AN EMBEDDING INTO THE TURING DEGREES

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Abstract. We show that any locally countable partial ordering of size at most continuum and height at most 3, that has at most $\aleph_1$ many elements of depth 3, is embeddable into the Turing degrees.

1. Introduction

In 1963, Sacks conjectured [6, (C4)] that if P is a partial ordering of size at most continuum which is locally countable (every point has only countably many predecessors), then P is embeddable into the Turing degrees. Sacks showed that the conjecture holds assuming the continuum hypothesis, or assuming Martin's axiom. Whether the conjecture is a theorem of ZFC remains open.

Indeed, the conjecture remains open when restricted to well-founded partial orderings. There is a universal locally countable, well-founded partial ordering of size $\mathfrak{c} = 2^{\aleph_0}$: it can be built as a collection of ω_1 many levels, with each countable type over a collection of countable levels realised in each level above. Even restricting to finite levels results in open problems. Define a partial ordering to have *height at most* n if any chain in it has size at most n. Among locally countable partial orderings of height n and size at most $\mathfrak{c}$ there are again universal ones. An example, that can be found in [4, 3] (where it is credited to Higuchi), is as follows:

- $\mathbb{H}^0$ is an anti-chain of size $\mathfrak{c}$.
- $\mathbb{H}^{n+1}$ is obtained from $\mathbb{H}^n$ by adding, for every countable infinite subset A of the n^{th} level of $\mathbb{H}^n$, a point p_A which is above all points in A, the predecessors of any $a \in A$, and no other elements of $\mathbb{H}^n$.

Then $\mathbb{H}^n$ is universal for partial orderings of height at most $n+1$ and size $\leqslant \mathfrak{c}$. Kumar and Raghavan [5] showed that $\mathbb{H}^1$ is embeddable into the Turing degrees. They also defined a generalisation: for a cardinal κ, let $\mathbb{H}^n_\kappa$ be defined as $\mathbb{H}^n$ is, except that $\mathbb{H}^0_\kappa$ is an antichain of size κ. Kumar [4] showed that $\mathbb{H}^2_\omega$ is embeddable into the Turing degrees; the result for $\mathbb{H}^1 = \mathbb{H}^1_\mathfrak{c}$ follows, as it is isomorphic to the top 2 layers of $\mathbb{H}^2_\omega$. We remark that Kumar and Raghavan's work was also motivated by previous work by Higuchi, Lempp, Raghavan, and Stephan [1], who studied $\mathbb{H}^1$ in relation to the order dimension of the Turing degrees.

It is still open, whether $\mathbb{H}^2 = \mathbb{H}^2_\mathfrak{c}$ is always embeddable into the Turing degrees (this is Question 2.6 in [5]). Higuchi and Lutz [2] explained why this problem is difficult: they showed that there is a Borel embedding of $\mathbb{H}^1_\mathfrak{c}$ into the Turing degrees, but there is no Borel embedding of $\mathbb{H}^2_\mathfrak{c}$ into the Turing degrees: indeed, there is no embedding of $\mathbb{H}^2_\mathfrak{c}$ into the Turing degrees in which the image of the lowest level contains (the Turing degrees of reals in) a perfect set. It follows (assuming the

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consistency of some large cardinals) that Sacks's conjecture cannot be a theorem of ZF. For a survey of this and related questions, see [3].

In this paper, we show:

Theorem 1.1. $\mathbb{H}^2_{\omega_1}$ *is embeddable into the Turing degrees.*

The partial ordering $\mathbb{H}^2_{\omega_1}$ is again universal for a class of partial orderings. If P is a partial ordering, the the *depth* of an element $p \in P$ is the supremum of the sizes of chains in P with least element p.

Proposition 1.2. *Every partial ordering P satisfying:*

(i) $|P| \leqslant \mathfrak{c}$;
(ii) *P is locally countable;*
(iii) *The height of P is at most 3; and*
(iv) *P has at most $\aleph_1$ many elements of depth 3,*[1]

is embeddable into $\mathbb{H}^2_{\omega_1}$.

As a result, Theorem 1.1 implies that each such linear ordering is embeddable into the Turing degrees.

Proof of Proposition 1.2. We think of $\mathbb{H}^2_{\omega_1}$ as consisting of three disjoint levels: the lowest level is (some set of size) ω_1; the second, $[\omega_1]^{\aleph_0}$ (note the collection of *infinite* countable subsets of ω_1); the third, $[[\omega_1]^{\aleph_0}]^{\aleph_0}$. The ordering on $\mathbb{H}^2_{\omega_1}$ is the reflexive, transitive closure of the relation consisting of (x, A), where x is an element of the first or second level, A is an element of the next level up, and $x \in A$.

Let P be a partial ordering as discussed. For $i = 1, 2, 3$, let P_i be the collection of elements of P of depth i; so $\{P_1, P_2, P_3\}$ is a partition of P.

Fix disjoint countable and infinite sets $A, B \subseteq \omega_1$, and injective functions $f\colon P_2 \to [A]^{\aleph_0}$ and $g\colon P_1 \to [B]^{\aleph_0}$.

We define an embedding $p\colon P \to \mathbb{H}^2_{\omega_1}$ by steps. First, we define $p \restriction P_3$ to be any injective function from P_3 to $\omega_1 \setminus (A \cup B)$.

Then, for $y \in P_2$, we let

$$p(y) = \{p(x) : x <_P y\} \cup f(y);$$

and finally, for $z \in P_1$, we let

$$p(z) = \{p(y) : y <_P z \ \&\ y \in P_2\} \cup \{\{p(x) : x <_P z \ \&\ x \in P_3\} \cup g(z)\}.$$

To show that this is an embedding, first we check that p is injective; since it maps level-to-level, it suffices to see that $p \restriction P_1$, $p \restriction P_2$, and $p \restriction P_3$ are all injective. This uses the injectivity of f and g (and the fact that $p(x) \notin A \cup B$ for $x \in P_3$, and $p(y) \subseteq \omega_1 \setminus B$ for $y \in P_2$). That p preserves $<_P$ is immediate from its definition. Preservation of $\not<_P$ is done by cases. For example, suppose that $x \in P_3$, $z \in P_1$, and $p(x) < p(z)$. Then there is some $c \in p(z)$ such that $p(x) \in c$. If $c = p(y)$ for some $y <_P z$ in P_2, then $p(x) \in p(y)$ implies $x <_P y$, as $p(x) \notin f(y)$. Otherwise, $c = \{p(w) : w <_P z \ \&\ x \in P_3\} \cup g(z)$; since $p(x) \notin g(z)$, we must have $x <_P z$. The other cases are easier. □

In the rest of the paper, we prove Theorem 1.1.

[1] Note that (iv) is weaker than saying that P has at most $\aleph_1$ many minimal elements; there can be minimal elements of depth 2 or of depth 1.

2. Embedding $\mathbb{H}^2_{\omega_1}$

Our main construction will yield the following.

Theorem 2.1. *There are:*

- *For $\alpha < \omega_1$, a real $A_\alpha \in 2^\omega$;*
- *For all limit $\delta < \omega_1$ and all $\mathcal{B} \subseteq \{A_\alpha : \alpha < \delta\}$, a real $C^\delta_{\mathcal{B}} \in 2^\omega$, which is a Turing upper bound of $\mathcal{B}$,*

such that whenever:

- *$\delta_1, \dots, \delta_n$ are limit ordinals; and for $i \leqslant n$, $\mathcal{B}_i \subseteq \{A_\alpha : \alpha < \delta_i\}$; and*
- *$\alpha_1, \dots, \alpha_k < \omega_1$ are such that for all $j \leqslant k$, $A_{\alpha_j} \notin \mathcal{B}_1 \cup \dots \cup \mathcal{B}_n$,*

the set $\{A_{\alpha_1}, \dots, A_{\alpha_k}, C^{\delta_1}_{\mathcal{B}_1}, \dots, C^{\delta_n}_{\mathcal{B}_n}\}$ is Turing independent.

We now show how Theorem 2.1 implies Theorem 1.1. We use the presentation of $\mathbb{H}^2_{\omega_1}$ that was described in the proof of Proposition 1.2 above (the disjoint union of ω_1, $[\omega_1]^{\aleph_0}$, and $[[\omega_1]^{\aleph_0}]^{\aleph_0}$).

As the image of the lowest level, we take the Turing degrees of the reals A_α.

For the second level, for each $I \in [\omega_1]^{\aleph_0}$, let δ be the least limit ordinal $> \sup I$ and let $C_I = C^\delta_{\mathcal{B}}$, where $\mathcal{B} = \{A_\alpha : \alpha \in I\}$. We map $I \in \mathbb{H}^2_{\omega_1}$ to the Turing degree of C_I.

Theorem 2.1 implies that both levels are antichains, and that $C_I \geqslant_{\mathrm{T}} A_\alpha$ if and only if $\alpha \in I$.

We construct the third level by modifying an argument of Kumar in [3, Lemma 3.5].

Recall that X is a *Sacks upper bound* of an Turing ideal $\mathcal{J}$ if X is an upper bound of $\mathcal{J}$, and for all $Y \leqslant_{\mathrm{T}} X$ outside $\mathcal{J}$ there is some $Z \in \mathcal{J}$ such that $X \equiv_{\mathrm{T}} (Y, Z)$.

For $J \in [[\omega_1]^{\aleph_0}]^{\aleph_0}$, let $\mathcal{D}_J$ be the Turing ideal generated by $\{C_I : I \in J\}$.

We observe that for any such J, if X is any Sacks upper bound of $\mathcal{D}_J$, then:

- for all $I \in [\omega_1]^{\aleph_0} \smallsetminus J$, $C_I \nleqslant_{\mathrm{T}} X$;
- for all $\alpha \in \omega_1 \smallsetminus \bigcup J$, $A_\alpha \nleqslant_{\mathrm{T}} X$.

