BOREL WADGE CLASSES AND SELIVANOV'S FINE HIERARCHY I: EXTENDING TO THE HYPERARITHMETIC

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ABSTRACT. We show how to extend Selivanov's fine hierarchy using descriptions of Borel Wadge classes. We give a game characterisation of containment between classes. We show that every class in the extended fine hierarchy has an admissible description, and use this to calculate heights in the hierarchy.

1. INTRODUCTION

Descriptive set theory and computability theory study hierarchies of classes, that at first appear merely analogues, but in fact can be considered as manifestations of the same basic concept in different settings. The most prominent example is the Borel hierarchy of $\boldsymbol{\Sigma}^0_\alpha$ subsets of Polish spaces (for all countable ordinals α), and the hyperarithmetic hierarchy of Σ^0_α subsets of $\mathbb{N}$ (for all computable ordinals α). The underlying connection here is that the lightface (effective) classes Σ^0_α can be defined not only for $\mathbb{N}$, but also for subsets of any computably presented Polish space. They can be relativised to oracles z (and z-computable ordinals α). We then get $\boldsymbol{\Sigma}^0_\alpha = \bigcup \left\{\Sigma^0_\alpha(z) : \alpha \text{ is } z\text{-computable}\right\}$. This enables us to apply effective methods to study Borel sets. Among prominent results in the effective theory, or using effective methods, are Louveau's separation theorem [Lou80], the Harrington-Kechris-Louveau dichotomy [HKL90], and the G_0-dichotomy [KST99].

A hierarchy finer than the Borel / hyperarithmetic one is defined using differences of sets in the Borel hierarchy. In set theory this is known as the Hausdorff, or Lavrentiev, difference hierarchy $D_\eta(\boldsymbol{\Sigma}^0_\alpha)$. In computability, this is the Ershov hierarchy [Es68, Ers68, Ers70]; see also [Put65]. The prominent result regarding the difference hierarchy in set theory is the Hausdorff-Kuratowski theorem, that $\boldsymbol{\Delta}^0_{\alpha+1} = \bigcup_{\eta<\omega_1} D_\eta(\boldsymbol{\Sigma}^0_\alpha)$. The analogous result, due to Ershov, is that every Δ^0_2 set is Σ^{-1}_η for some notation η for a computable ordinal (Σ^{-1}_η is Ershov's notation for $D_\eta(\Sigma^0_1)$). The effective version is more "fragile", as the class $D_\eta(\Sigma^0_1)$ depends on the notation (computable presentation) of η, rather than just its order-type.

The finest hierarchy of them all is due to Wadge [Wad84]. Motivated by "many-one" reducibility in computability, Wadge defined a subset A of Baire space to be reducible to B if A is a continuous pre-image of B. A *Wadge class* is a collection of sets closed under taking continuous pre-images. The structure of Borel Wadge classes under containment is surprisingly regular; it is semi-well-ordered, with alternating self-dual and non-self-dual classes. Wadge was able to calculate the order-type of the non-self-dual classes (after identifying such a class with its dual). He also showed that all such classes are the result of applying a Borel ω-ary Boolean operation to the class of open sets.

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In computability, the analogue of the Wadge hierarchy was defined by Selivanov [Sel83, Sel89], which he called the *fine hierarchy*. To avoid the problem of dependence on ordinal notation, Selivanov looked at generalisations of the finite difference hierarchy. The hierarchy was first defined using generalised jump operations in [Sel83]. This resembles Wadge's use of Kuratowski $(\mu, 0)$-homeomorphisms, which we now know can be thought of as relativised iterated Turing jumps. In [Sel89], Selivanov gave an inductive definition of the hierarchy (of length ε_0), using (essentially) jumps, and some fixed Boolean operations, most notably the BiSep operation (two sided separated unions). In [Sel95], an equivalent definition is given using typed Boolean terms. Selivanov also noted a close relationship between the fine hierarchy and the Wagner hierarchy in automata theory [Wag79, Sel02]. For a survey, see [Sel08]. Selivanov also generalised his hierarchy beyond subsets of spaces, to k-partitions and functions to BQOs [Sel11, Sel20], and to general Polish and quasi-Polish spaces [Sel21].

In this paper, we show how to naturally extend Selivanov's hierarchy beyond the arithmetic, all the way up the hyperarithmetic sets. To do this, we give a new way to define the hierarchy, using descriptions of Borel Wadge classes that were introduced in [DGHTTa] and systematically investigated in [GT]. These are descriptions that generalise ones given by Louveau [Lou83] and by Louveau and Saint Raymond [LSR88b]. The levels of the extended hierarchy will be the ones that have finite descriptions.

We then give a game characterisation of containment between classes, which is a modification of one given for Borel Wadge classes in [GT]. Using this characterisation we can explain why the fine hierarchy satisfies Wadge's semi-linear ordering principle. Essentially, this is due to the determinacy of finite games. We also use this game characterisation to explain why the hierarchy is well-founded.

To calculate the height of a class in the fine hierarchy, we introduce the notion of an *admissible* description, and show that every class in the hierarchy has such a description. Admissible descriptions directly give us information about the class described, for example, if it is at a successor or limit level.

The class descriptions we introduce utilise Montalbán's method of *true stages* (a non-effective version was introduced independently in [DSR07]). This allows us to computably approximate sets at all levels in the extended fine hierarchy: our descriptions are inherently *dynamic*. This is useful when performing priority arguments. In the sequel to this paper [GQT], we do exactly that, to give a complete answer to the question of which levels of the hierarchy contain new Turing degrees.

2. Preliminaries: described classes

To define and analyse the classes of the extended fine hierarchy, we use the class descriptions introduced in [DGHTTa, GT]. These are used to define classes of subsets of Baire space. We can, however, identify the natural numbers as a subspace of Baire space (identify n with the infinite sequence n^∞). To simplify notation we use true stage relations on $\omega + 1$ rather than $\omega^{\leqslant\omega}$, that were developed in [GT22], and adapt the class descriptions to this setting.

2.1. True stage relations.

True stage relations. The true stage relations allow us to computably approximate transfinite iterations of the Turing jump.

A *concrete computable ordinal* is a computable well-ordering of a computable subset of $\mathbb{N}$, in which the successor relation and collection of limit points are both computable. For concrete computable ordinals α and β we write $\alpha < \beta$ if α is an initial segment of β. For every concrete computable ordinal α we obtain a partial ordering $\leqslant_\alpha$ on $\omega + 1$ with a variety of pleasing properties. In particular:

(i) $\leqslant_0$ is the usual ordering $\leqslant$ on $\omega + 1$.
(ii) If $\alpha < \beta$ then $s \leqslant_\beta t$ implies $s \leqslant_\alpha t$.
(iii) $(\omega + 1, \leqslant_\alpha)$ is a tree, with root 0.
(iv) $\{s : s <_\alpha \omega\}$ is the unique infinite path of $(\omega, \leqslant_\alpha)$.
(v) The restriction of $\leqslant_\alpha$ to ω is computable, uniformly in α.
(vi) A set $A \subseteq \mathbb{N}$ is $\Sigma^0_{1+\alpha}$ if and only if its characteristic function 1_A has a computable α-*enumeration*: a computable function $f\colon \mathbb{N} \times \omega \to \{0, 1\}$ satisfying, for all $x \in \mathbb{N}$:
  - If $s \leqslant_\alpha t$ and $f(x, s) = 1$ then $f(x, t) = 1$;
  - $1_A(x) = \lim\{f(x, s) : s <_\alpha \omega\}$.

In (vi), and below, we write $\mathbb{N} \times \omega$ (rather than $\mathbb{N}^2$) to indicate that the first input x is an element of the "space" $\mathbb{N}$, and the second represents a stage of the approximation.

We will use the following corollary of property (vi).

Proposition 2.1. *Let α be a concrete computable ordinal. Let $h\colon \mathbb{N} \to \mathbb{N}$ be $\Delta^0_{1+\alpha}$-measurable (for all n ,$h^{-1}[\{n\}]$ is $\Delta^0_{1+\alpha}$, uniformly in n). Then h admit an "α-decision procedure": a computable function $f\colon \mathbb{N} \times \omega \to \mathbb{N} \cup \{?\}$ satisfying:*

- *If $s \leqslant_\alpha t$ and $f(x, s) \in \mathbb{N}$ then $f(x, t) = f(x, s)$;*
- *For all x, for all but finitely many $s <_\alpha \omega$, $f(x, s) = h(x)$.*

That is, for a while, $f(x, s)$ could be ?, indicating that we are not yet sure what $h(x)$ is;; but once some value is guessed for $h(x)$, we never change our mind again. Along the α-true stages (the stages $s <_\alpha \omega$), we eventually guess the correct value.

Proof. For each n, let $A_n = h^{-1}[\{n\}]$. Let g_n be uniformly computable α-enumerations of A_n. Let $x \in \mathbb{N}$. We define $f(x, s)$ by recursion on $s < \omega$. If there is some $r <_\alpha s$ such that $f(x, r) \in \mathbb{N}$, then we let $f(x, s) = f(x, r)$. Otherwise, if there is some $n \leqslant s$ such that $g_n(x, s) = 1$ then we let $f(x, s) = n$ for the least such n. If there is no such n then we let $f(x, s) =?$. ☐

We remark that (vi), and so Proposition 2.1, are uniform: given a $\Sigma^0_{1+\alpha}$ index of A, we can effectively compute an α-enumeration of A.

Definition 2.2. A *computable α-approximation* is a function $f\colon \mathbb{N} \times \omega \to \mathbb{N}$ such that for all x, $\lim\{f(x, s) : s <_\alpha \omega\}$ exists. The function approximated by f is the one taking x to that stable value.

The following is a "higher limit lemma". It is proved in [DGHTTb, Proposition 3.6].

Proposition 2.3. *A function $g\colon \mathbb{N} \to \mathbb{N}$ has a computable α-approximation if and only if it is $\Delta^0_{1+\alpha+1}$-measurable, meaning that for all n, $g^{-1}\{n\}$ is $\Delta^0_{1+\alpha+1}$, uniformly in n, equivalently $\Sigma^0_{1+\alpha+1}$, uniformly in n.*

We will require a particular type of α-approximations, that generalises the notion of a α-enumeration. Let $n \geqslant 1$. An (α, n)-*enumeration* is a α-approximation f such that for all x,

- $f(x, 0) = 0$; and

- For all s,
$$\#\left\{t \leqslant_\alpha s : f(x,t) \neq f(x,t^{-\alpha})\right\} \leqslant n,$$
where $t^{-\alpha}$ is t's predecessor in the tree $(\omega, <_\alpha)$.

An $(\alpha, 1)$-enumeration is simply an α-enumeration (with the added requirement that $f(x, 0) = 0$, which is an easy modification).

Proposition 2.4. *A set has an (α, n)-enumeration if and only if it is $D_n(\Sigma^0_{1+\alpha})$.*

Proof. Suppose that C has an (α, n)-enumeration f. For $k \leqslant n$ let A_k be the set of $x \in \mathbb{N}$ such that there are $s_0 <_\alpha s_1 <_\alpha s_2 <_\alpha \cdots <_\alpha s_k <_\alpha \omega$ such that $f(x, s_i) \neq f(x, s_{i+1})$ for all $i < k$. Then each A_k is $\Sigma^0_{1+\alpha}$, $\mathbb{N} = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$, and $C = (A_1 \backslash A_2) \cup (A_3 \backslash A_4) \cup \cdots$, showing that C is $D_n(\Sigma^0_{1+\alpha})$. In the other direction, let $C = (A_1 \backslash A_2) \cup (A_3 \backslash A_4) \cup \cdots$, with $A_i \in \Sigma^0_{1+\alpha}$ and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$. For $i = 1, \ldots, n$, let g_i be a α-enumeration of A_i. For simplicity let $g_0(x, s) = 1$ for all x and s. We may also assume that $g_i(x, s) = 1$ implies $g_{i-1}(x, s) = 1$ when $i > 0$. We let $f(x, s) = 0$ if the greatest i such that $g_i(x, s) = 1$ is even, $f(x, s) = 1$ otherwise. Then f is an (α, n)-enumeration of C. □

2.2. **Class descriptions.** A (computable) *class description* Γ consists of a well-founded computable tree $T_\Gamma \subset \omega^{<\omega}$, computably labelled as follows:

(i) If $\sigma \in T_\Gamma$ is not a leaf of T_Γ, then σ is labelled by a pair $(\xi_\sigma, \eta_\sigma)$ where ξ_σ and η_σ are concrete computable ordinals, and $\eta_\sigma \geqslant 1$;
(ii) If $\sigma \in T_\Gamma$ is a leaf of T_Γ then σ is labelled by a value $\Gamma(\sigma) \in \{0, 1\}$.

We use the term *internal node* of a tree T to denote a node of T that is not a leaf of T. We require that if $\sigma \leqslant \tau$ are both internal nodes of T_Γ then $\xi_\sigma \leqslant \xi_\tau$. We let $o(\Gamma) = \xi_{\langle\rangle}$ be the ξ-label of the root $\langle\rangle$ of T_Γ, unless T_Γ consists only of the root, in which case we set $o(\Gamma) = \omega_1$ (where ω_1 is treated as a formal symbol). We similarly set $\xi_\sigma = \omega_1$ for a leaf σ of T_Γ. We let $\eta^\Gamma = \eta_{\langle\rangle}$.

The idea is that at an internal node σ we need to choose one of its children. The leftmost child is considered a *default*, the one we choose initially. We can change our mind about the child we are choosing, but the "number" of mind-changes is bounded by η_σ: every time we change our mind, we need to decrease the ordinal. The ordinal ξ_σ tells us at what *level* we conduct this approximation. Informally, this means that the approximation is computable from $\varnothing^{(\xi_\sigma)}$. Technically, we use the true stage relations. Our intention is formalised using the notion of a Γ-name.

If Γ is a computable class description, then a (computable) Γ*-name* N consists of a choice, for each internal $\sigma \in T_\Gamma$, of a pair of functions (f_σ, β_σ), uniformly computable given σ, both defined on $\mathbb{N} \times \omega$, satisfying the following:

(i) For each $(x, s) \in \mathbb{N} \times \omega$, $f_\sigma(x, s)$ is a child of σ on T_Γ, and $\beta_\sigma(x, s) \leqslant \eta_\sigma$;
(ii) For each $x \in \mathbb{N}$ and $s, t \in \omega$, if $s \leqslant_{\xi_\sigma} t$ then:
  - $\beta_\sigma(x, t) \leqslant \beta_\sigma(x, s)$; and
  - if $f_\sigma(x, t) \neq f_\sigma(x, s)$ then $\beta_\sigma(x, t) < \beta_\sigma(x, s)$.
(iii) For each $(x, s) \in \mathbb{N} \times \omega$, if $\beta_\sigma(x, s) = \eta_\sigma$ then $f_\sigma(x, s)$ is the leftmost child of σ on T_Γ (the default outcome of σ).

Let N be a Γ-name. The definition ensures that for all internal $\sigma \in T_\Gamma$, for all $x \in \mathbb{N}$, the limit $\lim\{f_\sigma(x, s) : s <_{\xi_\sigma} \omega\}$ exists, and we denote it by $f_\sigma(x)$. Since T_Γ is well-founded, for each $x \in \mathbb{N}$, the sequence $\sigma_0(x) = \langle\rangle$, $\sigma_1(x) = f_{\sigma_0(x)}(x)$, $\sigma_2(x) = f_{\sigma_1(x)}(x)$, ..., terminates in a leaf of T_Γ that we denote by $\ell^N(x)$. We then let $N(x)$ be the value $\Gamma(\ell^N(x))$ assigned by Γ to this leaf. The subset of $\mathbb{N}$ named by N is the set whose characteristic function is $x \mapsto N(x)$.

Definition 2.5. Let Γ be a computable class description. The *class described by* Γ is the collection of all subsets of $\mathbb{N}$ that are named by computable Γ-names.

We will abuse notation and use Γ to denote both the description and the class that it described, even though a given class may have many different descriptions. A collection of subsets of $\mathbb{N}$ is a *described class* if it is the class described by some class description.

The simplest examples are the class descriptions Γ consisting only of the root $\langle\rangle$, labelled either 0 or 1. The former gives the class $\{\varnothing\}$, and the latter the class $\{\mathbb{N}\}$. The next simplest example is a tree consisting of the root and two children. The root is labelled with some ξ and $\eta = 1$, the leftmost child is labelled 0 and the other one 1 (see Fig. 1). The resulting class is $\Sigma^0_{1+\xi}$. Replacing $\eta = 1$ with any η, the resulting class is $D_\eta(\Sigma^0_{1+\xi})$ of iterated differences of $\Sigma^0_{1+\xi}$ sets.

