EFFECTIVE SEPARATION AND REDUCTIONS PROPERTIES OF BOREL WADGE CLASSES

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Abstract. We use our descriptions of Borel Wadge classes from [DGHTTa] to characterise those Borel Wadge classes that have the separation property, and those that have the reduction property. Our analysis shows that both properties are equivalent to their effective versions. To do so, we give a characterisation of containment between Borel Wadge classes based on their descriptions, and give a direct proof that all such classes admit admissible descriptions.

1. Introduction

In [LSR88a], Louveau and Saint Raymond gave a characterisation of those non-self-dual Borel Wadge classes that have the separation property, and those that have the reduction property. Their work is based on Louveau's classification of Borel Wadge classes ([Lou83]), which extends Wadge's work ([Wad84]).

In [DGHTTa], together with Day and Harrison-Trainor, we defined a new system of descriptions of Borel Wadge classes, which is effective in nature. It is based on Montalbán's "true stage" method. This method was first applied in descriptive set theory by Day, Downey and Westrick [DDW] and by Day and Marks [DM]. See [DGHTTb], a survey, in which the authors use the technique to give a new proof of Louveau's separation theorem.

Here, we use the class descriptions from [DGHTTa] to give intuitive characterisations of both the separation and reduction properties for Borel Wadge classes. These characterisations flesh out the dynamic intuition behind these properties: both rely on a "stage comparison" argument. The standard argument for the reduction property of the class of c.e. subsets of $\mathbb{N}$ is: run simultaneous enumerations of two c.e. sets A and B. When a number n enters $A \cup B$, if it first enters A, put it on the A-side (enumerate it into a c.e. $A_0 \subseteq A$), otherwise put it on the B-side. The result is a pair (A_0, B_0) reducing (A, B), meaning, $A_0 \subseteq A$, $B_0 \subseteq B$, $A \cup B = A_0 \cup B_0$, and $A_0 \cap B_0 = \varnothing$.

The same argument applies to open subsets of Baire space. Using the true stage machinery, we can extend this argument to all classes $\mathbf{\Sigma}^0_\alpha$, as follows. Let $A, B \subseteq \mathcal{N}$ be $\mathbf{\Sigma}^0_\alpha$. After relativising to an oracle, we may assume that $\alpha < \omega_1^{\mathrm{ck}}$ and that $A, B \in \Sigma^0_\alpha$. Let ξ be the ordinal such that $\alpha = 1 + \xi$. Then there are computable sets $U, V \subseteq \omega^{<\omega}$, upwards closed in $\preccurlyeq_\xi$, such that $A = [U]_\xi$ and $B = [V]_\xi$, meaning that $A = \{x \in \mathcal{N} : (\exists \sigma \prec_\xi x)\, \sigma \in U\}$ (and similarly for B). Here $\preccurlyeq_\xi$ is the ξ-true stage relation given by a particular computable copy of ξ. Now define computable $U_0 \subseteq U$ and $V_0 \subseteq V$ by letting $\sigma \in U_0$ if the least $\tau \preccurlyeq_\xi \sigma$ with $\tau \in U \cup V$ belongs to U; we let $\sigma \in V_0$ if the least $\tau \preccurlyeq_\xi \sigma$ with $\tau \in U \cup V$ belongs to $V \backslash U$. Then $A_0 = [U_0]_\xi$ and $B_0 = [V_0]_\xi$ are Σ^0_α sets that reduce (A, B). In brief: B_0 is the set of $x \in B$ such that $x \in B$ is witnessed before $x \in A$ is; a witness is a ξ-true stage for x that places x in B.

In this paper we show that an analysis of this kind can be carried out for all non-self-dual Borel Wadge classes. Informally:

- A non-self-dual Borel Wadge class has the separation property if and only if some (equivalently, every) description Γ of the class has default outcome "in" (we say it is of Π-*type*, the dual of a Σ-*type*).
- A non-self-dual Borel Wadge class has the reduction property if and only if some description Γ of the class is hereditarily of Σ-type, meaning that all of the classes Γ_s used in the construction of Γ have Σ-type.

Further, these characterisations show that the separation property is equivalent to the *effective separation property*, which states that a separator can be obtained effectively from the pair needing separation. Similarly, the reduction property for a Borel Wadge class is equivalent to the effective reduction property, however in this case, we may need to relativise to a Turing cone. The base of the cone can be taken to be Δ^1_1 relative to any given description of the class.

Along the way, we describe clopen games that characterise containment between non-self-dual Borel Wadge classes, and similarly, games that characterise the separation and reduction properties. An effective version of these games is used in up-coming work on Selivanov's fine hierarchy. Further, our game characterisation of containment between classes allows us to give direct translations of class descriptions into "admissible" class descriptions, which was hitherto done only indirectly.

Our characterisations are analogous to those provided by Louveau and Saint Raymond in [LSR88a]. The methods are fundamentally different, though. In particular, their argument uses Borel determinacy, whereas as in [DGHTTa], ours can be carried out in the system $\mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{IND}$.

2. Class descriptions

We shall use the true stage relations and class descriptions that were developed in [DGHTTa]. Let us recall the main notions.

We work with Baire space $\mathcal{N} = \omega^\omega$. A (concrete) computable ordinal is a computable well-ordering of a computable subset of $\mathbb{N}$, in which the successor relation and collection of limit points are both computable. For concrete computable ordinals α and β we write $\alpha < \beta$ if α is an initial segment of β.

For every concrete computable ordinal α we obtain a partial ordering $\preccurlyeq_\alpha$ with a variety of pleasing properties, (denoted TSP(1)–TSP(7) in [DGHTTa]). In particular, $(\omega^{\leqslant\omega}, \preccurlyeq_\alpha)$ is a tree, with root $\langle\rangle$ (the empty sequence); $\preccurlyeq_0$ is usual string extension $\preccurlyeq$; the relations are nested: if $\beta < \alpha$ then $\preccurlyeq_\alpha$ implies $\preccurlyeq_\beta$. For all $x \in \mathcal{N}$, $\{\sigma \in \omega^{<\omega} : \sigma \prec_\alpha x\}$ is the unique infinite path in $(\{\sigma : \sigma \prec x\}, \preccurlyeq_\alpha)$. And most importantly: a set $A \subseteq \mathcal{N}$ is $\Sigma^0_{1+\alpha}$ if and only if there is a c.e. (or computable) set $U \subseteq \omega^{<\omega}$ such that $A = [U]_\alpha = \{x \in \mathcal{N} : (\exists \sigma \prec_\alpha x)\, \sigma \in U\}$. These relations can be relativised to oracles z, in which case we write $\preccurlyeq^z_\alpha$.

Informally, the idea is that we can associate with each finite sequence σ, a guess about finitely many entries of the α^{th} iterated Turing jump of reals extending σ. The relation $\sigma \preccurlyeq_\alpha \tau$ for finite τ means that the τ guesses extend the σ guesses; the relation $\sigma \prec_\alpha x$ for infinite x means that σ guesses correctly about the iterated jump of x. While the true stage machinery is required for the definition of the class descriptions, we will see that our game characterisations will free us from directly using this machinery when analysing containment between classes, and the reduction and separation properties.

The presentation of $\Sigma^0_{1+\alpha}$ sets (as those which are generated by computable sets of strings using $\preccurlyeq_\alpha$) extends to a characterisation of a corresponding class of approximated functions. An *α-approximation* of a function $F\colon \mathcal{N} \to \mathbb{N}$ is a function $f\colon \omega^{<\omega} \to \mathbb{N}$ such that for all $x \in \mathcal{N}$, the sequence $\langle f(\sigma) : \sigma \prec_\alpha x\rangle$ is eventually constant with value $F(x)$. Generalizing the case $\alpha = 0$, we have that a function $F\colon \mathcal{N} \to \mathbb{N}$ is $\Sigma^0_{1+\alpha+1}$-measurable if and only if it has a computable α-approximation (see [DGHTTa, Prop. 2.14] or [DGHTTb, Prop. 3.6]).

A *class description* is a labelled tree Γ satisfying the following:

(i) the underlying tree $T_\Gamma \subseteq \omega^{<\omega}$ is well-founded;
(ii) for a leaf s of T_Γ, $\Gamma(s) \in \{0,1\}$;
(iii) for a non-leaf $s \in T_\Gamma$, $\Gamma(s)$ is a pair $(\xi_s, \eta_s) = (\xi^\Gamma_s, \eta^\Gamma_s)$ of (concrete) ordinals, with $\eta_s \geqslant 1$.

We require that $\xi_s \leqslant \xi_t$ if $s \preccurlyeq t$. A class description Γ is also equipped with an oracle y^Γ that computes Γ (including all the ordinals ξ_s and η_s, uniformly in s).

A class description is a template for defining nested approximations, that give decision procedures for sets in the described classes. A *Γ-name* will determine, for each real x, a leaf s of T_Γ, and x will be an element of the named set if the Γ-label of s is 1. If t is a non-leaf of T_Γ, and it has been determined that t is an initial segment of the leaf corresponding to x, then the label (ξ_s, η_s) tells us that in order to find which child of t on T_Γ is an initial segment of the leaf, we apply ξ_s many Turing jumps to x, and then computably approximate the choice of a child using an η_s-c.e. process: we first need to choose the leftmost child of t, which is a default child; we can then change our mind, but each time that we do, we need to decrease the counter ordinal, which started at η_s.

More formally, if Γ is a class description, then a Γ-name N consists of an oracle $z = z^N \geqslant_{\mathrm{T}} y^\Gamma$ computing N, and for each non-leaf $s \in T_\Gamma$, a pair $(f_s, \beta_s) = (f^N_s, \beta^N_s)$, such that f_s is a ξ_s-approximation of a function choosing children of t, with β_s being a witness for the convergence of the approximation. That is:

(1) for all $\sigma \in \omega^{<\omega}$, $f_s(\sigma)$ is a child of s on T_Γ, and $\beta_s(\sigma) \leqslant \eta^\Gamma_s$;
(2) if $\sigma \preccurlyeq^z_{\xi_s} \tau$ then $\beta_s(\sigma) \geqslant \beta_s(\tau)$, and if in addition, $f_s(\sigma) \neq f_s(\tau)$, then $\beta_s(\sigma) > \beta_s(\tau)$; and
(3) if $\beta_s(\sigma) = \eta^\Gamma_s$ then $f_s(\sigma)$ is the leftmost child of s on T_Γ.

For each such s, for each $x \in \omega^\omega$, the conditions above ensure that the sequence $\langle f_s(\sigma) : \sigma \prec^z_{\xi_s} x\rangle$ stabilizes to some value, which we denote by $f_s(x) = f^N_s(x)$. (Similarly, the sequence of ordinals $\langle \beta_s(\sigma) : \sigma \prec^z_{\xi_s} x\rangle$ stabilizes to a value denoted by $\beta_s(x) = \beta^N_s(x)$.) For each $x \in \omega^\omega$, we can then recursively define a sequence $s_0, s_1, \ldots$ of nodes on T_Γ, starting with s_0 being the root, and letting $s_{k+1} = f_{s_k}(x)$. This terminates in a leaf $\ell(x) = \ell^N(x)$ of T_Γ; the set named by N is the collection of x for which the Γ-label of $\ell(x)$ is 1.

Notation 2.1. To keep notation clean, for a Γ-name N, we will occasionally let N denote the set named by N, which we will also identify with its characteristic function (this was denoted by F^N in [DGHTTa]).

Remark 2.2. In [DGHTTa] we needed to consider names of partial functions on Baire space; these are not required in the current paper, and so we only defined total Γ-names.

For an oracle $z \geqslant_{\mathrm{T}} y^\Gamma$, we let $\Gamma(z)$ be the collection of all sets named by Γ-names N with $z^N = z$; we let

$$\boldsymbol{\Gamma} = \bigcup \{\Gamma(z) \,:\, z \geqslant_{\mathrm{T}} y^\Gamma\}\,.$$

The collection $\boldsymbol{\Gamma}$ is a non-self-dual Borel Wadge class, which has a universal set. This can be seen by the fact that we can effectively list z-computable approximations (f_s, β_s) as above, much as we can give effective lists of all η-c.e. sets; the default child allows us to convert partial approximations to total ones while preserving the limit. That $\boldsymbol{\Gamma}$ is closed under taking continuous preimages follows from the fact that we can effectively translate between true stage relations; this also shows that if $w \geqslant_{\mathrm{T}} z \geqslant_{\mathrm{T}} y^\Gamma$ then $\Gamma(z) \subseteq \Gamma(w)$, uniformly. For details, see [DGHTTa, Prop. 3.10,3.14].

A $\Gamma(z)$-*name* is a Γ-name N such that $z^N = z$.

The definitions so far may seem abstract, but examples can explain the intuition behind them. Perhaps the simplest examples are the descriptions of the classes $\boldsymbol{\Sigma}^0_{1+\alpha}$ and $\boldsymbol{\Pi}^0_{1+\alpha}$ (Fig. 1). To approximate membership in a $\boldsymbol{\Sigma}^0_{1+\alpha}$ set, we first take α many jumps, start with the default value "out", and are allowed to change our mind once, to the value "in". The dual class is similar, except that the default value is "in". We will encounter further examples below.

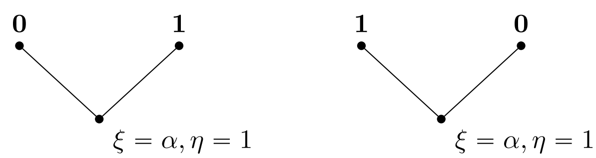

FIGURE 1. The simplest descriptions of $\boldsymbol{\Sigma}^0_{1+\alpha}$ and $\boldsymbol{\Pi}^0_{1+\alpha}$.

Notation 2.3. The labels of nodes on T_Γ do not play any role in the determination of the classes described; the only distinction is between the default child of a node, and all the rest. It will be convenient to assume that for any class description and any non-leaf $s \in T_\Gamma$, the default child of s is $s\hat{\ }0$.

Associated with class descriptions are the following concepts.