For let $Y = C_I$ (in the first case) or $Y = A_\alpha$ (in the second case). For any finite tuple $I_1, \dots, I_k \in J$, the set $\{Y, C_{I_1}, \dots, C_{I_k}\}$ is Turing independent. Hence, $Y \notin \mathcal{D}_J$, and further, since J is infinite, for all $Z \in \mathcal{D}_J$ there is some $W \in \mathcal{D}_J$ such that $W \nleqslant_{\mathrm{T}} (Z, Y)$. This implies that $Y \nleqslant_{\mathrm{T}} X$.

We also observe that if $J_1, J_2 \in [[\omega_1]^{\aleph_0}]^{\aleph_0}$ and X is any upper bound of $\mathcal{D}_{J_1}$, then $X \notin \mathcal{D}_{J_2}$. Again, this is because J_1 is infinite; no $Z \in \mathcal{D}_{J_2}$ can compute all $W \in \mathcal{D}_{J_1}$.

So it suffices to choose, for each $J \in [[\omega_1]^{\aleph_0}]^{\aleph_0}$, a Sacks upper bound of $\mathcal{D}_J$, such that the collection of all chosen upper bounds is a Turing antichain.

Enumerate $[[\omega_1]^{\aleph_0}]^{\aleph_0}$ as $(J_\gamma : \gamma < 2^{\aleph_0})$. By recursion on γ, We choose a Sacks upper bound X_γ for $\mathcal{D}_{J_\gamma}$. Let $\gamma < 2^{\aleph_0}$, and suppose that $X_{\gamma'}$ were chosen for all $\gamma' < \gamma$.

There is a perfect set P of Sacks upper bounds of $\mathcal{D}_{J_\gamma}$ ([6]; see also [3, Fact 2.7]). We claim that some $X \in P$ is Turing incomparable with each $X_{\gamma'}$ (for $\gamma' < \gamma$). Fix $\gamma' < \gamma$. Of course only countably many $X \in P$ are computable from $X_{\gamma'}$. On the other hand, since $X_{\gamma'} \notin \mathcal{D}_{J_\gamma}$, if $X_{\gamma'} \leqslant_{\mathrm{T}} X$ for some $X \in P$, then there is some $Z \in \mathcal{D}_{J_\gamma}$ such that $X \equiv_{\mathrm{T}} (X_{\gamma'}, Z)$. Since $\mathcal{D}_{J_\gamma}$ is countable, this shows that there are only countably many $X \in P$ that compute $X_{\gamma'}$. Overall, for each $\gamma' < \gamma$, $X_{\gamma'}$ is Turing comparable with only countably many $X \in P$. Since $|P| = 2^{\aleph_0}$ and $|\gamma| \cdot \aleph_0 < 2^{\aleph_0}$, an $X_\gamma \in P$ as required can be chosen.

3. Adding generic upper bounds

We will use notions of forcing where the conditions are compact subsets of (essentially) Cantor space.

Definition 3.1. Let $\mathbb{P}$ be a countable notion of forcing. We say that $\mathbb{P}$ is *forcing with compact sets* if for some 0-dimensional effectively compact space $X_{\mathbb{P}}$, the conditions in $\mathbb{P}$ are nonempty closed subsets of $X_{\mathbb{P}}$, ordered by containment.

Example 3.2. The simplest example is Cohen forcing, denoted by $\mathbb{C}$, consisting of all clopen subsets of $X = 2^{\omega}$.

3.1. Restricted genericity.

We will use a restricted form of genericity for our notions of forcing.

Definition 3.3. Let $\mathbb{P}$ be a notion of forcing with compact sets. Let $Z \in 2^{\omega}$ be an oracle. A real $G \in X_{\mathbb{P}}$ is $\mathbb{P}$*-generic relative to* Z if for every $U \subseteq X_{\mathbb{P}}$ that is $\Sigma^0_1(Z)$, either

(i) $G \in U$; or
(ii) there is some $P \in \mathbb{P}$ such that $G \in P$ and $P \cap U = \emptyset$.

Let us make a few remarks.

1. This is a restricted notion of genericity. There may be dense subets D of $\mathbb{P}$ such that for no oracle Z does $\mathbb{P}$-genericity relative to Z ensure that G is an element of some $P \in D$. See Remark 3.13.

2. The definition applies to all oracles Z, even when Z does not compute all the conditions in $\mathbb{P}$ (let alone a presentation of $\mathbb{P}$ itself).

3. For an open set U and $G \in X_{\mathbb{P}}$, $G \in U$ if and only if there is some clopen D such that $G \in D$ and $D \subseteq U$. Usually, all clopen subsets of $X_{\mathbb{P}}$ are in $\mathbb{P}$, so the genericity requirement can be reformulated as follows: for every $U \subseteq X_{\mathbb{P}}$ that is $\Sigma^0_1(Z)$ there is some $P \in \mathbb{P}$ such that $G \in \mathbb{P}$ and either $P \subseteq U$ or $P \subseteq U^{\complement}$.

4. Our notion of genericity is invariant under computable isomorphism between spaces. For example, if $X_{\mathbb{P}} = (2^{\omega})^2$, we can also regard $\mathbb{P}$ as a notion of forcing adding elements of 2^{ω}, by taking joins.

Example 3.4. For Cohen forcing $\mathbb{C}$, for all Z, $G \in 2^{\omega}$ is $\mathbb{C}$-generic relative to Z if and only if it is 1-generic relative to Z.

3.2. A product theorem.

If $\mathbb{P}$ and $\mathbb{Q}$ are notions of forcing with compact sets, then so is $\mathbb{P} \times \mathbb{Q}$ (where $X_{\mathbb{P}\times\mathbb{Q}} = X_{\mathbb{P}} \times X_{\mathbb{Q}}$). The following product theorem generalises the familiar one for Cohen 1-genericity (see [8]), and is analogous to the one for Martin-Löf randomness, named after van Lambalgen, [7]. Both are effectivisations of a product theorem for forcing in set theory. Care needs to be taken since we are not assuming that the conditions themselves are all computable. For computational purposes, we identify a closed set Q with the collection of clopen subsets that it intersects. We will only use the "hard" part of the theorem (part (b)) for Cohen forcing.

Theorem 3.5. *Let $\mathbb{P}$ and $\mathbb{Q}$ be notions of forcing with compact sets; let $Z \in 2^{\omega}$ be an oracle. Let $G \in X_{\mathbb{P}}$ and $H \in X_{\mathbb{Q}}$.*

(a) *If (G, H) is $\mathbb{P} \times \mathbb{Q}$-generic relative to Z then H is $\mathbb{Q}$-generic relative to (G, Z) and G is $\mathbb{P}$-generic relative to (H, Z).*

(b) *If H is $\mathbb{Q}$-generic relative to (G,Z), and for all $Q \in \mathbb{Q}$, G is $\mathbb{P}$-generic relative to (Z,Q), then (G,H) is $\mathbb{P}\times\mathbb{Q}$-generic relative to Z.*

Proof. For (a), by symmetry, it suffices to show that H is $\mathbb{Q}$-generic relative to (G,Z). Let $U \subseteq X_{\mathbb{Q}}$ be $\Sigma^0_1(G,Z)$. Then there is some $V \subseteq X_{\mathbb{P}} \times X_{\mathbb{Q}}$ which is $\Sigma^0_1(Z)$ and such that $U = V^G = \{Y \in X_{\mathbb{Q}} : (G,Y) \in V\}$. Suppose that $H \notin U$. So $(G,H) \notin V$. By assumption, there is some $P \times Q \in \mathbb{P}\times\mathbb{Q}$ such that $(G,H) \in P \times Q$ and $P \times Q \subseteq V^{\complement}$. Since $G \in P$, $Q \subseteq U^{\complement}$.

For (b), let $V \subseteq X_{\mathbb{P}} \times X_{\mathbb{Q}}$ be $\Sigma^0_1(Z)$; suppose that $(G,H) \notin V$. So $H \notin V^G$, and V^G is $\Sigma^0_1(G,Z)$. Since H is $\mathbb{Q}$-generic relative to (G,Z), there is some $Q \in \mathbb{Q}$ such that $H \in Q$ and $Q \cap V^G = \emptyset$.

Let S be the union of all clopen $C \subseteq X_{\mathbb{P}}$ such that for some clopen $D \subseteq X_{\mathbb{Q}}$, we have $C \times D \subseteq V$ and $D \cap Q \neq \emptyset$. Since V is $\Sigma^0_1(Z)$, S is $\Sigma^0_1(Z,Q)$. Since $Q \cap V^G = \emptyset$, $\{G\} \times Q \cap V = \emptyset$, and this shows that $G \notin S$. By the assumption on G, there is some $P \in \mathbb{P}$ such that $G \in P$ and $P \subseteq S^{\complement}$. Then $(G,H) \in P \times Q$ and $P \times Q \subseteq V^{\complement}$. □

3.3. **Turing independence.** We show that genercitiy for products implies Turing independence.

