaf of S_i for some $i < j$; in particular, a leaf of S_i for some $i > 0$ can also be a leaf of S_0.

The behaviour of an x.

Definition 3.4. Let $j \leqslant n$, and let $\nu \in L$. We say that *all of ν's guesses are correct up to the leaves of S_j* if for every $\mu < \nu$:

- If $g_\nu(\mu)$ is a leaf of S_i for some $i \leqslant j$, then $g_\nu(\mu)$ and $\ell^N(x_\mu)$ both extend the same leaf of S_j. (Recall that $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n$).
- If $i \geqslant j$, then $\ell^N(x_\mu)$ extends $g_\nu(\mu)$.

If $j < n$, this means that *all of ν's guesses are correct up to α_j*: for all $\mu < \nu$, for every $\sigma\hat{\ }m \leqslant \ell^N(x_\mu)$, if $\xi_\sigma \leqslant \alpha_j$ and $\sigma \prec g_\nu(\mu)$ then $\sigma\hat{\ }m \leqslant g_\nu(\mu)$. Of course, every ν guesses correctly up to the leaves of S_n.

The fact that Γ^{++} is acceptable, i.e., that all η_σ-ordinals are 1, means that the movement of guesses only goes from left to right. That is, for any internal σ, if $s <_{\xi_\sigma} t$ then the outcome $f^N_\sigma(y,s)$ is not to the right of $f^N_\sigma(y,t)$. So for every j, if σ is an internal node of S_j, and $s <_{\alpha_j} t$, then the guess of which leaf of S_j is extended by $\ell^N(y)$ at stage s cannot be to the right of the guess at stage t.

This implies that the property "all of g_ν's guesses are correct up to the leaves of S_j" is $D_2(\Sigma^0_{1+\alpha_j})$. While all approximations are to the left of g_ν's guesses, and at least one is strictly to the left, we do not believe the statement; once all approximations agree with g_ν's guesses, we believe; then once at least one has moved strictly to the right, we stop believing again.

Note that for $i < j$, since $\alpha_i > \alpha_j$, $D_2(\Sigma^0_{1+\alpha_j}) \subset \Delta^0_{1+\alpha_i}$. Thus, if $s <_{\alpha_i} \omega$, then at stage s we simply know the truth of the given statement. Similarly, the set of elements which are eventually not witnesses (either never selected as a witness, or chosen as a witness and then injured) is Σ^0_1, since we choose witnesses large, and thus $\Delta^0_{1+\alpha_i}$ for any $i < n-1$ (recall that $\alpha_{n-1} = 0$.)

So as long as we believe that x is a witness, we will define $N(x)$ as described within the basic strategy, subject to the following modifications. Say that a node σ believes that $x = x_\nu$ for some $\nu \in L$. If $\xi_\sigma > 0$, σ will know that this is correct; if $\xi_\sigma = 0$, σ will only currently believe this to be correct. Let $\xi_\sigma = \alpha_i$ (so $i < n$). Then σ "knows" whether g_ν's guesses are correct up to the leaves of S_{i+1}.

- If g_ν's guesses are not correct up to the leaves of S_{i+1}, then σ takes an outcome towards a leaf labelled with **0**.
- Otherwise, we consider g_ν's guesses up to the leaves of S_i:
  - If all approximations are currently to the left of g_ν's guesses, at least one strictly to the left, then σ takes the default outcome.
  - If all approximations currently extend g_ν's guesses, then σ behaves as described within the basic strategy.
  - If at least one approximation is strictly to the right of g_ν's guess, then σ takes a non-default outcome.

Note that in the first case, the choice **0** is arbitrary. All we need to do is guarantee some fixed outcome, **0** or **1**. Since Γ is efficient, both are always available.

Predicting previous witnesses' behaviour.

Suppose that x_ν is a witness, and g_ν is correct up to the leaves of S_0, i.e., all of its guesses are correct. We will show that x_ν can correctly predict the outcomes of all previous witnesses, and can thus perform a correct search for the strings τ which it needs. Note however that "outcome" here means the value $N(x_\mu)$; we will not necessarily be able to predict the leaf $\ell^N(x_\mu)$.

For any witness $x_\mu < x_\nu$, if μ is on an earlier tree than ν, then $\mu \in Y_\Diamond$. Thus $g_\nu(\mu)$ is a leaf of S_0, so by assumption $\ell^N(x_\mu) = g_\nu(\mu)$.

Suppose instead μ is on the same tree Q as ν. By our ordering of witnesses, it must be that μ is lexicographically to the left of ν. Then $g_\nu(\mu)$ is a leaf of some S_i

with $i > 0$. So all we know is that $\ell^N(x_\mu)$ extends $g_\nu(\mu)$. In this case we will not be able to always fully determine $\ell^N(x_\mu)$. However, the modifications we described above will give us enough information to predict the label of $\ell^N(x_\mu)$, i.e., $N(x_\mu)$. We will need the following.

Claim 3.4.1. Let $k \in \mathbb{N}$, and let μ, ν be leaves of Q_k, with μ to the left of ν. Let $\rho = \mu \wedge \nu$ by the longest common initial segment of μ and ν, and let $i = |\rho|$. Then there is some $\lambda < \mu$ such that $g_\mu(\lambda)$ and $g_\nu(\lambda)$ *disagree within* S_i, i.e., $\sigma = g_\mu(\lambda) \wedge g_\nu(\lambda)$ satisfies $\xi_\sigma \leqslant \alpha_i$.

Proof. Say $\mu \geqslant \rho\hat{\ }m$ and $\nu \geqslant \rho\hat{\ }l$ where $m < l$. So g_ν extends $g_{\rho\hat{\ }l}$ and g_μ extends $g_{\rho\hat{\ }m}$. Since $m \neq l$, there is some $\lambda \in Y_\rho \backslash \operatorname{dom} g_\rho$ such that $g_{\rho\hat{\ }m}(\lambda) \neq g_{\rho\hat{\ }l}(\lambda)$. Both of these will be leaves of S_i, and so incomparable. □

We will also need another special property of Γ^{++}, that follows from Γ and its subclasses being of one of the forms in Figs. 1 and 2.

Claim 3.4.2. Let σ be an internal node of $T_{\Gamma^{++}}$. One of the following hold:

(i) Following non-default outcomes from σ, we pass through internal nodes τ with $\xi_\tau = \xi_\sigma$, until we reach a unique leaf of T_Γ.
(ii) Following any choice of non-default outcomes from σ, we reach an internal node τ with $\xi_\tau > \xi_\sigma$.

The uniqueness in (i) means that σ has a unique non-default outcome $\sigma\hat{\ }m$, that node has a unique non-default outcome, and so on; and further, in (i), before we reach the leaf, the ξ-ordinal does not increase.