- The *ordinal level* of a class description Γ, denoted by $o(\Gamma)$, is the ξ-ordinal $\xi^\Gamma_\diamondsuit$ at the root of T_Γ. This is defined unless the root is also the leaf of T_Γ (in which case the class described is either $\{\varnothing\}$ or $\{\mathcal{N}\}$); we then set $o(\Gamma) = \omega_1$.
  When $o(\Gamma) < \omega_1$ we let $\eta^\Gamma = \eta^\Gamma_\diamondsuit$ denote the η-ordinal specified by Γ at the root of T_Γ.
- The *dual* $\check{\Gamma}$ of a class description Γ is obtained from Γ by flipping the values $\Gamma(s)$ at the leaves. The described class is indeed the dual of $\boldsymbol{\Gamma}$.
- If Γ is a class description and $s \in T_\Gamma$ then Γ_s is the class description obtained by setting s to be the new root and taking Γ above s: $\Gamma_s(t) = \Gamma(s\hat{\ }t)$. The various classes Γ_s are those which are used in a recursive construction of Γ (starting with the leaves).

We will use "definition by cases". For sets A and X, a class description Γ, and an oracle $z \geqslant_{\mathrm{T}} y^\Gamma$, we say that $A \upharpoonright X \in \Gamma(z)$ if there is some $B \in \Gamma(z)$ such that $A \cap X = B \cap X$. For sequences (A_n) and (X_n), we say that $A_n \upharpoonright X_n \in \Gamma(z)$ uniformly if with oracle z, given n, we can compute a $\Gamma(z)$-name N_n for a set B_n with $A_n \cap X_n = B_n \cap X_n$.

Proposition 2.4. *Let Γ be a class description, and let $z \geqslant_{\mathrm{T}} y^\Gamma$. Suppose that:*

- *$(X_n)_{n\in\omega}$ is a partition of $\mathcal{N}$ into sets which are uniformly $\Delta^0_{1+o(\Gamma)}(z)$; and*
- *$A \subseteq \mathcal{N}$ is a set such that $A \upharpoonright X_n \in \Gamma(z)$, uniformly.*

Then $A \in \Gamma(z)$.

This proposition follows from [DGHTTa, Prop. 3.17]. The proof, however, is easy, so we give a direct one.

Proof. For simplicity of notation, assume that z is computable. Let $\alpha = o(\Gamma)$. By the true stage properties mentioned above, there is a sequence of uniformly computable sets $U_n \subseteq \omega^{<\omega}$ with $X_n = [U_n]_\alpha$; we may assume that the sets U_n are pairwise incomparable under $\preccurlyeq_\alpha$, and that the union $\bigcup U_n$ is also computable. The nestedness of the true stage relations, together with the requirement that α is an initial segment of ξ^Γ_s for all non-leaf $s \in T_\Gamma$, imply that for all such s, for all n, $[U_n]_{\xi_s} = [U_n]_\alpha = X_n$.

Let N_n be a uniformly computable sequence of Γ-names, with N_n naming a set A_n such that $A_n \cap X_n = A \cap X_n$. Define a new $\Gamma(z)$-name M by taking the "disjoint union" of the names N_n according to (U_n): for each non-leaf s of T_Γ we define f^M_s and β^M_s as follows: for each $\sigma \in \omega^{<\omega}$, if σ has no predecessor in any U_n then we set $\beta^M_s(\sigma) = \eta_s$ and $f^M_s(\sigma) = s\hat{\ }0$ (the default); otherwise, for some unique n, σ has a predecessor in U_n, and then we set $f^M_s(\sigma) = f^{N_n}_s(\sigma)$ and $\beta^M_s(\sigma) = \beta^{N_n}_s(\sigma)$. ☐

We now introduce terminology that did not appear in [DGHTTa], but mentioned in the introduction. If Γ is a class description, then as T_Γ is well-founded, it has a leftmost leaf. This leaf of T_Γ is in some sense the ultimate default outcome: the default of the default of the default. . . .

Definition 2.5. Let Γ be a class description; let s be the leftmost leaf of T_Γ. We say that Γ is of Π-*type* if $\Gamma(s) = 1$, and Γ is of Σ-*type* if $\Gamma(s) = 0$.

Every description is either of Σ-type or of Π-type. A class description Γ has Σ-type if and only if its dual $\check{\Gamma}$ has Π-type. If $o(\Gamma) < \omega_1$ (i.e., if T_Γ is not just the root) then Γ and Γ_0 have the same type. The natural descriptions of $\mathbf{\Sigma}^0_{1+\alpha}$ and $\mathbf{\Pi}^0_{1+\alpha}$ (Fig. 1) are of Σ-type and Π-type, respectively, justifying the name.

For Wadge classes $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$, we write $\mathbf{\Gamma} < \mathbf{\Lambda}$ when $\mathbf{\Gamma} \subseteq \Delta(\mathbf{\Lambda}) = \mathbf{\Lambda} \cap \check{\mathbf{\Lambda}}$. For class descriptions Γ and Λ we write $\Gamma \subseteq \Lambda$ if $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ effectively: $y^\Gamma \geqslant_{\mathrm{T}} y^\Lambda$ and uniformly, given $z \geqslant_{\mathrm{T}} y^\Gamma$ and a $\Gamma(z)$-name N, we can compute a $\Lambda(z)$-name M, equivalent to N, in the sense that they name the same set.

Remark 2.6. Computability considerations are important for the definition of class descriptions, as they rely on the true-stage relations, which are inherently "lightface". We will also be interested in the effective versions of the separation and reduction properties, and there too we will need to keep track of which oracle we are working with. However, if we are willing to increase the complexity of the oracle as necessary, then boldface considerations suffice. For example, if Γ and Λ are class descriptions, and $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$, then $\Gamma \subseteq \Lambda$ *on a cone*: there is some oracle $w \geqslant_{\mathrm{T}} y^\Gamma, y^\Lambda$ such that after changing the Γ-oracle to w we have $\Gamma \subseteq \Lambda$. (This follows from results in [DGHTTa], but will also follow from our game characterisation of containment in the next section.) Technically, changing the oracle means replacing Γ with a new class description Γ' which is identical to Γ except that $y^{\Gamma'} = w$. This does not change the boldface class: $\mathbf{\Gamma} = \mathbf{\Gamma}'$ as for all $z \geqslant_{\mathrm{T}} w$, $\Gamma(z) = \Gamma'(z)$.

Below, we will often assume that a sufficiently strong oracle is being used, and ignore the difference between Γ and Γ'. See Remark 3.3.

The main result we use from [DGHTTa] is:

Theorem 2.7. *Every non-self-dual Borel Wadge class has a description.*

See [DGHTTa, Thm. 6.8].

3. A clopen game characterisation of containment

Wadge's semi-linear-ordering principle says that for Borel Wadge classes $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$, either $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ or $\mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}$. In this section we attempt to answer the question: given two class descriptions Γ and Λ, how can we tell whether $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ or not? An answer of sorts is given by Lemma 6.1 of [DGHTTa]. There, we devise a closed game G_Λ and show that $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ if and only if player I has a winning strategy in the game $G_\Lambda(H_\Lambda, H_\Gamma, H_{\check{\Gamma}})$, where H_Γ is a universal set for $\mathbf{\Gamma}$, and similarly for H_Λ. We now devise a much simpler game that is: (i) clopen, rather than closed; and (ii) relies only on the descriptions Γ and Λ, and not on their universal sets.

The leaf selection game. The main ingredient in the containment game is an auxiliary "leaf selection game" that we describe first. We need the following definition.

Definition 3.1. For a class description Γ with $o(\Gamma) < \omega_1$, let

$$S_\Gamma = \{\langle\rangle\} \cup \left\{t \in T_\Gamma \,:\, \xi^\Gamma_{t^-} = o(\Gamma)\right\},$$

where t^- is the predecessor of t on T_Γ. This is a subtree of T_Γ. The non-leaves of S_Γ are precisely those $s \in T_\Gamma$ with $\xi^\Gamma_s = o(\Gamma)$. Note that if $s \in S_\Gamma$ is not a leaf of S_Γ, then all the children of s on T_Γ are also on S_Γ.

An S_Γ-*position* p consists of a choice, for each *non-leaf* $s \in S_\Gamma$, of:

(i) a child $c_s = c^p_s$ of s on S_Γ; and
(ii) an ordinal $\eta^p_s \leqslant \eta^\Gamma_s$,

subject to the following restriction:

- If $\eta^p_s = \eta^\Gamma_s$ then $c^p_s = s\hat{\ }0$ is the default child of s.
- For all but finitely many non-leaves $s \in S_\Gamma$, $\eta^p_s = \eta^\Gamma_s$.

The second restriction is in place so that there are only countably many positions.

For two S_Γ-positions p and q, we let $q \leqslant p$ if for every non-leaf s of S_Γ,

(iii) $\eta^q_s \leqslant \eta^p_s$, and further, if $c^q_s \neq c^p_s$ then $\eta^q_s < \eta^p_s$.

The *initial* S_Γ-position is the position p determined by, for every non-leaf s of S_Γ, $\eta^p_s = \eta^\Gamma_s$ (which forces $c^p_s = s\hat{\ }0$).

Every S_Γ-position p determines a leaf t^p of S_Γ, by following the choices from the root, much like the definition of a leaf $\ell^N(x)$ of T_Γ used to compute the set named by a Γ-name N: for every non-leaf $s \prec t^p$, $c^p_s \preccurlyeq t^p$.

Now let Γ and Λ be two class descriptions, and suppose that $\xi = o(\Gamma) = o(\Lambda) < \omega_1$. In the game $G_{\texttt{leaf}}(\Gamma, \Lambda)$, two players, 1 and 2, take turns choosing positions $p[-1], p[0], p[1], p[2], \ldots$, satisfying:

(a) for odd k, $p[k]$ (played by player 1) is an S_Γ-position, and for even k, $p[k]$ (played by player 2) is an S_Λ-position;
(b) $p[-1]$ is the initial S_Γ-position, and $p[0]$ is the initial S_Λ-position;
(c) For all $k \geqslant 1$, $p[k] \leqslant p[k-2]$.

For each k we write $t[k] = t^{p[k]}$.

- A choice $p[k]$ (for $k \geqslant 1$) is called a *pass* if $t[k] = t[k-2]$.

Note that if no other legal move is possible, a player can always choose $p[k] = p[k-2]$, which is, of course, a pass.

- The play ends when one player passes immediately after the other player passed.
- The *outcome* of the play of the leaf selection game is the pair of leaves $(t[k_1], t[k_2])$, where k_j is the last round at which player j played.

Remark 3.2. Every play of the leaf selection game is finite: the child $c_{s_0}[k]$ of the root s_0 of S_Γ must stabilise to some s_1, and then the child $c_{s_1}[k]$ must stabilise, and so on.

The containment game. For two class descriptions Γ and Λ, the game $G_{\texttt{cont}}(\Gamma, \Lambda)$ is played between two players, 1 and 2. During the game, player 1 traverses a path up T_Γ, from the root to some leaf; player 2 does the same on T_Λ.

For every round k of a play of the game, the players choose nodes $s_1[k] \in T_\Gamma$ and $s_2[k] \in T_\Lambda$. We start with $s_1[0] = s_2[0] = \langle\rangle$ being the roots of the respective trees. Suppose that $s_1[k]$ and $s_2[k]$ have already been chosen. At round $k+1$:

(1) If $\xi^\Gamma_{s_1[k]} \neq \xi^\Lambda_{s_2[k]}$ then the player i with the smaller ordinal $\xi_{s_i[k]}$ chooses $s_i[k+1]$ to be some child of $s_i[k]$ on the corresponding tree T_Γ or T_Λ, whereas the other player j does not move: $s_j[k+1] = s_j[k]$.
(2) If $\xi^\Gamma_{s_1[k]} = \xi^\Lambda_{s_2[k]} = \xi < \omega_1$ then the two players play the leaf selection game $G_{\texttt{leaf}}(\Gamma_{s_1[k]}, \Lambda_{s_2[k]})$. The pair of nodes that are the outcome of the play of the leaf selection game are then chosen as $s_1[k+1]$ and $s_2[k+1]$.
(3) Henceforth, for a leaf s of T_Γ, we set $\xi^\Gamma_s = \omega_1$, and similarly for Λ. Hence, (1) implies that once a player reaches a leaf, they stop moving, and the other player must work their way up the tree until they get to a leaf.

The game ends with two leaves $s_1 = s_1[k] \in T_\Gamma$ and $s_2 = s_2[k] \in T_\Lambda$. Player 2 wins the play if

$$\Lambda(s_2) = \Gamma(s_1).$$

The containment game can be coded by a clopen subset of $\mathcal{N}$, and so is determined.

Remark 3.3. In the description of the games, we have implicitly identified concrete ordinals (well-orderings of subsets of $\mathbb{N}$) with their order-types (set theoretic, von-Neumann ordinals). The games do not use the true stage relations, or involve any computability for that matter, and so we didn't need concrete ordinals. Below we will use the game to define Γ- or Λ-names, and these, of course, require the true stage relations. We will work relative to an oracle that can compute the game (and a winning strategy for one of the players). This means that the oracle can compare all the ordinals involved. By [DGHTTa, Prop. 2.20], when we work with such an oracle, we may assume that the concrete ordinals appearing in both Γ and Λ are all initial segments of one long ordinal (they are all comparable as concrete ordinals). The resulting true stage relations are then all nested. We can also unambiguously speak of the concrete ordinal $\xi + 1$, for any ordinal ξ involved.

Note that since two hyperarithmetic ordinals are hyperarithmetically comparable, and the containment game is clopen, we can find such an oracle which is hyperarithmetic in $y^\Gamma \oplus y^\Lambda$.

The following, together with clopen determinacy and the fact that every non-self-dual Borel Wadge class has a description, implies Wadge's semi-linear-ordering principle for such classes.

Theorem 3.4. *Let Γ and Λ be class descriptions.*

(a) *Player 2 has a winning strategy in the game $G_{\mathtt{cont}}(\Gamma, \Lambda)$ if and only if $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$.*
(b) *Player 1 has a winning strategy in the game $G_{\mathtt{cont}}(\Gamma, \Lambda)$ if and only if $\boldsymbol{\Lambda} \subseteq \check{\boldsymbol{\Gamma}}$.*

To prove Theorem 3.4, it suffices to prove the following two propositions:

Proposition 3.5. *If player 2 has a winning strategy in the game $G_{\mathtt{cont}}(\Gamma, \Lambda)$ then $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$.*

Proposition 3.6. *If player 1 has a winning strategy in $G_{\mathtt{cont}}(\Gamma, \Lambda)$, then player 2 has a winning strategy in $G_{\mathtt{cont}}(\Lambda, \check{\Gamma})$.*

This suffices, since the game $G_{\mathtt{cont}}(\Gamma, \Lambda)$ is determined, and the class $\boldsymbol{\Gamma}$ has a universal set (so $\boldsymbol{\Gamma} \nsubseteq \check{\boldsymbol{\Gamma}}$).