Lemma 3.6. *Let $\mathbb{P}$ be a notion of forcing with compact sets, and suppose that no singleton is in $\mathbb{P}$. Suppose that G is $\mathbb{P}$-generic relative to Z. Then $G \nleqslant_{\mathrm{T}} Z$.*

Proof. Let $Y \leqslant_{\mathrm{T}} Z$. Then $X_{\mathbb{P}} \setminus \{Y\}$ is $\Sigma^0_1(Z)$. Since $\mathbb{P}$ does not contain the singleton $\{Y\}$, No $P \in \mathbb{P}$ is disjoint from $X_{\mathbb{P}} \setminus \{Y\}$. Hence $G \in X_{\mathbb{P}} \setminus \{Y\}$, i.e., $G \neq Y$. □

Remark 3.7. In general, we will not be able to obtain Turing incomparabilty between G and Z; indeed, in our main application, we will want G to be an upper bound for some ideal, and generic relative to all oracles in that ideal. We can't avoid computing Z since some conditions compute Z. This is the only obstacle: suppose that G is $\mathbb{P}$-generic relative to Z, but that $Z \leqslant_{\mathrm{T}} G$; say $Z = \Phi(G)$ for a Turing functional Φ. Let $U = \{Y \in X_{\mathbb{P}} : \Phi(Y) \perp Z\}$. Since $G \notin U$, let $P \in \mathbb{P}$ such that $G \in P$ and $P \subseteq U^{\complement}$. Since $\Phi(G)$ is total, $\{Z\} = \{\Phi(Y) : Y \in P\}$, i.e., Z is a $\Pi^0_1(P)$ singleton, so $Z \leqslant_{\mathrm{T}} P$. We will not need this result.

Proposition 3.8. *Let $\mathbb{P}_1, \dots, \mathbb{P}_n$ be notions of forcing with compact sets, and suppose that no $\mathbb{P}_i$ contains a singleton. Let Z be an oracle.*

If $(G_1,\dots,G_n)$ is $\mathbb{P}_1 \times \cdots \times \mathbb{P}_n$-generic relative to Z, then $\{G_1,\dots,G_n\}$ is Turing independent relative to Z.

Proof. It suffices to show that $G_1 \nleqslant_{\mathrm{T}} (G_2,\dots,G_n,Z)$. By Theorem 3.5, G_1 is $\mathbb{P}_1$-generic relative to $(G_2,\dots,G_n,Z)$. The result follows from Lemma 3.6. □

3.4. **Adding upper bounds of countable ideals.** The following notion of forcing adds a generic upper bound to a countable Turing ideal.

Definition 3.9. Let $\mathcal{A} \subseteq 2^{\omega}$ be countable. We let $\mathbb{P}_{\mathcal{A}}$ be the collection of partial functions x from ω to 2 satisfying:

- for almost all n, $\operatorname{dom} x^{[n]}$ is empty;
- for all n, either $\operatorname{dom} x^{[n]}$ is finite, or $\operatorname{dom} x^{[n]} = \omega$, in which case $x^{[n]} \in \mathcal{A}$.

For $x, y \in \mathbb{P}_{\mathcal{A}}$, y extends x if $x \preccurlyeq y$ (y extends x as a function).

For $x \in \mathbb{P}_{\mathcal{A}}$ we let $[x]$ be the collection of all $X \in 2^\omega$ such that $x \prec X$. Since $x \preccurlyeq y$ if and only if $[y] \subseteq [x]$, we can think of $\mathbb{P}_{\mathcal{A}}$ as forcing with compact sets.

Remark 3.10. $\mathbb{P}_\emptyset$ is Cohen forcing.

We will need to work with finite products: suppose that $\bar{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_m)$ is a finite tuple of countable subsets of 2^ω. We let

$$\mathbb{P}_{\bar{\mathcal{A}}} = \mathbb{P}_{\mathcal{A}_1} \times \cdots \times \mathbb{P}_{\mathcal{A}_m}.$$

For $\bar{x} = (x_1, \dots, x_m) \in \mathbb{P}_{\bar{\mathcal{A}}}$ we let $[\bar{x}] = [x_1] \times \cdots \times [x_m] \subseteq (2^\omega)^m$.

3.5. **Anticipating future conditions.** In our construction, we will need to construct reals while preparing them to be part of generic tuples. The typical situation is the following. Let $\delta_1 < \delta_2 \leqslant \delta_3$ be limit ordinals. Recall that we aim to construct reals A_α, and for $i = 1, 2, 3$, for some $\mathcal{B}_i \subseteq \{A_\alpha : \alpha < \delta_i\}$, an upper bound C_i of the ideal generated by $\mathcal{B}_i$. We want to make $\{C_1, C_2, C_3\}$ Turing independent; we will do this by ensuring that the triple (C_1, C_2, C_3) is $\mathbb{P}_{(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)}$-generic, and then invoke Proposition 3.8. We construct C_1 at stage δ_1 of the construction, by which we have only defined the reals A_α for $\alpha < \delta_1$. On the other hand, $\mathcal{B}_2$ and $\mathcal{B}_3$ will likely contain A_α for various $\alpha \geqslant \delta_1$. So at stage δ_1 we need to construct C_1 while preparing it to be generic relative to conditions in $\mathbb{P}_{(\mathcal{B}_2, \mathcal{B}_3)}$, that haven't been defined yet. At stage δ_1 we can imagine these conditions, except that columns in which such A_α appear, we leave undefined. However, we also need to "remember" which of these erased columns were the same. This gives us closed sets a bit more complicated than those in some $\mathbb{P}_{\bar{\mathcal{A}}}$. A mild complication is that we also need tuples such as $(C_1, C_2, C_3, A_\alpha)$ to be $\mathbb{P}_{(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)} \times \mathbb{C}$-generic, where $\alpha \geqslant \delta_1$ (but $A_\alpha \notin \mathcal{B}_2 \cup \mathcal{B}_3$). Then at stage δ_1 we will need to "leave room" for A_α as well.

Definition 3.11. Let $m \geqslant 1$. We let $\mathbb{S}(m)$ be the collection of pairs $\mathbf{p} = (\bar{y}^{\mathbf{p}}, a^{\mathbf{p}})$, where:

- $\bar{y}^{\mathbf{p}} = (y_1^{\mathbf{p}}, \dots, y_m^{\mathbf{p}})$, with each $y_i^{\mathbf{p}}$ a partial function from ω to 2;
- $a^{\mathbf{p}}$ is a finite set, whose elements are pairwise disjoint, finite subsets of $\{1, \dots, m\} \times \omega$, such that: for all $b \in a^{\mathbf{p}}$, for all $(i, n), (j, k) \in b$, $(y_i^{\mathbf{p}})^{[n]} = (y_j^{\mathbf{p}})^{[k]}$, and is finite.

With each $\mathbf{p} \in \mathbb{S}(m)$ we associate a closed set $P(\mathbf{p}) \subseteq (2^\omega)^m$, defined to be the collection of tuples $\bar{X} = (X_1, \dots, X_m) \in (2^\omega)^m$ satisfying:

- $\bar{X} \in [\bar{y}^{\mathbf{p}}]$;
- for all $b \in a$, for all $(i, n), (j, k) \in b$, $X_i^{[n]} = X_j^{[k]}$.

Observe that the restrictions imposed on $\mathbf{p}$ imply that $P(\mathbf{p})$ is nonempty. For $\mathbf{p}, \mathbf{q} \in \mathbb{S}(m)$, we let $\mathbf{q} \leqslant \mathbf{p}$ if $P(\mathbf{q}) \subseteq P(\mathbf{p})$. Note that this is equivalent to:

- for all $i \leqslant m$, $y_i^{\mathbf{p}} \preccurlyeq y_i^{\mathbf{q}}$;
- for all $b \in a^{\mathbf{p}}$ there is some $b' \in a^{\mathbf{q}}$ such that $b \subseteq b'$.

If $\mathcal{A} \subseteq 2^\omega$ and $m \geqslant 1$ then we let $\mathbb{S}(\mathcal{A}, m)$ denote the collection of all $\mathbf{p} \in \mathbb{S}(m)$ for which, for each $i = 1, \dots, m$, $y_i^{\mathbf{p}} \in \mathbb{P}_{\mathcal{A}}$.

The empty condition of length m, which we denote by $\mathbf{0}_m$, is the condition such that $y_i^{\mathbf{0}_m}$ is the empty (nowhere defined) function, and $a^{\mathbf{0}_m} = \emptyset$.

We will need to concatenate $\mathbb{S}$-conditions. Suppose that $\mathbf{p} \in \mathbb{S}(m)$ and $\mathbf{q} \in \mathbb{S}(k)$. We let $\mathbf{p}\hat{\ }\mathbf{q} \in \mathbb{S}(m + k)$ be defined as follows:

- $\bar{y}^{\mathbf{p}\hat{\ }\mathbf{q}} = (\bar{y}^{\mathbf{p}}, \bar{y}^{\mathbf{q}})$;
- $a^{\mathbf{p}\hat{\ }\mathbf{q}}$ is the union of $a^{\mathbf{p}}$ and the "k-shift" of $a^{\mathbf{q}}$, namely the collection of sets $\{(m+i,n) : (i,n) \in b\}$ for each $b \in a^{\mathbf{q}}$.

We will need to consider "mild extensions". For $\mathbf{p}, \mathbf{q} \in \mathbb{S}(m)$, we write $\mathbf{p} =^* \mathbf{q}$ if:

- for all $i \leqslant m$, $y_i^{\mathbf{p}} =^* y_i^{\mathbf{q}}$;
- $a^{\mathbf{p}} = a^{\mathbf{q}}$.

Remark 3.12. A variant of $\mathbb{P}_{\mathcal{A}}$ is defined using an existing enumeration of $\mathcal{A}$. A *coding constraint* is a sequence $\mathcal{X} = (X_n)$ where each X_n is either a real in 2^ω, or the symbol $*$, indicating no coding allowed in the n^{th} column. An $\mathcal{X}$-condition is a partial function $\alpha\colon \omega \to 2$ such that for almost all n, $\operatorname{dom} \alpha^{[n]}$ is empty, and for all n, either $\operatorname{dom} \alpha^{[n]}$ is finite, or $\operatorname{dom} \alpha^{[n]} = \omega$, $X_n \neq *$, and $\alpha_n =^* X_n$. Thus, the coding constraint tells us where a real is to be coded, but may leave room for Cohen reals to be interleaved in the generic. Let $\tilde{\mathbb{P}}_{\mathcal{X}}$ be the collection of $\mathcal{X}$-conditions. If $X_n = *$ for all n then $\tilde{\mathbb{P}}_{\mathcal{X}}$ is equivalent to Cohen forcing.