FIGURE 1. The simplest descriptions of $\Sigma^0_{1+\xi}$ and $\Pi^0_{1+\xi}$.

The dual description and class. The *dual* $\check{\Gamma}$ of a class description Γ is the class description obtained from Γ by exchanging all labels at the leaves. The resulting described class is the collection of complements of elements of the class described by Γ.

We let $\Delta(\Gamma) = \Gamma \cap \check{\Gamma}$ be the class of sets A for which both A and its complement are in Γ.

Subclasses. Let Γ be a class description and let $\sigma \in T_\Gamma$. The subclass Γ_σ is the class obtained by restricting to the tree above σ: $T_{\Gamma_\sigma} = \{\tau : \sigma\hat{\ }\tau \in T_\Gamma\}$ and the label of τ on Γ_σ is the label of $\sigma\hat{\ }\tau$ on Γ. Observe that $o(\Gamma_\sigma) = \xi^\Gamma_\sigma$.

We can think of Γ as the class constructed from the classes Γ_n (for $n \in T_\Gamma$) via an $(o(\Gamma), \eta^\Gamma)$-approximation method.

For a Γ-name N and $\sigma \in T_\Gamma$ we also let N_σ be the Γ_σ-name defined by $f^{N_\sigma}_\tau = f^N_{\sigma\hat{\ }\tau}$, and similarly for β_τ.

2.3. **Described classes are principal pointclasses.** Every described class is a lightface (effective) pointclass. The following is essentially proved in [DGHTTa], but in the setting of $\mathbb{N}$ is particularly simple.

Proposition 2.6. *Let Γ be a computable class description. For all $A, B \subseteq \mathbb{N}$, if $A \leqslant_m B$ and $B \in \Gamma$ then $A \in \Gamma$.*

Proof. Let g be a computable function such that $g^{-1}[B] = A$, and let N be a Γ-name of B. A Γ-name M of A is defined by letting, for every internal $\sigma \in T_\Gamma$, $f^M_\sigma(x,s) = f^N_\sigma(g(x),s)$ and $\beta^M_\sigma(x,s) = \beta^N_\sigma(g(x),s)$. □

Similarly:

Proposition 2.7. *Let Γ be a computable class description. For all $A, B \in \Gamma$, $A \oplus B \in \Gamma$. Indeed, the following are equivalent for a sequence of sets (A_n):*

(1) *There are uniformly computable Γ-names N_n with N_n a name for A_n;*
(2) $\bigoplus_n A_n \in \Gamma$.

We say that (A_n) are uniformly in Γ.

Note that it is *not* the case that a described class Γ is always closed under taking unions or intersections, the simplest counter-example being $D_2(\Sigma^0_1)$.

In [DGHTTa], it is shown that every described boldface class has a universal set. The same construction holds in the discrete setting.

Proposition 2.8. *Let Γ be a described class. There is an* acceptable listing of the sets in Γ*: a sequence $A_0, A_1, \dots$ of sets, uniformly in Γ, such that if $B_0, B_1, \dots$ is any sequence of sets uniformly in Γ, then there is a computable function g such that for all n, $B_n = A_{g(n)}$.*

As a result, the effective pointclass Γ is principal: there is a set $B \in \Gamma$ such that $\Gamma = \{A : A \leqslant_m B\}$.

Proof. As mentioned, the proof of [DGHTTa, Lemma 3.13] applies, using Lemma 3.12 of that paper. Namely, we can uniformly, given $\sigma \in T_\Gamma$ and a partial computable approximation $(g_\sigma, \alpha_\sigma)$, extend that approximation to a total computable approximation (f_σ, β_σ) as required, which has the same limit as the given partial approximation, if the latter happens to be total. This is enabled by the fact that there is a default outcome: as long as we do not see any value given for $g_\sigma(x, 0)$, we choose the default outcome (with ordinal value η_σ); as $g_\sigma(x, s)$ reveals more values, we copy them, with delay. ☐

Corollary 2.9. *For any computable class description Γ, the class Γ is non-self-dual.*

Proof. If (A_n) is an acceptable listing of the sets in Γ, then $A = \bigoplus_n A_n$ is universal for Γ, and the diagnoal argument shows that $A \notin \check{\Gamma}$. ☐

2.4. **Definition by cases.** The analogue of the following proposition is proved in [DGHTTa]:

Proposition 2.10. *Let Γ be a computable class description. Suppose that:*

- *(X_n) is a partition of $\mathbb{N}$ into uniformly $\Delta^0_{1+o(\Gamma)}$ sets;*
- *(A_n) is a sequence of subsets of $\mathbb{N}$, uniformly in Γ.*

Define $A \subseteq \mathbb{N}$ by letting $A \upharpoonright X_n = A_n \upharpoonright X_n$. Then $A \in \Gamma$.

Proof. The proof of [GT, Proposition 2.4] holds. Informally, we say that every internal node "eventually knows" which set X_n a given input x is in. More formally, let g be an $o(\Gamma)$-decision procedure for the the function taking $x \in \mathbb{N}$ to the n such that $x \in X_n$ (Proposition 2.1). Let (N_n) be Γ-names for (A_n), uniformly computable. We define a Γ-name N for A by taking the "disjoint union" of (N_n) using g. Namely, for internal $\sigma \in T_\Gamma$, x and s:

- If $g(x, s) = n \in \mathbb{N}$ then we let $f^N_\sigma(x, s) = f^{N_n}_\sigma(x, s)$ and $\beta^N_\sigma(x, s) = \beta^{N_n}_\sigma(x, s)$.
- If $g(x, s) =?$ then we let $f^N_\sigma(x, s)$ be the default child of σ and $\beta^N_\sigma = \eta_\sigma$.

Since $o(\Gamma) \leqslant \xi_\sigma$, the nestedness of the true stage relations (property (ii)) implies that when $s \leqslant_{\xi_\sigma} t$ and $g(x,s) = n$, we have $g(x,t) = n$, so N is indeed a Γ-name; and similarly, that for all $x \in X_n$, $g(x,s) = n$ for all but finitely many $s \prec_{\xi_\sigma} \omega$. □

Corollary 2.11. *Let Γ be a computable class description, and suppose that $\varnothing, \mathbb{N} \in \Gamma$. Then $\Delta^0_{1+o(\Gamma)} \subseteq \Gamma$, and furthermore, Γ is closed under taking unions and intersections with $\Delta^0_{1+o(\Gamma)}$ sets.*

2.5. **Ordinal invariance.** To define the true stage relations, and in general, to use ordinals in computability, we need concrete ordinals. If α and α' are two concrete computable ordinals of the same order-type, then they may fail to be computably isomorphic. Nonetheless, we can computably translate between the true-stage relations involved:

Proposition 2.12. *Suppose that α and α' are isomorphic concrete computable ordinals. There is a computable function $h\colon \omega \to \omega$ satisfying:*

(i) For all $s,t < \omega$, if $s \leqslant_\alpha t$ then $h(s) \leqslant_{\alpha'} h(t)$; and
(ii) $\{h(s) : s <_\alpha \omega\} = \{t : t <_{\alpha'} \omega\}$.

Such a function h can be calculated uniformly, given α and α'.

The reason is that even if α and α' are not computably isomorphic, the iterated jumps $\varnothing^{(\alpha)}$ and $\varnothing^{(\alpha')}$ are Turing equivalent, uniformly. As a result, the $\Sigma^0_{1+\alpha}$ sets are the same as the $\Sigma^0_{1+\alpha'}$ sets, again uniformly. For more details see [DGHTTa, Proposition 2.20].

The uniformity shows that the choice of concrete copies of the ordinals ξ_σ does not affect the class defined by a description:

Proposition 2.13. *Let Γ and Γ' be two class descriptions. Suppose that:*

(i) $T_\Gamma = T_{\Gamma'}$;
(ii) for every leaf σ of T_Γ, $\Gamma(\sigma) = \Gamma'(\sigma)$; and
(iii) for every internal σ, $\eta^\Gamma_\sigma = \eta^{\Gamma'}_\sigma$ and $\mathrm{otp}(\xi^\Gamma_\sigma) = \mathrm{otp}(\xi^{\Gamma'}_\sigma)$.

Then Γ and Γ' define the same class.

Note that we cannot relax condition (iii) to $\mathrm{otp}(\eta^\Gamma_\sigma) = \mathrm{otp}(\eta^{\Gamma'}_\sigma)$. Here the presentation matters, as the names use the particular copies of the η-ordinals, rather than the associated true stage relations. To translate names effectively, we would need uniformly computable isomorphisms between η^Γ_σ and $\eta^{\Gamma'}_\sigma$.

2.6. Σ **and** Π **classes.** To differentiate between classes within a dual pair $\{\Gamma, \check{\Gamma}\}$, we use the following definition:

Definition 2.14. A computable class description Γ has Σ-*type* if the label of the leftmost leaf of T_Γ is 0; otherwise it has Π-*type*.

The leftmost leaf of T_Γ is the "ultimate default" (the default outcome of the default outcome of the default outcome...) The notation generalises that for the classes $\Sigma^0_{1+\xi}$ and $\Pi^0_{1+\xi}$ (Fig. 1).

In [GT], it is shown that restricted to "efficient" descriptions (to be discussed later), all descriptions of a particular class have the same type, thus we can talk about a described class having type Π or type Σ. It is also shown that a described class has the separation property if and only if it is a Π-class. A more complicated condition characterises the classes with the reduction property: those are the classes that have *hereditarily* Σ-*type* descriptions (for all internal $\sigma \in T_\Gamma$, Γ_σ has Σ-type).

2.7. **Finite descriptions.**

Definition 2.15. A class description Γ is *finite* if T_Γ is a finite tree, and for all internal $\sigma \in T_\Gamma$, $\eta_\sigma < \omega$.

Note that we do not require that the ordinals ξ_σ be finite. A class is *finitely described* if it has a finite description.

Theorem 2.16. *The finitely described classes form a semi-well-ordered hierarchy that extends the Selivanov fine hierarchy. The height of the hierarchy is* ω_1^{ck}.

In fact, the classes in the fine hierarchy are precisely those classes that have a finite description in which every ξ_σ-ordinal is finite as well. We let the *extended fine hierarchy* denote the collection of all finitely described classes, partially ordered by inclusion.

Remark 2.17. If $\eta_\sigma = n$ is finite, then in specifying a name N, we don't need to explicitly define β_σ^N; it suffices to ensure that f_σ^N does not change more than n times, as in the definition of an (α, n)-enumeration above. However, sometimes it will be useful to nonetheless specify β_σ^N.

3. Comparing classes

For two class descriptions Γ and Λ, we write:

- $\Gamma \subseteq \Lambda$, if the class defined by Γ is contained in the class defined by Λ;
- $\Gamma \equiv \Lambda$ if $\Gamma \subseteq \Lambda$ and $\Lambda \subseteq \Gamma$;[1]
- $\Gamma < \Lambda$ if $\Gamma \subseteq \Delta(\Lambda)$.

Note that the existence of universal sets implies that every containment is effective: if $\Gamma \subseteq \Lambda$ then there is a computable procedure translating Γ-names into equivalent Λ-names.

Lemma 3.1. For any computable class description Γ, and any $\sigma \in T_\Gamma$, $\Gamma_\sigma \subseteq \Gamma$.

Proof. Let N be a Γ_σ-name. We extend N to a Γ-name M that names the same set, by letting, for $\tau \in T_{\Gamma_\sigma}$, $(f^M_{\sigma\hat{\ }\tau}, \beta^M_{\sigma\hat{\ }\tau}) = (f^N_\tau, \beta^N_\tau)$; for $\rho \in T_\Gamma$ such that $\rho \prec \sigma$, we let, for all x and s, $f^M_\rho(x,s)$ be the child of ρ extended by σ, and $\beta^M_\rho(x,s) = 0$; for $\rho \in T_\Gamma$ incomparable with σ, it doesn't matter how we define (f^M_ρ, β^M_ρ). □

3.1. **The tree S_Γ.**

In [GT], the authors define the *containment game* $G_{\texttt{cont}}(\Gamma, \Lambda)$ that characterises containment between the described *boldface* classes. It is a clopen game. They use determinacy of such games to show that the described Wadge classes are semi-linearly ordered. The arguments in that paper can be carried over to the current setting, provided that the games and the winning strategies are computable. When restricted to finite classes, all games are finite, and so have finite winning strategies, and therefore, computable ones.

In the current paper we present a simplification of the argument for the setting of finite class descriptions. The games we present are not technically finite, but we will observe that they are essentially finite, with finite positional strategies.

We remark that much of what we do here can be extended to computable class descriptions that are not finite. However, the semi-linear-ordering principle will fail in general.

[1] We do not write $\Gamma = \Lambda$, to emphasise that this is equality of classes, not of descriptions.

Definition 3.2. For a class description Γ with $o(\Gamma) < \omega_1$, let

$$S_\Gamma = \{\langle\rangle\} \cup \left\{\tau \in T_\Gamma \,:\, \xi^\Gamma_{\tau^-} = o(\Gamma)\right\},$$

where τ^- is the predecessor of τ on T_Γ. This is a subtree of T_Γ. The internal nodes of S_Γ are precisely those nodes $\sigma \in T_\Gamma$ with $\xi_\sigma = o(\Gamma)$. The leaves of S_Γ are those nodes $\sigma \in T_\Gamma$ (internal or not) that are minimal with respect to $\xi_\sigma > o(\Gamma)$. Again recall that for leaves σ of T_Γ we set $\xi_\sigma = \omega_1$, so every leaf of T_Γ has a predecessor which is a leaf of S_Γ, possibly itself.

Lemma 3.3. Let Γ be a computable class description with $o(\Gamma) < \omega_1$. Let N be a computable Γ-name. For a leaf σ of S_Γ, the set

$$\left\{x \in \mathbb{N} \,:\, \ell^N(x) \succcurlyeq \sigma\right\}$$

is $\Delta^0_{1+o(\Gamma)+1}$, uniformly in σ.

Proof. Follows from Proposition 2.3; for every internal $\sigma \in T_\Gamma$, for any child ρ of σ, the set of x such that $f^N_\sigma(x) = \rho$ is $\Delta^0_{1+\xi_\sigma+1}$, uniformly. □

The game characterisation of containment also yields information about containment and subclasses; see Proposition 3.12 below. For now, we observe the following.

Lemma 3.4. Let Γ and Λ be computable class descriptions. Suppose that $o(\Gamma) < o(\Lambda)$. Then $\Gamma \subseteq \Lambda$ if and only if for every leaf σ of S_Γ, $\Gamma_\sigma \subseteq \Lambda$, uniformly.

The containment being uniform means that given σ and a Γ_σ-name M we can compute a Λ-name M equivalent to Λ (one that names the same set). For example, Lemma 3.1 is uniform in σ.

For this lemma and its proof, and similarly below, we appeal to Proposition 2.13 and therefore blur the distinction betweeen concrete computable ordinals and their order-types. That is, the lemma holds also when $\text{otp}(o(\Gamma)) < \text{otp}(o(\Lambda))$.

Proof. In the easier direction we use Lemma 3.1. In the other direction, let $A \in \Gamma$; let N be a Γ-name for A. For each leaf σ of S_Γ, let $X_\sigma = \left\{n \in \mathbb{N} \,:\, \ell^N(x) \succcurlyeq \sigma\right\}$. Then (X_σ) is a partition of $\mathbb{N}$ into uniformly $\Delta^0_{1+o(\Gamma)+1}$ sets. By assumption, for every leaf σ of S_Γ, the set A_σ named by N_σ is in Λ, uniformly. Since $o(\Lambda) > o(\Gamma)$, Λ is closed under definition by cases at level $o(\Gamma)+1$ (Proposition 2.10); note that $A = A_\sigma$ on X_σ. □

3.2. **The leaf selection game.** The main tool for comparing classes at the same ordinal level is a "leaf selection" game. The game presented here is simpler than the one presented in [GT], as we do not need to worry about passing and the termination of the game.

Let Γ be a computable class description with $o(\Gamma) < \omega_1$. An S_Γ-*position* p consists of a choice, for each internal node σ of S_Γ, of

(i) a child $c_\sigma = c^p_\sigma$ of σ on S_Γ; and
(ii) an ordinal $\eta^p_\sigma \leqslant \eta^\Gamma_\sigma$,

subject to the following restrictions:

- For all internal $\sigma \in S_\Gamma$, if $\eta^p_\sigma = \eta^\Gamma_\sigma$ then c^p_σ is the default child of σ; and
- For all but finitely many internal $\sigma \in S_\Gamma$, $\eta^p_\sigma = \eta^\Gamma_\sigma$.