We start with the first proposition.

Proof of Proposition 3.5. Let $\mathfrak{S}$ be a winning strategy for player 2 in the game $G_{\mathtt{cont}}(\Gamma, \Lambda)$. We show that $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$ effectively: let z be an oracle that computes $\mathfrak{S}$ and the game (as discussed in Remark 3.3); we show that $\Gamma(z) \subseteq \Lambda(z)$, uniformly. This means that given any $\Gamma(z)$-name N, we can, with the aid of z, compute a $\Lambda(z)$-name M which is equivalent to N, meaning that they both name the same set.

Roughly, the idea of transforming N into M is, for every $x \in \mathcal{N}$, to run the approximation to $N(x)$ as a play for player 1, and to let $M(x)$ follow the strategy $\mathfrak{S}$. We present this construction as the result of effective transfinite recursion on the complexity of the pair (Γ, Λ).

There are four cases.

Case I: $o(\Gamma) = o(\Lambda) = \omega_1$, so $\boldsymbol{\Gamma}, \boldsymbol{\Lambda} \in \{\{\varnothing\}, \{\mathcal{N}\}\}$. The game $G_{\mathtt{cont}}(\Gamma, \Lambda)$ ends before it even begins, and player 2 winning it means that $\Gamma = \Lambda$.

Case II: $o(\Gamma) > o(\Lambda)$. In this case, player 2 makes the first move in the game, and so the strategy $\mathfrak{S}$ selects an outcome n (a child of the root). After this first move, the rest of the strategy is a winning strategy for player 2 in the game $G_{\mathtt{cont}}(\Gamma, \Lambda_n)$. By induction, $\Gamma(z) \subseteq \Lambda_n(z)$. The result follows from $\Lambda_n(z) \subseteq \Lambda(z)$.

Case III: $o(\Gamma) < o(\Lambda)$. In this case, player 1 makes the first move in the game. For each child n of the root on T_Γ, the strategy $\mathfrak{S}_n$ for player 2 that is played by following $\mathfrak{S}$ after player 1 played n, is a winning strategy for player 2 in the game $G_{\mathtt{cont}}(\Gamma_n, \Lambda)$. By induction, $\Gamma_n(z) \subseteq \Lambda(z)$, uniformly.

Let N be a Γ-name of a set A. For each n on T_Γ, let

$$H_n = \left\{x \in \mathcal{N} \,:\, \ell^N(x) \geqslant n\right\}.$$

For each n, H_n is $\Delta^0_{1+o(\Gamma)+1}(z)$: the function $f^N_{\langle\rangle}$ has a z-computable $o(\Gamma)$-approximation, and so is $\Delta^0_{1+o(\Gamma)+1}(z)$-measurable.[1] Since $o(\Lambda) \geqslant o(\Gamma)+1$, $H_n \in \Delta^0_{1+o(\Lambda)}(z)$. Each name N_n shows that $A \upharpoonright H_n \in \Gamma_n(z)$, so $A \upharpoonright H_n \in \Lambda(z)$, uniformly. By Proposition 2.4, $N \in \Lambda(z)$.

Case IV: $o(\Gamma) = o(\Lambda) = \xi < \omega_1$. The game $G_{\texttt{cont}}(\Gamma,\Lambda)$ starts with the leaf selection game $G_{\texttt{leaf}}(\Gamma,\Lambda)$, played on the trees S_Γ and S_Λ.

Let N be a $\Gamma(z)$-name of a set A; we will design an equivalent $\Lambda(z)$-name M. For simplicity of notation, we omit mentioning the oracle z in true stage relations. We assume that for all non-leaf $s \in T_\Gamma$, $\beta^N_s(\langle\rangle) = \eta^\Gamma_s$ (redefining $\beta^N_s(\langle\rangle) = \eta^\Gamma_s$ and $f^N_s(\langle\rangle)$ to be the default child $s\hat{\ }0$ of s on T_Γ does not violate the required properties of Γ-names, and does not change the limit values $f^N_s(x)$ for any $x \in \mathcal{N}$).

For each $\sigma \in \omega^{<\omega}$ we will define a sequence of moves for player 1 in the game $G_{\texttt{leaf}}(\Gamma,\Lambda)$. Player 2 will follow the strategy $\mathfrak{S}$. We let $p(\sigma)[-1], p(\sigma)[0], \dots$ denote the resulting play. We write $t(\sigma)[k]$ for $t^{p(\sigma)[k]}$ and similarly write $c_s(\sigma)[k]$ and $\eta_s(\sigma)[k]$. Let $k(\sigma)$ be the last round of the play. We define a round number $m(\sigma)$:

- if $k(\sigma)$ is even (the play ends with a pass by player 2), let $m(\sigma) = k(\sigma) - 2$;
- if $k(\sigma)$ is odd (the play ends with a pass by player 1), let $m(\sigma) = k(\sigma) - 1$.

In other words, $m(\sigma)$ is the round preceding the last pass made by player 1. In particular, by the end of this round, the play has not yet ended.

Let $q(\sigma)$ be the S_Γ-position defined by choosing, for all non-leaf s of S_Γ,

(1) $c^{q(\sigma)}_s = f^N_s(\sigma)$;
(2) $\eta^{q(\sigma)}_s = \beta^N_s(\sigma)$.

The definition of Γ-names implies that if $\sigma \preccurlyeq_\xi \tau$ then $q(\tau) \leqslant q(\sigma)$. The assumption on $\beta^N_s(\langle\rangle)$ implies that $q(\langle\rangle)$ is the initial S_Γ-position.

The definition of the play for σ is done by induction on $|\sigma|_\xi$, the number of proper $\prec_\xi$-predecessors of σ.

- If $\sigma = \langle\rangle$ then player 1 keeps playing $q(\langle\rangle)$.

Suppose that $\sigma \neq \langle\rangle$; let σ^- be the immediate $\prec_\xi$-predecessor of σ.

- In the play for σ, player 1 first follows all the moves $p(\sigma^-)[k]$ for $k < m(\sigma^-)$. From round $m(\sigma^-)+1$ onwards, player 1 keeps playing $q(\sigma)$. (Since $m(\sigma^-)$ is even, we do not need to specify player 1's play at that round.)

This play is legal for player 1 since $q(\sigma) \leqslant q(\sigma^-)$. Note that since $q(\langle\rangle) = p(\langle\rangle)[-1]$ is the initial S_Γ-position, every move by player 1 in the play for $\sigma = \langle\rangle$ is a pass.

Note that it is possible that $\sigma \prec_\xi \tau$ but that $m(\sigma) = m(\tau)$: if $\ell^N(\sigma)$ and $\ell^N(\sigma^-)$ extend the same leaf of S_Γ, then the play $p(\sigma)[m(\sigma)+1] = q(\sigma)$ is a pass.

We define, for non-leaf $s \in S_\Lambda$, the functions f^M_s and β^M_s. For $\sigma \in \omega^{<\omega}$,

- we let $f^M_s(\sigma) = c_s(\sigma)[m(\sigma)]$ and $\beta^M_s(\sigma) = \eta_s(\sigma)[m(\sigma)]$.

[1] In fact, the associated "time-keeping" function $\beta^N_{\langle\rangle}$ shows that the set H_0 is $\check{D}_\eta(\Sigma^0_{1+o(\Gamma)})(z)$, and $D_\eta(\Sigma^0_{1+o(\Gamma)})(z)$ for non-default childrenn. Here $\eta = \eta^\Gamma_{\langle\rangle}$. See the proof of Proposition 4.6 below.

If $\sigma \prec_\xi \tau$, then $m(\sigma) \leqslant m(\tau)$ and the play for τ extends the play for σ after $m(\sigma)$; it follows that $p(\tau)[m(\tau)] \leqslant p(\tau)[m(\sigma)] = p(\sigma)[m(\sigma)]$. This implies that f_s^M and β_s^M obey the rules for building a Λ-name M.

To define M, it suffices to define M_r for every leaf r of S_Λ that is reached by any σ. Let $r(\sigma) = t(\sigma)[m(\sigma)]$ be the leaf of S_Λ which is the outcome of the play for σ; let $u(\sigma)$ denote the outcome on the Γ-side, which is the leaf of S_Γ extended by $\ell^N(\sigma)$. Once we define the rest of M, we will have $\ell^M(\sigma) \geqslant r(\sigma)$.

For $x \in \mathcal{N}$, define $u(x)$ analogously, and let x^* be the shortest $\sigma \prec_\xi x$ such that for all $s \prec u(x)$, $\beta_s^N(\sigma) = \beta_s^N(x)$. For each $\sigma \in \omega^{<\omega}$ let

$$Q_\sigma = \{x \in \mathcal{N} \,:\, x^* = \sigma\}\,.$$

The sets Q_σ are $\Pi^0_{1+\xi}(z)$, and so $\Delta^0_{1+\xi+1}(z)$, uniformly. For each leaf r of S_Λ, $\xi^\Lambda_r > \xi$, so these sets are $\Delta^0_{1+\xi^\Lambda_r}(z)$ (when $\xi^\Lambda_r < \omega_1$).

For each $\sigma \in \omega^{<\omega}$, continuing with $\mathfrak{S}$ after the play for σ in $G_{\texttt{leaf}}(\Gamma, \Lambda)$ is a winning strategy for player 2 in the game $G_{\texttt{cont}}(\Gamma_{u(\sigma)}, \Lambda_{r(\sigma)})$. By induction, $\Gamma_{u(\sigma)}(z) \subseteq \Lambda_{r(\sigma)}(z)$, uniformly.

For each leaf r of S_Λ, let

$$P_r = \bigcup \{Q_\sigma \,:\, r = r(\sigma)\}\,.$$

For each σ, the name $N_{u(\sigma)}$ witnesses $A \upharpoonright Q_\sigma \in \Gamma_{u(\sigma)}(z)$ (recall that A is the set named by N), and so by induction, $A \upharpoonright Q_\sigma \in \Lambda_{r(\sigma)}(z)$. By Proposition 2.4, for each r we can find a Λ_r-name M_r witnessing $A \upharpoonright P_r \in \Lambda_r(z)$. This defines M. Now for each r,

$$P_r = \left\{x \in \mathcal{N} \,:\, \ell^M(x) \geqslant r\right\},$$

So M names the set A. □

Proof of Proposition 3.6. Let $\tilde{\mathfrak{S}}$ be a winning strategy for player 1 in the game $G_{\texttt{cont}}(\Gamma, \Lambda)$. We define a winning strategy $\mathfrak{S}$ for player 2 in the game $G_{\texttt{cont}}(\Lambda, \check{\Gamma})$ by strategy stealing. In fact, we can almost let $\mathfrak{S} = \tilde{\mathfrak{S}}$. However, in a leaf selection sub-game, we need to correct for the fact that player 1 moves first.

More formally, we will define a strategy $\mathfrak{S}$ for player 2 in $G_{\texttt{cont}}(\Lambda, \check{\Gamma})$ such that for every sequence of moves for player 1 in that game, which will result in a sequence $\langle s_1[k], s_2[k]\rangle$ of positions in the play of the game, there is a sequence of moves for player 2 in the game $G_{\texttt{cont}}(\Gamma, \Lambda)$, such that if player 1 responds with $\tilde{\mathfrak{S}}$, the resulting sequence of moves will be $\langle s_2[k], s_1[k]\rangle$.

Recall the two cases from the definition of the containment game, depending on whether the relevant ordinals ξ agree or disagree. In case (1) of the game, we let $\mathfrak{S}$ do exactly what $\tilde{\mathfrak{S}}$ does in reaction to the same moves by the opponent.

In case (2), let $G_{\texttt{leaf}}(\Lambda_{t_1}, \check{\Gamma}_{t_2})$ be a sub-game occuring when player 2 follows $\mathfrak{S}$. The game $G_{\texttt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$ is played in the corresponding play of $G_{\texttt{cont}}(\Gamma, \Lambda)$ when player 1 plays $\tilde{\mathfrak{S}}$. If the outcome of the latter is (s_2, s_1), we want the outcome of the former to be (s_1, s_2). It would seem that player 1 moving first would be even better for us now; the strategy $\mathfrak{S}$ could be one step ahead. The danger is that the play may end prematurely. This only happens if the first move by $\tilde{\mathfrak{S}}$ in $G_{\texttt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$ is a pass. Hence, we consider two cases. Let $x_1, x_3, x_5, \ldots$ be the play by player 1 in $G_{\texttt{leaf}}(\Lambda_{t_1}, \check{\Gamma}_{t_2})$. We let the reaction by $\mathfrak{S}$ to be $y_2, y_4, y_6, \ldots$ defined as follows:

- If the first move of $\tilde{\mathfrak{S}}$ in $G_{\texttt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$ is not a pass, then we ignore the first move of player 1 in $G_{\texttt{leaf}}(\Lambda_{t_1}, \check{\Gamma}_{t_2})$. In $G_{\texttt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$, we let player 2 play $x_3, x_5, x_7, \ldots$ and let $y_2, y_4, y_6, \ldots$ be player 1's response according to $\tilde{\mathfrak{S}}$ (so the play in $G_{\mathtt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$ is $y_2, x_3, y_4, x_5, y_6, x_7, \ldots$).

- If the first move of $\tilde{\mathfrak{S}}$ in $G_{\mathtt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$ is a pass, but x_1 is not a pass, then we let player 2 play $x_1, x_3, x_5, \ldots$ in $G_{\mathtt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$, and list the $\tilde{\mathfrak{S}}$ response as pass, $y_2, y_4, y_6, \ldots$ (so the $G_{\mathtt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$ play is pass, $x_1, y_2, x_3, y_4, \ldots$).
- If the first move of $\tilde{\mathfrak{S}}$ in $G_{\mathtt{leaf}}(\Gamma_{t_2}, \Lambda_{t_1})$ is a pass, and x_1 is a pass, then y_2 is a pass. $\square$

We record corollaries of Theorem 3.4, which were essentially observed during its proof.