An advantage of working with coding constraints is that they are closed under taking finite products: if $\mathcal{X}_1$ and $\mathcal{X}_2$ are two coding constraints, and $\mathcal{X}$ is some join of $\mathcal{X}_1$ and $\mathcal{X}_2$, then $\tilde{\mathbb{P}}_{\mathcal{X}}$ is equivalent to $\tilde{\mathbb{P}}_{\mathcal{X}_1} \times \tilde{\mathbb{P}}_{\mathcal{X}_2}$.

On the other hand, if we were to work with coding-constraint forcing instead, we would need to modify Definition 3.11 to deal with $=^*$. The $\mathbb{S}$-conditions would need to remember which pairs of columns are $=^*$, and for each pair, state from which point they are equal, and how they are filled up to that point.

Remark 3.13. Coding constraint forcing gives a good illustration of the restricted nature of genericity that we defined. Suppose, for example, that $\mathcal{X}$ is a coding constraint (say with no $*$ entries), and that infinitely many of the X_n end with 0^ω. Then the set of $\mathcal{X}$-conditions in which some column ends in $11^\frown 0^\omega$ is dense in $\tilde{\mathbb{P}}_{\mathcal{X}}$. However, for any oracle Z, there is some G, $\tilde{\mathbb{P}}_{\mathcal{X}}$-generic relative to Z, that does not meet this dense set.

4. The construction

We prove Theorem 2.1. During the construction, we will choose, for each $\alpha < \omega_1$, a real A_α. By stage δ of the construction, we will have defined A_α for all $\alpha < \delta$, and we let

$$\mathcal{A}_\delta = \{A_\alpha : \alpha < \delta\}.$$

Further, for all finite $c \subseteq \delta$, we write $c = \{\alpha_1 < \alpha_2 < \cdots < \alpha_{|c|}\}$, and we let

$$A_c = (A_{\alpha_1}, \ldots, A_{\alpha_{|c|}}).$$

For the "second layer", at limit stages δ, we will approximate the sets $C^\delta_{\mathcal{B}}$ (for all $\mathcal{B} \subseteq \mathcal{A}_\delta$) by defining a tree of conditions x^δ_σ for $\sigma \in 2^{<\omega}$. Here σ will code an initial segment of the characteristic function of $\mathcal{B}$ as a subset of $\mathcal{A}_\delta$ (based on some ω-enumeration of $\mathcal{A}_\delta$), and $C^\delta_{\mathcal{B}}$ will extend x^δ_σ when $\mathcal{B}$ extends σ.

Towards making the desired tuples generic, we will try to meet a collection of requirements. For the following definitions, suppose that A_α have been chosen for all $\alpha < \delta$.

Definition 4.1. Let $\delta = 0$ or $\delta < \omega_1$ be a limit ordinal. A *δ-requirement* is a tuple $(n, \bar{\bar{\delta}}, \bar{m}, c, U, \mathbf{p})$ satisfying:

(i) $n \geqslant 0$;

(ii) $\bar{\delta}$ is an increasing sequence $\delta_0 < \delta_1 < \delta_2 < \cdots < \delta_n$ of non-successor ordinals, with $\delta_n = \delta$ and $\delta_0 = 0$ (hence $n = 0$ iff $\delta = 0$);
(iii) $\bar{m} = (m_1, \ldots, m_n, m_{\mathrm{fut}}) \in \omega^{n+1}$, where:
  - for $i = 1, \ldots, n$, $m_i > 0$;
  - if $n = 0$ then $m_{\mathrm{fut}} > 0$;

(iv) c is a finite subset of δ (so if $\delta = 0$ then $c = \emptyset$);
(v) $U \subseteq (2^\omega)^{m_1+m_2+\cdots+m_n+|c|+m_{\mathrm{fut}}}$ is Σ^0_1;
(vi) $\mathbf{p} \in \mathbb{S}(\mathcal{A}_\delta, m_{\mathrm{fut}})$ (if $m_{\mathrm{fut}} = 0$ then $\mathbf{p}$ is the vacuous condition).
(vii) For all $\alpha \in c$, A_α is not a column of $y_i^{\mathbf{p}}$ (for any $i = 1, \ldots, m_{\mathrm{fut}}$).

The meaning of this is the following. At stage $\delta = \delta_n$ we are looking at tuples of the form:

$$(C^{\delta_1}_{\bar{\mathcal{B}}_1}, C^{\delta_2}_{\bar{\mathcal{B}}_2}, \ldots, C^{\delta_n}_{\bar{\mathcal{B}}_n}, A_c)$$

where $\bar{\mathcal{B}}_i = (\mathcal{B}_{i,1}, \mathcal{B}_{i,2}, \ldots, \mathcal{B}_{i,m_i})$ is a tuple of subsets of $\mathcal{A}_{\delta_i}$ and $C^{\delta_i}_{\bar{\mathcal{B}}_i} = (C^{\delta_i}_{\mathcal{B}_{i,1}}, \ldots, C^{\delta_i}_{\mathcal{B}_{i,m_i}})$. We are trying to take another step toward making this tuple generic, by trying to meet or avoid U. However, when $m_{\mathrm{fut}} > 0$, we are actually planning for this tuple to be part of a longer tuple (with m_{fut} more components), and $\mathbf{p}$ is our guess of the structure of the rest of this tuple. When $n = 0$, we are just planning for the future. The last restriction, (vii), is because we cannot make $(A_\alpha, C^\delta_{\mathcal{B}})$ generic when $A_\alpha \in \mathcal{B}$; see Remark 4.4 below.

4.1. **Restricting conditions.** Let $\delta < \omega_1$ and let $\bar{x} = (x_1, \ldots, x_m) \in (\mathbb{P}_{\mathcal{A}_\delta})^m$.

For $\gamma < \delta$ we define $\mathbf{q}(\bar{x}, \gamma) \in \mathbb{S}(m)$ by erasing occurrences of A_α for $\alpha \geqslant \gamma$, and remembering which columns are equal. That is:

- $\bar{y}^{\mathbf{q}(\bar{x},\gamma)} = (y_1, \ldots, y_m)$, where y_i is defined as follows: for each column n,
  - If $x_i^{[n]} = A_\alpha$ for some $\alpha \geqslant \gamma$, then $y_i^{[n]}$ is everywhere undefined.
  - Otherwise, $y_i^{[n]} = x_i^{[n]}$.
- For all $\alpha \in [\gamma, \delta)$, let

$$b_\alpha = \left\{ (i, n) \in \{1, \ldots, m\} \times \omega : \ \ \& \ x_i^{[n]} = A_\alpha \right\}.$$

  We let $a^{\mathbf{q}(\bar{x},\gamma)}$ be the collection of all b_α which are nonempty.

Observe that $\mathbf{q}(\bar{x}, \gamma) \in \mathbb{S}(\mathcal{A}_\gamma, m)$, and that $[\bar{x}] \subseteq P(\mathbf{q}(\bar{x}, \gamma))$.

We extend the previous definition as follows. For $\mathbf{p} \in \mathbb{S}(\mathcal{A}_\delta, m)$ and $\gamma < \delta$ we let $\mathbf{q}(\mathbf{p}, \gamma)$ be defined by:

- $\bar{y}^{\mathbf{q}(\mathbf{p},\gamma)} = \bar{y}^{\mathbf{q}(\bar{y}^{\mathbf{p}},\gamma)}$;
- $a^{\mathbf{q}(\mathbf{p},\gamma)} = a^{\mathbf{p}} \cup a^{\mathbf{q}(\bar{y}^{\mathbf{p}},\gamma)}$.

Recall that if (i, n) is an element of any element of $a^{\mathbf{p}}$, then $(y_i^{\mathbf{p}})^{[n]}$ is finite, and so $\bigcup a^{\mathbf{p}}$ and $\bigcup a^{\mathbf{q}(\bar{y}^{\mathbf{p}},\gamma)}$ are disjoint; so $\mathbf{q}(\mathbf{p}, \gamma) \in \mathbb{S}(\mathcal{A}_\gamma, m)$.

Finally, we define the restriction of a δ-*requirement*. Let $\delta > 0$ and let $\mathfrak{r}$ be a δ-requirement (so $n^{\mathfrak{r}} > 0$). Let $\gamma = (\delta_{n-1})^{\mathfrak{r}}$. Let $\bar{x} \in (\mathbb{P}_{\mathcal{A}_\delta})^{(m_n)^{\mathfrak{r}}}$. We define

$$\mathbf{q}(\mathfrak{r}, \bar{x}) = \mathbf{0}_{|c^{\mathfrak{r}} \cap [\gamma,\delta)|} {}^\frown \mathbf{q}(\bar{x}, \gamma) {}^\frown \mathbf{q}(\mathbf{p}^{\mathfrak{r}}, \gamma).$$

Definition 4.2. Let $\mathfrak{r}$ be a δ-requirement, and suppose that $n^{\mathfrak{r}} > 0$. Let $\gamma = (\delta_{n-1})^{\mathfrak{r}}$. Let $\bar{x} \in (\mathbb{P}_{\mathcal{A}_\delta})^{(m_n)^{\mathfrak{r}}}$.