Proof. Suppose first that σ is a node of T_Γ or $T_{\check{\Gamma}}$ (viewed as subtrees of $T_{\Gamma^{++}}$). Then Γ_σ has one of the forms from Figs. 1 and 2. In the second case, $o(\Theta) < \omega_1$ (as $\Lambda < \Theta$), which guarantees that (ii) holds. In the first case, if $o(\Theta) < \omega_1$ then again (ii) holds; if $o(\Theta) = \omega_1$ then (i) holds.

Otherwise, σ is either the root of $T_{\Gamma^{++}}$, or the non-default child of the root. If $o(\Gamma) > 0$ then (ii) holds. If $o(\Gamma) = 0$, consider again the form of Γ. Then $o(\Theta) < \omega_1$, for otherwise, Γ has the form in Fig. 1, and then $\delta(\Gamma) < \omega$, which we assumed is not the case. Hence, (ii) holds for σ. □

Definition 3.5. Let $\nu \in L$. We define a function $h_\nu \colon Y_\nu \to \{0,1\}$ as follows. Let $\mu < \nu$.

If $g_\nu(\mu)$ is a a leaf of S_0, i.e., of $T_{\Gamma^{++}}$, then we let $h_\nu(\mu)$ be the label that Γ^{++} gives this leaf.

Suppose that $g_\nu(\mu)$ is not a leaf of S_0. Then μ and ν lie on the same tree Q_k. Let $j < n$ be the greatest such that there is some $\lambda < \mu$ such that $g_\mu(\lambda)$ and $g_\nu(\lambda)$ disagree within S_j. By Claim 3.4.1, $j \geqslant |\mu \wedge \nu|$. Let $\sigma = g_\nu(\mu)$. Note that σ is a leaf of $S_{|\mu\wedge\nu|+1}$, so $\xi_\sigma \geqslant \alpha_{|\mu\wedge\nu|}$. It follows that $\xi_\sigma \geqslant \alpha_j$.

If $\xi_\sigma > \alpha_j$, we let $h_\nu(\mu) = 0$.

Suppose that $\xi_\sigma = \alpha_j$. There are three cases.

(a) For every $\lambda < \mu$, $g_\nu(\lambda)$ is lexicographically to the left of $g_\mu(\lambda)$ within S_j (and so at least one of these strictly so). Consider the leftmost leaf of S_j extending σ) (i.e., the leaf reached by beginning at σ and repeatedly taking default children until a leaf of S_j is reached). If this is a leaf of S_0, let $h_\nu(\mu)$ be the label of this leaf. Otherwise, let $h_\nu(\mu) = 0$.

(b) Suppose that there is some $\lambda < \mu$ such that $g_\nu(\lambda)$ is lexicographically strictly to the right of $g_\mu(\lambda)$ within S_j. Consider which case in Claim 3.4.2 above holds for the tree $T_{\Gamma^{++}}$ above σ.
- In case (i), let $h_\nu(\mu)$ be the label of the rightmost leaf of S_j extending σ (this will be a leaf of S_0).
- In case (ii), let $h_\nu(\mu) = 0$.

We note that the finiteness of Y_ν ensures that $\nu \mapsto h_\nu$ is computable. We will verify later that if g_ν is correct, then indeed for all $\mu < \nu$, $h_\nu(\mu) = N(x_\mu)$. Informally: suppose that $\sigma = g_\nu(\mu)$ is not a leaf of S_0, so we only know that $\ell^N(x_\mu)$ extends σ. If $\xi_\sigma > \alpha_j$ then σ and its extensions know that g_μ is wrong somewhere within S_j, and are instructed to take an outcome towards a leaf labelled with $\mathbf{0}$, which matches the definition $h_\nu(\mu) = 0$.

Suppose that $\xi_\sigma = \alpha_j$. If every $\lambda \in Y_\mu$ has $\ell^N(x_\lambda)$ to the left of $g_\mu(\lambda)$ within S_j (at least one strictly so), then all nodes τ with $\xi_\tau = \alpha_j$ will remain on their default outcome. So $\ell^N(x_\mu)$ will be sent to the leftmost leaf of S_j extending σ. If this is a leaf of S_0, then this gives us the value of $N(x_\mu)$ and of $h_\nu(\mu)$, by construction. If this is not a leaf of S_0, then this is some internal node ρ with $\xi_\rho > \alpha_j$, so x_μ will be sent from this node to a leaf labelled with $\mathbf{0}$, as in the previous case. Again, this matches our definition of $h_\nu(\mu)$.

If there is some λ such that $\ell^N(x_\lambda)$ is strictly to the right of $g_\mu(\lambda)$ within S_j, then all nodes τ with $\xi_\tau = \alpha_j$ will eventually discover this, and switch to a non-default outcome. In case (i) for σ, this determined $\ell^N(x_\mu)$, whose label is $h_\nu(\mu)$. In case (ii) we again are guaranteed to reach some internal node ρ with $\xi_\rho > \alpha_j$, which again, guarantees a $\mathbf{0}$ outcome.

3.2. **The full construction.** At each stage s, we will have some $\nu_s \in L$, and:
- a witness $x_{\nu,s} \in \mathbb{N}$ for each $\nu < \nu_s$;
- a string $\pi_{\nu,s} \in \{0,1,\star\}^{<\omega}$ for each $\nu \leqslant \nu_s$;
- for each $\nu < \nu_s$, possibly, strings $\tau_{0,\nu,s}$ and $\tau_{1,\nu,s}$, and $\rho_{0,\nu,s}$ and $\rho_{1,\nu,s}$.

The witnesses x_ν will respect the priority ordering: if $\mu < \nu < \nu_s$ then $x_{\mu,s} < x_{\nu,s}$. For $\nu < \nu_s$ we will have $|\pi_{\nu,s}| = x_{\nu,s}$.

The string $\pi_{\nu,s}$ indicates the restraint on ν that is imposed by $\mu < \nu$. The $\star$ in the restraint string indicate the locations of witnesses x_μ, which cannot be restrained, so ν will need to use h_ν to guess these values. That is, $\pi_{\nu,s}(y) = \star$ if and only if $y = x_{\mu,s}$ for some $\mu < \nu$. So if $\pi_{\nu,s}$ is defined, we then define $\hat{\pi}_{\nu,s}$ as follows: for $y < |\pi_{\nu,s}|$,

$$\hat{\pi}_{\nu,s}(y) = \begin{cases} \pi_{\nu,s}(y), & \text{if } \pi_{\nu,s}(y) \in \{0,1\}; \\ h_\nu(\mu), & \text{if } y = x_{\mu,s}. \end{cases}$$

For $\nu < \nu_s$, we will say that ν is (at stage s) at one of 5 *steps*. Say such ν is working for some requirement $R_{e,i,j}$. We say that ν *requires attention* at stage $s+1$ if one of the following holds:
- ν is at step (1) at stage s, and there are strings $\tau_0, \rho_0 \in 2^{<\omega}$ of length $\leqslant s$ such that $\Phi_i^{\rho_0}(x_{\nu,s}) = 0$, and $\Phi_j^{\hat{\pi}_{\nu,s} \hat{} \star \hat{} \tau_0} \succcurlyeq \rho_0$.[2]
- ν is at step (3) at stage s, and there are strings $\tau_1, \rho_1 \in 2^{<\omega}$ of length $\leqslant s$ such that $\tau_1 \succcurlyeq \tau_0$, $\Phi_i^{\rho_1}(x_{\nu,s}) = 1$, and $\Phi_j^{\hat{\pi}_{\nu,s} \hat{} 1 \hat{} \tau_1} \succcurlyeq \rho_1$.