Of course the latter condition always holds if Γ is a finite class description. Its purpose is to ensure that there are only countably many S_Γ-positions when S_Γ is infinite. If Γ is a finite class description, then there are only finitely many S_Γ-positions (again note that this holds even if the ξ_σ-ordinals are infinite).

For two S_Γ-positions p and q, we let $q \leqslant p$ if for every internal node σ of S_Γ,

(iii) $\eta^q_\sigma \leqslant \eta^p_\sigma$, and further, if $c^q_\sigma \neq c^p_\sigma$ then $\eta^q_\sigma < \eta^p_\sigma$.

The *initial* S_Γ-position is the position p determined by setting $\eta^p_\sigma = \eta^\Gamma_\sigma$ for all internal $\sigma \in S_\Gamma$.

Every S_Γ-position p determines a leaf τ^p of S_Γ, by following the choices from the root upwards, much like the definition of the leaf $\ell^N(x)$ of T_Γ used to compute the value $N(x)$. Namely, τ^p is the unique leaf τ of S_Γ determined by $c^p_\sigma \prec \tau$ for all $\sigma \prec \tau$.

We let $\mathcal{P}_\Gamma$ denote the collection of all S_Γ-positions, ordered by $\leqslant$.

Lemma 3.5. The relation "$p < q$ and $\tau^p \neq \tau^q$" on $\mathcal{P}_\Gamma$ is well-founded.

Proof. Let $p_1, p_2, \dots$ be an infinite sequence with $p_{k+1} \leqslant p_k$. For each σ, $(\eta_\sigma)^{p_k}$ is non-increasing, so stabilises to some value; it follows that $(c^{p_k}_\sigma)$ stabilises to some value c_σ. Let $\sigma_0 = \langle\rangle$ and $\sigma_{i+1} = c_{\sigma_i}$; this sequences ends with a leaf τ^* of S_Γ, and for all but finitely many k, $\tau^{p_k} = \tau^*$. □

Let Γ and Λ be two class descriptions, and suppose that $\xi = o(\Gamma) = o(\Lambda) < \omega_1$. In the game $G_{\texttt{leaf}}(\Gamma, \Lambda)$, two players, 1 and 2, take turns choosing positions:

$$\begin{array}{cccccccc} p[1] & & p[2] & & p[3] & & \dots & \\ & q[1] & & q[2] & & q[3] & & \dots \end{array}$$

(so player 1 plays $p[1], p[2], \dots$ and player 2 plays $q[1], q[2], \dots$), satisfying:

- each $p[k]$ is an S_Γ-position, and each $q[k]$ is an S_Λ-position;
- for each $k \geqslant 1$, $p[k+1] \leqslant p[k]$ and $q[k+1] \leqslant q[k]$.

We write $\sigma[k] = \tau^{p[k]}$ and $\rho[k] = \tau^{q[k]}$. By Lemma 3.5, the sequences $(\sigma[k])$ and $(\rho[k])$ both stabilise at a pair of leaves (σ^*, ρ^*) of S_Γ and S_Λ. This is the outcome of the play of the game.

Note that for computable Γ and Λ, the game $G_{\texttt{leaf}}(\Gamma, \Lambda)$ is computable (the partial orderings $\mathcal{P}_\Gamma$ and $\mathcal{P}_\Lambda$ are computable). However, neither player may have a useful computable strategy. We will show that when Γ and Λ are finite, such strategies exist.

Definition 3.6.

(a) A *containment strategy* for player 2 in $G_{\texttt{leaf}}(\Gamma, \Lambda)$ is a strategy that ensures an outcome (σ, ρ) such that $\Gamma_\sigma \subseteq \Lambda_\rho$.

(b) A *containment strategy* for player 1 in $G_{\texttt{leaf}}(\Gamma, \Lambda)$ is a strategy that ensures an outcome (σ, ρ) such that $\Lambda_\rho \subseteq \Gamma_\sigma$.

Lemma 3.7. Let Γ and Λ be computable class descriptions with $o(\Gamma) = o(\Lambda) < \omega_1$. Player 2 has a (computable) containment strategy in $G_{\texttt{leaf}}(\Gamma, \Lambda)$ if and only if player 1 has a (computable) containment strategy in $G_{\texttt{leaf}}(\Lambda, \Gamma)$.

Proof. Suppose that player 2 has a containment strategy $\mathfrak{S}$ in $G_{\texttt{leaf}}(\Gamma, \Lambda)$. In $G_{\texttt{leaf}}(\Lambda, \Gamma)$, given a play $p[1], p[2], \dots$ for player 2, player 1 can respond by using the strategy $\mathfrak{S}$ against the play $p[0], p[1], p[2], \dots$ for player 1 in $G_{\texttt{leaf}}(\Gamma, \Lambda)$, where $p[0]$ is the initial S_Γ-position.

Suppose that player 1 has a containment strategy $\mathfrak{T}$ in $G_{\texttt{leaf}}(\Lambda,\Gamma)$. In $G_{\texttt{leaf}}(\Gamma,\Lambda)$, given a play $p[1], p[2], \dots$ for player 1, player 2 can respond by using the strategy $\mathfrak{T}$ against the same play $p[1], p[2], \dots$ for player 1 in $G_{\texttt{leaf}}(\Gamma,\Lambda)$. (In this case the strategy for player 2 always ignores the most recent move by player 1, but eventually reaches the same outcome.) □

Lemma 3.8. Let Γ and Λ be finite class descriptions with $o(\Gamma) = o(\Lambda) < \omega_1$. Suppose that player 2 has a computable containment strategy in $G_{\texttt{leaf}}(\Gamma,\Lambda)$. Then $\Gamma \subseteq \Lambda$.

Proof. This is the main part of the proof of [GT, Proposition 3.5]. We simplify the argument. Let N be a computable Γ-name; we devise a Λ-name M naming the same set. For a leaf σ of S_Γ, let X_σ be the set of $x \in \mathbb{N}$ such that $\ell^N(x) \succcurlyeq \sigma$.

Fix a leaf ρ of S_Λ. Since $\xi^\Lambda_\rho > o(\Lambda) = o(\Gamma)$, by Proposition 2.10 and Lemma 3.3, there is a Λ_ρ-name M_ρ such that for all σ such that $\Gamma_\sigma \subseteq \Lambda_\rho$, for all $x \in X_\sigma$, $M_\rho(x) = N_\sigma(x) = N(x)$. Note that we are using the finiteness of S_Γ to obtain uniformity of containment (and being able to "tell" if $\Gamma_\sigma \subseteq \Lambda_\rho$ or not); we will use the finiteness of S_Λ to get that the names M_ρ are uniformly computable.

The names M_ρ define the approximations for M on all nodes that are not internal on S_Λ. Thus, to define the name M, it remains to define f^M_σ and β^M_σ for all internal $\sigma \in S_\Lambda$. This is done using a computable containment strategy $\mathfrak{S}$ for player 2. Fix $x \in \mathbb{N}$. For each $s < \omega$ let p_s be the S_Γ-position defined, for all internal $\sigma \in S_\Gamma$, by $c^{p_s}_\sigma = f^N_\sigma(x,s)$ and $\eta^{p_s}_\sigma = \beta^N_\sigma(x,s)$. The notion of position and ordering of positions ensures that each p_s is indeed an S_Γ-position (if $\eta^p_\sigma = \eta_\sigma$ then c^p_σ is the default outcome), and that $p_t \leqslant p_s$ when $s \leqslant_{o(\Gamma)} t$.

For each s, consider the partial play $p_{s_0}, p_{s_1}, \dots, p_{s_k}$, where $0 = s_0 \leqslant_{o(\Gamma)} s_1 \leqslant_{o(\Gamma)} \dots \leqslant_{o(\Gamma)} s_k = s$ is the enumeration of the stages $r \leqslant_{o(\Gamma)} s$. Thus, $k = |s|_{o(\Gamma)} + 1$, where $|s|_{o(\Gamma)}$ is the height of s on the tree $(\omega, \leqslant_{o(\Gamma)})$. The strategy $\mathfrak{S}$ gives a response $q_{s_0}, q_{s_1}, \dots, q_{s_k}$; note that q_{s_i} only depends on $p_{s_0}, \dots, p_{s_i}$. For internal $\sigma \in S_\Lambda$ we let $f^M_\sigma(x,s) = c^{q_s}_\sigma$ and $\beta^M_\sigma(x,s) = \eta^{q_s}_\sigma$. The fact that $\mathfrak{S}$ always responds with a legal play implies that $(f^M_\sigma(x,-), \beta^M_\sigma(x,-))$ obey the rules for properly defining a Λ-name. The fact that it is a successful strategy implies that $\ell^M(x)$ extends a leaf $\rho(x)$ of S_Λ such that $\Gamma_{\sigma(x)} \subseteq \Lambda_{\rho(x)}$; by the definition of $M_{\rho(x)}$ we then get

$$N(x) = N_{\sigma(x)} = M_{\rho(x)} = M(x)$$

as required. □

Remark 3.9. Lemma 3.8 can be extended to computable classe descriptions that are not necessarily finite. We need to add the assumption that $\Gamma_\sigma \subseteq \Lambda_\rho$ is uniform. More specifically, it suffices to have a *partial* computable function that gives, for each pair (σ,ρ) such that $\Gamma_\sigma \subseteq \Lambda_\rho$, a Λ_ρ-name for N_σ. In the definition of the name M_ρ, we don't actually need to know if $\Gamma_\sigma \subseteq \Lambda_\rho$ or not: we simply instruct M, at and above ρ, to keep taking the default outcome, until ρ discovers which X_σ the number x belongs to, and further, the partial procedure gives us some Λ_ρ-name. This holds when player 2 has a computable winning strategy in the containment game $G_{\texttt{cont}}(\Gamma,\Lambda)$ from [GT].

Next, we show that for finite class descriptions, $G_{\texttt{leaf}}(\Gamma,\Lambda)$ is effectively determined.

Lemma 3.10. Let Γ, Λ be finite class descriptions with $o(\Gamma) = o(\Lambda) < \omega_1$. Exactly one of the following holds:

(1) Player 2 has a computable containment strategy in $G_{\texttt{leaf}}(\Gamma, \Lambda)$.
(2) Player 2 has a computable strategy in $G_{\texttt{leaf}}(\Lambda, \Gamma)$, that ensures an outcome (ρ, σ) such that $\Gamma_\sigma \nsubseteq \Lambda_\rho$.

Proof. This is because the partial orderings $\mathcal{P}_\Gamma$ and $\mathcal{P}_\Lambda$ (of all S_Γ and S_Λ-positions) are *finite*, and the outcome of the game is decided by the final positions only. So the game is almost a finite game, so Zermelo determinacy holds. For the current game: define the notion of a "good pair" of positions $(q, p) \in \mathcal{P}_\Lambda \times \mathcal{P}_\Gamma$ by induction on the number of predecessors $q' < q$ and $p' < p$ in $\mathcal{P}_\Lambda$ and $\mathcal{P}_\Gamma$. Namely, suppose that the notion has been defined for all pairs (q', p') where $q' \leqslant q$ and $p' < p$. Then we say that (q, p) is good if $\Gamma_{\tau^p} \subseteq \Lambda_{\tau^q}$, and further, for all $p' < p$ there is some $q' \leqslant q$ such that (q', p') is good.

Let p_0 be the initial S_Γ-position. There are two possibilities. If there is some $q \in \mathcal{P}_\Lambda$ such that (q, p_0) is good, then player 1 has a computable containment strategy in $G_{\texttt{leaf}}(\Lambda, \Gamma)$: first move to q, and then keep moving to ensure that we are in a good position. As mentioned above (Lemma 3.7), this means that player 2 has a computable containment strategy in $G_{\texttt{leaf}}(\Gamma, \Lambda)$.

Otherwise, player 2 has a computable "anti-containment" strategy as in (2) above. Call a pair (q, p) "bad" if for all $q' \leqslant q$, the pair (q', p) is not good. By assumption, the pair (q_0, p_0) of initial positions is bad. Also if (q, p) is bad then for all $q' \leqslant q$, (q', p) is bad. Call a pair (q, p) "very bad" if it is bad, and further, $\Gamma_{\tau^p} \nsubseteq \Lambda_{\tau^q}$. By definition, if (q, p) is bad, then it is very bad, or there is some $p' < p$ such that (q, p') is bad. Hence (by induction on the predecessors of p), if (q, p) is bad then there is some $p' \leqslant p$ such that (q, p') is very bad. Hence, player 2 can keep responding with moves that keep the game situation at very bad positions. $\square$

We now obtain the semi-linear-ordering principle for the extended fine hierachy:

Proposition 3.11. *If Γ and Λ are finite class descriptions, then either $\Gamma \subseteq \Lambda$ or $\Lambda \subseteq \check{\Gamma}$.*

Note that this proposition, together with $\Gamma \nsubseteq \check{\Gamma}$, imply the familiar pattern for the fine hierarchy: for any two finite Γ and Λ, either $\Gamma < \Lambda$, or $\Lambda < \Gamma$, or $\Gamma = \Lambda$, or $\Gamma = \check{\Lambda}$.

Proof. We prove the proposition by induction on the complexity of the pair (Γ, Λ). In particular, if $o(\Gamma) < \omega_1$, then we assume that for all leaves σ of S_Γ, for all $\tau \in \Lambda$, either $\Gamma_\sigma \subseteq \Lambda_\rho$, or $\Lambda_\rho \subseteq \check{\Gamma}_\sigma$; and similarly, if $o(\Lambda) < \omega_1$, then we assume that for all leaves ρ of S_Λ, for all $\sigma \in \Gamma$, either $\Gamma_\sigma \subseteq \Lambda_\rho$ or $\Lambda_\rho \subseteq \check{\Gamma}_\sigma$.

The induction starts with any pair (Γ, Λ) such that $o(\Gamma) = o(\Lambda) = \omega_1$. In this case, both Γ and Λ are one of $\{\varnothing\}$, $\{\mathbb{N}\}$, and the proposition in this case is immediate.

Let Γ and Λ be class descriptions, and suppose that either $o(\Gamma) < \omega_1$ or $o(\Lambda) < \omega_1$; without loss of generality, assume that $o(\Gamma) < \omega_1$. There are two cases.

First, suppose that $o(\Gamma) < o(\Lambda)$ (including the case $o(\Lambda) = \omega_1$). In this case we use the induction hypothesis for all pairs (Γ_σ, Λ), where σ is a leaf of S_Γ. If there is some such σ such that $\Gamma_\sigma \nsubseteq \Lambda$, then by induction, $\Lambda \subseteq \check{\Gamma}_\sigma$; by Lemma 3.1, $\Lambda \subseteq \check{\Gamma}$. Otherwise, by Lemma 3.4, $\Gamma \subseteq \Lambda$.

Next, suppose that $o(\Gamma) = o(\Lambda)$ (so both are $< \omega_1$). If player 2 has a computable containment strategy in $G_{\texttt{leaf}}(\Gamma, \Lambda)$, then by Lemma 3.8, $\Gamma \subseteq \Lambda$. Otherwise, by Lemma 3.10, and by induction, player 2 has a computable containment strategy in $G_{\texttt{leaf}}(\Lambda, \check{\Gamma})$, so by Lemma 3.8, $\Lambda \subseteq \check{\Gamma}$. □

We summarise our findings.

Proposition 3.12. *Let Γ and Λ be finite class descriptions.*

(a) *If $o(\Gamma) < o(\Lambda)$ then $\Gamma \subseteq \Lambda$ if and only if for every leaf σ of S_Γ, $\Gamma_\sigma \subseteq \Lambda$.*
(b) *If $o(\Gamma) > o(\Lambda)$, then $\Gamma \subseteq \Lambda$ if and only if there is some leaf τ of S_Λ such that $\Gamma \subseteq \Lambda_\tau$.*
(c) *If $o(\Gamma) = o(\Lambda)$, then $\Gamma \subseteq \Lambda$ if and only if player 2 has a computable containment strategy in $G_{\texttt{leaf}}(\Gamma, \Lambda)$, if and only if player 1 has a computable containment strategy in $G_{\texttt{leaf}}(\Lambda, \Gamma)$.*

Proof. (1) is Lemma 3.4. For (2), in the harder direction, if there is no leaf τ of S_Λ such that $\Gamma \subseteq \Lambda_\tau$, then by Proposition 3.11, for all leaves τ of S_Λ, $\Lambda_\tau \subseteq \check{\Gamma}$; by (1), $\Lambda \subseteq \check{\Gamma}$; it follows that $\Gamma \nsubseteq \Lambda$, as $\Gamma \nsubseteq \check{\Gamma}$ (Corollary 2.9).