We say that a γ-requirement $\mathfrak{s}$ is a *predecessor* of $(\mathfrak{r}, \bar{x})$ if:

- $n^{\mathfrak{s}} = n^{\mathfrak{r}} - 1$;
- $\bar{\delta}^{\mathfrak{s}} = (\delta^{\mathfrak{r}}_0, \ldots, (\delta_{n-1})^{\mathfrak{r}})$;

- for $i = 1, \dots, n^{\mathfrak{r}} - 1$, $m_i^{\mathfrak{s}} = m_i^{\mathfrak{r}}$;
- $m_{\text{fut}}^{\mathfrak{s}} = |c^{\mathfrak{r}} \cap [\gamma, \delta)| + (m_n)^{\mathfrak{r}} + m_{\text{fut}}^{\mathfrak{r}}$;
- $c^{\mathfrak{s}} = c^{\mathfrak{r}} \cap \gamma$;
- $U^{\mathfrak{s}} = U^{\mathfrak{r}}$;
- $\mathbf{p}^{\mathfrak{s}} \leqslant \mathbf{q}(\mathfrak{r}, \bar{x})$ and $\mathbf{p}^{\mathfrak{s}} =^* \mathbf{q}(\mathfrak{r}, \bar{x})$.

Here the idea is that $\mathfrak{s}$ is a requirement that can anticipate $(\mathfrak{r}, \bar{x})$. During stage δ, we will have constructed $\bar{x}$ as an approximation to an $m_n^{\mathfrak{r}}$-tuple of sets $C_{\mathcal{B}}^{\delta}$. In stepping back to stage γ, we need to let $\mathfrak{s}$ know about the "γ-part of $\bar{x}$", namely, $\mathbf{q}(\bar{x}, \gamma)$, stripping away A_α for $\alpha \in [\gamma, \delta)$ (but remembering which stripped columns are equal). We also leave an empty space for $A_{c^{\mathfrak{r}} \cap [\gamma, \delta)}$. Also, we incorporate the future from $\mathfrak{r}$'s point of view into $\mathfrak{s}$'s view of the future, again stripping A_α's that $\mathfrak{s}$ cannot know about—thus, $\mathfrak{s}$ is supplied with $\mathbf{q}(\mathbf{p}^{\mathfrak{r}}, \gamma)$. We do not require $\mathbf{p}^{\mathfrak{s}} = \mathbf{q}(\mathfrak{r}, \bar{x})$, only the weaker condition $\mathbf{p}^{\mathfrak{s}} \leqslant \mathbf{q}(\mathfrak{r}, \bar{x})$ and $\mathbf{p}^{\mathfrak{s}} =^* \mathbf{q}(\mathfrak{r}, \bar{x})$, in order to exploit the genericity of the A_α's; this is used in the proof of Lemma 4.11 below.

Remark 4.3. Condition (vii) of Definition 4.1 impies that $(\mathfrak{r}, \bar{x})$ has a predecessor if and only if for all $\alpha \in c^{\mathfrak{r}} \cap \gamma$, A_α is not a column of any x_i.

Remark 4.4. Note that in passing from $\mathfrak{r}$ to $\mathfrak{s}$, if there is some $\alpha \in c \cap [\gamma, \delta)$ such that A_α is a column of some x_i, then this fact is not recorded by $\mathfrak{s}$. This is because we will only try to meet $\mathfrak{r}$ in situations when no such equality occurs. Meeting the requirement entails making the pair $(A_\alpha, C_{\mathcal{B}}^{\delta})$ generic, where $\alpha \in c$ and $C_{\mathcal{B}}^{\delta}$ extends some x_i. But if A_α is already a column of x_i, this means that $A_\alpha \in \mathcal{B}$, and recall that we are trying to make $C_{\mathcal{B}}^{\delta}$ an upper bound of $\mathcal{B}$. So we do not want (and cannot hope) to make $\{C_{\mathcal{B}}^{\delta}, A_\alpha\}$ Turing independent. This complements condtition (vii) of Definition 4.1.

Remark 4.5. Applying Definition 4.2 repeatedly, we see that our envisioned tuple of A_α's and $C_{\mathcal{B}}^{\delta}$'s from above needs to be re-ordered as

$$\left(A_{c \cap [\delta_0, \delta_1)}, C_{\bar{\mathcal{B}}_1}^{\delta_1}, A_{c \cap [\delta_1, \delta_2)}, C_{\bar{\mathcal{B}}_2}^{\delta_2}, \dots, A_{c \cap [\delta_{n-1}, \delta_n)}, C_{\bar{\mathcal{B}}_n}^{\delta_n}\right).$$

4.2. **The construction.** Recall that we will construct, for each $\alpha < \omega_1$, a real A_α, and for limit $\delta < \omega_1$ and $\sigma \in 2^{<\omega}$, a partial function x_σ^δ. In addition, we will define:

- For all $\alpha < \omega_1$, an oracle Z_α; and
- For non-successor $\delta < \omega_1$, for all $e < \omega$, a tuple of partial functions $\bar{w}^{\delta,e}$.

Let us elaborate on the latter. For non-successor δ, we will define an ω-list $\mathfrak{r}^{\delta,0}, \mathfrak{r}^{\delta,1}, \dots$ of δ-requirements (with many repetitions). For brevity, we will let

$$\mathfrak{r}^{\delta,e} = (n^{\delta,e}, \bar{\gamma}^{\delta,e}, \bar{m}^{\delta,e}, c^{\delta,e}, U^{\delta,e}, \mathbf{p}^{\delta,e}).$$

Further, we write $\bar{y}^{\delta,e}$ for $\bar{y}^{\mathbf{p}^{\delta,e}}$ and $a^{\delta,e}$ for $a^{\mathbf{p}^{\delta,e}}$.

For each e, $\bar{w}^{\delta,e}$ will be an $m_{\text{fut}}^{\delta,e}$-tuple of partial functions $(w_1^{\delta,e}, w_2^{\delta,e}, \dots, w_{m_{\text{fut}}^{\delta,e}}^{\delta,e})$, whose role is to "prompt" future tuples; if those extend $\bar{w}^{\delta,e}$, this will help decide $U^{\delta,e}$. We will ensure:

- For all $i = 1, \dots, m_{\text{fut}}^{\delta,e}$, $w_i^{\delta,e}$ is a finite extension of $y_i^{\delta,e}$;
- $[\bar{w}^{\delta,e}] \cap P(\mathbf{p}^{\delta,e}) \neq \emptyset$, which in light of (a), amounts to ensuring that if $(i, n), (j, k)$ belong to some element of $a^{\delta,e}$, then $(w_i^{\delta,e})^{[n]}$ is consistent with $(w_j^{\delta,e})^{[k]}$.

Let $\delta < \omega_1$, and suppose that the construction has been performed for all stages $\alpha < \delta$.

1. We determine Z_δ. We choose Z_δ to be sufficiently strong so that it computes a copy of δ, and based on this copy, computes, uniformly in $\alpha < \delta$, both A_α and Z_α, and for limit $\gamma < \delta$, the construction at stage γ, uniformly in such γ. We start with $Z_0 = \emptyset'$.

2. We let A_δ be Cohen generic relative to Z_δ.

3. When $\delta = 0$, we perform a preparatory construction. Note that the 0-requirements are essentially tuples $(m_{\text{fut}}, U, \mathbf{p})$ where $m_{\text{fut}} > 0$, $U \subseteq (2^\omega)^{m_{\text{fut}}}$ is Σ^0_1, and $\mathbf{p} \in \mathbb{S}(\mathcal{A}_0, m_{\text{fut}})$. Since $\mathcal{A}_0 = \emptyset$, the latter means that each $y_i^{\mathbf{p}}$ is a finite function.

Thus, we can let $(\mathfrak{r}^{0,e})$ be a computable list of all 0-requirements, with each requirement appearing infinitely often on the list. For each e we define a tuple $\bar{w}^{0,e}$ as follows. We ask if:

$$P(\mathbf{p}^{0,e}) \cap U^{0,e} = \emptyset.$$

If so, we let $\bar{w}^{0,e} = \bar{y}^{0,e}$. If not, then we let $\bar{w}^{0,e}$ be an $m_{\text{fut}}^{0,e}$-tuple of finite functions from ω to 2 such that:

- Each $w_i^{0,e}$ is a finite extension of $y_i^{0,e}$;
- $[\bar{w}^{0,e}] \cap P(\mathbf{p}^{0,e}) \neq \emptyset$; and
- $[\bar{w}^{0,e}] \subseteq U^{0,e}$.

4. If δ is a limit ordinal, we perform the stage δ construction as follows.

Based on the Z_δ-computable copy of δ, let $\langle A_e^\delta : e < \omega\rangle$ be an enumeration of $\mathcal{A}_\delta$. Note that Z_δ can enumerate $\mathbb{S}(\mathcal{A}_\delta, m)$ for all m (uniformly), and thus, enumerate all δ-requirements. So we create a Z_δ-computable list of δ-requirements $\mathfrak{r}^{\delta,0}, \mathfrak{r}^{\delta,1}, \dots$, so that each δ-requirement appears infinitely often on the list.

The stage δ construction is performed in ω-many steps. At the beginning of step e, we will have defined:

- a partial function x_σ^δ for all $\sigma \in \{0,1\}^{\leqslant e}$; and
- $\bar{w}^{\delta,e'}$ for all $e' < e$.

We start with $x_{\langle\rangle}^\delta$ being the empty function.

Let $e < \omega$ and suppose that the construction has been performed up to the beginning of step e. We then consider the e^{th} requirement on the list. In what follows, we omit all superscripts δ and e from all objects considered or constructed at step e. So $n = n^{\delta,e}$, $c = c^{\delta,e}$, and so on. When we consider parameters of other requirements we will write the superscripts in full.

Recall that $\delta > 0$ implies $n > 0$.