Corollary 3.7. *Let Λ and Γ be class descriptions.*

(a) *If $o(\boldsymbol{\Gamma}) > o(\boldsymbol{\Lambda})$, then $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$ if and only if $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}_n$ for some $n \in T_\Lambda$.*
(b) *If $o(\boldsymbol{\Gamma}) < o(\boldsymbol{\Lambda})$, then $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$ if and only if for all $n \in T_\Gamma$, $\boldsymbol{\Gamma}_n \subseteq \boldsymbol{\Lambda}$.*
(c) *If $o(\boldsymbol{\Gamma}) = o(\boldsymbol{\Lambda}) < \omega_1$, then $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$ if and only if there is a strategy $\mathfrak{S}$ for player 2 in the game $G_{\mathtt{leaf}}(\Gamma, \Lambda)$, such that for any play for player 1 that ends in some leaf t_1 of S_Γ, replying using $\mathfrak{S}$ yields a leaf t_2 of S_Λ such that $\boldsymbol{\Gamma}_{t_1} \subseteq \boldsymbol{\Lambda}_{t_2}$.*

Here is a simple example.

Lemma 3.8. Let Γ be a class description of Σ-type; suppose that $o(\Gamma) < \omega_1$ and that $\boldsymbol{\Gamma} \neq \{\varnothing\}$. Then $\boldsymbol{\Sigma}^0_{1+o(\Gamma)} \subseteq \boldsymbol{\Gamma}$.

Proof. Let $\xi = o(\Gamma)$ and let Θ be the simple description of $\boldsymbol{\Sigma}^0_{1+\xi}$ (Fig. 1). The game $G_{\mathtt{cont}}(\Theta, \Gamma)$ begins with a leaf selection game $G_{\mathtt{leaf}}(\Theta, \Gamma)$. The strategy for player 2 is to pass if player 1 passes. Since Γ has Σ-type, the leftmost leaf s of S_Γ satisfies $\{\varnothing\} \subseteq \boldsymbol{\Gamma}_s$. If player 1 does not pass, their only possible move is to choose the 1-child of the root of Θ and pass henceforth. In this case, player 2 chooses some leaf t of S_Γ such that $\mathcal{N} \in \boldsymbol{\Gamma}_t$; there must be one since $\boldsymbol{\Gamma} \neq \{\varnothing\}$. $\square$

It follows that if $\boldsymbol{\Gamma} \neq \{\varnothing\}, \{\mathcal{N}\}$ then $\boldsymbol{\Delta}^0_{1+o(\Gamma)} \subseteq \boldsymbol{\Gamma}$; this also follows from Proposition 2.4.

The same argument shows: if Γ has Σ-type, $o(\Gamma) < \omega_1$, and $\boldsymbol{\Gamma} \neq \{\varnothing\}$, then $D_{\eta^\Gamma}(\boldsymbol{\Sigma}^0_{1+o(\Gamma)}) \subseteq \boldsymbol{\Gamma}$, where $\eta^\Gamma = \eta^\Gamma_{\langle\rangle}$ is the η-ordinal at the root of T_Γ; see Fig. 2.

FIGURE 2. The simplest descriptions of $D_\eta(\boldsymbol{\Sigma}^0_{1+\xi})$ and $\check{D}_\eta(\boldsymbol{\Sigma}^0_{1+\xi})$.

Remark 3.9. Suppose that $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$ and that $o(\Gamma) = o(\Lambda) < \omega_1$. Then player 2 has a strategy $\mathfrak{S}$ in the leaf selection game $G_{\mathtt{leaf}}(\Gamma, \Lambda)$ which is *prompt*, meaning that for any play $p[-1], p[0], p[1], \ldots$ where player 2 follows $\mathfrak{S}$, for every odd $k \geqslant 1$ such that $p[k+1]$ is defined, we have $\boldsymbol{\Gamma}_{t[k]} \subseteq \boldsymbol{\Lambda}_{t[k+1]}$. The idea is that if this is not the case, then instead of playing $p[k+1]$, player 2 can imagine that player 1 keeps passing, until such a stage at which $\mathfrak{S}$ gives an adequate response, and then play that response.

In greater detail, let $\mathfrak{T}$ be a strategy for player 2 as in Corollary 3.7(c). To describe $\mathfrak{S}$, we consider a play $p[-1], p[0], \dots$ of the game in which player 1's moves are given; we explain how player 2 responds. To do that, we run a (possibly) different play $q[-1], q[0], q[1], \dots$ of the same game, in which we specify player 1's moves, and player 2 follows $\mathfrak{T}$. To do so, to each even k for which $p[k]$ is defined, we will match a corresponding even round $l(k)$ in the auxiliary game; l will be strictly increasing, and we will choose $p[k] = q[l(k)]$. We start with $l(0) = 0$. Now suppose that $k \geqslant 1$ is odd, that $l(k-1)$ is defined, and player 1 is now playing some $p[k]$. The auxiliary game has been played up to round $l(k-1)$. We set $q[l(k-1)+1] = p[k]$. Henceforth, we let player 1 pass in the auxiliary game, while player 2 follows $\mathfrak{T}$, until some odd round $n \geqslant l(k-1)+1$ at which $\mathbf{\Gamma}_{q[n]} \subseteq \mathbf{\Lambda}_{q[n+1]}$. Such a round must occur by the assumption on $\mathfrak{T}$. We let $l(k+1) = n+1$ and $p[k+1] = q[n+1]$. Note that the move $q[l(k-1)+1] = p[k]$ is legal for player 1 in the auxiliary game because $q[l(k-1)-1] = q[l(k-3)+1] = p[k-2]$, and $p[k] \leqslant p[k-2]$. Similarly, $p[k+1] = q[l(k+1)]$ is legal for player 2 in the main game.

4. Efficient, monotone, and admissible class descriptions

Some class descriptions are wasteful. Suppose, for example, that Γ is a class description, $\xi < o(\Gamma)$, and that Θ is a class description with $o(\Theta) = \xi$, and $\Theta_n = \Gamma$ for every child n of the root on T_Θ. Then by Corollary 3.7, $\mathbf{\Theta} = \mathbf{\Gamma}$; making extra choices at the root of T_Θ does not help make more complicated sets, essentially because these choices happen "at a lower level", namely ξ; as the root of T_Γ operates at a higher ordinal level, it can divine the result of these choices. It will be useful to use names in which such a situation does not occur.

Recall that for a collection $\mathcal{C}$ of Borel Wadge classes, by the semi-linear-oredering principle, the following are equivalent: (1) $\mathcal{C}$ does not contain a class maximal under $\subseteq$; (2) for every $\mathbf{\Gamma} \in \mathcal{C}$ there is some $\mathbf{\Lambda} \in \mathcal{C}$ such that $\mathbf{\Gamma} \subseteq \check{\mathbf{\Lambda}}$. The definition of efficiency, below, states that at each step, the choice among classes at the next step is non-trivial (there is no maximal choice), and that the containments in duals that witness this fact are provided effectively.

Definition 4.1. A class description Γ is *efficient* if:

- For all non-leaf $s \in T_\Gamma$, for every child t of s, there is some child r of s such that $\Gamma_t \subseteq \check{\Gamma}_r$.
- For all $s, t \in T_\Gamma$, either $\Gamma_s \subseteq \Gamma_t$ or $\Gamma_t \subseteq \check{\Gamma}_s$, uniformly.

The second condition means that given any pair (s, t), the oracle y^Γ can tell which containment $\mathbf{\Gamma}_s \subseteq \mathbf{\Gamma}_t$ or $\mathbf{\Gamma}_t \subseteq \check{\mathbf{\Gamma}}_s$ holds, and that these containments are effective.

Shortly, we will show that all non-self-dual Borel Wadge classes have efficient descriptions (this also follows from the work in [DGHTTa]). Indeed, we will consider a much stronger notion. For now, we note that efficient descriptions determine the ordinal level of a class.

Proposition 4.2. *Suppose that Γ is an efficient class description, Λ is a class description, and that $\mathbf{\Gamma} = \mathbf{\Lambda}$. Then $o(\Gamma) \geqslant o(\Lambda)$.*

Proof. Suppose that $o(\Gamma) < o(\Lambda)$. Since $\mathbf{\Lambda} \subseteq \mathbf{\Gamma}$, by Corollary 3.7, $\mathbf{\Lambda} \subseteq \mathbf{\Gamma}_n$ for some child n of the root on T_Γ. Since Γ is efficient, there is another child m such that $\mathbf{\Gamma}_n \subseteq \check{\mathbf{\Gamma}}_m$. Since $\mathbf{\Gamma}_m \subseteq \mathbf{\Gamma}$, it follows that $\mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}$, but then we cannot have $\mathbf{\Lambda} = \mathbf{\Gamma}$. □

This allows us to define the ordinal level of a non-self-dual Borel Wadge class, as $o(\Gamma)$ for any efficient description Γ of the class; we write $o(\mathbf{\Gamma})$. Louveau and Saint Raymond noted that this ordinal level can also be characterised in terms of definitions by cases: Proposition 2.4 is optimal. Say that a Wadge class $\mathbf{\Theta}$ is *closed under definition by cases at level* ξ if for all $A \subseteq \mathcal{N}$, for every partition of $\mathcal{N}$ into $\mathbf{\Delta}^0_{1+\xi}$ sets (X_n), if for all n, $A \upharpoonright X_n \in \mathbf{\Theta}$, then $A \in \mathbf{\Theta}$.

Proposition 4.3. *A non-self-dual Borel Wadge class $\mathbf{\Theta} \neq \{\varnothing\}, \{\mathcal{N}\}$ is closed under definition by cases at level ξ if and only if $\xi \leqslant o(\mathbf{\Theta})$.*

Of course if $\mathbf{\Theta} = \{\varnothing\}$ or $\{\mathcal{N}\}$ then it is closed under definition by cases at every level $\xi < \omega_1 = o(\mathbf{\Theta})$.

Proof. That $\mathbf{\Theta}$ is closed under definition by cases at level $o(\mathbf{\Theta})$ follows from Proposition 2.4, using any efficient description Θ of $\mathbf{\Theta}$. On the other hand, if Θ is such a description, let N be a Θ-name for a set A, universal for $\mathbf{\Theta}$. For each child n of the root on T_Θ, let X_n be the collection of $x \in \mathcal{N}$ such that $\ell^\Theta(x) \geqslant n$. As above, the sets X_n are $\mathbf{\Delta}^0_{1+o(\mathbf{\Theta})+1}$. For each n, $A \upharpoonright X_n \in \mathbf{\Theta}_n$ (as is witnessed by the name N_n); efficiency implies that $A \upharpoonright X_n \in \check{\mathbf{\Theta}}$. If $\mathbf{\Theta}$ were closed under definition by cases at level $o(\mathbf{\Theta}) + 1$, then we would have $A \in \check{\mathbf{\Theta}}$, and being universal, it is not. □

Definition 4.4. A class description Γ is *monotone* if for all non-leaf $s \in T_\Gamma$, for all $n \in \mathbb{N}$, $s\hat{\ }n \in T_\Gamma$, and $\Gamma_{s\hat{\ }n} \subseteq \check{\Gamma}_{s\hat{\ }(n+1)}$, uniformly in n and s.

These are the descriptions used in [DGHTTa]. Every monotone description is efficient.

4.1. **Admissible descriptions.** The paper [DGHTTa] used the notion of an *acceptable* class description, which is a monotone class description in which every $\eta_s = 1$. Unfortunately, to properly classify those classes with the reduction property, we must move to a different sort of description.

Definition 4.5. A class description Γ is *admissible* if it is efficient, and for all non-leaf $s \in T_\Gamma$, for every child t of s other than the default one, $\xi^\Gamma_s < \xi^\Gamma_t$.

In general, descriptions only require $\xi^\Gamma_s \leqslant \xi^\Gamma_t$; in admissible descriptions, equality is permitted only for the default outcome. Acceptable descriptions are closer in spirit to “type 2 descriptions” from [LSR88b]. Admissible descriptions are closer to “type 1 descriptions” discussed in [Lou83].

4.2. **The utility of admissible descriptions.** One important common property of both acceptable and admissible class descriptions is that non-default outcomes $t = s\hat{\ }n$, in some sense, “know” the limit behaviour of f_s. That is, for an acceptable or admissible class description Γ and a Γ-name N, we may make the simplifying assumption that for a non-default child t of a node $s \in T_\Gamma$, for all σ, if $\beta^N_t(\sigma) < \eta^\Gamma_t$ then $f^N_s(\sigma) = t$. In other words, f^N_t does not begin to act (possibly moving away from its default outcome) until it is certain that f^N_s has converged to t.

For acceptable descriptions, this is because if $f^N_s(\sigma) = t$ then $\beta^N_s(\sigma) = 0$, so $f^N_s(\tau) = t$ when $\sigma \preccurlyeq^z_{\xi_s} \tau$. Thus, f^N_t can begin acting as soon as it sees f^N_s take the value t.

For admissible descriptions, the fact that t is working with a higher ordinal, specifically that $\xi_t \geqslant \xi_s + 1$, allows it to comprehend the eventual behaviour of f^N_s.[2] Specifically, there is a z-computable set $X \subseteq \omega^{<\omega}$, $\prec^z_{\xi_t}$-upwards closed, such that: for $\sigma \preccurlyeq^z_{\xi_t} \tau$ with $\sigma \in X$, $f^N_s(\sigma) = f^N_s(\tau)$; and $[X]_{\xi_t} = \mathcal{N}$. Then f^N_t can defer any action until it reaches a $\sigma \in X$ with $f^N_s(\sigma) = t$.

Here is a related example. In [DGHTTa], we gave a class description for the class $\mathrm{BiSep}(\mathbf{\Sigma}^0_{1+\xi}, \mathbf{\Gamma}, \mathbf{\Lambda})$ of two-sided separated unions. Using admissible descriptions, we can extend it to some classes $\mathrm{BiSep}(D_\eta(\mathbf{\Sigma}^0_{1+\xi}), \mathbf{\Gamma}, \mathbf{\Lambda})$, as follows. Let ξ be an ordinal; let Λ and Γ be class descriptions, and suppose that:

- $\Lambda < \Gamma$;
- $\xi \leqslant o(\Lambda)$; and
- $\xi < o(\Gamma)$.

Let η be an ordinal. Define a new class description Υ by setting:

- $o(\Upsilon) = \xi$;
- $\eta^\Upsilon = \eta$;
- The children of the root are 0,1, and 2 (with 0 being the default), and:
  - $\Upsilon_0 = \Lambda$;
  - $\Upsilon_1 = \Gamma$;
  - $\Upsilon_2 = \check{\Gamma}$.