For (3), one direction is the combination of Lemma 3.8 and Lemma 3.7. In the other direction, if player 2 does not have a computable containment strategy for $G_{\texttt{leaf}}(\Gamma, \Lambda)$, then by Lemma 3.10 and Proposition 3.11, player 2 has an effective containment strategy for $G_{\texttt{leaf}}(\Lambda, \check{\Gamma})$, and we conclude that $\Lambda \subseteq \check{\Gamma}$; again, this implies that $\Lambda \nsubseteq \Gamma$. □

Example 3.13. The two class descriptions in Fig. 2 describe the same class $D_2(\Sigma^0_1)$ of differences of c.e. sets. Let Γ denote the description on the left and Λ the description on the right. Note that $S_\Gamma = T_\Gamma$ and $S_\Lambda = T_\Lambda$. Thus, a containment strategy for player 2 is one which guarantees an outcome in which both leaves have the same label, **0** or **1**. In $G_{\texttt{leaf}}(\Gamma, \Lambda)$, both players start with the defaults, labelled **0**, and player 1 cannot change labels more than twice, showing how player 2 can always move to match the label of the leaf chose by player 1. In $G_{\texttt{leaf}}(\Lambda, \Gamma)$, if player 1 shifts to the **1**-outcome, player 2 chooses the rightmost leaf labelled **1** on T_Γ (choosing the other one would be a bad move, since no further changes would then be allowed).

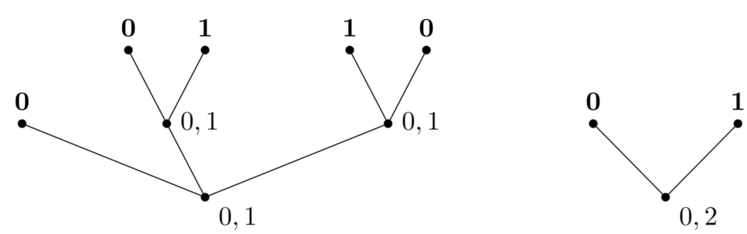

FIGURE 2. Two descriptions of $D_2(\Sigma^0_1)$

3.3. The successor class.

Definition 3.14. Let Γ be a class description. We let Γ^+ denote the class description obtained by letting the root have three children, 0, 1 and 2; the first is a leaf of T_{Γ^+}, labelled **0**; we set $\Gamma^+_1 = \Gamma$ and $\Gamma^+_2 = \check{\Gamma}$. See Fig. 3.

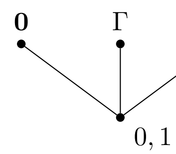

FIGURE 3. The successor class Γ^+.

The following proposition says that for a finite description Γ, the pair consisting of Γ^+ and its dual is the successor of the pair $\{\Gamma, \check{\Gamma}\}$ in the extended fine hierarchy.[2]

Proposition 3.15. *Let Γ be a finite class description. Then $\Gamma < \Gamma^+$, and for any finite class description Λ, if $\Gamma < \Lambda$ then $\Gamma^+ \subseteq \Lambda$ or $\Gamma^+ \subseteq \check{\Lambda}$.*

Proof. That $\Gamma, \check{\Gamma} \subseteq \Gamma^+$ follows from Lemma 3.1. Let Λ be a finite class description, and suppose that $\Gamma < \Lambda$. Suppose that Λ is a Σ-type description (Definition 2.14); we show that $\Gamma^+ \subseteq \Lambda$. It follows that if Λ has Π-type then $\Gamma^+ \subseteq \check{\Lambda}$.

There are three cases.

Suppose that $o(\Lambda) > 0$. For every leaf σ of S_{Γ^+} we have $\Gamma^+_\sigma \subseteq \Gamma$ or $\Gamma^+_\sigma \subseteq \check{\Gamma}$; by assumption, in either case, $\Gamma^+_\sigma \subseteq \Lambda$. Then $\Gamma^+ \subseteq \Lambda$ by Proposition 3.12(a).[3]

If $o(\Lambda) = 0 < o(\Gamma)$, then there are some leaves $\tau, \check{\tau}$ of S_Λ with $\Gamma \subseteq \Lambda_\tau$ and $\check{\Gamma} \subseteq \Lambda_{\check{\tau}}$. A containment strategy for player 2 in the game $G_{\texttt{leaf}}(\Gamma^+, \Lambda)$ is to remain on the default outcomes until player 1 makes a change at the root. Since Λ has Σ-type, its ultimate default outcome is $\mathbf{0}$, so player 2 is covering, as long as this situation persists. If player 1 switches at the root to the Γ child, then player 2 immediately switches all necessary nodes such that it is selecting τ as its leaf. If player 1 instead switches at the root to the $\check{\Gamma}$ child, then player 2 immediately switches to $\check{\tau}$. By Proposition 3.12(c), $\Gamma^+ \subseteq \Lambda$.

If $o(\Lambda) = o(\Gamma) = 0$, then the argument is a more complicated version of the previous one. Player 2 has a containment strategy $\mathfrak{S}_0$ in the game $G_{\texttt{leaf}}(\Gamma, \Lambda)$, and a containment strategy $\mathfrak{S}_1$ in the game $G_{\texttt{leaf}}(\check{\Gamma}, \Lambda)$. Player 2's strategy in $G_{\texttt{leaf}}(\Gamma^+, \Lambda)$ is to remain on default outcomes until player 1 makes a change at the root. Again, since Λ has Σ-type, this is a success if player 1 never changes. If player 1 switches to the Γ child, then player 2 begins playing $\mathfrak{S}_0$. If player 2 switches to the $\check{\Gamma}$ child, then player 2 begins playing $\mathfrak{S}_1$. □

3.4. **Efficient descriptions, and the ordinal level of a class.** Let Γ be a class description, with $o(\Gamma) < \omega_1$. Suppose that there is some leaf σ of S_Γ such that for every leaf τ of S_Γ we have $\Gamma_\tau \subseteq \Gamma_\sigma$. Then as $o(\Gamma_\sigma) > o(\Gamma)$, by Proposition 3.12 (and Lemma 3.1), $\Gamma \equiv \Gamma_\sigma$. Such a class description is "wasteful". In [GT, Definition 4.1], the authors introduce the notion of an *efficient description*, one which is not wasteful. The definition is based on the following observation. Let $\mathcal{C}$ be a collection of finitely described classes. Proposition 3.11 implies that exactly one of the following holds: (1) $\mathcal{C}$ contains a $\subseteq$-greatest element; (2) For all $\Gamma \in \mathcal{C}$ there is some $\Lambda \in \mathcal{C}$ with $\Gamma \subseteq \check{\Lambda}$.

[2]Note that by Example 3.13, this implies that the pair consisting of $D_2(\Sigma^0_1)$ and its dual, is the successor of the pair $\{\Sigma^0_1, \Pi^0_1\}$.

[3]Note that in this case, we did not use the fact that Λ has Σ-type, and indeed, the argument shows that $\Gamma^+ < \Lambda$. Later, we will see that such Λ must have limit order-type in the extended fine hierarchy.

In the context of finite descriptions, we define the following:

Definition 3.16. A finite class description Γ is *efficient* if $o(\Gamma) = \omega_1$, or $o(\Gamma) < \omega_1$ and for every leaf σ of S_Γ there is some leaf τ of S_Γ such that $\Gamma_\sigma \subseteq \check{\Gamma}_\tau$.[4]

The pleasing properties of efficient descriptions hold in the current context as well. For example, they determine the ordinal level of a finitely described class:

Proposition 3.17. *If Θ and Γ are finite class descriptions, with Γ efficient, and $\Theta \equiv \Gamma$, then $o(\Theta) \leqslant o(\Gamma)$.*

The proof is the same as that of the analogous [GT, Proposition 4.2]: suppose that $o(\Gamma) < o(\Lambda)$. Since $\Lambda \subseteq \Gamma$, by Proposition 3.12, there is a leaf σ of S_Γ such that $\Lambda \subseteq \Gamma_\sigma$. Since Γ is efficient, there is a leaf τ of S_Γ such that $\Gamma_\sigma \subseteq \check{\Gamma}_\tau$, and $\check{\Gamma}_\tau \subseteq \check{\Gamma}$; so $\Lambda \subseteq \check{\Gamma}$, contradicting $\Gamma \subseteq \Lambda$.

This allows us to unambiguously define the ordinal level of a finitely described class. As in [GT, Proposition 4.3], the ordinal level has a characterisation due to Louveau and Saint Raymond [LSR88a]: it is the greatest ξ such that Γ is closed under definitions by cases at level ξ (Proposition 2.10); the proof is identical.

3.5. **Well-foundedness of the extended fine hierarchy.** To prove Theorem 2.16, it remains to show that the extended fine hierarchy is well-founded. We will give two arguments. For the second, we will define a "normal form" for descriptions, and will be able to directly calculate the ordinal rank of each class from these special descriptions, called "admissible".

The first proof is a direct proof, using Proposition 3.12. We first remark that in fact, we can deduce well-foundedness directly from the results in [GT], and the fact that the Wadge hierarchy is well-founded: using the containment characterisation in [GT] for described Wadge classes, and an equivalent characterisation in the effective setting, we see that if $\cdots < \Gamma_2 < \Gamma_1 < \Gamma_0$ is an infinite descending sequence in the extended fine hierarchy, then the associated boldface classes are an infinite descending sequence $\cdots < \boldsymbol{\Gamma}_2 < \boldsymbol{\Gamma}_1 < \boldsymbol{\Gamma}_0$ in the Wadge degrees, which is impossible.

This argument is unsatisfying. The proof of well-foundedness for Borel Wadge classes relies on heavy tools, such as Borel determinacy, and universal sets for the boldface classes. It seems that there should be a direct, "local" argument. We give such a proof. The same proof can be also used to show that the Wadge hierarchy is well-founded. The final part of the proof is similar to the Martin-Monk argument, in its use of Baire category. However, the bulk of the argument seems different.

First proof that the extended fine hierarchy is well-founded. Suppose, for a contradiction, that there is a sequence $\Gamma^0, \Gamma^1, \dots$ of finite class descriptions such that $\Gamma^{n+1} < \Gamma^n$ for all n. (We use superscripts to accommodate nodes / stage numbers as subscripts.)

For all n, we let $\Gamma^{n,1} = \Gamma^n$ and $\Gamma^{n,0} = \check{\Gamma}^n$.

---

[4]This definition is slightly different from [GT, Definition 4.1]. The latter requires the property to be *hereditary*, namely, for every $\sigma \in T_\Gamma$, Γ_σ is efficient as well. This difference is not important to us right now. The latter also defines efficiency in terms of children of the root, rather than leaves of S_Γ. This is a bit different; we will return to this when discussing *admissible* descriptions below.

Since this proof is not effective, we blur the distinction between a concrete ordinal $\xi_\sigma^{\Gamma^n}$ and its order-type. We therefore let

$$\xi^* = \sup\{\xi_\sigma^{\Gamma^n} : n \in \mathbb{N}, \sigma \in T_{\Gamma^n}, \ \& \ \xi_\sigma^{\Gamma^n} < \omega_1\} + 1,$$

which is a countable ordinal.

Fix $X \in 2^\omega$. We define an array of nodes $\sigma_\alpha^n \in T_{\Gamma^n}$ for $n \in \mathbb{N}$ and $\alpha \leqslant \xi^*$ by transfinite recursion on α. To simplify notation, we let $\Gamma_\alpha^n = \Gamma_{\sigma_\alpha^n}^n$ and $\xi_\alpha^n = \xi_{\sigma_\alpha^n}^{\Gamma^n} = o(\Gamma_\alpha^n)$. We similarly define $\Gamma_\alpha^{n,i} = \Gamma_{\sigma_\alpha^n}^{n,i}$ for $i = 0, 1$.

We will ensure that for all n and $\alpha \leqslant \beta \leqslant \xi^*$,

(i) $\sigma_\alpha^n \leqslant \sigma_\beta^n$;
(ii) $\xi_\alpha^n \geqslant \alpha$; and
(iii) $\Gamma_\alpha^{n+1} \subseteq \Gamma_\alpha^{n,X(n)}$.

We start with σ_0^n being the root of T_{Γ^n}. The containment property (iii) follows from the assumption that $\Gamma^{n+1} \subseteq \Delta(\Gamma^n)$.

If $\beta \leqslant \xi^*$ is a limit ordinal, and σ_α^n were defined for all $\alpha < \beta$, then for each n, by (i), and by the fact that T_{Γ^n} is finite (well-founded would be enough), the sequence $(\sigma_\alpha^n)_{\alpha<\beta}$ stabilises; we let σ_β^n be that stable value. Note that (iii) holds at β since for all n there is some $\alpha < \beta$ such that both $\sigma_\beta^{n+1} = \sigma_\alpha^{n+1}$ and $\sigma_\beta^n = \sigma_\alpha^n$. Also (ii) holds since it holds at each $\alpha < \beta$.

Let $\alpha < \xi^*$ and suppose that σ_α^n have been defined for all n. We define $\sigma_{\alpha+1}^n$.

If $\xi_\alpha^n > \alpha$ then we let $\sigma_{\alpha+1}^n = \sigma_\alpha^n$. It thus remains to define $\sigma_{\alpha+1}^n$ for all n such that $\xi_\alpha^n = \alpha$. For such n we will choose $\sigma_{\alpha+1}^n$ to be a leaf of $S_{\Gamma_\alpha^n}$; this will ensure that (ii) holds for n and $\alpha + 1$, as we will have $\xi_{\alpha+1}^n > \xi_\alpha^n$.

Let I be any maximal interval of $\mathbb{N}$ such that $\xi_\alpha^n = \alpha$ for all $n \in I$. There are two cases.

If I is finite, let $m = \max I$. We choose $\sigma_{\alpha+1}^n$ for $n \in I$ by reverse recursion on n. The maximality of I and m implies that $\xi_\alpha^{m+1} > \alpha$. Hence by Proposition 3.12(b), we can choose $\sigma_{\alpha+1}^m \succ \sigma_\alpha^m$ to be a leaf of $S_{\Gamma_\alpha^m}$ such that $\Gamma_\alpha^{m+1} \subseteq \Gamma_{\alpha+1}^{m,X(m)}$. If $n \in I$, $n < m$, and $\sigma_{\alpha+1}^{n+1}$ was defined, then by Proposition 3.12(c), we can find a leaf $\sigma_{\alpha+1}^n$ of $S_{\Gamma_\alpha^n}$ such that $\Gamma_{\alpha+1}^{n+1} \subseteq \Gamma_{\alpha+1}^{n,X(n)}$.

The more complicated case is when I is infinite, i.e., is a final segment of $\mathbb{N}$. In this case we need to play the leaf selection games. By (iii) and Proposition 3.12(c), fix, for each $n \in I$, a containment strategy $\mathfrak{S}_n$ for player 1 in the game $G_{\mathtt{leaf}}(\Gamma_\alpha^{n,X(n)}, \Gamma_\alpha^{n+1})$.

For each n in I, we define a descending sequence of $S_{\Gamma_n^\alpha}$ positions, $p_0^n \geqslant p_1^n \geqslant p_2^n \geqslant \dots$ starting with the first move given by $\mathfrak{S}_n$. We then "steal moves": we let p_1^n be the move given by $\mathfrak{S}_n$ in response to the move p_0^{n+1}, regarded as a move for player 2 in the game $G_{\mathtt{leaf}}(\Gamma_\alpha^{n,X(n)}, \Gamma_\alpha^{n+1})$. In general, if p_k^n is defined for all $n \in I$, then we let p_{k+1}^n be the move given by $\mathfrak{S}_n$ in response to $p_0^{n+1}, \dots, p_k^{n+1}$.

For each n, the sequence of leaves $(\tau^{p_k^n})$ stabilises to a leaf $\sigma_{\alpha+1}^n$ of $S_{\Gamma_\alpha^n}$.

We check that (iii) holds at $\alpha + 1$. Let $n \in \mathbb{N}$. There are four cases:

- If $\xi_\alpha^n, \xi_\alpha^{n+1}$ are both $> \alpha$, then this follows from (iii) holding at stage α.
- If $\xi_\alpha^n = \xi_\alpha^{n+1} = \alpha$ then n and $n + 1$ belong to the same interval I. If I is finite, then (iii) follows by construction. If I is infinite, this follows from the success of the strategy $\mathfrak{S}_n$.