Let $\bar{\tau}^0, \bar{\tau}^1, \dots, \bar{\tau}^{t-1}$ be an enumeration of all m_n-tuples of distinct $\sigma \in \{0,1\}^e$. By recursion, for all $k = 0, 1, \dots, t$, we define:

- for all $\sigma \in \{0,1\}^e$, a partial function u_σ^k; and
- A tuple $\bar{w}^k = (w_1^k, \dots, w_{m_{\text{fut}}}^k)$ of partial functions from ω to 2.

We start with $u_\sigma^0 = x_\sigma$ and $\bar{w}^0 = \bar{y}^{\mathbf{p}}$.

Let $k < t$, and suppose that $\bar{w}^k$ and all u_σ^k have been defined. We will ensure that $[\bar{w}^k] \cap P(\mathbf{p}) \neq \emptyset$.

Enumerate $\bar{\tau}^k$ as $(\tau_1^k, \dots, \tau_{m_n}^k)$. For brevity, for $i \leqslant m_n$ we let $u_i^k = u_{\tau_i^k}^k$; let $\bar{u}^k = (u_1^k, \dots, u_{m_n}^k)$.

Since $n > 0$, let $\gamma = \gamma_{n-1}$; let $c' = c \cap [\gamma, \delta)$.

We first check if we need to tend to this instance of the requirement. We say that (δ, e) is *active* at k if for all $\alpha \in c$, A_α is not a column of any u_i^k (for $i = 1, \dots, m_n$). If (δ, e) is inactive at k then we do nothing: we let $\bar{w}^{k+1} = \bar{w}^k$ and $u_\sigma^{k+1} = u_\sigma^k$ for all σ.

Suppose that (δ, e) is active at k. Let

$$\bar{z}^k = (A_{c'}, \bar{u}^k, \bar{w}^k).$$

Note that the length of $\bar{z}^k$ is $|c'|+m_n+m_{\text{fut}}$, which is $m_{\text{fut}}^{\mathfrak{s}}$, where $\mathfrak{s}$ is any predecessor of $(\mathfrak{r}^{\delta,e}, \bar{u}^k)$.

We say that $d < \omega$ is *suitable* at (δ, e, k) if:

- $d \geqslant e$;
- $\mathfrak{r}^{\gamma,d}$ is a predecessor of $(\mathfrak{r}^{\delta,e}, \bar{u}^k)$; and
- the tuple $\bar{w}^{\gamma,d}$ is consistent with $\bar{z}^k$.

Below (in Lemma 4.11) we will show that there is some d which is suitable for (δ, e, k). Choose d to be the least such.

- for $i \leqslant m_n$ we let $u_{\tau_i^k}^{k+1} = u_i^k \cup w_{|c'|+i}^{\gamma,d}$, and
- we let $\bar{w}^{k+1} = (w_{|c'|+m_n+1}^{\gamma,d} \cup w_1^k, \dots, w_{|c'|+m_n+m_{\text{fut}}}^{\gamma,d} \cup w_{m_{\text{fut}}}^k)$.

For $\sigma \in \{0,1\}^e$ that is not τ_i^k for any i, we let $u_\sigma^{k+1} = u_\sigma^k$.

At the end of step e, we let, for each $\sigma \in \{0,1\}^e$:

- $x_{\sigma\hat{\ }0}^\delta = u_\sigma^t$;
- $x_{\sigma\hat{\ }1}^\delta$ be a condition extending u_σ^t by setting, for some large r, the r^{th} column of $x_{\sigma\hat{\ }1}^\delta$ to be A_e^δ.

We also let $\bar{w}^{\delta,e} = \bar{w}^t$.

This completes step e of the δ-construction, and so, the description of the construction as a whole.

4.3. **Verification: the construction makes sense.** For the first part of the verification, we need to show that the construction can be carried out as described. Namely: for non-successor δ and $e < \omega$,

(a) If $\delta > 0$, then for all $\sigma \in \{0,1\}^{e+1}$, x_σ^δ is a function, and $x_{\sigma\hat{\ }0}^\delta$ is a finite extension of x_σ^δ.
(b) For all $i = 1, \dots, m_{\text{fut}}^{\delta,e}$, $w_i^{\delta,e}$ is a function, which is a finite extension of $y_i^{\delta,e}$.
(c) $[\bar{w}^{\delta,e}] \cap P(\mathbf{p}^{\delta,e}) \neq \emptyset$.
(d) If $\delta > 0$, for all $k < t^{\delta,e}$, if (δ, e) is active at k, then some d is suitable for (δ, e, k).

We start with $\delta = 0$:

Lemma 4.6. *(b) and (c) hold when $\delta = 0$; further, for each e, $\bar{w}^{0,e}$ is a tuple of finite functions.*

Proof. As noted above, since $\mathcal{A}_0 = \emptyset$, every $y_i^{0,e}$ is a finite function. Note that by definition, for all $\mathbf{p} \in \mathbb{S}(m)$, $[\bar{y}^{\mathbf{p}}] \supseteq P(\mathbf{p})$ and $P(\mathbf{p}) \neq \emptyset$, ensuring that (c) holds in the case that $\bar{w}^{0,e} = \bar{y}^{0,e}$. □

Let δ be a countable limit ordinal and $e < \omega$. Suppose that the construction was performed successfully up to step e of stage δ, and that (a)–(d) above hold for all non-successor $\gamma < \delta$ and for (δ, e') for all $e' < e$. We will show that step e of the construction can be performed successfully, and that (a)–(d) hold for (δ, e). To do this, by induction on $k \leqslant t^{\delta,e}$, we show:

(1) For all $\sigma \in \{0,1\}^e$, $u^{\delta,e,k}_\sigma$ is a function, which is a finite extension of x^δ_σ.
(2) For all $i \leqslant m^{\delta,e}_{\text{fut}}$, $w^{\delta,e,k}_i$ is a function, which is a finite extension of $y^{\delta,e}_i$.
(3) $[\bar{w}^{\delta,e,k}] \cap P(\mathbf{p}^{\delta,e}) \neq \emptyset$.
(4) If (δ, e) is active at k, then some d is suitable for (δ, e, k).

We observe that for $k = 0$, (1)–(3) hold by induction, and the choice of $\bar{w}^{\delta,e,0} = \bar{y}^{\delta,e}$. Let $k < t^{\delta,e}$, and suppose that (1)–(3) hold at k.

Lemma 4.7. *If (4) holds at (δ, e, k), then (1)–(3) hold at $k + 1$.*

Proof. If (δ, e) is inactive at k, then (1)–(3) follow by induction. Suppose otherwise; let d be the least which is suitable for (δ, e, k). As in the construction, we omit the superscripts δ, e.

Since $\bar{w}^{\gamma,d}$ is consistent with $\bar{z}^k$, for all $i \leqslant m_n$, $u^{k+1}_{\tau^k_i}$ is a function, and so is w^{k+1}_j for all $j \leqslant m_{\text{fut}}$. For $i \leqslant m_n$, $y^{\gamma,\delta}_{|c'|+i}$ is a finite extension of u^k_i (after all A_α columns for $\alpha \geqslant \gamma$ are stripped away), and similarly, for $i \leqslant m_{\text{fut}}$, $y^{\gamma,\delta}_{|c'|+m_n+i}$ is a finite extension of w^k_i. By induction ((b) above), each $w^{\gamma,d}_j$ is a finite extension of $y^{\gamma,d}_j$. This gives us (1) and (2) at $k + 1$.

By induction, $[\bar{w}^{\gamma,d}] \cap P(\mathbf{p}^{\gamma,d}) \neq \emptyset$. Let $b \in a^{\delta,e}$. By the definition of $\mathbf{q}(\mathfrak{r}^{\delta,e}, \bar{u}^k)$, the $(|c'| + m_n)$-shift of b is in $a^{\gamma,d}$. Thus, for for all $(i, o), (i', o') \in b$, $(w^{\gamma,d}_{|c'|+m_n+i})^{[o]}$ and $(w^{\gamma,d}_{|c'|+m_n+i'})^{[o']}$ are compatible. Also by induction, $(w^k_i)^{[o]}$ and $(w^k_{i'})^{[o']}$ are compatible. It follows that $(w^{k+1}_i)^{[o]}$ and $(w^{k+1}_{i'})^{[o']}$ are compatible. This gives (3) at $k + 1$. □

It remains to show that (4) holds at (δ, e, k). To do so, we need a few lemmas which will also be useful later.

Lemma 4.8. *For all non-successor $\gamma < \delta$, Z_γ computes the stage γ construction.*

Proof. For $\gamma = 0$, $Z_0 = \emptyset'$ can determine if $U^{0,e} \cap P(\mathbf{p}^{0,e}) \neq \emptyset$. For a limit γ, the entire construction is computable from Z_γ. □

Lemma 4.9. *For any finite $c \subset \delta$, the tuple A_c is Cohen generic relative to $Z_{\min c}$.*

Proof. This follows from the product theorem for Cohen forcing (part (b) of Theorem 3.5), and the fact that $Z_\alpha \geqslant_{\mathrm{T}} Z_\beta, A_\beta$ when $\alpha > \beta$. □

Lemma 4.10. *For all σ,*

$$x^\delta_\sigma \in \mathbb{P}_{\{A^\delta_r \,:\, r<|\sigma| \;\;\&\;\; \sigma(r)=1\}}.$$

In particular, $x^\delta_\sigma \in \mathbb{P}_{\mathcal{A}_\delta}$.

Proof. By induction on the length of σ. The lemma holds for $\sigma = \langle\rangle$. Suppose that the lemma holds for some σ; let $e = |\sigma|$. By (a) above, $x^\delta_{\sigma\hat{\ }0}$ is a finite extension of x^δ_σ, so the lemma holds for $\sigma\hat{\ }0$ as well; and by construction, we only add A^δ_e to $x^\delta_{\sigma\hat{\ }1}$, so the lemma holds for $\sigma\hat{\ }1$ as well. □

Lemma 4.11. *(4) holds at k.*

Proof. We use the notation of the construction. Let $\gamma = \gamma_{n-1}$; let $\mathbf{q} = \mathbf{q}(\mathfrak{r}, \bar{u}^k)$; let $m^* = |c'| + m_n + m_{\text{fut}}$. Recall that we let $\bar{z}^k = (A_{c'}, \bar{u}^k, \bar{w}^k)$. We assume that (δ, e) is active at k.