If Λ and Γ are efficient, then so is Υ; if Λ and Γ are admissible, then so is Υ.

Proposition 4.6. $\mathbf{\Upsilon} = \mathrm{BiSep}(D_\eta(\mathbf{\Sigma}^0_{1+\xi}), \mathbf{\Gamma}, \mathbf{\Lambda})$ *is the class of sets of the form* $(C_1 \cap A_1) \cup (C_2 \cap A_2) \cup ((C_1 \cup C_2)^\complement \cap B)$, *where* C_1 *and* C_2 *are disjoint* $D_\eta(\mathbf{\Sigma}^0_{1+\xi})$ *sets,* $A_1 \in \mathbf{\Gamma}$, $A_2 \in \check{\mathbf{\Gamma}}$, *and* $B \in \mathbf{\Lambda}$.

Proof. In the easier direction, let N be an Υ-name of a set F; let $z = z^N$. For each $n = 0, 1, 2$, let

$$C_n = \left\{x \,:\, \ell^N(x) \geqslant n\right\}.$$

These sets form a partition of $\mathcal{N}$, in particular, C_1 and C_2 are disjoint. The sets C_1 and C_2 are both $D_\eta(\Sigma^0_{1+\xi})(z)$. To see this, recall ([DGHTTb, Prop. 3.8]) that a set $E \subseteq \mathcal{N}$ is $D_\eta(\Sigma^0_{1+\xi})(z)$ if and only if there is a z-computable ξ-approximation $g\colon \omega^{<\omega} \to \{0, 1\}$ of the characteristic function 1_E, equipped with an ordinal function $\gamma\colon \omega^{<\omega} \to \eta + 1$ witnessing the convergence of g, with default outcome 0, i.e., $\gamma(\sigma) = \eta \implies g(\sigma) = 0$. Let $g_1(\sigma) = 1 \iff f^N(\sigma) = 1$ and $g_2(\sigma) = 1 \iff f^N(\sigma) = 2$; then (g_1, β^N) and (g_2, β^N) show that C_1 and C_2 are both $D_\eta(\Sigma^0_{1+\xi})$. Here, as usual, $f^N = f^N_{\langle\rangle}$ and $\beta^N = \beta^N_{\langle\rangle}$.

The names N_n for $n = 0, 1, 2$ define sets A_1, A_2 and A_3; since $\Upsilon_1 = \Gamma$ we have $A_1 \in \mathbf{\Gamma}$, and similarly, $A_2 \in \check{\mathbf{\Gamma}}$ and $A_0 \in \mathbf{\Lambda}$. Finally, $F \cap C_n = A_n \cap C_n$, so the sets A_1, A_2, A_0, C_1, C_2 show that $F \in \mathrm{BiSep}(D_\eta(\mathbf{\Sigma}^0_{1+\xi}), \mathbf{\Upsilon}, \mathbf{\Lambda})$.

In the easier direction, we had an "excess of ordinals $< \eta$"; it was easy to show that $C_1, C_2 \in D_\eta(\Sigma^0_{1+\xi})$. In the other direction we have to work harder. We are given disjoint $C_1, C_2 \in D_\eta(\Sigma^0_{1+\xi})(z)$, $B \in \Lambda(z)$, $A_1 \in \Gamma(z)$, $A_2 \in \check{\Gamma}(z)$, and we need

[2]This was already used in the proof of Proposition 3.5, in constructing the names M_r.

to come up with an Υ-name M for a set F such that $F = A_1$ on C_1, $F = A_2$ on C_2, and $F = B$ on $(C_1 \cup C_2)^\complement$.

Fix approximations (g_1, γ_1) and (g_2, γ_2) witnessing that $C_1, C_2 \in D_\eta(\Sigma^0_{1+\xi})(z)$. Our opponent, in some sense, has double the "amount of ordinal space" to make changes compared to us: they can change $g_1(x)$ and pay by decreasing γ_1, and then change g_2 and pay by decreasing γ_2. We define a single β^M.

If the opponent makes changes and currently $g_1(\sigma) = g_2(\sigma) = 1$ then we can wait for a further change, since we know that C_1 and C_2 are disjoint. But consider the following scenario: the opponent puts x into C_1 ($g_1(\sigma) = 1$ for some $\sigma \prec^z_\xi x$), then takes it out ($g_1(\tau) = 0$ for a longer $\tau \prec^z_\xi x$, and note that $\sigma \prec^z_\xi \tau$). The opponent paid by decreasing γ_1 twice ($\gamma_1(\tau) < \gamma_1(\sigma) < \eta$); but γ_2 still has maximal value η. If we followed the opponent, our ordinal β^M is now γ_1. The opponent now puts x in and out of C_2. They have larger ordinals to play with, and so can defeat us.

The solution is: when the opponent makes the second change and takes x out of C_1, we do not follow them. From now on, we commit to play either outcome 1 or 2, and never return to the default outcome. We change the outcome when we must: x goes out of C_1 and into C_2. Such a change, or a change back, must be accompanied by a decrease of γ_1. If x goes into C_2 before it goes into C_1, we follow γ_2 instead. If the opponent takes x out of C_1 and does not place it into C_2, we use the fact that $\Lambda \subseteq \Gamma$ to emulate the set B rather than A_1 on x. The fact that $o(\Gamma) = \xi_1^\Upsilon$ is greater than ξ allows the outcome 1 to correctly determine whether $x \in C_1$ or not, and so know which one of B or A_1 to evaluate on x.

In detail: since C_1 and C_2 are both $\Delta^0_{1+o(\Gamma)}$, and since $\Lambda < \Gamma$, there is a $\Gamma(z)$-name M_1 and a $\check{\Gamma}(z)$-name M_2 such that $M_i = A_i$ on C_i and $M_i = B$ outside $C_1 \cup C_2$ (Proposition 2.4). Let M_0 be a $\Lambda(z)$-name for B. To define M, it remains to define f^M and β^M. Let $\sigma \in \omega^{<\omega}$. If $\gamma_1(\sigma) = \gamma_2(\sigma) = \eta$ then let $\beta^M(\sigma) = \eta$ and $f^M(\sigma) = 0$. Otherwise, let $\tau \preccurlyeq^z_\xi \sigma$ be shortest such that either $\gamma_1(\tau) < \eta$ or $\gamma_2(\tau) < \eta$; say $\gamma_1(\tau) < \eta$; the other case is symmetric. We let $\beta^M(\sigma) = \gamma_1(\sigma)$. Let σ^- be the longest proper $\prec^z_\xi$-initial segment of σ (the predecessor of σ on the tree $(\omega^{<\omega}, \preccurlyeq^z_\xi)$). If $\gamma_1(\sigma) < \gamma_1(\sigma^-)$ then we set $f^M(\sigma) = 1$ if $g_1(\sigma) = 1$, otherwise $f^M(\sigma) = 2$. If $\gamma_1(\sigma) = \gamma_1(\sigma^-)$ then $f^M(\sigma) = f^M(\sigma^-)$. That is, we move only when γ_1 allows us to. For $x \in C_1 \cup C_2$, $f^M(x) = i \iff x \in C_i$ (consider the last $\sigma \prec^z_\xi x$ at which γ_1 changed). If $x \notin C_1 \cup C_2$ we may have $f^M(x) \neq 0$, but in any case, we still have $M(x) = B(x)$. □

4.3. **Containment in admissibly decribed classes.** With admissible descriptions, a leaf selection game is simplified: non-default children of the root are necessarily leaves of the S-tree. We obtain useful criteria for containment.

Lemma 4.7. If Γ is admissible and $o(\Gamma_0) = o(\Gamma)$ then there is some n with $\Gamma_0 < \Gamma_n$.

Proof. Since Γ is efficient, there is some n such that $\Gamma_0 \subseteq \check{\Gamma}_n$. Since Γ_0 is non-self-dual, $n \neq 0$. Since Γ is admissible, $o(\Gamma_n) > o(\Gamma)$. Since Γ_0 is efficient, Proposition 4.2 implies that $\Gamma_n \neq \check{\Gamma}_0$, so $\Gamma_0 < \Gamma_n$. □

Proposition 4.8. *Let Γ and Λ be class descriptions. Suppose that:*

- $o(\Gamma) = o(\Lambda)$*;*
- *For all $n \in T_\Gamma$ there is some $m \in T_\Lambda$ such that $\mathbf{\Gamma}_n \subseteq \mathbf{\Lambda}_m$;*
- $\eta^\Gamma < \eta^\Lambda$*;*

- *Λ is admissible.*

Then $\mathbf{\Gamma} < \mathbf{\Lambda}$.

Proof. The assumptions imply: for every $n \in T_\Gamma$ there is some $m \in T_\Lambda$ which is a leaf of S_Λ and such that $\mathbf{\Gamma}_n \subseteq \mathbf{\Lambda}_m$. For if $m \in T_\Lambda$ is not a leaf of S_Λ then $m = 0$ and $o(\Lambda_0) = o(\Lambda)$, so Lemma 4.7 applies.

We observe that since Λ is efficient, all the assumptions apply to the pair $(\Gamma, \check{\Lambda})$ as well, so it suffices to show that $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$. We describe a strategy for player 2 in the leaf selection game $G_{\mathtt{leaf}}(\Gamma, \Lambda)$ as in Corollary 3.7(c).

In this game, write $c[k]$ and $\eta[k]$ for $c_\diamondsuit[k]$ and $\eta_\diamondsuit[k]$. We will ensure that for all odd $k \geqslant 1$, if $p[k+1]$ is defined, then $\eta[k+1] \geqslant \eta[k]$, $c[k+1]$ is a leaf of S_Λ (so $t[k+1] = c[k+1]$), and $\mathbf{\Gamma}_{c[k]} \subseteq \mathbf{\Lambda}_{c[k+1]}$. This suffices, since $\mathbf{\Gamma}_{t[k]} \subseteq \mathbf{\Gamma}_{c[k]}$ (as $c[k] \leqslant t[k]$).

Let $k \geqslant 1$ be odd; suppose that player 1 played $p[k]$, and that the game has not yet ended.

If $k \geqslant 3$ and $c[k] = c[k-2]$, then player 2 passes.

Suppose that $k = 1$, or that $k \geqslant 3$ and $c[k] \neq c[k-2]$. In this case, $\eta[k] < \eta[k-1]$: this follows from $p[k] \leqslant p[k-2]$ and $\eta[k-1] \geqslant \eta[k-2]$ when $c[k] \neq c[k-2]$; otherwise, $k = 1$, and this follows from $\eta[0] = \eta^\Lambda < \eta^\Gamma = \eta[-1]$.

In this case, therefore, we can set $\eta[k+1] = \eta[k]$ and choose $c[k+1]$ as we like; as discussed, we choose $c[k+1]$ to be some leaf of S_Λ satisfying $\mathbf{\Gamma}_{c[k]} \subseteq \mathbf{\Lambda}_{c[k+1]}$. □

Proposition 4.9. *Let Γ and Λ be class descriptions. Suppose that:*

- *$o(\Gamma) = o(\Lambda)$;*
- *For all $n \in T_\Gamma$ there is some $m \in T_\Lambda$ such that $\mathbf{\Gamma}_n \subseteq \mathbf{\Lambda}_m$;*
- *$\eta^\Gamma \leqslant \eta^\Lambda$;*
- *$\mathbf{\Gamma}_0 \subseteq \mathbf{\Lambda}_0$*
- *Λ is admissible.*

Then $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$.

Proof. This is similar to the proof of Proposition 4.8. The only difference is that as long as player 1 plays $c[k] = 0$ and does not decrease $\eta[k]$, player 2 cannot choose some $m > 0$ with $\mathbf{\Gamma}_0 \subseteq \mathbf{\Lambda}_m$, since she does not have the "ordinal space" to do so: we only have $\eta^\Lambda \geqslant \eta^\Gamma$, not strict inequality. Instead, player 2 can set $c[k+1] = 0$ and play according to a winning strategy in $G_{\mathtt{cont}}(\Gamma_0, \Lambda_0)$. If player 1 ever decreases $\eta[k]$, then player 2 can revert to the strategy above. □

Corollary 4.10. *Suppose that Γ and Λ are both admissible, and that $o(\Gamma) = o(\Lambda) < \omega_1$. Then $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ if and only if one of the following holds:*

(1) *For some $m \in T_\Lambda$, $\bigcup_{n \in T_\Gamma} \mathbf{\Gamma}_n \subseteq \mathbf{\Lambda}_m$;*
(2) *$\bigcup_{n \in T_\Gamma} \mathbf{\Gamma}_n = \bigcup_{m \in T_\Lambda} \mathbf{\Lambda}_m$, and either*
    - *$\eta^\Gamma < \eta^\Lambda$; or*
    - *$\eta^\Gamma = \eta^\Lambda$ and $\mathbf{\Gamma}_0 \subseteq \mathbf{\Lambda}_0$.*

Proof. Suppose that (1) holds. By Lemma 4.7, we may assume that $m > 0$. In $G_{\mathtt{leaf}}(\Gamma, \Lambda)$, player 2 immediately chooses $c[k] = m$ (he can set $\eta[k] = 0$). Note that in this case, $\mathbf{\Gamma} < \mathbf{\Lambda}$. If (2) holds then $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ follows from Propositions 4.8 and 4.9.

In the other direction, suppose that $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$, and that (1) does not hold. By the semi-linear ordering principle, and the fact that both Γ and Λ are efficient, we have

$\bigcup \mathbf{\Lambda}_m \subseteq \bigcup \mathbf{\Gamma}_n$. Since it is not the case that $\mathbf{\Lambda} < \mathbf{\Gamma}$, (1) fails in the other direction, and so in fact $\bigcup \mathbf{\Lambda}_m = \bigcup \mathbf{\Gamma}_n$. Again, since $\mathbf{\Lambda} < \mathbf{\Gamma}$ fails, we have $\eta^\Gamma \leqslant \eta^\Lambda$. Suppose that these ordinals are equal. If $\mathbf{\Gamma}_0 \nsubseteq \mathbf{\Lambda}_0$ then $\mathbf{\Lambda}_0 \subseteq \check{\mathbf{\Gamma}}_0$, but then we get $\mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}$, which again is not the case. □

4.4. **The ubiquity of admissible descriptions.** Theorem 6.8 of [DGHTTa] states that every non-self-dual Borel Wadge class has an acceptable description. We will need the analogous result for admissible descriptions:

Theorem 4.11. *Every non-self-dual Borel Wadge class has an admissible description.*

In general, we do not expect a class to have a description which is simultaneously acceptable and admissible.