- If $\alpha = \xi^n_\alpha < \xi^{n+1}_\alpha$ then again this is by construction, as in this case n is the maximal element of its interval I.
- If $\xi^n_\alpha > \xi^{n+1}_\alpha = \alpha$ then this follows from (iii) holding at stage α, together with the choice of $\xi^{n+1}_{\alpha+1}$ to be a leaf of $S_{\Gamma^{n+1}_\alpha}$, and Proposition 3.12(a).

At the end of this construction we obtain nodes $\sigma^n_{\xi^*} \in T_{\Gamma^n}$ with $\xi_{\sigma^n_{\xi^*}} \geqslant \xi^*$; by choice of ξ^*, this means that $\xi_{\sigma^n_{\xi^*}} = \omega_1$. Let $\ell(X,n)$ be the label of $\sigma^n_{\xi^*}$ on T_{Γ^n}. By (iii), $\ell(X,n) = \ell(X,n+1)$ if and only if $X(n) = 0$.

We observe that in the construction above, each choice σ^n_α depended only on Γ^n and Γ^{n+1}, and $X(n)$. This implies:

(∗): If $X, Y \in 2^\omega$, $n \in \mathbb{N}$, and $X \upharpoonright [n,\infty) = Y \upharpoonright [n,\infty)$ then $\ell(X,n) = \ell(Y,n)$.

For $i = 0,1$ let B_i be the set of $X \in 2^\omega$ such that $\ell(X,0) = i$. The sets B_0 and B_1 are Borel: essentially, they can be decided by roughly ξ^* many Turing jumps of X (and the sequence of descriptions (Γ_n)). On the other hand, they fail the property of Baire, as in the Martin-Monk argument. If $X, Y \in 2^\omega$ and $X \triangle Y = \{n\}$ then $l(X,n+1) = l(Y,n+1)$ (by (∗)), but by reverse induction on $m \leqslant n$ we see that $l(X,m) = 1 - l(Y,m)$. Hence, $X \in B_0 \leftrightarrow Y \in B_1$. □

4. Admissible descriptions

Among all finite descriptions, we specify a collection of particularly nice descriptions that are easier to analyse. We adapt the definition [GT, Definition 4.5] to finite classes. We will then show that this restricted collection suffices to describe all finitely describable classes.

Definition 4.1. A finite class description Γ is *admissible* if for all internal $\sigma \in T_\Gamma$:

(i) For any non-default child τ of σ on T_Γ, $\xi_\tau > \xi_\sigma$; and
(ii) For any child $\sigma\hat{}n$ of σ there is some child $\sigma\hat{}m$ of σ such that $\Gamma_{\sigma\hat{}n} \subseteq \check{\Gamma}_{\sigma\hat{}m}$.

Note that we allow $\xi_\tau = \xi_\sigma$, where τ is the default child of σ. Note also that the definition implies that if Γ is admissible, then for every $\sigma \in T_\Gamma$, Γ_σ is admissible as well (Γ is "hereditarily admissible").

Notation 4.2. For admissible descriptions, we will denote the default outcome by 0, i.e., the default child of σ will always be $\sigma\hat{}0$.

Lemma 4.3. Every admissible class description is efficient.

Proof. Let Γ be admissible, and let σ be a leaf of S_Γ. Let n be the child of the root that σ extends. By assumption, there is a child m of the root such that $\Gamma_n \subseteq \check{\Gamma}_m$, so $\Gamma_\sigma \subseteq \check{\Gamma}_m$. Note that $n \neq m$. If m is a leaf of S_Γ then we are done. Otherwise, $m = 0$ is the default child of the root, so $n \neq 0$. Hence $o(\Gamma_n) > o(\Gamma)$, and $\sigma = n$. Since m is not a leaf of S_Γ, $o(\Gamma_0) = o(\Gamma)$. By Proposition 3.12(b), there is a leaf τ of S_{Γ_0} such that $\Gamma_n \subseteq \check{\Gamma}_\tau$; τ is also a leaf of S_Γ. □

Definition 4.4. For two finite descriptions Γ and Λ, an ordinal $\xi \leqslant o(\Lambda), o(\Gamma)$, and $n < \omega$, we let

$$\mathrm{SU}_{\xi,n}(\Gamma,\Lambda)$$

be the class description Θ defined by declaring the children of the root to be 0,1, and 2 (so 0 is the default), and setting $\Theta_0 = \Lambda$, $\Theta_1 = \Gamma$, $\Theta_2 = \check{\Gamma}$, $o(\Theta) = \xi$ and $\eta^\Theta = n$ (see Fig. 4).

If:

(i) $\Lambda \subseteq \Gamma$;
(ii) $o(\Gamma) > \xi$ and $o(\Lambda) \geqslant \xi$; and
(iii) Λ and Γ are admissible,

then $\mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$ is admissible as well.

Lemma 4.5. If Θ is admissible, then there are Λ, Γ, ξ and n as above such that $\Theta \equiv \mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$.

Proof. By the semi-linear ordering principle, and the fact that T_Θ is finite, there are children n and m of the root on T_Θ such that letting $\Gamma = \Theta_n$, we have $\Theta_m \equiv \check{\Gamma}$, and for all leaves σ of S_Θ we have $\Theta_\sigma \subseteq \Gamma$ or $\Theta_\sigma \subseteq \check{\Gamma}$. Since n and m cannot both be the default outcome, we have $o(\Gamma) > o(\Theta)$. Let k denote the default child of the root in T_Θ; let $\Lambda = \Theta_k$. Let $\xi = o(\Theta)$ and $n = \eta^\Theta$. We claim that $\Theta \equiv \mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$.

In one direction, in the game $G_{\texttt{leaf}}(\Theta, \mathrm{SU}_{\xi,n}(\Gamma, \Lambda))$, while player 1 plays above the default outcome, player 2 extends the default outcome 0 and matches the moves of player 1. If player 1 moves away from the default outcome, player 2 chooses either n or m, i.e., either Γ or $\check{\Gamma}$, to contain the class played by player 1; the η-ordinal is matched with player 1.

In the other direction, in the game $G_{\texttt{leaf}}(\mathrm{SU}_{\xi,n}(\Gamma, \Lambda), \Theta)$, player 2 can always match the classes played by player 1. □

By the semi-linear-ordering property, there are two possibilities:

- $\Lambda < \Gamma$; or
- $\Lambda = \Gamma$ or $\Lambda = \check{\Gamma}$.

In the second case we have $o(\Lambda) = o(\Gamma) > \xi$. By Proposition 3.12(c), in this case, we can then omit one of the non-default outcomes and obtain the description seen in Fig. 5 (or its dual).

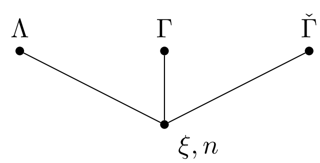

FIGURE 4. The class description $\mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$.

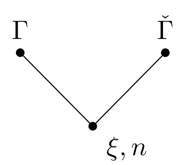

FIGURE 5. The class description equivalent to $\mathrm{SU}_{\xi,n}(\Gamma, \Gamma)$ (when $o(\Gamma) > \xi$).

We will later give "Boolean interpretations" of these two kinds of classes, in terms of the operations of one- and two-sided separated unions.

Definition 4.6. An admissible class description Γ is *very admissible* if for all internal $\sigma \in T_\Gamma$, $\Gamma_{\sigma\hat{}0} \subseteq \Gamma_{\sigma\hat{}1}$ and $\Gamma_\sigma = \mathrm{SU}_{\xi_\sigma,\eta_\sigma}(\Gamma_{\sigma\hat{}1}, \Gamma_{\sigma\hat{}0})$.

Proposition 4.11 and Lemma 4.5 imply that every finitely described class has a very admissible description. Note that Γ is very admissible if and only if $\check{\Gamma}$ is very admissible.

4.1. **Containment and equivalence between admissible descriptions.** Admissibility allows us to easily characterise containment between classes. For a finite class description Γ, let

$$\mathcal{C}(\Gamma) = \bigcup \{\Gamma_\sigma \,:\, \sigma \text{ is a leaf of } S_\Gamma\}$$

be the union of the classes Γ_σ for the leaves σ of S_Γ. If Γ is efficient then $\mathcal{C}(\Gamma) = \Gamma_\sigma \cup \check{\Gamma}_\sigma$ for some leaf σ. For two finite efficient class descriptions Γ and Λ we write $\mathcal{C}(\Gamma) < \mathcal{C}(\Lambda)$ if $\mathcal{C}(\Gamma) \subsetneq \mathcal{C}(\Lambda)$; this is equivalent to having some leaf τ of S_Λ such that for all leaves $\sigma \in S_\Gamma$, $\Gamma_\sigma \subseteq \Lambda_\tau$, equivalently, $\Gamma_\sigma < \Lambda_\tau$. Note that for efficient Γ we have $\mathcal{C}(\Gamma) = \mathcal{C}(\check{\Gamma})$. If Γ is admissible, then we can choose the witnessing σ to be a non-default child n of the root.

Proposition 4.7. *Suppose Γ and Λ are finite admissible descriptions with $o(\Gamma) = o(\Lambda)$. Then $\Gamma \subseteq \Lambda$ if and only if*

- $\mathcal{C}(\Gamma) < \mathcal{C}(\Lambda)$*; or*
- $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$*, and:*
  - $\eta^\Gamma < \eta^\Lambda$*; or*
  - $\eta^\Gamma = \eta^\Lambda$ *and* $\Gamma_0 \subseteq \Lambda_0$*.*

Proof. We first prove the forward direction. We use Proposition 3.12(c). Again we use the fact that any non-default outcome of the root is a leaf of the corresponding S-tree.

First, suppose that $\mathcal{C}(\Gamma) < \mathcal{C}(\Lambda)$. A containment strategy for player 2 is to choose any outcome n such that $\mathcal{C}(\Lambda) = \Lambda_n \cup \check{\Lambda}_n$.

Suppose then that $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$. Let n_0 and n_1 be outcomes of the root on T_Λ such that $\mathcal{C}(\Lambda) = \Lambda_{n_0} \cup \Lambda_{n_1}$ (so $\Lambda_{n_1} = \check{\Lambda}_{n_0}$). Since $o(\Lambda_{n_i}) > o(\Lambda)$, both n_0 and n_1 are leaves of S_Λ. For every child m of the root on T_Γ, either $\Gamma_m \subseteq \Lambda_{n_0}$ or $\Gamma_m \subseteq \Lambda_{n_1}$.

Suppose that $\eta^\Gamma < \eta^\Lambda$. The strategy for player 2 is to first choose n_i (either n_0 or n_1) so that $\Gamma_0 \subseteq \Lambda_{n_i}$. Player 2 decreases the ordinal at the root by 1. Henceforth, the ordinal at the root of Λ is at least as large as that at the root of Γ; player 2 always chooses either n_0 or n_1. If player 1 "goes out of cover", i.e., chooses some leaf σ of S_Γ such that Γ_σ is not contained in the current class player by player 2, then it must be that player 1 made a change at the root, enabling player 2 to move from n_0 to n_1 or the other way round.

Suppose that $\eta^\Gamma = \eta^\Lambda$ and that $\Gamma_0 \subseteq \Lambda_0$. While player 1 chooses leaves of S_Γ extending 0, player 2 responds in kind: if $o(\Lambda_0) > o(\Lambda)$ then player 1 just chooses the outcome 0; if $o(\Gamma_0) > o(\Gamma) = o(\Lambda)$ but $o(\Lambda_0) = o(\Lambda)$ then player 1 just chooses some leaf τ of S_{Λ_0} such that $\Gamma_0 \subseteq \Lambda_\tau$ (Proposition 3.12(b)); if $o(\Gamma_0) = o(\Lambda_0) = o(\Gamma)$ then player 2 plays a containment strategy in the game $G_{\mathtt{leaf}}(\Gamma_0, \Lambda_0)$. Once player 1 moves away from the default outcome 0, player 2 can alternate between the outcomes n_0 and n_1 as necessary, keeping the ordinals at the root equal.

In the reverse direction, first note that if $\mathcal{C}(\Gamma) < \mathcal{C}(\Lambda)$, or $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$ and $\eta^\Gamma < \eta^\Lambda$, or $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$ and $\eta^\Gamma = \eta^\Lambda$ and $\Gamma_0 < \Lambda_0$, then the same conditions hold for $(\check{\Gamma}, \Lambda)$, so by the forward direction, $\Gamma < \Lambda$. Hence, if $\Gamma \nleq \Lambda$ and $\Lambda \nleq \Gamma$, then it must be the case that $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$ and $\eta^\Gamma = \eta^\Lambda$ and $\Gamma_0 \equiv \check{\Lambda}_0$; taking the dual and using the forward direction again, we see that in this case, $\Lambda \equiv \check{\Gamma}$, so $\Gamma \nleq \Lambda$. □

Corollary 4.8. *If Γ and Λ are finite admissible descriptions, then $\Gamma \equiv \Lambda$ if and only if:*

- $o(\Gamma) = o(\Lambda)$*;*
- $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$*;*
- $\eta^\Gamma = \eta^\Lambda$*; and*
- $\Gamma_0 \equiv \Lambda_0$*.*

Proof. $o(\Gamma) = o(\Lambda)$ by Proposition 3.17 (and Lemma 4.3). The rest follows by applying Proposition 4.7 in both directions. □

Remark 4.9. The Σ/Π type of an admissibly described class is well-defined: if Γ and Λ are admissible, and $\Gamma \equiv \Lambda$, then Γ has Σ-type if and only if Λ has Σ-type. This follows from $\Lambda_0 \equiv \Gamma_0$, and induction along the leftmost paths.

As in [GT], we can show that for admissible Γ, the class Γ has the separation property if and only if it has Π-type.

Remark 4.10. Corollary 4.8 implies that a very admissible description of a class (Definition 4.6) is almost unique. The only freedom to vary the description is when $\Lambda < \Gamma$ are very admissible; in that case, the descriptions $\Theta = \mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$ and $\Theta' = \mathrm{SU}_{\xi,n}(\check{\Gamma}, \Lambda)$ describe the same class. That is, we can exchange Θ_1 and Θ_2. By Remark 4.9, if we wish to specify a unique description, we can require that in this case, Θ_1 has Σ-type.

4.2. Ubiquity of admissible descriptions.

Proposition 4.11. *Every finitely describable class has a finite admissible description.*

Let Θ be a finite class description. How do we go about finding an admissible description equivalent to Θ? The main idea is the following. Consider all the classes Θ_σ, where σ is a leaf of S_Θ. Among these classes we can identify a maximal pair: some Γ such that $\Theta_\sigma \subseteq \Gamma$ or $\Theta_\sigma \subseteq \check{\Gamma}$ for all such σ. In most cases we will show that $\Theta \equiv \mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$, where $\xi = o(\Gamma)$. What is n? In light of Proposition 3.12, we will let n be the length of the longest descending sequence of S_Γ positions $p_1, p_2, \dots, p_n$ that alternates between Γ and $\check{\Gamma}$. There will be three possibilities:

1. Only Γ appears as Θ_σ (and $\check{\Gamma}$ does not): in this case $\Theta \equiv \Gamma$.
2. For some $n > 1$, there is an S_Θ-play alternating between Γ and $\check{\Gamma}$ of length n, starting with Γ, but all such alternating plays starting with $\check{\Gamma}$ have length at most $n-1$. In this case $\Theta \equiv \mathrm{SU}_{\xi,n-1}(\Gamma, \Gamma)$.
3. There is no such preference for Γ over $\check{\Gamma}$ (or the other way round): there is a $\Gamma/\check{\Gamma}$ alternating sequence of length n starting with Γ, and one of the same length starting with $\check{\Gamma}$, but no alternating sequences (of either kind) of length $n+1$. In this case $\Theta \equiv \mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$ for some $\Lambda < \Gamma$.

The main challenge will be to identify the class Λ in the third case. Naively, we would think that we should take the class obtained from Θ by removing all leaves τ of S_Θ with Θ_τ equivalent to one of Γ and $\check{\Gamma}$. Consider, however, trying to show that $\mathrm{SU}_{\xi,n}(\Lambda, \Gamma) \subseteq \Theta$. In the leaf-selection game, while player 1 extends the default at the root, player 2 can copy his moves. But once player 1 moves away, player 2 still needs to be able to produce a $\Gamma/\check{\Gamma}$ alternating play of length n (by choosing appropriate leaves of S_Θ). We know that there is such a play; but there is no reason to believe that such a play is still available to us after the moves made at the first part of the game. Ideally, we would need to restrict ourselves to a class Λ determined by some collection of leaves of S_Θ such that every S_Λ-play can be extended to a $\Gamma/\check{\Gamma}$ alternating play of length n. It is not clear, though, how to identify such a collection of leaves, and further, why player 2 would be able to win the game in the other direction (showing $\Theta \subseteq \mathrm{SU}_{\xi,n}(\Gamma,\Lambda)$).