[2]Note that this is a slightly different usage for τ_0 compared to the basic strategy.

In the following construction, unless otherwise stated, we let $x_{\mu,s} = x_{\mu,s-1}$, and similarly for π, ρ_i, τ_i.

Stage $s = 0$: We let ν_0 be the strongest (least) $\nu \in L$. We let $\pi_{\nu_0,0} = \langle\rangle$.

Stage $s > 0$: Suppose that no ν requires attention at stage s. Let $\mu = \nu_{s-1}$. We let $x_{\mu,s} = |\pi_{\mu,s-1}|$. We place $x_{\mu,s}$ at step (1). We let ν_s be μ's successor in L; we let $\pi_{\nu_s,s} = \pi_{\mu,s-1}\hat{\ }\star$.

For every $\lambda < \mu$, a witness for a requirement $R_{e,i,j}$:

- If λ is at step (2) at stage $s-1$, and for every y with $\rho_{0,\lambda,s}(y) = 1$, $f^{M_e}_{\langle\rangle}(y,s)$ is a non-default child of the root, then we move λ to step (3) at stage s.
- If λ is at step (4) at stage $s-1$, and $f^{M_e}_{\langle\rangle}(z,s)$ is a non-default child of the root, where z is least with $\rho_{0,\lambda,s}(z) \neq \rho_{1,\lambda,s}(z)$, then we move λ to step (5) at stage s.

Suppose that some $\mu < \nu_{s-1}$ requires attention at stage s. Let μ be the strongest such. Initialise all $\nu > \mu$ by making $x_{\nu,s}$, and all associated strings, undefined. We let ν_s be μ's successor in L. We act according to the step in which $x_{\mu,s-1}$ is.

- If $x_{\mu,s-1}$ is at step (1), let $\rho_{0,\mu,s}$ and $\tau_{0,\mu,s}$ be the discovered ρ_0 and τ_0. We place $x_{\mu,s}$ at step (2). By extending, we may assume that $|\tau_0|$ is fresh (large). We let $\pi_{\nu_s,s} = \pi_{\mu,s}\hat{\ }\star\hat{\ }\tau_0$.
- If $x_{\mu,s-1}$ is at step (3), let $\rho_{1,\mu,s}$ and $\tau_{1,\mu,s}$ be the discovered ρ_1 and τ_1. We place $x_{\mu,s}$ at step (4). By extending, we may assume that $|\tau_1|$ is fresh (large). We let $\pi_{\nu_s,s} = \pi_{\mu,s}\hat{\ }\star\hat{\ }\tau_1$.

Every $\mu \in L$ is injured only finitely often, and so $x_{\mu,s}$ and $\pi_{\mu,s}$ will reach final values, which we denote x_μ and π_μ. Similarly, if after last initialised, μ reaches step (2), we denote the last chosen strings as $\tau_{0,\mu}$ and $\rho_{0,\mu}$, and similarly for τ_1 and ρ_1.

We now turn to defining the computable name N. We employ some higher-level decision procedures. The map $\mu \mapsto (\pi_\mu, x_\mu)$ is Δ^0_2. So is the map taking π to the last step k_μ that μ reaches (after last initialised). By [GQT, Prop. 2.1]:

- we let $p\colon \mathbb{N} \times \omega \to 2^{<\omega}$ be a 1-decision procedure for $\mu \mapsto (\pi_\mu, x_\mu, k_\mu)$.

Further, by the recursion theorem, during the construction of N, we have access to a computable index of the construction. This gives us, for all internal $\sigma \in T_{\Gamma^{++}}$, a $\Delta^0_{\xi_\sigma+1}$-index of the function $y \mapsto f^N_\sigma(y)$ (the limit value of $f^N_\sigma(y,s)$ on the ξ_s-true stages). By the uniformity of [GQT, Prop. 2.1]:

- for every internal $\sigma \in T_{\Gamma^{++}}$, we let c_σ be a $(\xi_\sigma + 1)$-decision procedure for $y \mapsto f^N_\sigma(y)$.

Fix $y \in \mathbb{N}$. By induction on $s < \omega$, we define $f^N_\sigma(y,s)$ for every internal $\sigma \in T_{\Gamma^{++}}$. At a stage s, if $y \geqslant |\pi_{\nu_s,s}|$, then we let $f^N_\sigma(y,s)$ be the default child of σ.

For $y < |\pi_{\nu_s,s}|$, we define all $f^N_\sigma(y,s)$ by induction on y. Thus, when we are working for y, we have $\ell^N(y',s)$ for all $y' < y$.

Let $r = s^{-\xi_\sigma}$ be the $<_{\xi_s}$-predecessor of s. We will define $f^N_\sigma(y,s)$ by recursion on $|\sigma|$. We will explicitly define $f^N_\sigma(y,s)$ if we have already determined that $\sigma \prec \ell^N(y,s)$, i.e., if $\sigma = \sigma_{|\sigma|}(y,s)$, where $\sigma_0(y,s) = \langle\rangle$ and $\sigma_{k+1}(y,s) = f^N_{\sigma_k(y,s)}(y,s)$. For all other σ, we simply let $f^N_\sigma(y,s) = f^N_\sigma(y,r)$.

So suppose that we have already determined that $\sigma \prec \ell^N(y,s)$. We proceed by cases.

Case I: $\sigma = \langle\rangle$ is the root of $T_{\Gamma^{++}}$. Suppose that $y = x_{\mu,s}$ for some $\mu < \nu_s$. If μ is at step (3) or higher at stage s, then we let $f^N_{\langle\rangle}(y,s)$ be the non-default child of the root; if it is at stage (2) or lower, we let $f^N_{\langle\rangle}(y,s)$ be the default child of the root.

If $y \neq x_{\mu,s}$ for all $\mu < \nu_s$, then we let $f^N_{\langle\rangle}(y,s)$ be the non-default child of the root.

Case II: σ is the non-default child of the root of $T_{\Gamma^{++}}$.