Proof. The argument is an elaboration on that for [GQT, Prop. 4.1], which discusses finite class descriptions. Given a class description Θ, we examine the classes $\mathbf{\Theta}_s$ for the leaves s of S_Θ (recall the notation S_Θ from the leaf selection game). By induction, they all have admissible descriptions. We know that these classes are semi-linearly ordered. In the simpler case, among these classes there is one which is maximal under containment; then Θ is equivalent to that class. Otherwise, we will construct a description Ξ equivalent to Θ by setting $o(\Xi) = o(\Theta)$ and the classes $\mathbf{\Xi}_n$ where n is a non-default to be the various classes $\mathbf{\Theta}_s$ above; the assumption that we are not in the easier case implies that Ξ will be efficient, and the fact that we are taking classes $\mathbf{\Theta}_s$ for s a *leaf* of S_Θ implies that $o(\Theta_s) > o(\Theta)$, so Ξ will in fact be admissible. The difficulty, though, is to identify the class $\mathbf{\Xi}_0$, and to find the ordinal η^Ξ, telling us how many times we can change our mind at the root.

The main idea is to look at possible collections of S_Θ-positions; these could be used by a player in a game $G_{\mathtt{leaf}}(\Theta, \Upsilon)$ or $G_{\mathtt{leaf}}(\Upsilon, \Theta)$ for some Υ. With each position we will associate an ordinal rank, which measures how much leeway a player still has, after playing this position, to keep playing any class Θ_s. The ordinal η^Ξ will be the maximal rank occuring, which will correspond to the rank of the starting position. The class Ξ_0 will be obtained by considering all S_Θ-positions of this maximal rank; we will show that it is equivalent to an admissibly described class.

As in [GQT], we need to extend the notation S_Θ. Let ξ be a countable ordinal. For a class description Θ with $o(\Theta) \geqslant \xi$ we define $S_{\Theta,\xi}$ as follows:

- If $o(\Theta) = \xi$ then $S_{\Theta,\xi} = S_\Theta$;
- If $o(\Theta) > \xi$ then $S_{\Theta,\xi}$ consists only of the root of T_Θ.

Note that both cases can be defined together as in the original definition of S_Θ, replacing $o(\Theta)$ by ξ. $S_{\Theta,\xi}$-positions are defined as in Definition 3.1; when $o(\Theta) > \xi$, there is just one $S_{\Theta,\xi}$ position p, determined by taking t^p to be the root of T_Θ. Note that these notions make sense even when $o(\Theta) = \omega_1$.

Fixing ξ, in this proof, we let $\mathcal{P}$ and $\mathcal{Q}$ denote nonempty collections of $S_{\Theta,\xi}$-positions, for some Θ, that are upwards closed: if $p \in \mathcal{P}$ and $q \geqslant p$ then $q \in \mathcal{P}$. (Recall the partial order defined on positions in Definition 3.1.)

Let Θ and Ξ be class descriptions with ordinal levels $\geqslant \xi$; let $\mathcal{P}$ be a nonempty, upwards closed collection of $S_{\Theta,\xi}$-positions, and let $\mathcal{Q}$ be such a collection of $S_{\Xi,\xi}$-positions. The game $G(\mathcal{P}, \mathcal{Q})$ is defined as the game $G_{\mathtt{leaf}}(\Theta, \Xi)$, except that the trees used are $S_{\Theta,\xi}$ and $S_{\Xi,\xi}$, and further, player 1 is only allowed to choose positions from $\mathcal{P}$, while player 2 must choose positions from $\mathcal{Q}$. We write

$$\mathcal{P} \leqslant \mathcal{Q}$$

if player 2 has a strategy in the game $G(\mathcal{P}, \mathcal{Q})$ which guarantees an outcome (s, t) satisfying $\mathbf{\Theta}_s \subseteq \mathbf{\Xi}_t$. We write $\mathcal{P} \equiv \mathcal{Q}$ if $\mathcal{P} \leqslant \mathcal{Q}$ and $\mathcal{Q} \leqslant \mathcal{P}$. Corollary 3.7 implies:

Claim 4.11.1. For a class Θ with $o(\Theta) \geqslant \xi$, let $\mathcal{P}_\Theta$ denote the collection of all $S_{\Theta,\xi}$-positions. If $o(\Theta), o(\Xi) \geqslant \xi$, then $\mathbf{\Theta} \subseteq \mathbf{\Xi}$ if and only if $\mathcal{P}_\Theta \leqslant \mathcal{P}_\Xi$.

(Observe that Corollary 3.7 covers all cases, whether $o(\Theta) = \xi$ or $o(\Theta) > \xi$, and similarly for Ξ.) We therefore write Θ in place of $\mathcal{P}_\Theta$, and so write $\Theta \leqslant \mathcal{Q}$, $\mathcal{P} \equiv \Xi$, etc.

Theorem 4.11 follows from:

Claim 4.11.2. Let Θ be a class description with $\xi = o(\Theta)$. For any nonempty, upwards closed collection $\mathcal{P}$ of S_Θ-positions, there is an admissible class description Ξ with $\mathcal{P} \equiv \Xi$.

The notation implies that $o(\Xi) \geqslant \xi$.

For brevity, for an S_Θ-position p, let $\Theta_p = \Theta_{t^p}$. Claim 4.11.2 is proved by a double induction: first on the complexity of Θ, then on a $\subseteq$-upper bound on the collection of classes $\mathbf{\Theta}_p$ for $p \in \mathcal{P}$: let

$$C(\mathcal{P}) = \{\Theta_p \,:\, p \in \mathcal{P}\}.$$

the induction hypothesis for $\mathcal{P}$ is that Claim 4.11.2 holds for all sets $\mathcal{Q}$ of S_Θ-positions for which there is some $\Gamma \in C(\mathcal{P})$ such that for all $\Lambda \in C(\mathcal{Q})$, $\mathbf{\Lambda} \subseteq \check{\mathbf{\Gamma}}$. This relation is well-founded.

Fix a class description Θ; let $\xi = o(\Theta)$, which we may assume is countable. By induction, we assume that for evey leaf t of S_Θ, Θ_t is admissible. Proposition 4.2 implies that after replacing Θ_t be an admissible equivalent, the ordinal level cannot decrease; this means that after such replacement, the tree S_Θ does not change. Fix a nonempty, upwards closed collection $\mathcal{P}$ of S_Θ-positions. As usual, we assume that we have relativised to a sufficiently strong oracle, so that all containments between classes Θ_t are effective, uniformly, and all ordinals involved are comparable; see Remark 3.3.

We dispose of the easy case first.

Claim 4.11.3. Suppose that there is some maximal $\Gamma \in C(\mathcal{P})$: for all $\Gamma' \in C(\mathcal{P})$, $\Gamma' \subseteq \Gamma$. Then $\mathcal{P} \equiv \Gamma$.

Proof. Player 2 easily wins both $G(\mathcal{P}, \Gamma)$ and $G(\Gamma, \mathcal{P})$, using constant plays. $\square$

For the rest of the proof, suppose that the hypothesis of Claim 4.11.3 fails. By the semi-linear-ordering principle for described classes, this implies:

($*$): For every $\Gamma \in C(\mathcal{P})$ there is some $\Gamma' \in C(\mathcal{P})$ with $\check{\Gamma} \subseteq \Gamma'$.

We define an ordinal rank on S_Θ-positions $p \in \mathcal{P}$. As is often the case, by induction on ordinals β we define when $\mathrm{rk}(p) \geqslant \beta$:

(i) For every $p \in \mathcal{P}$, $\mathrm{rk}(p) \geqslant 0$.
(ii) $\mathrm{rk}(p) \geqslant \beta + 1$ if for every $\Gamma \in C(\mathcal{P})$ there is some $q \in \mathcal{P}$ with $q \leqslant p$, $\Gamma \subseteq \Theta_q$, and $\mathrm{rk}(q) \geqslant \beta$.

(iii) For a limit ordinal λ, $\mathrm{rk}(p) \geqslant \lambda$ if for all $\beta < \lambda$, $\mathrm{rk}(p) \geqslant \beta$.

By induction on β we observe that if $\mathrm{rk}(p) \geqslant \beta+1$ then $\mathrm{rk}(p) \geqslant \beta$, so that the collection of β such that $\mathrm{rk}(p) \geqslant \beta$ is an initial segment of the ordinals. We write $\mathrm{rk}(p) = \beta$ if $\mathrm{rk}(p) \geqslant \beta$ but $\mathrm{rk}(p) \ngeqslant \beta + 1$.

Claim 4.11.4. Every $p \in \mathcal{P}$ has a countable rank.

Proof. If not, let $p \in \mathcal{P}$ with $\mathrm{rk}(p) \geqslant \omega_1$. Since $\mathcal{P}$ is countable, this implies that $\mathrm{rk}(p) \geqslant \omega_1 + 1$. By $(*)$, we can find some $q \leqslant p$ with $\mathrm{rk}(q) \geqslant \omega_1$ and $\check{\Theta}_p \subseteq \Theta_q$; in particular, $t^p \neq t^q$. Proceeding, we obtain an inifnite sequence of positions, none of which is a "pass" in the leaf selection game, contradicting Remark 3.2. □

By induction on β we observe that $q \leqslant p$ implies $\mathrm{rk}(q) \leqslant \mathrm{rk}(p)$. In an ideal world, this would be strict: if $q < p$ then $\mathrm{rk}(q) < \mathrm{rk}(p)$. At least, this would be good to have when q makes a choice that cannot be covered by p, i.e., when $\Theta_q \nsubseteq \Theta_p$. Sadly, this may fail. The following will suffice:

Claim 4.11.5. For every $p \in \mathcal{P}$ there is some $\Gamma \in C(\mathcal{P})$ such that $\Theta_p \subseteq \Gamma$, and for all $q \leqslant p$, if $\Theta_q \nsubseteq \Gamma$ then $\mathrm{rk}(q) < \mathrm{rk}(p)$.

Proof. Let $\Gamma' \in C(\mathcal{P})$ witness that $\mathrm{rk}(p)$ is not greater than it actually is: for all $q \leqslant p$, if $\Gamma' \subseteq \Theta_q$ then $\mathrm{rk}(q) < \mathrm{rk}(p)$. In particular, $\Gamma' \nsubseteq \Theta_p$; by SLO, $\Theta_p \subseteq \check{\Gamma}'$. By $(*)$, choose $\Gamma \in C(\mathcal{P})$ with $\check{\Gamma}' \subseteq \Gamma$. Again by SLO, if $q \leqslant p$ and $\Theta_q \nsubseteq \Gamma$ then $\check{\Gamma} \subseteq \Theta_q$ and then $\Gamma' \subseteq \Theta_q$, so $\mathrm{rk}(q) < \mathrm{rk}(p)$. □

Let p_0 be the initial S_Θ-position (all ordinals maximal and all choices are default); then $p_0 \geqslant p$ for all $p \in \mathcal{P}$. By assumption on $\mathcal{P}$, $p_0 \in \mathcal{P}$, and so

$$\eta = \mathrm{rk}(p_0)$$

is maximal among all ranks of elements of p.

Claim 4.11.6. $\eta \geqslant 1$.

Proof. Let $\Gamma \in C(\mathcal{P})$; so $\Gamma = \Theta_p$ for some p; since $p \leqslant p_0$, this shows that $\mathrm{rk}(p_0) \geqslant 1$. □

We let

$$\mathcal{Q} = \{p \in \mathcal{P} \,:\, \mathrm{rk}(p) = \eta\}.$$

So $\mathcal{Q}$ is nonempty, and since the rank is monotone, $\mathcal{Q}$ is upwards closed.

Claim 4.11.7. There is some $\Gamma \in C(\mathcal{P})$ such that for all $\Gamma' \in C(\mathcal{Q})$, $\Gamma' \subseteq \check{\Gamma}$.

Proof. Suppose not. By the semi-linear-ordering principle, for all $\Gamma \in C(\mathcal{P})$ there is some $\Gamma' \in C(\mathcal{Q})$ such that $\Gamma \subseteq \Gamma'$, i.e., there is some $p \in \mathcal{P}$ with $\mathrm{rk}(p) = \eta$ and $\Gamma \subseteq \Theta_p$. But then $\mathrm{rk}(p_0) \geqslant \eta + 1$. □

By induction, there is some admissible Λ (with $o(\Lambda) \geqslant \xi$) such that $\Lambda \equiv \mathcal{Q}$.

Claim 4.11.8. Let Γ be given by Claim 4.11.7; then $\check{\Lambda} \subseteq \Gamma$.

Proof. By Claim 4.11.1, it suffices to show that $\mathcal{Q} \leqslant \check{\Gamma}$. Since $o(\Gamma) > \xi$ (as $\Gamma = \Theta_t$ for some leaf t of S_Θ), this is witnessed by constant plays. □

Since $C(\mathcal{P})$ is countable, fix a list $\Gamma_1, \Gamma_2, \ldots$ enumerating $C(\mathcal{P})$. We define a class description Ξ as follows:

- $o(\Xi) = \xi$;
- $\eta^{\Xi} = \eta$;
- For $n \geqslant 1$, $\Xi_n = \Gamma_n$;
- $\Xi_0 = \Lambda$.

Then $(*)$, together with Claim 4.11.8 (and the assumption that each Γ_n is efficient) ensure that Ξ is efficient. Since each Γ_n is Θ_t for some leaf t of S_Θ, $o(\Gamma_n) > \xi$ for all $n \geqslant 1$. By the assumption that each Γ_n is admissible, we see that Ξ is admissible as well.

Claim 4.11.9. $\Xi \equiv \mathcal{P}$.