Let r be the set of $\alpha \in [\gamma, \delta)$ such that A_α appears as a column of some u_i^k or w_j^k. Since (δ, e) is active at k, and by (vii) of Definition 4.1 (applied to $\mathfrak{r}^{\delta,e}$), r and c are disjoint, and so r and c' are disjoint.

By Lemma 4.9, the tuple $A_{r\cup c'}$ is Cohen generic relative to Z_γ.

First, observe that $(\mathfrak{r}, \bar{u}^k)$ has predecessors: to do so, we need to check that $\mathbf{q}$ satisfies (vii) of Definition 4.1 with respect to $c \cap \gamma$. Let $i \leqslant m^*$, and suppose that some A_α is a column of $y_i^{\mathbf{q}}$. Then $i > |c'|$, as $y_i^{\mathbf{q}}$ is the empty function when $i \leqslant |c'|$. If $i = |c'| + j$ for some $j \leqslant m_n$, then $y_i^{\mathbf{q}}$ is the result of stripping various A_β's from u_j^k. Since (δ, e) is active at k, $\alpha \notin c$, so $\alpha \notin c \cap \gamma$. If $i = |c'| + m_n + j$ for some $j \leqslant m_{\text{fut}}$, then $y_i^{\mathbf{q}}$ is the result of stripping various A_β's from w_j^k, so (vii) for $\mathfrak{r}^{\delta,e}$ ensures that $\alpha \notin c$.

For a sequence of finite functions $\bar{\mu} = (\mu_\alpha)_{\alpha\in r\cup c'}$ we define $\mathbf{q}(\bar{\mu}) \in \mathbb{S}(m^*)$ by letting:

- $a^{\mathbf{q}(\bar{\mu})} = a^{\mathbf{q}}$;
- for all $i = 1, \dots, |c'|$, $y_i^{\mathbf{q}(\bar{\mu})} = \mu_\alpha$ where $z_i^k = A_\alpha$. That is, $y_i^{\mathbf{q}(\bar{\mu})} = \mu_{\alpha_i}$, where $c' = \{\alpha_1 < \cdots < \alpha_{|c'|}\}$.
- for all $i = |c'| + 1, \dots, m^*$, for all l,
  - if $(z_i^k)^{[l]} = A_\alpha$ for some $\alpha \in r$, then $(y_i^{\mathbf{q}(\bar{\mu})})^{[l]} = \mu_\alpha$;
  - Otherwise, $(y_i^{\mathbf{q}(\bar{\mu})})^{[l]} = (z_i^k)^{[l]}$.

In other words, $\mathbf{q}(\bar{\mu})$ is defined like $\mathbf{q}$, except that instead of erasing all of A_α for $\alpha \geqslant \gamma$, we write μ_α in the appropriate coordinate or column. Since $y_i^{\mathbf{q}(\bar{\mu})}$ is a finite extension of $y_i^{\mathbf{q}}$ (for all $i \leqslant m^*$), $\mathbf{q}(\bar{\mu})$ also satisfies (vii) of Definition 4.1 with respect to $c \cap \gamma$. Hence, $\mathbf{q}(\bar{\mu}) \in \mathbb{S}(\mathcal{A}_\gamma, m^*)$ appears in γ-requirements which are predecessors of $\mathfrak{r}^{\delta,e}$; note that $\mathbf{q}(\bar{\mu}) \leqslant \mathbf{q}$ and $\mathbf{q}(\bar{\mu}) =^* \mathbf{q}$ for any $\bar{\mu}$.

For each such $\bar{\mu}$, let $d(\bar{\mu})$ be the least d such that:

- $d \geqslant e$;
- $\mathfrak{r}^{\gamma,d}$ is a predecessor of $(\mathfrak{r}^{\delta,e}, \bar{u}^k)$;
- $\mathbf{p}^{\gamma,d} = \mathbf{q}(\bar{\mu})$.

Let $\bar{\mu}$ be such a tuple, and let $d = d(\bar{\mu})$. For all $i \leqslant |c'|$, since $y_i^{\gamma,d} = y_i^{\mathbf{q}(\mu)} = \mu_\alpha$ for some $\alpha \in c$, and (by induction) $w_i^{\gamma,d}$ is a finite extension of $y_i^{\gamma,d}$, we see that $w_i^{\gamma,d}$ is a finite function; we let $\nu_\alpha = w_i^{\gamma,d}$. For $\alpha \in r$ let b_α be the set of pairs (i, l) where $|c'| < i \leqslant m^*$ and $(z_i^k)^{[l]} = A_\alpha$. Then for all $\alpha \in r$, for all $(i, l) \in b_\alpha$, $(y_i^{[\gamma,d]})^{[l]} = \mu_\alpha$, so $(w_i^{\gamma,d})^{[l]}$ is a finite function. Further, since $b_\alpha \in a^{\mathbf{q}} = a^{\mathbf{q}(\bar{\mu})}$, and since by induction, $[\bar{w}^{\gamma,d}] \cap P(\mathbf{q}(\bar{\mu})) \neq \emptyset$, for all $(i_1, l_1), (i_2, l_2) \in b_\alpha$, $(w_{i_1}^{\gamma,\delta})^{[l_1]}$ and $(w_{i_2}^{\gamma,\delta})^{[l_2]}$ are compatible. We let $\nu_\alpha = \bigcup \left\{ (w_i^{\gamma,d})^{[l]} \, : \, (i, l) \in b_\alpha \right\}$. Then:

- For all $\alpha \in r \cup c'$, ν_α is a finite function and $\nu_\alpha \succcurlyeq \mu_\alpha$;
- $\bar{\nu} = (\nu_\alpha)_{\alpha\in r\cup c'}$ "ensures extending $\bar{w}^{\gamma,d}$ in the correct places": for all $i \leqslant m^*$
  - if $i \leqslant |c'|$, and $z_i^k = A_\alpha$ (where $\alpha \in c'$), then $\nu_\alpha = w_i^{\gamma,d}$;
  - if $i > |c'|$ and l is such that $(z_i^k)^{[l]} = A_\alpha$ for some $\alpha \in r$, then $\nu_\alpha \succcurlyeq (w_i^{\gamma,d})^{[l]}$.

By Lemma 4.8, the search for $d(\bar{\mu})$ is computable in Z_γ. As a result, by the genericity of $A_{r\cup c'}$, there is some $\bar{\mu}$ such that the resulting $\bar{\nu}$ is extended by $A_{r\cup c'}$. Then $d(\bar{\mu})$ is suitable for (δ, e, k): to show that $\bar{w}^{\gamma,d}$ is compatible with $\bar{z}^k$, let $i \leqslant m^*$.

- If $i \leqslant |c'|$ then $z_i^k = A_\alpha$ for some $\alpha \in c'$, and $w_i^{\gamma,d} = \nu_\alpha \prec A_\alpha$.
- If $i > |c'|$, let $l < \omega$.
  - If $(z_i^k)^{[l]} = A_\alpha$ for some $\alpha \in r$, then $(w_i^{\gamma,d})^{[l]} = \nu_\alpha \prec A_\alpha$.
  - Otherwise, $(w_i^{\gamma,d})^{[l]} \succcurlyeq (y_i^{\gamma,d})^{[l]} = (z_i^k)^{[l]}$. □

This concludes the induction above: (1)–(4) hold at k, and so (a)–(d) hold at (δ, e). The entire construction can be carried out as described.

4.4. **Verification: genericity.** For any limit $\delta < \omega_1$ and any $\mathcal{B} \subseteq \mathcal{A}_\delta$, we define $f_{\mathcal{B}}^\delta \in 2^\omega$ by $f_{\mathcal{B}}^\delta(e) = 1$ iff $A_e^\delta \in \mathcal{B}$. We then let

$$C_{\mathcal{B}}^\delta = \bigcup \left\{ x_\sigma^\delta \,:\, \sigma \prec f_{\mathcal{B}}^\delta \right\}.$$

Theorem 2.1 follows from Proposition 3.8 and the following lemma.

Lemma 4.12. *Let:*

- $\delta_1 < \delta_2 < \cdots < \delta_n$ *be limit ordinals, and*
- *for* $l = 1, \ldots, n$, $\bar{\mathcal{B}}_l = (\mathcal{B}_{l,1}, \ldots, \mathcal{B}_{l,|\bar{\mathcal{B}}_l|})$ *be a tuple of distinct subsets of* $\mathcal{A}_{\delta_l}$*; and*
- c *be a finite subset of* ω_1*, such that* $\{A_\alpha \,:\, \alpha \in c\}$ *is disjoint from* $\bigcup_{l\leqslant n} \bigcup_{j\leqslant|\bar{\mathcal{B}}_l|} \mathcal{B}_{l,j}$.