The non-default children of σ are identified with the roots of T_Γ and $T_{\check\Gamma}$, as are the non-default children of the root of T_{Γ^+}.

Suppose that $y = x_{\mu,s}$ for some $\mu < \nu_s$, a witness for a requirement $R_{e,i,j}$. If μ is at step (4) or lower at stage s, then we let $f^N_{\langle\rangle}(y,s)$ be the default child of the root. Suppose that it is at step (5). Let z be the least such that $\rho_{0,\mu,s}(z) \neq \rho_{1,\mu,s}$. If $z = 0$, let $f^N_\sigma(y,s)$ be the opposite of $f^{M_e}_{\langle\rangle}(z,s)$. If $z = 1$, let $f^N_\sigma(y,s)$ be the same as $f^{M_e}_{\langle\rangle}(z,s)$.

If $y \neq x_{\mu,s}$ for all $\mu < \nu_s$, and $f^N_\sigma(y,r)$ is a non-default child of σ, then we let $f^N_\sigma(y,s) = f^N(y,r)$. If $f^N_\sigma(y,r)$ is a default child of σ, then we let $f^N_\sigma(y,s)$ be a non-default child of σ, say the root of Γ.

Case III: σ is an internal node of T_Γ or $T_{\check\Gamma}$, and $\xi_\sigma = 0$.

Suppose that $y = x_{\mu,s}$ for some $\mu < \nu_s$. There are four sub-cases:

(i) For all $\lambda < \mu$, $\ell^N(x_{\lambda,s},s)$ does not lie to the right of $g_\nu(\lambda)$ within S_{n-1}, and for at least one such λ, $g_\nu(\lambda)$ lies strictly to the left of $\ell^N(x_{\lambda,s},s)$ within S_{n-1}. In this case we let $f^N_\sigma(y,s)$ be the default child of σ.

(ii) For all $\lambda < \mu$, $g_\mu(\lambda)$ and $\ell^N(x_{\lambda,s},s)$ agree up to the leaves of S_{n-1}, and $x_{\mu,s}$ had not reached step (5): we let $f^N_\sigma(y,s)$ be the default child of σ.

(iii) For all $\lambda < \mu$, $g_\nu(\lambda)$ and $\ell^N(x_{\lambda,s},s)$ agree up to the leaves of S_{n-1}, and $x_{\mu,s}$ is currently at step (5): let $z = z_{\mu,s}$ be the least with $\rho_{0,\mu,s}(z) \neq \rho_{1,\mu,s}(z)$.
- If $z = 1$, let $f^N_\sigma(y,s) = f^{M_e}_\sigma(y,s)$, where σ is identified with the node on the same tree as σ (either T_Γ or $T_{\check\Gamma}$).
- If $z = 0$, let $f^N_\sigma(y,s) = f^{M_e}_\sigma(y,s)$, where σ is identified with the node on the opposite tree to σ.

(iv) There is some $\lambda < \mu$ such that $\ell^N(x_{\lambda,s},s)$ lies to the right of $g_\mu(\lambda)$ within S_{n-1}: if $f^N_\sigma(y,r)$ is a non-default child of σ, let $f^N_\sigma(y,s) = f^N_\sigma(y,r)$. Otherwise, let $f^N_\sigma(y,s)$ be some non-default child of σ.

Suppose that $y \neq x_{\mu,s}$ for all $\mu < \nu_s$. If $f^N_\sigma(y,r)$ is a non-default child of σ, let $f^N_\sigma(y,s) = f^N_\sigma(y,r)$. Otherwise, let $f^N_\sigma(y,s)$ be some non-default child of σ.

Case IV: σ is an internal node of T_Γ or $T_{\check\Gamma}$, and $\xi_\sigma > 0$.

We say that *σ's guesses are aligned up to y at stage s* if:

- If $\mu \leqslant \nu_s$ is least with $|\pi_{\mu,s}| > y$, then for all $\lambda \leqslant \mu$, if $\lambda < \nu_s$ then $p(\lambda,s) = (\pi_{\lambda,s}, x_{\lambda,s}, k_{\lambda,s})$, and if $\lambda = \nu_s$, then the first entry of $p(\nu_s,s)$ is $\pi_{\nu_s,s}$.
- For all $y' < y$, for all internal $\tau \in T_{\Gamma^{++}}$ with $\xi_\tau < \xi_\sigma$, $c_\tau(y',s) = f^N_\tau(y',s)$.

If this is not the case, we let $f^N_\sigma(y,s) = f^N_\sigma(y,r)$. Suppose that the guesses do align.

Let $\xi_\sigma = \alpha_i$, where $i < n-1$.

Suppose that $y = x_{\mu,s}$ for some $\mu < \nu_s$. If there is some $\lambda < \mu$ such that $g_\mu(\lambda)$ and $\ell^N(x_{\lambda,s}, s)$ disagree within S_{i+1}, then we let ζ be the leftmost leaf of $T_{\Gamma^{++}}$ extending σ labelled by $\mathbf{0}$; we let $f^N_\sigma(y,s) \preccurlyeq \zeta$.

Suppose that for all $\lambda < \mu$, $g_\mu(\lambda)$ and $\ell^N(x_{\lambda,s}, s)$ agree within S_{i+1}. There are four sub-cases, similar to case III above:

(i) For all $\lambda < \mu$, $\ell^N(x_{\lambda,s}, s)$ does not lie to the right of $g_\nu(\lambda)$ within S_i, and for at least one such λ, $g_\nu(\lambda)$ lies strictly to the left of $\ell^N(x_{\lambda,s}, s)$ within S_i. In this case we let $f^N_\sigma(y,s)$ be the default child of σ.

(ii) For all $\lambda < \mu$, $g_\mu(\lambda)$ and $\ell^N(x_{\lambda,s}, s)$ agree up to the leaves of S_i, and $x_{\mu,s}$ had not reached step (5): we let $f^N_\sigma(y,s)$ be the default child of σ.

(iii) For all $\lambda < \mu$, $g_\mu(\lambda)$ and $\ell^N(x_{\lambda,s}, s)$ agree up to the leaves of S_i, and $x_{\mu,s}$ is currently at step (5): let $z = z_{\mu,s}$ be the least with $\rho_{0,\mu,s}(z) \neq \rho_{1,\mu,s}(z)$.
- If $z = 1$, let $f^N_\sigma(y,s) = f^{M_e}_\sigma(y,s)$, where σ is identified with the node on the same tree as σ (either T_Γ or $T_{\check{\Gamma}}$).
- If $z = 0$, let $f^N_\sigma(y,s) = f^{M_e}_\sigma(y,s)$, where σ is identified with the node on the opposite tree to σ.