Proof. We play the games for both directions. In $G(\mathcal{P}, \Xi)$, as long as player 1 keeps playing $q \in \mathcal{Q}$, player 2 chooses the default at the root of T_Ξ, and uses her winning strategy in the game $G(\mathcal{Q}, \Lambda)$ (note that this covers both cases $o(\Lambda) = \xi$ and $o(\Lambda) > \xi$). Once player 1 leaves $\mathcal{Q}$, playing some position p with $\mathrm{rk}(p) < \eta$, player 2 chooses an outcome n with Γ_n witnessing Claim 4.11.5 for p; player 2 decreases the ordinal at the root to $\mathrm{rk}(p)$. From then on, player 2 moves only if forced (if the current Γ_n does not contain Θ_q for the current position q played by player 1). When forced to move, player 2 matches the ordinal rank of the position chosen by player 1, and chooses a sufficiently large Γ_n given by Claim 4.11.5. These choices ensure that when forced to move, the ordinal indeed drops.

In $G(\Xi, \mathcal{P})$, as long as player 1 remains above the default outcome of the root, player 2 plays their winning strategy in $G(\Lambda, \mathcal{Q})$. Once player 1 moves off the default outcome, and is currently presenting some ordinal $\beta < \eta$ and outcome n, player 2 can respond with some position $p \in \mathcal{P}$ with $\Gamma_n \subseteq \Theta_p$ and $\mathrm{rk}(p) \geqslant \beta$; the definition of rank allows it to proceed. ☐

This completes the proof of Claim 4.11.2, and so of Theorem 4.11. ☐

Remark 4.12. Proposition 3.34 of [DGHTTa] allows us to directly transform a monotone class description into an equivalent acceptable class description. It does not seem possible to mimick the same argument to transform monotone class descriptions into admissible ones. Hence, we cannot use [DGHTTa, Thm. 6.8] to prove Theorem 4.11.

However, the proof of [DGHTTa, Thm. 6.8] can be adapted to give another proof of Theorem 4.11. An analogue of [DGHTTa, Thm. 4.4]: every admissible class description is classified, holds. The main change in the proof is in [DGHTTa, Prop. 4.12]; one has to consider three cases, depending on whether $\eta = 1$ (the acceptable case), $\eta > 1$ is a successor, or η is a limit. Note that the proof of [DGHTTa, Prop. 4.13] is naturally suited to admissible descriptions.

[DGHTTa, Prop. 3.34] allows us to use [DGHTTa, Thm. 5.1] to show its analogue for admissible class descriptions. Then, following the work in Section 6 of [DGHTTa] completes a proof of Theorem 4.11.

4.5. **Admissible monotone descriptions.** The proof of Theorem 4.11 can be easily adjusted to show that every non-self-dual Borel Wadge class has a description which is both admissible and monotone (Definition 4.4). In the definition of Ξ, instead of letting $\Gamma_1, \Gamma_2, \dots$ list all of $C(\mathcal{P})$, we let it list a cofinal sequence in $C(\mathcal{P})$ which is monotone ($\Gamma_n \subseteq \check{\Gamma}_{n+1}$). Indeed, we can reduce to two cases: either $C(\mathcal{P})$ has a maximal pair $\Theta, \check{\Theta}$, in which we can set $\Gamma_n = \Theta$ for odd $n > 0$ and $\Gamma_n = \check{\Theta}$ for even $n > 0$; or we can set $\Gamma_1 < \Gamma_2 < \cdots$.

However, it is also easy to effectively transfrom any admissible description into an equivalent description which is both admissible and monotone.

Proposition 4.13. *For any admissible class description* Γ *there is a monotone admissible class description* Λ *with* $\mathbf{\Gamma} = \mathbf{\Lambda}$*, effectively.*

Proof. Let Γ be admissible. By (effective transfinite) recursion, we may assume that for all children n of the root on T_Γ, Γ_n is admissible and monotone. We define a class description by letting $o(\Lambda) = o(\Gamma)$, $\eta^\Gamma = \eta^\Lambda$, and $\Lambda_0 = \Gamma_0$. Then, by recursion, having defined Λ_n, we let Λ_{n+1} be some Γ_m such that:

- $\Gamma_m \supseteq \check{\Lambda}_n$;
- if $n \in T_\Gamma$, then either $\Gamma_n \subseteq \Gamma_m$ or $\Gamma_n \subseteq \check{\Gamma}_m$;
- $o(\Gamma_m) > o(\Gamma)$.

Lemma 4.7 implies that such an m exists. Proposition 4.9 shows that $\mathbf{\Gamma} = \mathbf{\Lambda}$. □

5. Game characterisations of separation and reduction

While the containment game and Theorem 3.4 are interesting and useful in their own right, they also serve as a simple version of more general games, that we use to characterise the reduction and separation properties.

5.1. **The reduction game.** We will devise a game $G_{\mathtt{red}}(\Gamma)$ such that for any class description Γ, the class $\mathbf{\Gamma}$ has the reduction property if and only if player 2 has a winning strategy in the game. In the containment game $G_{\mathtt{cont}}(\Gamma, \Lambda)$, the idea is that player 1 plays a set $A \in \mathbf{\Gamma}$ and challenges player 2 to prove that this set is in $\mathbf{\Lambda}$. In the reduction game, player 1 plays two sets $A^0, A^1 \in \mathbf{\Gamma}$ and challenges player 2 to construct a pair of sets (B^0, B^1), both in $\mathbf{\Gamma}$, that reduce the pair (A^0, A^1), meaning that $B^i \subseteq A^i$, $B^0 \cap B^1 = \varnothing$, and $B^0 \cup B^1 = A^0 \cup A^1$. In this game, the players will each produce leaves on two copies of T_Γ, the labels of which represent the values $A^0(x), A^1(x)$ and $B^0(x), B^1(x)$. The winning positions for player 2 will correspond precisely to the requirements of reduction.

However, recall that the proof of Proposition 3.5 was inductive: it assumed the proposition held for pairs such as (Γ, Λ_n) or (Γ_t, Λ_r). The same argument will be applied for the reduction game, which means that we need to describe a more general game and a more general property, ones which are not restricted to just one class.

Definition 5.1.

(a) Let $\mathbf{\Gamma}^0, \mathbf{\Gamma}^1, \mathbf{\Lambda}^0, \mathbf{\Lambda}^1$ be pointclasses. The pair $(\mathbf{\Lambda}^0, \mathbf{\Lambda}^1)$ *reduces* $(\mathbf{\Gamma}^0, \mathbf{\Gamma}^1)$ if for every pair (A^0, A^1) with $A^i \in \mathbf{\Gamma}^i$, there is a pair (B^0, B^1) with $B^i \in \mathbf{\Lambda}^i$ that reduces (A^0, A^1).

(b) We say that a pointclass $\mathbf{\Lambda}$ reduces a pointclass $\mathbf{\Gamma}$ if $(\mathbf{\Lambda}, \mathbf{\Lambda})$ reduces $(\mathbf{\Gamma}, \mathbf{\Gamma})$. A pointclass $\mathbf{\Gamma}$ has the *reduction property* if $\mathbf{\Gamma}$ reduces $\mathbf{\Gamma}$.

We now describe the clopen game that captures the reduction relation between pairs of classes.

The extended leaf selection game. We will make use of the leaf selection game described above, except that now, each player may start with either one or two classes: we could play $G_{\mathtt{leaf}}(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$, or $G_{\mathtt{leaf}}(\Gamma^0; \Lambda^0, \Lambda^1)$, etc. At each round, the player i whose turn it is to play chooses positions on each of the trees that the player is playing on. For example, in $G_{\texttt{leaf}}(\Gamma^0, \Gamma^1; \Lambda^0)$, player 2 chooses an S_{Λ^0}-position on every even round, while at each odd round, player 1 chooses both an S_{Γ^0}-position and an S_{Γ^1}-position. A player has passed when *all* of their currently chosen leaves are the same as in the previous round. The outcome of the game is a choice of leaf on each tree involved in the game.

The reduction game. Let $\Gamma^0, \Gamma^1, \Lambda^0$ and Λ^1 be class descriptions. The reduction game $G_{\texttt{red}}(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$ is played between two players 1 and 2. Player 1 devises a path from the root to leaves on both T_{Γ^0} and T_{Γ^1}; player 2 does the same on T_{Λ^0} and T_{Λ^1}. At each round $k \geqslant 1$ of the game, player 1 defines nodes $s_1^j[k] \in T_{\Gamma^j}$ and player 2 nodes $s_2^j[k] \in T_{\Lambda^j}$. We start with $s_i^j[0]$ being the root of the corresponding tree. At round $k+1$, let $\xi_1^j[k] = o(\Gamma^j_{s_1^j[k]})$, and $\xi_2^j[k] = o(\Lambda^j_{s_1^j[k]})$; we let

$$\xi[k] = \min\{\xi_i^j[k] \,:\, i = 1, 2;\, j = 0, 1\}.$$

(1) If $\xi[k]$ occurs for only one of the players: for some $i \in \{1, 2\}$ we have $\xi_i^j[k] > \xi[k]$ for both j, then the other player $i' = 3 - i$ selects a child $s_{i'}^j[k+1]$ of $s_{i'}^j[k]$ on the corresponding tree, for each j such that $\xi_{i'}^j[k] = \xi[k]$.
(2) If $\xi[k]$ occurs for both players, then the players play the extended leaf selection game; player 1 plays with $\Gamma^j_{s_1^j[k]}$ for all $j \in \{0, 1\}$ for which $\xi_1^j[k] = \xi[k]$, and similarly for player 2.

The game ends with leaves s_i^j on the respective trees. Player 2 wins if the labels of the leaves agree with the requirements of reduction:

- for both $j = 0, 1$, if $\Lambda^j(s_2^j) = 1$ then $\Gamma^j(s_1^j) = 1$;
- $\Lambda^0(s_2^0)$ and $\Lambda^1(s_2^1)$ are not both 1;
- If $\Gamma^0(s_1^0) = 1$ or $\Gamma^1(s_1^1) = 1$, then $\Lambda^0(s_2^0) = 1$ or $\Lambda^1(s_2^1) = 1$.

Proposition 5.2. *Player 2 has a winning strategy in the game $G_{\texttt{red}}(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$ if and only if the pair $(\boldsymbol{\Lambda}^0, \boldsymbol{\Lambda}^1)$ reduces the pair $(\boldsymbol{\Gamma}^0, \boldsymbol{\Gamma}^1)$.*

Proof. Let $\mathfrak{S}$ be a winning strategy for player 2 in the game $G_{\texttt{red}}(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$; suppose that an oracle z is sufficiently powerful, as in Remark 3.3. Given $\Gamma^j(z)$-names N^j we devise $\Lambda^j(z)$-names M^j so that (M^0, M^1) reduces (N^0, N^1). This is done by effective transfinite recursion on the complexity of the quadruple $(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$. The argument is almost identical to that of the proof of Proposition 3.5. For example, in case III, suppose that $\xi = o(\Gamma^0)$ is smaller than the other ordinals $o(\Gamma^1)$, $o(\Lambda^0)$ and $o(\Lambda^1)$. So at the first move of the game, player 1 chooses a child n of the root on T_{Γ^0}. By induction, for each such n, the pair $(\Lambda^0(z), \Lambda^1(z))$ reduces $(\Gamma^0_n(z), \Gamma^1(z))$, uniformly; so there are $\Lambda^j(z)$-names M^j_n such that (M^0_n, M^1_n) reduces the pair (N^0_n, N^1). Since $o(\Lambda^j) > \xi$ for both $j = 0, 1$, we can merge these to $\Lambda^j(z)$-names M^j such that for all x, if $\ell^{N^0}(x) \geqslant n$ then $M^j(x) = M^j_n(x)$. If $o(\Gamma^0) = o(\Gamma^1) = \xi$ is smaller than both $o(\Lambda^j)$ then player 1 chooses children on both Γ^j, so we will have names $M^j_{n,m}$ for $n \in T_{\Gamma^0}$ and $m \in T_{\Gamma^1}$. The other cases of the proof are modified in the same way.

In the other direction, though, we do not have such a neat dichotomy. Indeed, it is not the case that if $(\boldsymbol{\Lambda}_0, \boldsymbol{\Lambda}_1)$ does not reduce $(\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_1)$ then $(\check{\boldsymbol{\Gamma}}_0, \check{\boldsymbol{\Gamma}}_1)$ reduces $(\boldsymbol{\Lambda}_0, \boldsymbol{\Lambda}_1)$. To understand the situation in general, consider that both containment and reduction can be viewed as specifying permissible lines in truth tables. In the containment case, there are four lines in total, one for each possible value of the pair

$(A(x), B(x))$, where A is player by player 1 and B by player 2. The two permissible lines are $(0,0)$ and $(1,1)$. That is, $B = A$ if for all x, either $(A(x), B(x)) = (0,0)$ or $(A(x), B(x)) = (1,1)$. The "anti-containment" property that is given by a winning strategy for player 1 in $G_{\mathtt{cont}}(\Gamma, \Lambda)$ is characterised by allowing the other possibilities $(0,1)$ and $(1,0)$, which happens to characterise equality with the complement.

In the reduction case, we have 16 lines in the truth table, and the permissible ones can be summarized by saying which values for $(B^0(x), B^1(x))$ are permissible, given $(A^0(x), A^1(x))$:

$$\begin{aligned}(0,0) &\mapsto (0,0);\\ (1,0) &\mapsto (1,0);\\ (0,1) &\mapsto (0,1);\\ (1,1) &\mapsto (0,1),(1,0).\end{aligned}$$

Using the strategy-stealing method for the (extended) leaf selection game described in the proof of Proposition 3.6, we see that if player 1 has a winning strategy in the game $G_{\mathtt{red}}(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$, then player 2 has a winning strategy in the game whose winning lines are the ones not permissible for reduction, however with exchanging the roles of A^j and B^j. By the version of Proposition 3.5 for this "anti-reduction" game, we see that in this case, the pair $(\mathbf{\Gamma}^0, \mathbf{\Gamma}^1)$ *anti-reduces* the pair $(\mathbf{\Lambda}^0, \mathbf{\Lambda}^1)$, meaning that for any $B^0 \in \mathbf{\Lambda}^0$ and $B^1 \in \mathbf{\Lambda}^1$ there are $A^0 \in \mathbf{\Gamma}^0$ and $A^1 \in \mathbf{\Gamma}^1$ such that for all $x \in \mathcal{N}$,

- If $(B^0(x), B^1(x)) = (0,0)$ then $(A^0(x), A^1(x)) \neq (0,0)$;
- If $(B^0(x), B^1(x)) = (1,0)$ then $(A^0(x), A^1(x)) \neq (1,0), (1,1)$;
- If $(B^0(x), B^1(x)) = (0,1)$ then $(A^0(x), A^1(x)) \neq (0,1), (1,1)$.