The solution is to look first not for a collection of leaves of S_Θ, but at the collection of S_Θ-*positions* that can be extended by maximal alternating sequences. Among the classes appearing in these, we can identify a maximal pair of classes $\Gamma', \check{\Gamma}'$, and the process can repeat.

For this reason, we will need to extend the leaf-selection game, to accommodate restrictions on the kind of positions we allow in the game.

To take the first case above into account (where the ordinal level of the admissible description increases), we need to extend the notation S_Θ. Let ξ be a computable ordinal. For a class description Θ with $o(\Theta) \geqslant \xi$, define $S_{\Theta,\xi}$ as follows:

- If $o(\Theta) = \xi$ then $S_{\Theta,\xi} = S_\Theta$;
- If $o(\Theta) > \xi$ then $S_{\Theta,\xi}$ consists only of the root of T_Θ.

Note that both cases can be defined together as in the original definition of S_Θ, replacing $o(\Theta)$ by ξ.

$S_{\Theta,\xi}$-positions are defined as in Definition 3.2; when $o(\Theta) > \xi$, there is just one $S_{\Theta,\xi}$ position p, determined by taking τ^p to be the root of T_Θ. Note that these notions apply even when $o(\Theta) = \omega_1$.

Fixing ξ, in this proof, we let $\mathcal{P}$ and $\mathcal{Q}$ denote nonempty collections of $S_{\Theta,\xi}$-positions, for some Θ, that are upwards closed: if $p \in \mathcal{P}$ and $q \geqslant p$ then $q \in \mathcal{P}$.

Let Θ and Ξ be class descriptions with ordinal levels $\geqslant \xi$; let $\mathcal{P}$ be a nonempty, upwards closed collection of $S_{\Theta,\xi}$-positions, and let $\mathcal{Q}$ be such a collection of $S_{\Xi,\xi}$-positions. The game $G_{\mathtt{leaf}}(\mathcal{P},\mathcal{Q})$ is defined as the game $G_{\mathtt{leaf}}(\Theta,\Xi)$, except that the trees used are $S_{\Theta,\xi}$ and $S_{\Xi,\xi}$, and further, player 1 is only allowed to choose positions from $\mathcal{P}$, while player 2 must choose positions from $\mathcal{Q}$. We write

$$\mathcal{P} \leqslant \mathcal{Q}$$

if player 2 has a computable containment strategy in the game $G_{\mathtt{leaf}}(\mathcal{P},\mathcal{Q})$: one which guarantees an outcome (σ,ρ) satisfying $\Theta_\sigma \subseteq \Xi_\rho$. We write $\mathcal{P} \equiv \mathcal{Q}$ if $\mathcal{P} \leqslant \mathcal{Q}$ and $\mathcal{Q} \leqslant \mathcal{P}$.

We let $\mathcal{P}_\Theta$ denote the collection of all $S_{\Theta,\xi}$-positions. Proposition 3.12 implies:

Claim 4.11.1. If $o(\Theta), o(\Xi) \geqslant \xi$, then $\Theta \subseteq \Xi$ if and only if $\mathcal{P}_\Theta \leqslant \mathcal{P}_\Xi$.

(Observe that Proposition 3.12 covers all cases, whether $o(\Theta) = \xi$ or $o(\Theta) > \xi$, and similarly for Ξ.) We therefore write Θ in place of $\mathcal{P}_\Theta$, and so write $\Theta \leqslant \mathcal{Q}$, $\mathcal{P} \equiv \Xi$, etc.

Proposition 4.11 follows from:

Claim 4.11.2. Let Θ be a finite class description with $\xi = o(\Theta)$. For any nonempty upwards-closed collection $\mathcal{P}$ of S_Θ-positions there is an admissible class description Ξ with $\mathcal{P} \equiv \Xi$.

The notation implies that $o(\Xi) \geqslant o(\Theta)$. Claim 4.11.1 shows that Claim 4.11.2 implies Proposition 4.11.

For brevity, for an S_Θ-position p, let $\Theta_p = \Theta_{\tau^p}$. Claim 4.11.2 is proved by a double induction: first on the complexity of Θ, and then on the size of $D(\mathcal{P})$, where

$$D(\mathcal{P}) = \{\Theta_p \,:\, p \in \mathcal{P}\}\,.$$

Let Θ be a finite class description. If $o(\Theta) = \omega_1$ then Θ is admissible. Suppose, then, that $\xi = o(\Theta) < \omega_1$. By induction, we assume that for every leaf τ of S_Θ, Θ_τ is admissible; Proposition 3.17 ensures that this does not change S_Θ. Fix a nonempty, upwards closed collection $\mathcal{P}$ of S_Θ-positions.

We dispose of the easy case first.

Claim 4.11.3. Suppose that there is some maximal $\Gamma \in D(\mathcal{P})$: for all $\Gamma' \in D(\mathcal{P})$, $\Gamma' \subseteq \Gamma$. Then $\mathcal{P} \equiv \Gamma$.

Proof. There are computable containment strategies for Player 2 in both $G(\mathcal{P}, \Gamma)$ and $G(\Gamma, \mathcal{P})$, using constant plays. □

We assume henceforth that the hypothesis of Claim 4.11.3 fails. By the semi-linear-ordering property of finitely described classes, we obtain a maximal pair of classes in $D(\mathcal{P})$: some Γ such that $\Gamma, \check{\Gamma} \in D(\mathcal{P})$, and for all $\Gamma' \in D(\mathcal{P})$, $\Gamma' \subseteq \Gamma$ or $\Gamma' \subseteq \check{\Gamma}$.

Call a descending sequence $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_k$ from $\mathcal{P}$ a $\Gamma/\check{\Gamma}$ *sequence* if $\Theta_{p_i} \equiv \Gamma$ for odd i and $\Theta_{p_i} \equiv \check{\Gamma}$ for even i. Similarly, such a descending sequence is a $\check{\Gamma}/\Gamma$ *sequence* if $\Theta_{p_i} \equiv \check{\Gamma}$ for odd i and $\Theta_{p_i} \equiv \Gamma$ for even i.

We let n be the greatest such that there are both: a $\Gamma/\check{\Gamma}$ sequence of length n and a $\check{\Gamma}/\Gamma$ sequence of length n. Our assumption implies that $n \geqslant 1$.

We let $\mathcal{Q}$ be the collection of all $q \in \mathcal{P}$ such that there are both $\Gamma/\check{\Gamma}$ sequence $p_1, \ldots, p_n$ of length n such that $p_1 \leqslant q$, and a $\check{\Gamma}/\Gamma$ sequence $r_1, \ldots, r_n$ of length n such that $r_1 \leqslant q$. By definition, $\mathcal{Q}$ is upwards closed. By choice of n, the initial S_Θ position is in $\mathcal{Q}$, so $\mathcal{Q}$ is nonempty.

Claim 4.11.4. $D(\mathcal{Q}) \subsetneq D(\mathcal{P})$.

Proof. We claim that either $\Gamma \notin D(\mathcal{Q})$ or $\check{\Gamma} \notin D(\mathcal{Q})$. Otherwise, let $q_1 \in \mathcal{Q}$ with $\Theta_{q_1} \equiv \Gamma$ and $q_2 \in \mathcal{Q}$ with $\Theta_{q_2} \equiv \check{\Gamma}$. Since $q_1 \in \mathcal{Q}$, (q_1) can be extended by a $\check{\Gamma}/\Gamma$ sequence of length n; similarly, q_2 can be extended by a $\Gamma/\check{\Gamma}$ sequence of length n. This shows that in $\mathcal{P}$ there are both a $\Gamma/\check{\Gamma}$ and a $\check{\Gamma}/\Gamma$ sequence of length $n+1$, contradicting the definition of n. □

By induction, there is some admissible class description Λ (with $o(\Lambda) \geqslant \xi$) satisfying $\mathcal{Q} \equiv \Lambda$.

Claim 4.11.5. $\Lambda \subseteq \Gamma$ or $\Lambda \subseteq \check{\Gamma}$.

Proof. Without loss of generality, suppose that $\check{\Gamma} \notin D(\mathcal{Q})$. Then $\Lambda \subseteq \Gamma$. To see this, by Claim 4.11.1, it suffices to show that $\mathcal{Q} \subseteq \Gamma$. Since $o(\Gamma) > \xi$, this is witnessed by constant plays, as $\Upsilon \subseteq \Gamma$ for all $\Upsilon \in D(\mathcal{Q})$. □

It follows that one of $\mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$ or $\mathrm{SU}_{\xi,n}(\check{\Gamma}, \Lambda)$ is admissible; without loss of generality, suppose that $\check{\Gamma} \notin D(\mathcal{Q})$, so $\Xi = \mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$ is admissible.

The following claim then concludes the proof of Claim 4.11.2, and so of Proposition 4.11.

Claim 4.11.6. $\mathcal{P} \equiv \Xi$.

Proof. In $G_{\mathtt{leaf}}(\mathcal{P}, \Xi)$, as long as player 1 plays within $\mathcal{Q}$, player 2 remains above the default at the root, and responds according to her containment strategy in the game $G_{\mathtt{leaf}}(\mathcal{Q}, \Lambda)$ (note that this covers both cases $o(\Lambda) = \xi$ and $o(\Lambda) > \xi$). Once player 1 leaves $\mathcal{Q}$, player 2 chooses either Γ or $\check{\Gamma}$, moving only when she must. The first step out of $\mathcal{Q}$ "breaks the tie" between Γ and $\check{\Gamma}$, so even if that move results in some $\Theta_p < \Gamma$, player 2 can examine all future possibilities and choose a safe option between Γ and $\check{\Gamma}$.

In $G_{\mathtt{leaf}}(\Xi, \mathcal{P})$, as long as player 1 remains above the default at the root, player 2 plays his containment strategy in $G(\Lambda, \mathcal{Q})$; once player 1 chooses either Γ or $\check{\Gamma}$, player 2 can move to an alternating sequence of length n with the appropriate start. ☐

4.3. **Ranked Boolean formulas.** Selivanov gave an equivalent definition of the fine hierarchy, using ranked Boolean formulas. We show that the extended hierarchy has a similar characterisation. We consider a ranked language of propositional logic. Each propositional variable is assigned a *rank* (or *level*), which is a computable ordinal. For a ranked variable v, we let $r(v)$ denote the rank of v.

A *ranked Boolean formula* is a (finite) propositional formula using ranked variables. For a ranked Boolean formula ψ, let:

- V_ψ denote the set of variables appearing in ψ; and
- $B_\psi \colon \{0,1\}^{V_\psi} \to \{0,1\}$ be the "truth table" of ψ.

Definition 4.12. Let ψ be a ranked Boolean formula. A *ψ-name* N consists of a choice of a $\Sigma^0_{1+r(v)}$ set A_v for every variable $v \in V_\psi$. The set named by a ψ-name N is the result of applying the truth-table B_ψ to this choice:

$$N(x) = B_\psi(A_{v_1}(x), A_{v_2}(x), \ldots, A_{v_m}(x))$$

where $v_1, \ldots, v_m$ are the variables appearing in ψ.

The *class* $\mathcal{C}_\psi$ *defined by* ψ is the collection of all sets that have ψ-names.

Note that by definition, if ψ and ψ' are logically equivalent (meaning $V_\psi = V_{\psi'}$ and $B_\psi = B_{\psi'}$) then $\mathcal{C}_\psi = \mathcal{C}_{\psi'}$.

Proposition 4.13. *A collection of subsets of* $\mathbb{N}$ *is a finitely described class if and only if it is* $\mathcal{C}_\psi$ *for some ranked Boolean formula* ψ.

In one direction, we give explicit names to the Boolean operations from which we can build all admissibly defined classes. We recall the following definitions (see for example [Lou83]).

- Let Ξ and Γ be classes. The class $\mathrm{Sep}(\Xi, \Gamma)$ is the class of all sets of the form $(A \cap C) \cup (B \cap C^{\complement})$, where $B \in \Gamma$, $A \in \check{\Gamma}$, and $C \in \Xi$.
- Let Ξ, Γ and Λ be classes. The class $\mathrm{BiSep}(\Xi, \Gamma, \Lambda)$ is the class of all sets of the form $(A \cap C_1) \cup (B \cap C_2) \cup (D \cap (C_1 \cup C_2)^{\complement})$, where $A \in \Gamma$, $B \in \check{\Gamma}$, $D \in \Lambda$ and $C_1, C_2 \in \Xi$ are disjoint.

Lemma 4.14. Let $\Theta = \mathrm{SU}_{\xi,n}(\Gamma, \Gamma)$, where Γ is a finite description with $o(\Gamma) > \xi$. The class described by Θ is $\mathrm{Sep}(D_n(\Sigma^0_{1+\xi}), \Gamma)$.

Proof. As mentioned above, Θ is equivalent to the class in Fig. 5, which has only two outcomes, 0 and 1. Let N be a Θ-name. We let C be the set of $x \in \mathbb{N}$ such that $\ell^N(x) \geqslant 1$. By Proposition 2.4, C is $D_n(\Sigma^0_{1+\xi})$; letting $A = N_1$ and $B = N_0$ shows that $N \in \mathrm{Sep}(D_n(\Sigma^0_{1+\xi}), \Gamma)$.

In the other direction, let $P \in \mathrm{Sep}(D_n(\Sigma^0_{1+\xi}), \Gamma)$, given by sets A, B, C as in the definition. Let N_0 be a Γ-name for B and N_1 be a $\check{\Gamma}$-name for A. To define a Θ-name N, we let $f^N_\diamondsuit$ be a ξ, n-enumeration of C (Proposition 2.4). ☐

For the following, we modify the definition of BiSep as follows:

- Let Ξ, Γ and Λ be classes. The class $\mathrm{BiSep}^*(\Xi, \Gamma, \Lambda)$ is the class of all sets of the form $(A \cap (C_1 \backslash C_2)) \cup (B \cap (C_2 \backslash C_1)) \cup (D \cap (C_1 \cup C_2)^\complement)$, where $A \in \Gamma$, $B \in \check{\Gamma}$, $D \in \Lambda$ and $C_1, C_2 \in \Xi$.

Lemma 4.15. Let $\Theta = \mathrm{SU}_{\xi,n}(\Gamma, \Lambda)$, where $o(\Gamma) > \xi$, $o(\Lambda) \geqslant \xi$, and $\Lambda < \Gamma$ are finite descriptions. The following coincide:

(1) $\mathrm{BiSep}(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda)$;
(2) $\mathrm{BiSep}^*(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda)$;
(3) The class described by Θ.

Proof. This extends [GT, Proposition 4.6]. We show three containments. One is immediate: $\mathrm{BiSep}(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda) \subseteq \mathrm{BiSep}^*(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda)$. Another is also fairly simple: $\Theta \subseteq \mathrm{BiSep}(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda)$. To see this, given a Θ-name N, for $i = 1, 2$ let $C_i = \{x \in \mathbb{N} : \ell^N(x) \geqslant i\}$, $A = N_1$, $B = N_2$, and $D = N_0$. Then C_1 and C_2 are both $D_n(\Sigma^0_{1+\xi})$, as witnessed by modifications of $f^N_\diamondsuit$, so the sets defined show that $N \in \mathrm{BiSep}(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda)$.