(iv) There is some $\lambda < \mu$ such that $\ell^N(x_{\lambda,s}, s)$ lies to the right of $g_\mu(\lambda)$ within S_i: if $f^N_\sigma(y,r)$ is a non-default child of σ, let $f^N_\sigma(y,s) = f^N_\sigma(y,r)$. Otherwise, let $f^N_\sigma(y,s)$ be some non-default child of σ.

Suppose that $y \neq x_{\mu,s}$ for all $\mu < \nu_s$. Let μ be least with $y < |\pi_{\mu,s}|$. Then $\pi_{\mu,s}(y) \in \{0,1\}$. Let ζ be the leftmost leaf of $T_{\Gamma^{++}}$ extending σ labelled by $\pi_{\mu,s}(y)$; we let $f^N_\sigma(y,s) \preccurlyeq \zeta$.

This completes the definition of N.

3.3. **Verification.** We say that y *is a witness at stage* s if $y = x_{\mu,s}$ for some $\mu < \nu_s$. The standard finite injury arguments yield:

Claim 3.5.1.
(a) If $s < t$ then $|\pi_{\nu_s,s}| < |\pi_{\nu_t,t}|$.
(b) If $y < |\pi_{\nu_s,s}|$ and y is not a witness at stage s, then for all $t \geqslant s$, y is not a witness at stage t.
(c) If $y = x_{\mu,s}$ (for some $\mu < \nu_s$), $t > s$ and $y \neq x_{\mu,t}$, then y is not a witness at stage t.

We need to show that N is legally defined. First, observe that when we ask for some leaf $\zeta \prec \sigma$ of T_Γ with a given label, such a leaf exists: since $\sigma \in T_\Gamma$ is internal, and Γ is efficient, σ will have extensions labelled $\mathbf{0}$ and extensions labelled $\mathbf{1}$.

We will prove, by induction on y, that $f^N_\sigma(y,-)$ satisfies Claim 3.5.3, which is the requirement for N being a name. Fix some y, and suppose that this has been shown for all $y' < y$.

Claim 3.5.2. Let σ be an internal node with $\xi_\sigma > 0$. The stages at which σ's guesses are aligned up to y are convex in $<_{\xi_\sigma}$: if $r <_{\xi_\sigma} s <_{\xi_\sigma} t$ and σ's guesses are aligned up to y at stages r and t, then they are at stage s as well.

Proof. This follows from the fact that for all μ, if $p(\mu,r) \neq ?$ and $r <_{\xi_\sigma} s$ then $p(\mu,s) = p(\mu,r)$ (as $\xi_\sigma \geqslant 1$), and if $\xi_\tau < \xi_\sigma$, $y' < y$, and $c_\tau(y',r) \neq ?$ then $c_\tau(y',s) = c_\tau(y',r)$ (as $\xi_\sigma \geqslant \xi_\tau + 1$). Thus, if the guesses are aligned up to y at stages r and t, then letting μ be least with $y < |\pi_{\mu,r}|$, we have $\pi_{\lambda,r} = \pi_{\lambda,t}$ for all $\lambda \leqslant \mu$, and $x_{\lambda,r} = x_{\lambda,t}$ for such $\lambda < \nu_r$. By Claim 3.5.1, we get $\pi_{\lambda,s} = \pi_{\lambda,r}$

and $x_{\lambda,s} = x_{\lambda,r}$ (as well as $p(\lambda,s) = p(\lambda,r)$). The same holds for the c_τ's, using the inductive assumption on $f^N_\tau(y',-)$; if $f^N_\tau(y',r) = f^N_\tau(y',t)$ then we must have $f^N_\tau(y',s) = f^N_\tau(y',r)$. □

The next claim finishes the verification that N is a legal name.

Claim 3.5.3. Let $\sigma \in T_{\Gamma^{++}}$ be internal, $s > 0$ be a stage, and let $r <_{\xi_s} s$. If $f^N_\sigma(y,r)$ is a non-default child of σ, then $f^N_\sigma(y,s) = f^N_\sigma(y,r)$.

Proof. By induction on r, we may assume that $r = s^{-\xi_\sigma}$ is the $<_{\xi_\sigma}$-predecessor of s. Since $f^N_\sigma(y,r)$ is non-default, $y < |\pi_{\nu_r,r}|$, and so $y < |\pi_{\nu_s,s}|$ (Claim 3.5.1). It follows that if y is a witness at stage s, say $y = x_{\mu,s}$, then $y = x_{\mu,r}$, indeed, x_μ is constant between stages r and s.

We check the cases for defining $f^N_\sigma(y,s)$.

Case I: In this case, $\sigma = \langle\rangle$ has a unique non-default child, and so it suffices to show that $f^N_\sigma(y,s)$ is not the default child of σ.

Toward a contradiction, suppose that it is. Then $y = x_{\mu,s}$ for some $\mu < \nu_s$; so x_μ is constant between stages r and s. It follows that μ's step at stage r is not greater than μ's step at stage s; this is impossible.

Case II: If y is not a witness at stage s then $f^N_\sigma(y,s) = f^N_\sigma(y,r)$ (as we are assuming that $f^N_\sigma(y,r)$ is non-default). Suppose that $y = x_{\mu,s}$. Again, $y = x_{\mu,r}$, and so, μ is at step (5) at stage r, and so, is at step (5) at stage s. The instructions are then the same at both stages.

Case III: As in the previous case, if y is not a witness at stage s then $f^N_\sigma(y,s) = f^N_\sigma(y,r)$. Suppose that $y = x_{\mu,s} = x_{\mu,r}$. Then for all $\lambda < \mu$, $x_{\lambda,r} = x_{\lambda,s}$. This implies that for all such λ, within S_{n-1}, $\ell^N(x_{\lambda,r},r)$ does not lie to the right of $\ell^N(x_{\lambda,s},s)$. This implies that the sub-case that holds at stage s is not smaller than the sub-case that holds at stage r.

Since $f^N_\sigma(y,r)$ is non-default, sub-cases (i) and (ii) do not hold at stage r. Hence, they do not hold at stage s either. If case (iv) holds at stage s, then $f^N_\sigma(y,s) = f^N_\sigma(y,r)$ is instructed. If case (iii) holds at stage s, then it holds at stage r as well, and so the outcome is the same.

Case IV: We may assume that σ's guesses are aligned up to y at stage s. Claim 3.5.2 implies that this must be the case at stage r as well, for otherwise (by induction on r), $f^N_\sigma(y,r)$ is the default child. Let μ be least with $y < |\pi_{\mu,r}|$. Since $p(\lambda,r) = p(\lambda,s)$ for $\lambda \leqslant \mu$, and $c_\tau(y',r) = c_\tau(y',s)$ when $\xi_\tau < \xi_\sigma$, all "ingredients" are the same at stages r and s: $\pi_{\lambda,s} = \pi_{\lambda,r}$ for $\lambda \leqslant \mu$ (and so μ is the least with $y < |\pi_{\mu,s}|$); and for all $y' < y$, $\ell^N(y'r)$ and $\ell^N(y',s)$ agree up to the leaves of S_{i+1}.