Then $(C_{\bar{\mathcal{B}}_1}^{\delta_1}, \ldots, C_{\bar{\mathcal{B}}_n}^{\delta_n}, A_c)$ *is* $\mathbb{P}_{\bar{\mathcal{B}}_1} \times \cdots \times \mathbb{P}_{\bar{\mathcal{B}}_n} \times \mathbb{C}^{|c|}$*-generic (relative to* $\emptyset$*).*

Proof. Let $m^* = |\bar{\mathcal{B}}_1| + \cdots + |\bar{\mathcal{B}}_n| + |c|$. Let $\delta_0 = 0$; for $l = 1, \ldots, n$, let

$$c_l = c \cap [\delta_{l-1}, \delta_l).$$

Recall (Remark 4.5) that we need to reorder our to-be-generic tuple as

$$\bar{X} = (A_{c_1}, C_{\bar{\mathcal{B}}_1}^{\delta_1}, \ldots, A_{c_n}, C_{\bar{\mathcal{B}}_n}^{\delta_n}).$$

Let $U \subseteq (2^\omega)^{m^*}$ be Σ_1^0; we need to show that either $\bar{X} \in U$, or there is some condition

$$\bar{q} \in \mathbb{C}^{|c_1|} \times \mathbb{P}_{\bar{\mathcal{B}}_1} \times \cdots \times \mathbb{C}^{|c_n|} \times \mathbb{P}_{\bar{\mathcal{B}}_n}$$

such that $\bar{X} \in [\bar{q}]$ and $[\bar{q}] \subseteq U^{\complement}$.

Fix some e^* sufficiently large so that for $l = 1, \ldots, n$, the strings

$$f_{\mathcal{B}_{l,1}}^{\delta_l} \restriction e^*, f_{\mathcal{B}_{l,2}}^{\delta_l} \restriction e^*, \ldots, f_{\mathcal{B}_{l,|\bar{\mathcal{B}}_l|}}^{\delta_l} \restriction e^*$$

are all distinct.

By reverse induction on $l = n, n-1, \ldots, 0$, we define $e_l \geqslant e^*$, and when $l \geqslant 1$, also some $k_l < t^{\delta_l, e_l}$. We start with choosing some $e_n \geqslant e^*$ such that:

- $n^{\delta_n, e_n} = n$;
- $\bar{\delta}^{\delta_n, e_n} = (\delta_0, \delta_1, \delta_2, \ldots, \delta_n)$;
- for $l = 1, \ldots, n$, $m_l^{\delta_n, e_n} = |\bar{\mathcal{B}}_l|$;
- $m_{\text{fut}}^{\delta_n, e_n} = 0$;
- $c^{\delta_n, e_n} = c$;
- $U^{\delta_n, e_n} = U$.

(Since $m_{\text{fut}}^{\delta_n,e_n} = 0$, $\mathbf{p}^{\delta_n,e_n}$ is the vacuous condition; so (vii) of Definition 4.1 holds vacuously.)

Now for $l \geqslant 1$, given $e_l \geqslant e^*$, let

$$\bar{\sigma}_l = (f^{\delta_l}_{\mathcal{B}_{l,1}} \upharpoonright e_l, f^{\delta_l}_{\mathcal{B}_{l,2}} \upharpoonright e_l, \dots, f^{\delta_l}_{\mathcal{B}_{l,|\bar{\mathcal{B}}_l|}} \upharpoonright e_l).$$

and let k_l be the $k < t^{\delta_l,e_l}$ such that $\bar{\tau}^{\delta_l,e_l,k} = \bar{\sigma}_l$.

Let $l \geqslant 1$ and suppose that e_l and k_l have been chosen, with $c^{\delta_l,e_l} = c \cap \delta_l$. By assumption, and by Lemma 4.10, for all $\alpha \in c^{e_l,\delta_l}$ and $i \leqslant |\bar{\mathcal{B}}_l|$, A_α is not a column of $x^{\delta_l}_{\sigma_l,i}$, and so of $u_i^{\delta_l,e_l,k_l}$. That is, (δ_l, e_l) is active at k_l. We thus let e_{l-1} be the least e which is suitable for (δ_l, e_l, k_l). Note that $e_{l-1} \geqslant e_l \geqslant e^*$. This concludes the definition of e_l for all $l \leqslant n$.

For $l = 1, \dots, n$, let

$$\bar{v}_l = \left(y^{\delta_l,e_l,k_l+1}_{\sigma_{l,1}}, y^{\delta_l,e_l,k_l+1}_{\sigma_{l,2}}, \dots, y^{\delta_l,e_l,k_l+1}_{\sigma_{l,|\bar{\mathcal{B}}_l|}}\right).$$

We observe:

- $C^{\delta_l}_{\bar{\mathcal{B}}_l} \in [\bar{v}_l]$, and $\bar{v}_l \in \mathbb{P}_{\bar{\mathcal{B}}_l}$.

For $l = 0, \dots, n-1$, write

$$\bar{w}^{\delta_l,e_l} = (\bar{s}_{l,l+1}, \bar{t}_{l,l+1}, \bar{s}_{l,l+2}, \bar{t}_{l,l+2}, \dots, \bar{s}_{l,n}, \bar{t}_{l,n}),$$

where $|\bar{s}_{l,j}| = |c_j|$ and $|\bar{t}_{l,j}| = |\bar{\mathcal{B}}_j|$.

Let $0 < j \leqslant n$. For all $i = 1, \dots, |\bar{\mathcal{B}}_j|$, by definition,

$$v_{j,i} \succcurlyeq t_{j,j-1,i},$$

i.e., $[\bar{v}_j] \subseteq [\bar{t}_{j,j-1}]$. Also, since $\bar{w}^{\delta_{j-1},e_{j-1}}$ is consistent with $\bar{z}^{\delta_j,e_j,k_j}$, $A_{c_j} \in [\bar{s}_{j,j-1}]$.

Further, if $j > l$, then the choice of $\bar{w}^{\delta_l,e_l}$ ensures that $[\bar{s}_{l,j}] \subseteq [\bar{s}_{l-1,j}]$ and $[\bar{t}_{l,j}] \subseteq [\bar{t}_{l-1,j}]$. Hence: for all $l = 1, \dots, n$,

- $[\bar{v}_l] \subseteq [\bar{t}_{0,l}]$; and
- $A_{c_l} \in [\bar{s}_{0,l}]$.

Further, observe that each $\bar{s}_{0,l}$ is finite. Let

$$\bar{q} = (\bar{s}_{0,1}, \bar{v}_1, \bar{s}_{0,2}, \bar{v}_2, \dots, \bar{s}_{0,n}, \bar{v}_n).$$

Then $\bar{q} \in \mathbb{C}^{|c_1|} \times \mathbb{P}_{\bar{\mathcal{B}}_1} \times \cdots \times \mathbb{C}^{|c_n|} \times \mathbb{P}_{\bar{\mathcal{B}}_n}$ and $\bar{X} \in [\bar{q}]$. Further, $[\bar{q}] \subseteq [\bar{w}^{0,e_0}]$. Note that this also implies $[\bar{q}] \subseteq [\bar{y}^{0,e_0}]$, as $[\bar{w}^{0,e_0}] \subseteq [\bar{y}^{0,e_0}]$. Further, for every $b \in a^{0,e_0}$, if $(i_1, o_1), (i_2, o_2) \in b$, then this is because $(q_{i_1})^{[o_1]} = (q_{i_2})^{[o_2]} = A_\alpha$ for some α. Putting these together, we get

$$[\bar{q}] \subseteq P(\mathbf{p}^{0,e_0}).$$

Now we can finish the proof:

- If $U \cap P(\mathbf{p}^{0,e_0}) \neq \emptyset$, then as $[\bar{w}^{0,e_0}] \subseteq U$ and $[\bar{q}] \subseteq [\bar{w}^{0,e_0}]$, we have $\bar{X} \in U$.
- If not, then as $[\bar{q}] \subseteq P(\mathbf{p}^{0,e_0})$, $[\bar{q}] \subseteq U^{\complement}$.

Thus, $\bar{q}$ is a condition as required. □

Remark 4.13. Where did we use the assumption that $\{A_\alpha : \alpha \in c\}$ is disjoint from $\bigcup_{l \leqslant n} \bigcup_{j \leqslant |\bar{\mathcal{B}}_l|} \mathcal{B}_{l,j}$? It would, at first look, appear that the proof of Lemma 4.11 relies on this fact (on (δ, e) being active at k). However, we could modify $\mathbb{S}$ so that the lemma applies in the inactive case as well: instead of just passing $\mathbf{q}$ as defined above to the predecessor requirement, we could also tell that requirement, for each $\alpha \in c'$, which columns of which z_i^k are equal to A_α. The argument of the lemma would then proceed in the same way. (Note that we would need to rescind condition (vii) of Definition 4.1).

It is only in the proof of Lemma 4.12 that the assumption is used, and it is perhaps not easy to see where. Under the proposed modification, when $\bar{X} \notin U$, we still get $P(\mathbf{p}^{0,e_0}) \subseteq U^{\complement}$. The rest of the argument is defining $[\bar{q}] \subseteq P(\mathbf{p}^{0,e_0})$ that is a cylinder: a product of 1-dimensional closed sets. In other words, a condition in the product of m^*-many notions of forcing. This allows us to apply Proposition 3.8, which relies on the product theorem (Theorem 3.5). The restrictions embodied in $a^{\mathbf{p}^{0,e_0}}$ prevent $P(\mathbf{p}^{0,e_0})$ from being a cylinder; they "cause an interaction" (or dependence) between different coordinates. However, when we fill in the missing A_α's (that is, pass to the conditions $\bar{v}_l$), we do get a cylinder; the dependency between coordinates is eliminated by completely filling in the columns that are required to be equal. In the proof above, we are allowed to do so, because the conditions in $\mathbb{P}_{\bar{\mathcal{B}}}$ allow us to fill in entire columns with such A_α. However, under the proposed modification, to obtain a cylinder, we would need to also fill in the various A_α for $\alpha \in c$. That is, the cylinder we would obtain would be, in the worst case,

$$\{A_{c_1}\} \times [\bar{v}_1] \times \cdots \times \{A_{c_n}\} \times [\bar{v}_n].$$

But A_{c_l} is not a condition in $\mathbb{C}^{|c_l|}$, whereas $\bar{s}_l$ is.

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