For the last containment, $\mathrm{BiSep}^*(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda) \subseteq \Theta$, we use the stage comparison argument. First note that since $\Lambda < \Gamma$, we cannot have $\Gamma = \{\varnothing\}$ or $\Gamma = \{\mathbb{N}\}$; and so, $\varnothing \in \Gamma, \check{\Gamma}$. Let $P \in \mathrm{BiSep}^*(D_n(\Sigma^0_{1+\xi}), \Gamma, \Lambda)$, witnessed by sets A, B, D, C_1, C_2. Let g_1 and g_2 be (ξ, n)-enumerations of C_1 and C_2 (Proposition 2.4). Let N_0 be a Λ-name for D. Since $o(\Gamma) > \xi$, and since $\Lambda < \Gamma$, by Proposition 2.10, we let N_1 be a Γ-name for the set

$$(A \cap (C_1 \backslash C_2)) \cup (D \cap (C_1 \cup C_2)^\complement);$$

That is, N_1 behaves as A on $C_1 \backslash C_2$, as D on $(C_1 \cup C_2)^\complement)$, and as $\varnothing$ on C_2. We similarly let N_2 be a $\check{\Gamma}$ name for $(B \cap (C_2 \backslash C_1)) \cup (D \cap (C_1 \cup C_2)^\complement)$. To specify N it remains to define $f^N_\diamondsuit$. For each x and s, if for all $t \leqslant_\xi s$ we have $g_1(x, s) = g_2(x, s) = 0$, then we set $f^N_\diamondsuit(x, s) = 0$. Otherwise, let $t(s)$ be the least $t \leqslant_\xi s$ such that either $g_1(x, s) = 1$ or $g_2(x, s) = 1$. If $g_1(x, t(s)) = 1$ set $i(s) = 1$, otherwise set $i(s) = 2$. Then set $f^N_\diamondsuit(x, s) = i(s)$ if $g_{i(s)}(x, s) = 1$, otherwise let $f^N_\diamondsuit(x, s) = 3 - i(s)$. If $x \in C_1 \backslash C_2$ then $N(x) = N_1(x) = A(x)$; if $x \in C_2 \backslash C_1$ then $N(x) = N_2(x) = B(x)$. If $x \in (C_1 \cup C_2)^\complement$ then we may have $\ell^N(x)$ extending either 0, 1 or 2; but in each case we will have $N(x) = D(x)$. If $x \in C_1 \cap C_2$ then $\ell^N(x)$ will extend either 1 or 2, but in either case we will have $N(x) = 0$. The number of mind changes of $f^N_\diamondsuit$ on any x will be bounded by n, as the number of mind changes of both g_1 and of g_2 is bounded by n. ☐

Proof of Proposition 4.13. First, we show that any finitely described class is $\mathcal{C}_\psi$ for some ranked Boolean formula ψ. We use Proposition 4.11 and Lemma 4.5, and use induction over the complexity of very admissible descriptions (Definition 4.6). The base of the induction are the class descriptions with $o(\Gamma) = \omega_1$, i.e., the classes $\{\varnothing\}$ and $\{\mathbb{N}\}$. For these, we allow the Boolean formulas $\top$ and $\bot$, that have no variables.

For the inductive step, we are given a very admissible class description $\Theta = \mathrm{SU}_{\xi,n}(\Gamma,\Lambda)$, and by induction assume that we have ranked Boolean formulas ψ_Λ, ψ_Γ, and $\psi_{\check{\Gamma}}$, that define the same classes as Λ, Γ and $\check{\Gamma}$. We assume that these three formulas share no common variables (this is why we did not take $\psi_{\check{\Gamma}} = \neg\psi_\Gamma$). If $\Lambda = \Gamma$ then we use Lemma 4.14; we take a formula φ that defines the class $D_n(\Sigma^0_{1+\xi})$, again with its own variables, and use φ, ψ_Γ and $\psi_{\check{\Gamma}}$ to write a formula that defines $\mathrm{Sep}(D_n(\Sigma^0_{1+\xi}),\Gamma)$. We used distinct variables so that we could choose sets $B \in \Gamma$, $A \in \check{\Gamma}$, and $C \in D_n(\Sigma^0_{1+\xi})$ independently of each other.

If $\Lambda < \Gamma$ then we use two formulas φ_1 and φ_2, each with their own variables, both defining $D_n(\Sigma^0_{1+\xi})$, and use these and the other given formulas to write a formula that defines $\mathrm{BiSep}^*(D_n(\Sigma^0_{1+\xi}),\Gamma,\Lambda)$, and appeal to Lemma 4.15. Note that we can't use BiSep directly, since a ranked Boolean formula cannot force the chosen C_1 and C_2 to be disjoint.

It remains to show that each class $\mathcal{C}_\psi$ has a finite description. To see this, we show, in fact, that for any ranked Boolean formula ψ, letting $o(\psi) = \min\{r(v) : v \in V_\psi\}$, the class $\mathcal{C}_\psi$ has a finite description Γ with $o(\Gamma) \geqslant o(\psi)$.

We use induction on the number of variables of ψ. Again, the base of the induction are the formulas $\top$ and $\bot$, that have no variables, and we already observed that they define finitely described classes. (We can set $o(\top) = o(\bot) = \omega_1$.)

Let ψ be a ranked Boolean formula which has variables. Let $\xi = o(\psi)$. Let $v_0,\dots,v_{k-1}$ be a list of those $v \in V_\psi$ with $r(v_i) = \xi$. Let $W = V_\psi \backslash \{v_1,\dots,v_k\}$. For any $\rho \in 2^k = \{0,1\}^{\{0,1,\dots,k-1\}}$, let $\bar{v}^\rho = \bigwedge_{i<k} v_i^{\rho(i)}$, where of course $v^1 = v$ and $v^0 = \neg v$. Find ranked Boolean formulas ψ_ρ for each such ρ, with $V_{\psi_\rho} = W$, such that ψ is logically equivalent to

$$\bigvee_{\rho\in 2^k} (\bar{v}^\rho \wedge \psi_\rho).$$

By induction, for each ρ there is a finite description Γ_ρ of $\mathcal{C}_{\psi_\rho}$, with $o(\Gamma_\rho) > \xi$.

We let Γ be the class description obtained by first taking the full binary tree $2^{\leqslant k}$ of height k, with all internal nodes of labels $\xi_\sigma = \xi$ and $\eta_\sigma = 1$, and then attaching Γ_ρ at the node ρ. So the notation Γ_ρ is appropriate. Since $o(\Gamma_\rho) > \xi$, this is a valid class description. We claim that Γ is a description of $\mathcal{C}_\psi$.

To see this, first, let $B \in \mathcal{C}_\psi$, obtained by a choice of sets A_v for $v \in V_\psi$. For $\rho\colon \{1,\dots,k\} \to \{0,1\}$, let B_ρ be the set in $\mathcal{C}_{\psi_\rho}$ obtained by choosing the same A_v for $v \in W$. Let N_ρ be a Γ^ρ-name for B_ρ. Then a Γ-name M for B is obtained by setting, for each $\sigma \in 2^{<k}$, the node σ to follow $A_{v_{|\sigma|}}$. That is, if f_i is a ξ-enumeration of A_{v_i} (for $i < k$), then for each $\sigma \in \{0,1\}^i$ we let $f^M = f_i$.

The other direction is a bit trickier. We are given a Γ-name M. This gives a subname M_ρ for each $\rho \in 2^k$ (a Γ_ρ-name), and ξ-enumerations f^M_σ for $\sigma \in 2^{<k}$. One difficulty is that there is no reason to believe that f^M_σ and $f^M_{\sigma'}$ are enumerations of the same $\Sigma^0_{1+\xi}$ set when $|\sigma| = |\sigma'|$. Another is that while we can simulate each M_ρ by a ψ_ρ-name, all of the formulas ψ_ρ share the same variables (those in W), and the ψ_ρ-names need not agree on the assignments of these variables. Both of these problems can be summarised by saying that M is "too independent"; approximations in one node do not need to correspond to approximations in others.

The first problem is not actually a real problem; we can take unions and obtain the same effect. This works since the "movement" on 2^k is always left-to-right, as we set $\eta_\sigma = 1$ for $\sigma \in 2^{<k}$. For the second problem, we use the the higher ordinal level of the variables in W.

For $\rho \in 2^k$ let C_ρ be the set of $x \in \mathbb{N}$ such that $\ell^M(x) \geqslant \rho$. These sets are $\Delta^0_{1+\xi+1}$. For $i < k$, we let

$$A_{v_i} = \bigcup \{C_\rho \,:\, \rho(i) = 1\}.$$

These sets are $\Sigma^0_{1+\xi}$: $x \in A_{v_i}$ if *some* σ of length i sends x to the non-default outcome. More formally, A_{v_i} is the set of x such that for some $\sigma \in 2^i$, for some ξ-true stage s, we have $f^M(x,s) = 1$. The point now is that for all $x \in \mathbb{N}$, $x \in C_\rho$ if and only if the assignment $v_i \mapsto A_{v_i}$ makes $\bar{v}^\rho$ true on x.

For $\rho \in 2^k$, we are given sets $D^\rho_w \in \Sigma^0_{1+r(w)}$, for the variables $w \in W$, that give a ψ_ρ-name N_ρ equivalent to M_ρ. For $w \in W$ we then let

$$A_w = \bigcup_{\rho \in 2^k} (D^\rho_w \cap C_\rho).$$

Since $r(w) > \xi$, each set C_ρ is $\Sigma^0_{1+r(w)}$, and so D_w is $\Sigma^0_{1+r(w)}$. Then the assignment of A_{v_i} for $i < k$ and A_w for $w \in W$ gives a ψ-name that is equivalent to M. □

Remark 4.16. The proof of Proposition 4.11 can be used to directly translate from ranked Boolean formulas to admissible descriptions, rather than merely finite descriptions; instead of collections of S_Θ-positions, we look at "initial segments" of the truth table of ψ. It is more difficult, though, to pass from a finite class description to a ranked Boolean formula, without considering admissible descriptions first.

4.4. **The fine hierarchy.** Selivanov showed in [Sel95] that the classes in the fine hierarchy are precisely the classes $\mathcal{C}_\psi$, where ψ is a ranked Boolean formula where all the variables have finite ranks. An examination of the proofs of Proposition 4.11 and Proposition 4.13 shows that the translations between finite descriptions, admissible descriptions, and ranked Boolean formulas, all preserve ranks. Hence:

Proposition 4.17. *The following are equivalent for a finitely described class* Γ*:*

- Γ *has a finite description in which* ξ_σ *is finite for all internal* σ*;*
- Γ *has an admissible description in which* ξ_σ *is finite for all internal* σ*;*
- $\Gamma = \mathcal{C}_\psi$*, where* ψ *is a ranked Boolean formula in which all variables have finite ranks.*

Corollary 4.18. *The fine hierarchy forms an initial segment of the extended fine hierarchy.*

That is, if Γ is in the fine hierarchy, Λ is in the extended fine hierarchy, and $\Lambda \leqslant \Gamma$, then Λ is in the fine hierarchy.

Proof. All classes in the fine hierarchy consist of subclasses of the arithmetic sets. On the other hand, if Γ is in the extended fine hierarchy but not the fine hierarchy, then Γ has an admissible description in which we have $\xi_\sigma \geqslant \omega$ for some internal σ. That is, $\omega \leqslant o(\Gamma_\sigma) < \omega_1$. Since Γ_σ is efficient, by Corollary 2.11, $\Delta^0_\omega \subseteq \Gamma_\sigma$, so $\Delta^0_\omega \subseteq \Gamma$. □

5. The extended fine hierarchy

As Wadge did for Borel classes, and Selivanov for the fine hierarchy, we can calculate the ordinal height of described classes — the order-type of the class $\Gamma \cup \check{\Gamma}$ under proper containment. The level will be read recursively off admissible descriptions. This will give an alternative proof that the extended fine hierarchy is well-founded.

5.1. **The modified Veblen functions.** The heights of Borel Wadge classes are calculated using the ω_1-based Veblen functions. For example, the height of $\mathbf{\Sigma}^0_2$ is ω_1, of $\mathbf{\Sigma}^0_3$ is $\omega_1^{\omega_1}$, and so on. A similar phenomenon holds for the admissibly described classes, except that we use the ω-based Veblen functions. A slight modification at finite inputs allows for a uniform treatment.

We define a sequence of closed and unbounded classes of ordinals. We start with

$$C_0 = \{\alpha \,:\, (\forall \beta, \gamma < \alpha)\ \beta + \gamma < \alpha\}\,,$$

the class of ordinals closed under ordinal addition. The class C_0 contains 0, 1, and all infinite ordinal powers of ω (and in fact, 1 is a power of ω): $C_0 = \{0\} \cup \{\omega^\beta \,:\, \beta \geqslant 0\}$.

Now given the closed unbounded class C_γ, we let

$$C_{\gamma+1} = \{\alpha \in C_\gamma \,:\, \operatorname{otp}(C_\gamma \cap \alpha) = \alpha\}\,;$$

and for limit γ, we of course let

$$C_\gamma = \bigcap_{\beta<\gamma} C_\beta.$$

Note that for all γ, $0, 1 \in C_\gamma$. We let φ_γ be the increasing enumeration of the elements of C_γ.

Example 5.1. For all γ, $\varphi_\gamma(0) = 0$ and $\varphi_\gamma(1) = 1$. We have $\varphi_0(2) = \omega$, $\varphi_0(3) = \omega^2$, $\varphi_0(\omega) = \omega^\omega$, and in general, $\varphi_0(1 + \alpha) = \omega^\alpha$.

The class C_1 contains 0,1 and all ordinals α with $\omega^\alpha = \alpha$. In other words, all ordinals α such that $\varphi_0(\alpha) = \alpha$. Hence, $\varphi_1(2) = \varepsilon_0$, and in general, $\varphi_1(2 + \alpha) = \varepsilon_\alpha$.

In general, $C_{\gamma+1}$ is the class of all ordinals α with $\varphi_\gamma(\alpha) = \alpha$, and so $\varphi_2(2)$ is the least ordinal ζ with $\varepsilon_\zeta = \zeta$.

Γ_0 is the least ordinal ζ with $\zeta = \varphi_\zeta(2)$.[5]

Using the modified Veblen functions and Cantor's normal form for ordinals, we define "increment functions" θ_β defined as follows.

Definition 5.2. For an ordinal $\beta > 0$ write

$$\beta = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}$$

with $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_k$. We let

$$\theta_\beta = \varphi_{\alpha_1} \circ \varphi_{\alpha_2} \circ \cdots \circ \varphi_{\alpha_k}.$$

We also let θ_0 be the identity function on the ordinals.

If $\alpha > \alpha'$ then $C_\alpha = \operatorname{range} \varphi_\alpha$ consists of fixed points of $\varphi_{\alpha'}$, and so $\varphi_\alpha = \varphi_{\alpha'} \circ \varphi_\alpha$. This shows:

Lemma 5.3. For any two ordinals β and γ, $\theta_{\beta+\gamma} = \theta_\beta \circ \theta_\gamma$.

[5]Here we are using the notation "Γ_0" as is standard in the ordinal literature, not as we have been using it to denote a sub-description of a description Γ; this usage will not be used outside examples.

5.2. **Jumps of classes.**

Definition 5.4. For a computable class description Γ and a computable ordinal β, we let $\Gamma^{(\beta)}$ be the class description obtained from Γ by replacing each ξ^Γ_σ by $\beta+\xi^\Gamma_\sigma$.

Lemma 5.5. Let Γ be a computable class description.

(a) For all α and β, $(\Gamma^{(\beta)})^{(\alpha)} = \Gamma^{(\alpha+\beta)}$.
(b) For all $\sigma \in T_\Gamma$ and all β, $(\Gamma_\sigma)^{(\beta)} = (\Gamma^{(\beta)})_\sigma$.

For any class description Γ, because for each internal $\sigma \in T_\Gamma$ we have $\xi^\Gamma_\sigma \geqslant o(\Gamma)$, for all $\zeta \leqslant o(\Gamma)$ there is a (unique) class description $\Gamma^{(-\zeta)}$ satisfying $(\Gamma^{(-\zeta)})^{(\zeta)} = \Gamma$; for internal $\sigma \in T_\Gamma$, $\xi^{\Gamma^{(-\zeta)}}_\sigma$ is the unique ordinal ε satisfying $\zeta+\varepsilon = \xi^\Gamma_\sigma$. In particular, $o(\Gamma^{(-o(\Gamma))}) = 0$.

Lemma 5.6. For any two finite class descriptions Γ and Λ, and any computable β,

$$\Gamma \subseteq \Lambda \iff \Gamma^{(\beta)} \subseteq \Lambda^{(\beta)}.$$

Proof. By Lemma 5.5, $S_{\Gamma^{(\beta)}} = S_\Gamma$. The result then follows from Proposition 3.12, and induction on the complexity of the pair (Γ, Λ). ☐

As a result, Γ is admissible if and only if $\Gamma^{(\beta)}$ is admissible. Indeed:

$\Theta \equiv \mathrm{SU}_{\xi,n}(\Gamma,\Lambda)$ if and only if $\Theta^{(\beta)} \equiv \mathrm{SU}_{\beta+\xi,n}(\Gamma^{(\beta)},\Lambda^{(\beta)})$.

In particular, Γ is very admissible (Definition 4.6) if and only if $\Gamma^{(\beta)}$ is very admissible.

5.3. **Assigning heights to very admissible descriptions.** For every very admissible class description Γ, we define an ordinal $\delta(\Gamma) > 0$. We will verify that $\delta(\Gamma)$ is the height of the class defined by Γ (with a +1 offset for finite heights). The definition is by recursion on the complexity of the description.