In particular y is not a witness at stage s if and only if it is not a witness at stage r, and the leaf ζ that is worked towards will be the same at stage s and r. Similarly, if $y = x_{\mu,s}$ then $y = x_{\mu,r}$; if there is some $\lambda < \mu$ such that $\ell^N(x_{\lambda,s},s)$ and $g_\mu(\lambda)$ disagree within S_{i+1}, then the same disagreement will be found at stage r, and so again, we will get the same outcomes at both stages.

Suppose, on the other hand, that $y = x_{\mu,s}$ and that for all $\lambda < \mu$, $\ell^N(x_{\lambda,s},s)$ and $g_\mu(\lambda)$ agree up to the leaves of S_{i+1}. Since $r \leqslant_{\xi_\sigma} s$, know that $\ell^N(x_{\lambda,r},r)$ cannot lie to the right of $\ell^N(x_{\lambda,s},s)$ within S_i. The argument then is the same as in case III. □

We now work toward showing that each requirement is met.

Claim 3.5.4. Let σ be an internal node with $\xi_\sigma > 0$. For all $y \in \mathbb{N}$, for all but finitely many ξ_σ-true stages s, σ's guesses are aligned up to y at stage s.

Proof. This is because p, and c_τ for all τ with $\xi_\tau < \xi_\sigma$ are indeed ξ_σ-decision procedures; on ξ_σ-true stages, they only give us correct answers, and each question is eventually answered. For alignment up to y, there are only finitely many questions to ask.

In greater detail: fix $y \in \mathbb{N}$. Let μ be the least such that $y < |\pi_\mu|$. There is a stage s_0 by which the values of x_λ and π_λ have settled to their final values. So at each $s \geqslant s_0$, $\mu < \nu_s$ and μ is the least $\mu \leqslant \nu_\sigma$ with $y < |\pi_{\mu,s}|$. Further, there is a stage s_1 such that for all internal τ, for all $y' < y$, if $s \geqslant s_2$ is ξ_τ-true then $f^N_\tau(y,s) = f^N_\tau(y)$.

Also, there is a stage s_2 after which for all $\lambda \leqslant \mu$, for every $s \geqslant s_1$ that is 1-true, $p(\lambda, s) = (\pi_\lambda, x_\lambda, k_\lambda)$. And there is a stage s_3 such that for all internal τ, for all $y' < y$, if $s \geqslant s_2$ is $\xi_\tau + 1$-true, then $c_\tau(y', s) = f^N_\tau(y)$. Thus, for sufficiently large s, if s is ξ_σ-true, then σ's guesses are aligned up to y at stage s. □

Claim 3.5.5. Let $y \in \mathbb{N}$, and suppose that y is not a witness (for all $\mu \in L$, $y \neq x_\mu$). Then for all $\mu \in L$, if $y < |\pi_\mu|$, then $\pi_\mu(y) = N(y)$.

Proof. If $\mu < \nu$ then $\pi_\mu \prec \pi_\nu$, so it suffices to show this for the least μ such that $y < |\pi_\mu|$. As above, let s_0 be a stage by which for all $\lambda \leqslant \mu$, the values of x_λ and π_λ have settled to their final values. In particular, for all $s \geqslant s^*$, μ is the least such that $y < |\pi_{\mu,s}|$, and y is not a follower at stage s. Thus, at each stage $s \geqslant s^*$, if $\sigma \prec \ell^N(y,s)$ and ξ_σ, then $f^N_\sigma(y,s)$ is a non-default child of σ.

As we mentioned early in the discussion, since Γ has one of the forms from Figs. 1 and 2, this means that for all $s \geqslant s^*$, there is some $\sigma \prec \ell^N(y,s)$ such that $\xi_\sigma > 0$.

Let $s \geqslant s_0$ be ξ_σ-true for all internal σ, and sufficiently late, so that for all σ with $\xi_\sigma > 0$, σ's guesses are aligned up to $y{+}1$ at stage s. In particular, $\ell^N(y,s) = \ell^N(y)$. If $\sigma \prec \ell^N(y,s)$ and $\xi_\sigma > 0$ then at stage s, σ is instructed to take an outcome toward the leftmost leaf extending σ with label $\pi_\mu(y)$. Hence, the label of $\ell^N(y,s) = \ell^N(y)$ is $\pi_\mu(y)$, as required. □

Claim 3.5.6. Let $\nu \in L$, and suppose that all of ν's guesses are correct up to the leaves of S_0. Then for all $\mu < \nu$, $h_\nu(\mu) = N(x_\mu)$.

Proof. The argument was given, somewhat informally, after Definition 3.5. We show the more formal details of part of it.

Let $\mu < \nu$. If $g_\nu(\mu)$ is a leaf of S_0, i.e., of $T_{\Gamma^{++}}$, then $h_\nu(\mu)$ is the label of this leaf; by assumption, $g_\nu(\mu) = \ell^N(x_\mu)$.

Suppose, then, that $g_\nu(\mu)$ is not a leaf of S_0. Let $\sigma = g_\nu(\mu)$ and let j be as in the definition of $h_\nu(\mu)$. By assumption, $\sigma \prec \ell^N(x_\mu)$.

Let s be a late stage that is ξ_τ-true for all τ; so for all $\lambda \leqslant \nu$, $\ell^N(x_\lambda, s) = \ell^N(x_\lambda)$.

Suppose that $\xi_\sigma > \alpha_j$. Let $\lambda < \mu$ be such that $g_\mu(\lambda)$ and $g_\nu(\lambda)$ disagree within S_j. By assumption, $g_\nu(\lambda) \prec \ell^N(x_\lambda)$. So $g_\mu(\lambda)$ and $\ell^N(x_\lambda)$ disagree within S_j.

At stage s, σ, and all of its extensions below $\ell^N(x_\mu)$, is instructed to take an outcome towards a leaf labelled $\mathbf{0}$; so $N(x_\mu) = 0 = h_\nu(\mu)$, as required.

The rest of the argument is as above, following the discussion after Definition 3.5. □

Fix a requirement $R_{e,i,j}$. Say that it is the k^{th} requirement on the list.

Claim 3.5.7. There is some ν, a leaf of Q_k, whose guesses are correct up to the leaves of S_0.

Proof. This is because in the construction of Q_k, we take into account all possible functions. So the first entry of ν will be the m such that $g_m(\mu) = \ell^N(\mu)$ for all $\mu \in Y_{\langle\rangle}$, and so on. □

Let ν be given by Claim 3.5.7, and let $x = x_\nu$. This number x witnesses that the requirement was met. To see this, we assume, for a contradiction, that $\Phi_j^N = M_e$, and $\Phi_i^{M_e}(x) = N(x)$. We consider the last step that ν reaches (with the final value of x_ν, after last initialised). Let s_0 be a stage at which ν has reached that step. Note that Claim 3.5.5 and Claim 3.5.6 together imply that $\hat{\pi}_\nu \prec N$.

(1) In this case, at each $s \geqslant s_0$, the root $\langle\rangle$ of $T_{\Gamma^{++}}$ is instructed to take the default outcome which is labelled **0**, and so $N(x) = 0$. Then $\hat{\pi}_\nu\hat{\ }0 \prec N$; the assumption for a contradiction then shows that strings such as τ_0 and ρ_0 do exist, so x will be moved to step (2).

(2) In this and later cases, since τ_0 is part of π_μ (where μ is ν's successor in L), Claim 3.5.5 implies that τ_0 aligns with N; so $\hat{\pi}\hat{\ }(N(x))\hat{\ }\tau_0 \prec N$. At step (2), as in step (1), the default outcome is taken at the root, so $N(x) = 0$ in this case too. So $\rho_0 \prec M_e$. If $\rho(y) = 1$ then $M_e(y) = 1$ which means that the final outcome at the root of T_{Γ^+}, when calculating $M_e(y)$, must be the non-default outcome. So eventually, ν will be moved to step (3).

(3) In step (3) and later, the root of $T_{\Gamma^{++}}$ is instructed to take the non-default child, call it σ; at steps (3) and (4), σ is instructed to take the default outcome, labelled **1**, so in these cases, $N(x) = 1$. So $\hat{\pi}_\nu\hat{\ }1\hat{\ }\tau_0 \prec N$. As in case (1), this shows that τ_1 and ρ_1 will be discovered, so x will be moved to step (4).

(4) Let z be the least point of difference between ρ_0 and ρ_1. If step (4) is the last, then $M_e(z) = 0$; this implies that $\rho_0(z) = 0$, so $\rho_1(z) = 1$. However, at this step, $N(x) = 1$. As for τ_0, Claim 3.5.5 shows that N aligns with τ_1, so overall, $\hat{\pi}_\nu\hat{\ }1\hat{\ }\tau_1 \prec N$, whence $\rho_1 \prec M_e$; a contradiction.

(5) At this step we successfully diagonalise. In this case, the instructions at the non-default child of the root, and the fact that for every $\sigma \prec \ell^N(x)$, at every true stage, sub-case (iii) applies (in either case III or case IV of the construction, depending on ξ_σ), ensure that $\ell^N(x) \neq \ell^{M_e}(z)$ when $\rho_0(z) = 0$, and $\ell^N(x) = \ell^{M_e}(z)$ otherwise. As discussed above, both contradict the assumption.

This completes the proof of Theorem 3.1.

4. Strengthening the result

Our real goal is Theorem 1.1, which is stronger than Theorem 3.1. However, only minor modifications to the construction are required. Indeed, we only need to alter the basic strategy; all the details for implementing the strategies are the same.

Let Γ be a class with $\delta(\Gamma) > 0$ not the successor of a limit. If $\delta(\Gamma)$ is finite, then the result is known (there are n-c.e. sets not Turing equivalent to any set which is n-c.e. and co-n-c.e.) Hence, again, we assume that $\delta(\Gamma) \geqslant \omega$. We build a set in Γ not Turing equivalent to any in $\Delta(\Gamma)$.

There are two cases.

Case (a): $\delta(\Gamma)$ is the successor of a successor. This is very similar to the previous construction. Reverting to the notation above, we replace Γ by its double-predecessor; i.e., we build a set in Γ^{++} not Turing equivalent to any in $\Delta(\Gamma^{++})$. Now, (M_e) list Γ^{++}-names, and the requirements are:

$R_{d,e,i,j}$**:** If M_d is the complement of M_e, then it is not the case that $\Phi_i^N = M_e$ and $\Phi_j^{M_e} = N$.

The basic strategy for meeting such a requirement is the following. Let σ be the non-default child of the root of $T_{\Gamma^{++}}$.

(1) Choose a witness x; direct $N(x)$ to take the default outcome at the root. Search for strings τ_0 and ρ_0 with $\tau_0(x) = 0$, $\Phi_i^{\rho_0}(x) = 0$, and $\Phi_j^{\tau_0} \succcurlyeq \rho_0$. When found, restrain N to agree with τ_0.
(2) For every y with $\rho_0(y) = 1$, wait until $M_e(y)$ has moved off the default outcome at the root (of $T_{\Gamma^{++}}$ this time). For every y with $\rho_0(y) = 0$, wait until $M_d(y)$ has moved off the default outcome at the root.
(3) Direct $N(x)$ to take the non-default outcome at the root, and the default outcome above that. Search for strings τ_1 and ρ_1 with $\tau_1(x) = 1$, $\Phi_i^{\rho_1}(x) = 1$, $\Phi_j^{\tau_1} \succcurlyeq \rho_1$, and $\tau_1(y) = \tau_0(y)$ for every $y < |\tau_0|$ other than x. When found, restrain N to agree with τ_1. Fix z least such that $\rho_0(z) \neq \rho_1(z)$.
(4) If $\rho_1(z) = 0$, wait until $M_e(z)$ has moved off the default outcome of σ. If $\rho_1(z) = 1$, wait until until $M_d(z)$ has moved off the default outcome of σ.
(5)
   - If $\rho_1(z) = 0$, then let $N(x)$ follow $M_e(z)$, i.e., ensure that $\ell^N(x) = \ell^{M_e}(z)$, by copying the choices that M_e makes for z.
   - If $\rho_1(z) = 1$, then let $N(x)$ follow $M_d(z)$, i.e., ensure that $\ell^N(x) = \ell^{M_d}(z)$, by copying the choices that M_e makes for z.

Let us explain why this works, again by considering the last step that x reaches. Suppose, for a contradiction, that the requirement is not met: M_e is the complement of M_d, $\Phi_i^N = M_e$ and $\Phi_j^{M_e} = N$.

- If x never leaves step (1), then $N(x) = 0$; but then, strings τ_0 and ρ_0 will be found.
- If x never leaves step (2), then $N(x) = 0$, so $\tau_0 \prec N$, so $\rho_0 \prec M_e$. But then, if $\rho_0(y) = 1$, then $M_e(y) = 1$, so on y, M_e must move off the default outcome at the root. Since M_d is the complement of M_e, if $\rho_0(y) = 0$ then $M_d(y) = 1$, so on y, M_d must move off the default at the root.
- If x never leaves step (3), then $N(x) = 1$; so strings τ_1 and ρ_1 will be found.
- Suppose that x never leaves step (4). At this step we still set $N(x) = 1$, so $\rho_1 \prec M_e$.