(i) If $o(\Gamma) = \omega_1$ then $\delta(\Gamma) = 1$.
(ii) If $o(\Gamma) = 0$ then

$$\delta(\Gamma) = \delta(\Gamma_1)\cdot \eta^\Gamma + \delta(\Gamma_0).$$

That is, if $\Gamma = \mathrm{SU}_{\xi,n}(\Theta,\Lambda)$ then $\delta(\Gamma) = \delta(\Theta)\cdot n + \delta(\Lambda)$.
(iii) If $0 < o(\Gamma) < \omega_1$ then we let

$$\delta(\Gamma) = \theta_{o(\Gamma)}(\delta(\Gamma^{(-o(\Gamma))})).$$

Example 5.7. Let $\mathbf{0}$ denote the trivial class description of $\{\varnothing\}$, and let $\mathbf{1}$ be the trivial class description of $\{\mathbb{N}\}$. By definition, $\delta(\mathbf{0}) = \delta(\mathbf{1}) = 1$. Let $\Gamma = \mathrm{SU}_{0,1}(\mathbf{0},\mathbf{0})$; then $\delta(\Gamma) = \delta(\mathbf{0})\cdot 1 + \delta(\mathbf{0}) = 2$. Note that Γ names the class Σ^0_1. Similarly, for $n \geqslant 1$, let $\Gamma = \mathrm{SU}_{0,n}(\mathbf{0},\mathbf{0})$ be the description of the class of $D_n(\Sigma^0_1)$, the n-c.e. sets; then $\delta(\Gamma) = n+1$.

By induction on the complexity of Γ we observe:

Lemma 5.8. For any very admissible class description Γ, $\delta(\Gamma) = \delta(\check{\Gamma})$.

Lemma 5.3 implies:

Lemma 5.9. For any very admissible Γ and computable ordinal β,

$$\delta(\Gamma^{(\beta)}) = \theta_\beta(\delta(\Gamma)).$$

The range of any θ_β, for any $\beta > 0$, is contained in C_0, and so:

Lemma 5.10. For any very admissible class description Γ, if $o(\Gamma) > 0$ then $\delta(\Gamma)$ is a power of ω.

5.4. **Assigning heights to classes.** The following proposition implies that we can unambiguously define δ of a finitely described class.

Proposition 5.11. *For very admissible class descriptions Γ and Λ, if $\Gamma < \Lambda$ then $\delta(\Gamma) < \delta(\Lambda)$. If $\Gamma \equiv \Lambda$, then $\delta(\Gamma) = \delta(\Lambda)$.*

One step will require the following special case:

Lemma 5.12. Let $\Gamma = \mathrm{SU}_{\xi,n}(\Theta, \Lambda)$ be very admissible. Then $\delta(\Theta) < \delta(\Gamma)$.

Proof. If $\xi = 0$ then this follows from the definition, since $\delta(\Lambda) > 0$ (and $n \geqslant 1$). If $\xi > 0$, then as $\Gamma^{(-\xi)} = \mathrm{SU}_{0,n}(\Theta^{(-\xi)}, \Lambda^{(-\xi)})$ (Lemma 5.5), we obtain $\delta(\Theta^{(-\xi)}) < \delta(\Gamma^{(-\xi)})$; now apply Lemma 5.9, noting that θ_ξ is strictly increasing. □

Note that without Proposition 5.11, we cannot yet conclude that $\delta(\Lambda) < \delta(\Gamma)$.

Proof of Proposition 5.11. The proof is by induction on the complexity of the pair (Γ, Λ). Note that if the proposition has been proved for some pair (Γ, Λ), then by the semi-linear-ordering property, if $\Gamma \subseteq \Lambda$ then $\delta(\Gamma) \leqslant \delta(\Lambda)$.

Let $\xi = \min\{o(\Gamma), o(\Lambda)\}$. By Lemma 5.9, $\delta(\Gamma) = \theta_\xi(\Gamma^{(-\xi)})$ and $\delta(\Lambda) = \theta_\xi(\Lambda^{(-\xi)})$. Since θ_ξ is strictly increasing, by Lemma 5.6, it suffices to show the appropriate relation for $\delta(\Gamma^{(-\xi)})$ and $\delta(\Lambda^{(-\xi)})$. Thus we may assume $\xi = 0$. We assume that $\Gamma \subseteq \Lambda$. Now we consider the cases. Note that only in the last one we have equality of the classes.

If $0 = o(\Gamma) < o(\Lambda)$, then $\delta(\Gamma) = \delta(\Gamma_1) \cdot \eta^\Gamma + \delta(\Gamma_0)$. Since $\Gamma_0, \Gamma_1 \subseteq \Gamma < \Lambda$, we have $\delta(\Gamma_0), \delta(\Gamma_1) < \delta(\Lambda)$ by induction. Since $o(\Lambda) > 0$, by Lemma 5.10, $\delta(\Lambda)$ is closed under ordinal addition, so $\delta(\Gamma) < \delta(\Lambda)$.

If $o(\Gamma) > o(\Lambda) = 0$, then by Proposition 3.12, $\Gamma \subseteq \Lambda_1$ or $\Gamma \subseteq \check{\Lambda}_1$. Since $\delta(\check{\Gamma}_1) = \delta(\Gamma_1)$ (Lemma 5.8), by induction, $\delta(\Gamma) \leqslant \delta(\Lambda_1)$; now apply Lemma 5.12.

Suppose then that $o(\Gamma) = 0 = o(\Lambda)$. If $\Gamma_1 < \Lambda_1$, then by induction, $\delta(\Gamma_1) < \delta(\Lambda_1)$; since $\Gamma_0 \subseteq \Gamma_1$, we also have $\delta(\Gamma_0) < \delta(\Lambda_1)$. As $o(\Lambda_1) > 0$, $\delta(\Lambda_1)$ is closed under addition (Lemma 5.10), so $\delta(\Gamma) < \delta(\Lambda_1) < \delta(\Lambda)$.

Suppose that $\{\Gamma_1, \check{\Gamma}_1\} \equiv \{\Lambda_1, \check{\Lambda}_1\}$. Let $\gamma = \delta(\Gamma_1) = o(\Lambda_1)$ (by induction). Also by induction, $\delta(\Gamma_0) \leqslant \gamma$.

If $\eta^\Gamma < \eta^\Lambda$, then $\delta(\Gamma) = \gamma \cdot \eta^\Gamma + \delta(\Gamma_0) \leqslant \gamma \cdot (\eta^\Gamma + 1) < \delta(\Lambda)$ (as $\delta(\Lambda_0) > 0$).

Suppose that $\eta^\Gamma = \eta^\Lambda$. If $\Gamma_0 < \Lambda_0$ then by induction, $\delta(\Gamma_0) < \delta(\Lambda_0)$, and then by definition we obtain $\delta(\Gamma) < \delta(\Lambda)$; if $\Gamma_0 \equiv \Lambda_0$ then by induction, $\delta(\Gamma_0) = \delta(\Lambda_0)$, and then by definition we obtain $\delta(\Gamma) = \delta(\Lambda)$.

By Proposition 4.7, this covers all the possibilities. □

By the semi-linear-ordering property, we immediately get a converse.

Corollary 5.13. *For very admissible class descriptions Γ and Λ, if $\delta(\Gamma) < \delta(\Lambda)$ then $\Gamma < \Lambda$. If $\delta(\Gamma) = \delta(\Lambda)$, then $\Gamma \equiv \Lambda$ or $\Gamma \equiv \check{\Lambda}$.*

Example 5.14. Let α be a computable ordinal. Then $\Sigma^0_{1+\alpha}$ is the α-jump of Σ^0_1, and so by definition (and Example 5.7) $\delta(\Sigma^0_{1+\alpha}) = \theta_\alpha(2)$. So:

- $\delta(\Sigma^0_2) = \theta_1(2) = \varphi_0(2) = \omega$; $\delta(\Sigma^0_3) = \theta_2(2) = \varphi_0(\varphi_0(2)) = \omega^\omega$, and similarly, $\delta(\Sigma^0_4) = \omega^{\omega^\omega}$, and so on.
- $\delta(\Sigma^0_\omega) = \theta_\omega(2) = \varphi_1(2) = \varepsilon_0$; $\delta(\Sigma^0_{\omega+1}) = \theta_{\omega+1}(2) = \varphi_1(\varphi_0(2)) = \varphi_1(\omega) = \varepsilon_\omega$, the ω^{th} fixed point of $\beta \mapsto \omega^\beta$; $\delta(\Sigma^0_{\omega+2}) = \varepsilon_{\omega^\omega}$, and so on.
- $\delta(\Sigma^0_{\omega^2}) = \varphi_2(2)$ is the least fixed point of $\beta \mapsto \varepsilon_\beta$, $\delta(\Sigma^0_{\omega^2+\omega})$ is the $\varepsilon_0{}^{\text{th}}$ fixed point of $\beta \mapsto \varepsilon_\beta$, etc.

- The least α such that $\delta(\Sigma^0_\alpha) = \alpha$ is the ordinal Γ_0.

There are no gaps:

Proposition 5.15. *For every computable ordinal $\varepsilon > 0$ there is a finitely described class Γ with $\delta(\Gamma) = \varepsilon$.*

To show this, we will require the following:

Lemma 5.16. For any finitely described class Γ, $o(\Gamma) > 0$ if and only if $\delta(\Gamma)$ is a power of ω.

Proof. One direction is Lemma 5.10. The other direction is proved by induction on the complexity of a very admissible description of Γ. Suppose that $o(\Gamma) = 0$. By Proposition 5.11, $\delta(\Gamma_0) \leqslant \delta(\Gamma_1)$. Since $\delta(\Gamma_0) > 0$,

$$\delta(\Gamma_1) \leqslant \delta(\Gamma_1) \cdot \eta^\Gamma < \delta(\Gamma) \leqslant \delta(\Gamma_1) \cdot (\eta^\Gamma + 1) < \delta(\Gamma_1) \cdot \omega.$$

Since $o(\Gamma_1) > 0$, $\delta(\Gamma_1)$ is a power of ω, and so there are no powers of ω strictly between $\delta(\Gamma_1)$ and $\delta(\Gamma_1) \cdot \omega$; so $\delta(\Gamma)$ is not a power of ω. □

Proof of Proposition 5.15. We proceed by induction on ε. The base case $\varepsilon = 1$ is immediate.

For $\varepsilon > 1$, a straightforward induction shows that $\varepsilon \notin C_{\varepsilon+1}$, so fix the least γ with $\varepsilon \notin C_\gamma$. By construction, γ cannot be a limit ordinal.

If $\gamma = \beta + 1$, then ε is in the range of φ_β but is not a fixed point of φ_β, hence $\varepsilon = \varphi_\beta(\alpha)$ for some $\alpha < \varepsilon$. By induction, $\alpha = \delta(\Lambda)$ for some finitely described class Λ. Let $\Gamma = \Lambda^{(\omega^\beta)}$. By Lemma 5.9, $\delta(\Gamma) = \theta_{\omega^\beta}(\delta(\Lambda)) = \varphi_\beta(\alpha) = \varepsilon$.

If $\gamma = 0$, then ε is not a power of ω. There are two cases. One is when ε is finite. This case is covered by Example 5.7. We assume then that ε is infinite. Write $\varepsilon = \rho_1 \cdot n + \rho_0$, where ρ_1 is the largest power of ω less than ε and $0 < \rho_0 < \rho_1$. By the inductive hypothesis, there are classes Λ_0 and Λ_1 with $\delta(\Lambda_i) = \rho_i$ for $i \in \{0, 1\}$. By Lemma 5.16, $o(\Lambda_1) > 0$. By Corollary 5.13, $\Lambda_0 \subseteq \Lambda_1$ or $\Lambda_0 \subseteq \check{\Lambda}_1$. So one of $\mathrm{SU}_{0,n}(\Lambda_1, \Lambda_0)$ and $\mathrm{SU}_{0,n}(\check{\Lambda}_1, \Lambda_0)$ is very admissible, and is as required. □

Corollary 5.17. *For any admissibly described class Γ,*

$$\{\Lambda \cup \check{\Lambda} : \Lambda \text{ is admissible and } \Lambda < \Gamma\}$$

is well-ordered under $\subseteq$, and its order-type is:

- $\delta(\Gamma)$, *if $\delta(\Gamma)$ is infinite;*
- $\delta(\Gamma) - 1$, *if $\delta(\Gamma)$ is finite.*

Example 5.18. Recall that $\Gamma^+ = \mathrm{SU}_{0,1}(\Gamma, \mathbf{0})$ is the successor of Γ in the extended fine hierarchy (Proposition 5.19). Thus, by Corollary 5.17, $\delta(\Gamma^+) = \delta(\Gamma) + 1$. Note that this can be deduced by the definition if $o(\Gamma) > 0$ (and Γ is very admissible of Σ-type), but is less obvious when $o(\Gamma) = 0$, in which case Γ^+ is not admissible.

5.5. **A characterisation of limit classes.** In [DGHTTa], a classification of the ambiguous classes of boldface described pointclasses is given based on the ordinals appearing along the leftmost path of a description. This determines the ordertype of the height of the class in the Wadge ordering: successor, limit of countable cofinality, or limit of uncountable cofinality. For the effective hierarchy, the classification is a little simpler, since we don't have different cofinalities at limit points.

Proposition 5.19. *Let Γ be a very admissible class description, and suppose that $o(\Gamma) < \omega_1$. Let ρ^* be the leftmost leaf of T_Γ. Then $\delta(\Gamma)$ is a successor ordinal if and only if for every $\sigma \prec \rho^*$, $\xi^\Gamma_\sigma = 0$.*

Proof. This is proved by induction on the length of ρ^*. By Lemma 5.10, we may assume that $o(\Gamma) = 0$, so $\delta(\Gamma) = \delta(\Gamma_1) \cdot \eta^\Gamma + \delta(\Gamma_0)$.

If $|\rho^*| = 1$, i.e., if $o(\Gamma_0) = \omega_1$, then $\delta(\Gamma_0) = 1$ is a successor. Otherwise, since Γ is admissible, we have $o(\Gamma_1) < \omega_1$; by Lemma 5.10, $\delta(\Gamma_1)$ is a limit ordinal, and so $\delta(\Gamma)$ is a limit if and only if $\delta(\Gamma_0)$ is a limit. The result then follows by induction. □

Here is a consequence. As we will see in the sequel [GQT], this explains the fact that classes whose heights are successors of limits do not contain any new Turing degrees.

Proposition 5.20. *Let Γ be a finitely described class, and suppose that $\delta(\Gamma)$ is a limit ordinal. Then Γ is closed under unions and intersections with Δ^0_2 sets.*

Proof. If $o(\Gamma) > 0$ then we appeal to Corollary 2.11; so we assume that $o(\Gamma) = 0$. Let Γ be a very admissible description of the class. Let σ^* be the leftmost leaf of S_Γ (Definition 3.2), and ρ^* the leftmost leaf of T_Γ. By Proposition 5.19, $\sigma^* \precneq \rho^*$ as $0 < o(\Gamma_{\sigma^*}) < \omega_1$.

Since Γ_{σ^*} is admissible, $\varnothing, \mathbb{N} \in \Gamma_{\sigma^*}$. Also, if $\tau \neq \sigma^*$ is a leaf of S_Γ, then $\tau = \sigma\hat{\ }m$ for some $\sigma \prec \sigma^*$ and $m \in \{1,2\}$, and $\Gamma_{\sigma\hat{\ }0} \subseteq \Gamma_\tau$ or $\Gamma_{\sigma\hat{\ }0} \subseteq \check{\Gamma}_\tau$; in either case, $\varnothing, \mathbb{N} \in \Gamma_\tau$. By Corollary 2.11, for any leaf τ of S_Γ, Γ_τ is closed under unions and intersections with Δ^0_2 sets. For any Γ-name N and any Δ^0_2 set B, for any leaf τ of S_Γ, let M_τ be a Γ_τ name for $N_\tau \cup B$; for internal $\sigma \in S_\Gamma$, let $(f^M_s, \beta^M_s) = (f^N_s, \beta^N_s)$. Then M is a Γ-name for $N \cup B$. The same argument holds for intersections. □

Remark 5.21. Note that the argument above does not show that Γ is closed under definition by cases at the level Δ^0_2, and indeed cannot; we mentioned that if $o(\Gamma) = 0$ then Γ cannot be closed under definition by cases at this level. The point is that with more than one Γ-name, we would not know which leaf of S_Γ to take (essentially, the choice is between Γ_1 and $\check{\Gamma}_1$). Once we get to a leaf, we would know which one we should have taken, but by then it is too late.

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