  If $\rho_1(z) = 0$, then $\rho_0(z) = 1$, which means that on z, M_e already moved off the default at the root after step (2). The default child of σ (recall, this is the non-default child of the root of $T_{\Gamma^{++}}$) is labelled 1, whereas $M_e(z) = \rho_1(z) = 0$, so $\ell^{M_e}(z)$ cannot be the default child of σ; so on z, M_e must move off the default child of σ.

  Similarly, if $\rho_1(z) = 1$, then $M_e(z) = 1$, so $M_d(z) = 0$; but also, $\rho_0(z) = 0$, so at step (2), on z, M_d already moved off the default at the root and pointed at σ. So $\ell^{M_d}(z)$ must extend a non-default child of σ.
- Suppose that x reaches step (5). If $N(x) = 0$ then $\tau_0 \prec N$, and so $\rho_0 \prec M_e$, so $M_e(z) = \rho_0(z)$; if $N(x) = 1$ then $\tau_1 \prec N$, so $M_e(z) = \rho_1(z)$. In other words, $M_e(z) = \rho_{N(x)}(z)$.

If $\rho_1(z) = 0$ then we ensure that $N(x) = M_e(z)$, so $N(x) = \rho_{N(x)}(z)$. But in this case, $\rho_i(z) = 1 - i$ (for $i = 0, 1$), a contradiction.

If $\rho_1(z) = 1$ then we ensure that $N(x) = M_d(z) = 1 - M_e(z)$, so $N(x) = 1 - \rho_{N(x)}(z)$. But in this case, $\rho_i(z) = i$, a contradiction.

Case (b): $\delta(\Gamma)$ is a limit. In this case, we need the characterisation of classes at limit levels from [GQT, Prop. 5.19]. Starting with a very admissible description of the class, and applying the transformation that gives an acceptable description in the proof of Proposition 2.2, we obtain a description Γ, hereditarily as in Figs. 1 and 2, which has the extra property that every leaf extends some internal σ with $\xi_\sigma > 0$. In particular, the predecessor σ of the leftmost leaf has $\xi_\sigma > 0$; and we note that we get to σ by repeatedly taking the default outcome.

This means that σ knows what step a witness x will end up in, and so, while searches continue for strings, we can safely take the default outcomes up to σ, and then let it choose a **0** or a **1** outcome, based on their superior knowledge.

The basic strategy is as follows.

(1) Choose a witness x. Wait until we find strings τ_0 and ρ_0 with the usual properties: $\tau_0(x) = 0$, $\rho_0 \prec \Phi_i^{\tau_0}$, $\Phi_j^{\rho_0}(x) = 0$. While waiting, direct $\ell^N(x)$ to pass through σ.

(2) When such τ_0 and ρ_0 are found, search for strings τ_1 and ρ_1 as above. While waiting, direct $\ell^N(x)$ to pass through σ.

(3) When ρ_i and τ_i are found, let $z = |\rho_0 \wedge \rho_1|$ as before; if $\rho_1(z) = 0$ let $N(x) = M_e(z)$, otherwise let $N(x) = M_d(z)$, by copying all choices the given names make on z.

During steps (1) and (2), the node σ waits until $\varnothing'$ tells it what step x will reach. If σ finds that x will never leave step (1), then it chooses some leaf extending it labelled **0**. If σ finds that x will eventually move to step (2), but not to step (3), it will choose some leaf above labelled **1**. If it finds that step (3) will be reached, it just chooses the default.

The argument for the previous case shows that if we reach step (3), the requirement will be met.

References

[Ash86] C. J. Ash. Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees. *Trans. Amer. Math. Soc.*, 298(2):497–514, 1986.

[Coo71] Barry S. Cooper. *Degrees of unsolvability.* PhD thesis, Leicester University, 1971.

[DGHTT] Adam R. Day, Noam Greenberg, Matthew Harrison-Trainor, and Daniel Turetsky. An effective classification of Borel Wadge classes. Submitted.

[DSR07] Gabriel Debs and Jean Saint Raymond. Borel liftings of Borel sets: some decidable and undecidable statements. *Mem. Amer. Math. Soc.*, 187(876):viii+118, 2007.

[GQT] Noam Greenberg, Renrui Qi, and Daniel Turetsky. Borel Wadge classes and Selivanov's fine hierarchy I: extending to the hyperarithmetic. Submitted.

[GT] Noam Greenberg and Daniel Turetsky. Effective separation and reduction properties of Borel Wadge classes. Submitted.

[GT22] Noam Greenberg and Daniel Turetsky. Completeness of the hyperarithmetic isomorphism relation. *Advances in Mathematics*, 403:56 pages, 2022.

[JS84] Carl G. Jockusch, Jr. and Richard A. Shore. Pseudojump operators. II. Transfinite iterations, hierarchies and minimal covers. *J. Symbolic Logic*, 49(4):1205–1236, 1984.

[Kni90a] J. F. Knight. A metatheorem for constructions by finitely many workers. *J. Symbolic Logic*, 55(2):787–804, 1990.

[Kni90b] Julia F. Knight. Constructions by transfinitely many workers. *Ann. Pure Appl. Logic*, 48(3):237–259, 1990.

[Lou83] Alain Louveau. Some results in the Wadge hierarchy of Borel sets. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 28–55. Springer, Berlin, 1983.

[LSR88] Alain Louveau and Jean Saint Raymond. The strength of Borel Wadge determinacy. In *Cabal Seminar 81–85*, volume 1333 of *Lecture Notes in Math.*, pages 1–30. Springer, Berlin, 1988.

[Mon14] Antonio Montalbán. Priority arguments via true stages. *J. Symb. Log.*, 79(4):1315–1335, 2014.

[MSY20] Alexander G. Melnikov, Victor L. Selivanov, and Mars M. Yamaleev. Turing reducibility in the fine hierarchy. *Ann. Pure Appl. Logic*, 171(7):102766, 17, 2020.

[Sel85] V. L. Selivanov. The Ershov hierarchy. *Sibirsk. Mat. Zh.*, 26(1):134–149, 1985.

[Sel08] Victor L. Selivanov. Fine hierarchies and m-reducibilities in theoretical computer science. *Theoret. Comput. Sci.*, 405(1-2):116–163, 2008.

[SY18a] V. L. Selivanov and M. M. Yamaleev. On Turing degrees in refinements of the arithmetic hierarchy. *Algebra Logika*, 57(3):338–361, 2018.

[SY18b] Victor L. Selivanov and Mars M. Yamaleev. Extending Cooper's theorem to Δ^0_3 Turing degrees. *Computability*, 7(2-3):289–300, 2018.