The other direction of the current proposition is then proved by verifying:

$(*)$: If $(\mathbf{\Gamma}^0, \mathbf{\Gamma}^1)$ anti-reduces the pair $(\mathbf{\Lambda}^0, \mathbf{\Lambda}^1)$ then $(\mathbf{\Lambda}^0, \mathbf{\Lambda}^1)$ does not reduce the pair $(\mathbf{\Gamma}^0, \mathbf{\Gamma}^1)$.

To show this, we use universal sets for pairs. There are sets A^0 and A^1, universal for $\mathbf{\Gamma}^0 \times \mathbf{\Gamma}^1$: this means that $A^i \in \mathbf{\Gamma}^i$, and for all pairs $C^0 \in \mathbf{\Gamma}^0$ and $C^1 \in \mathbf{\Gamma}^1$ there is some $y \in \mathcal{N}$ such that

$$C^i = (A^i)^{[y]} = \left\{x \in \mathcal{N} \,:\, \langle y, x\rangle \in A^i\right\}$$

for $i = 0, 1$; here $(y, x) \mapsto \langle y, x\rangle$ is some computable "pairing function", an isomorphism between $\mathcal{N}^2$ and $\mathcal{N}$.

Suppose, for a contradiction, that (B^0, B^1) reduces (A^0, A^1), with $B^i \in \mathbf{\Lambda}^i$. Let $D^i = \left\{y \,:\, \langle y, y\rangle \in B^i\right\}$; and let (C^0, C^1) anti-reduce (D^0, D^1), with $C^i \in \mathbf{\Gamma}^i$. Then $y \in \mathcal{N}$ such that $C^i = (A^i)^{[y]}$ gives a contradiction, as no line in the truth table is allowed for $\langle y, y\rangle$. □

Example 5.3. Let α and $\eta \geqslant 1$ be ordinals. The class $D_\eta(\mathbf{\Sigma}^0_{1+\alpha})$ has the reduction property. To see this, let Γ be the simple description of this class (Fig. 2). The game $G_{\mathtt{red}}(\Gamma, \Gamma; \Gamma, \Gamma)$ is the game $G_{\mathtt{leaf}}(T^0, T^1; S^0, S^1)$ where T^0, T^1, S^0 and S^1 are all copies of T_Γ. To win, player 2, on the tree S^j, copies the moves of player 1 on T^j, except for when player 1 moves to two 1 outcomes; the move to the second is not matched.

On the other hand, the class $\check{D}_\eta(\mathbf{\Sigma}^0_{1+\alpha})$ does not have the reduction property; in fact, the class $\check{D}_{\eta+1}(\mathbf{\Sigma}^0_{1+\alpha})$ does not reduce the class $\check{D}_\eta(\mathbf{\Sigma}^0_{1+\alpha})$ (whereas the class

$D_{\eta+1}(\mathbf{\Sigma}^0_{1+\alpha})$ does reduce $\check{D}_\eta(\mathbf{\Sigma}^0_{1+\alpha})$, as $D_{\eta+1}(\mathbf{\Sigma}^0_{1+\alpha})$ has the reduction property). To see this, let Λ be the simple description of $D_{\eta+1}(\mathbf{\Sigma}^0_{1+\alpha})$; we show how player 1 wins the game $G_{\mathtt{red}}(\check{\Gamma}, \check{\Gamma}; \check{\Lambda}, \check{\Lambda})$. Again, this game is $G_{\mathtt{leaf}}(T^0, T^1; S^0, S^1)$ with $T^j = T_{\check{\Gamma}}$ and $S^j = T_{\check{\Lambda}}$.

Player 1 starts with a pass. We refer to children of the root by their labels, so 1 is the default child on both sides. To survive, player 2 must move at least one of his leaves to 0, say on S^0; he reduces his ordinal label $\eta^0_{2,\diamondsuit}$ to some value $\leqslant \eta$. Player 1 then moves to 0 on T^1 (with ordinal 0, say; player 1 will not move on T^1 again). Henceforth, on T^0, player 1 plays the opposite of what player 2 does on S^0, with the same ordinal label.

5.2. **The separation game.** Like reduction, for a game characterisation of separation, we need a more general property, involving more than one class.

Definition 5.4.

(a) Let $A^0, A^1, B^0, B^1 \subseteq \mathcal{N}$, with $A^0 \cap A^1 = \varnothing$. The pair (B^0, B^1) *separates* the pair (A^0, A^1) if $A^0 \subseteq B^0$, $A^1 \subseteq B^1$, and $B^1 = (B^0)^\complement$.

(b) Let $\mathbf{\Gamma}^0, \mathbf{\Gamma}^1, \mathbf{\Lambda}^0, \mathbf{\Lambda}^1$ be pointclasses. The pair $(\mathbf{\Lambda}^0, \mathbf{\Lambda}^1)$ *separates* $(\mathbf{\Gamma}^0, \mathbf{\Gamma}^1)$ if for every pair (A^0, A^1) of disjoint sets with $A^i \in \mathbf{\Gamma}^i$, there is a pair (B^0, B^1) with $B^i \in \mathbf{\Lambda}^i$ that separates (A^0, A^1).

(c) A pointclass $\mathbf{\Gamma}$ has the *separation property* if $(\mathbf{\Gamma}, \mathbf{\Gamma})$ separates $(\mathbf{\Gamma}, \mathbf{\Gamma})$.

The *separation game* $G_{\mathtt{sep}}(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$ is played exactly like the reduction game, except that the winning condition for player 2, upon producing leaves s^i_j on the respective trees, is:

- If $(\Gamma^0(s^0_1), \Gamma^1(s^1_1)) = (0,1)$ then $(\Lambda^0(s^0_2), \Lambda^1(s^1_2)) = (0,1)$;
- If $(\Gamma^0(s^0_1), \Gamma^1(s^1_1)) = (1,0)$ then $(\Lambda^0(s^0_2), \Lambda^1(s^1_2)) = (1,0)$;
- If $(\Gamma^0(s^0_1), \Gamma^1(s^1_1)) = (0,0)$ then $(\Lambda^0(s^0_2), \Lambda^1(s^1_2))$ is either $(0,1)$ or $(1,0)$.

Note that if player 1 ends up with $(\Gamma^0(s^0_1), \Gamma^1(s^1_1)) = (1,1)$ then player 2 wins regardless of the leaves they chose.

Proposition 5.5. *Player 2 has a winning strategy in the game* $G_{\mathtt{sep}}(\Gamma^0, \Gamma^1; \Lambda^0, \Lambda^1)$ *if and only if the pair* $(\mathbf{\Lambda}^0, \mathbf{\Lambda}^1)$ *separates the pair* $(\mathbf{\Gamma}^0, \mathbf{\Gamma}^1)$.

Proof. The same as the proof of Proposition 5.2. Note that for the forward direction, the definition of the winning condition when player 1 plays the outcome $(1,1)$ does not affect the proof, since we only need to verify the separation property when player 1 plays a pair of disjoint sets. However, this condition is important for the other direction, when player 1 has a winning strategy. The game is stated as it is because the resulting "anti-separation" property forces player 1 to play disjoint sets. It is defined by the truth-table function:

- $(0,1) \mapsto (1,0)$;
- $(1,0) \mapsto (0,1)$;
- $(0,0) \mapsto (0,0), (0,1), (1,0)$;
- $(1,1) \mapsto (0,0), (0,1), (1,0)$.

This means that the argument above when player 1 wins the game applies in this case as well. ☐

If player 2 wins the separation game, this gives us some information even when player 1 does not necessarily plays disjoint sets. We obtain the following strengthening of the separation property:

Definition 5.6. A Wadge class $\mathbf{\Gamma}$ has the *generalized separation property* if for any two $A^0, A^1 \in \mathbf{\Gamma}$, there are $B^0, B^1 \in \mathbf{\Gamma}$ which form a separation A^0 and A^1 off of $A^0 \cap A^1$. That is, for any $x \notin A^0 \cap A^1$:

- $x \in B^0 \leftrightarrow x \notin B^1$; and
- For $i < 2$, if $x \in A^i$, then $x \in B^i$.

Overall we see that a non-self-dual Borel Wadge class has the separation property if and only if it has the generalised separation property.

5.3. **Effective properties.** We recall that the proof of Proposition 3.5 is effective: if z computes both descriptions Γ and Λ and a winning strategy for player 2 in $G_{\mathtt{cont}}(\Gamma, \Lambda)$, then uniformly in $w \geqslant_{\mathrm{T}} z$ and a $\Gamma(w)$-name N we can compute a $\Lambda(w)$-name equivalent to N. Similarly, from the arguments for Propositions 5.2 and 5.5 we obtain:

- If z computes a description Γ and a winning strategy for player 2 in the game $G_{\mathtt{red}}(\Gamma, \Gamma; \Gamma, \Gamma)$, then uniformly given $w \geqslant_{\mathrm{T}} z$ and a pair N^0, N^1 of $\Gamma(w)$-names, we can compute a pair M^0, M^1 of $\Gamma(w)$-names for sets that reduce (N^0, N^1).
- If z computes a description Γ and a winning strategy for player 2 in the game $G_{\mathtt{sep}}(\Gamma, \Gamma; \Gamma, \Gamma)$, then uniformly given $w \geqslant_{\mathrm{T}} z$ and a pair N^0, N^1 of $\Gamma(w)$-names, we can compute a pair M^0, M^1 of $\Gamma(w)$-names for sets that separate (N^0, N^1) off of $N^0 \cap N^1$.

6. The separation property

We now characterise the classes that have the separation property. Our strategies will be computable in the descriptions, so we define:

Definition 6.1. A class description Γ has the *effective separation property* if uniformly, given a pair of Γ-names N^0 and N^1 of disjoint sets in $\mathbf{\Gamma}$, we can compute a pair M^0 and M^1 of Γ-names such that $N^i \subseteq M^i$ for $i = 0, 1$, and $M^0 = (M^1)^{\complement}$.

That is, if the separation can be performed effectively in the cone above y^Γ, where recall that y^Γ is the designated oracle computing Γ. We can similarly define the *effective generalised separation property.*

Proposition 6.2. *If Γ is a monotone class description of Π-type, then Γ has the effective generalised separation property.*

Proof. We describe a y^Γ-computable winning strategy for player 2 in the separation game $G_{\mathtt{sep}}(\Gamma, \Gamma; \Gamma, \Gamma)$. This is done by recursion on the length of the leftmost (ultimate default) leaf of T_Γ. Note that this is finite recursion, not transfinite.

The base case is when $o(\Gamma) = \omega_1$, that is, when T_Γ consists only of the root; by assumption, this root is labelled 1. The game finishes before it even begins, with player 2 winning.

Suppose that $o(\Gamma) < \omega_1$. Since Γ_0 has Π-type (recall that 0 is the default child of the root), by recursion, player 2 has a y^Γ-computable winning strategy $\mathfrak{S}$ in the game $G_{\mathtt{sep}}(\Gamma_0, \Gamma_0; \Gamma_0, \Gamma_0)$. The strategy for player 2 in the game $G_{\mathtt{sep}}(\Gamma, \Gamma; \Gamma, \Gamma)$ is as follows. At each step of the game (including the rounds of leaf selection subgames, such as the one starting the separation game for Γ), let the current leaves played by player 1 be t_0 and t_1, and the leaves played by player 2 be r_0 and r_1.

During the leaf selection game starting the separation game, let η_0 and η_1 denote the η-ordinals played at the roots by player 1.

As long as player 1 chooses leaves t_0, t_1 both extending 0, then player 2 also lets r_0 and r_1 extend 0. If $o(\Gamma_0) > o(\Gamma)$ then this means that player 1 passes in the first move of the game, so does player 2, and the leaf selection subgame ends with outcome $(0,0;0,0)$. Henceforth, player 2 follows its winning strategy $\mathfrak{S}$ in the rest of the game. If $o(\Gamma_0) = o(\Gamma)$ then S_{Γ_0} is the restriction of S_Γ to leaves extending 0, so as long as player 1 plays extensions of 0, player 2 can follow the strategy $\mathfrak{S}$. If the leaf selection subgame ends within S_{Γ_0}, then player 2 can continue with $\mathfrak{S}$.

Suppose that player 1 moves away from 0 at some step; say t_0 extends some outcome $m > 0$ of the root of T_Γ. From now on, player 2 commits to eumulating t_0 by r_0, and emulating the opposite value by r_1. Henceforth, t_1 is ignored. If t_1 is the leaf moved, then the argument is symmetric, replacing t_0 by t_1 below.

The emulation is done as follows. During the leaf-slection subgame, the η-ordinal played at the root for both r_0 and r_1 is the same as η_0, the η-ordinal played by player 1 for choosing t_0. At a step at which this η-ordinal decreases (such as the first step at which t_0 moved away from 0), player 2 observes the child m extended by the new value of t_0 (after a second move, this can again be 0).

Since Γ is monotone, at such a step we can choose a large n, not hitherto used, of the same parity as m. So $\Gamma_m \subseteq \Gamma_n \subseteq \check{\Gamma}_{n+1}$. As long as player 1 does not decrease η_0, we proceed as follows. Let $\mathfrak{S}_0$ be a winning strategy for player 2 in the game $G_{\mathtt{cont}}(\Gamma_m, \Gamma_n)$, and let $\mathfrak{S}_1$ be a winning strategy for player 2 in the game $G_{\mathtt{cont}}(\Gamma_n, \check{\Gamma}_{n+1})$. The general idea is to interpret t_0 as a move by player 1 in the game $G_{\mathtt{cont}}(\Gamma_m, \Gamma_n)$, and let r_0 be the response by player 2 following $\mathfrak{S}_0$; then, we interpret r_0 as a move by player 1 in the game $G_{\mathtt{cont}}(\Gamma_n, \check{\Gamma}_{n+1})$, and use $\mathfrak{S}_1$ to define r_1.

In greater detail, while the leaf selection sub-game of $G_{\mathtt{sep}}(\Gamma,\Gamma;\Gamma,\Gamma)$ continues and the η_0-ordinal does not decrease, t_0 keeps extending m, and we let r_0 extend n and r_1 extend $n+1$. Depending on the ξ-ordinals involved, this either determines r_0 or r_1 (if $o(\Gamma_n) > o(\Gamma)$) or $o(\Gamma_{n+1}) > o(\Gamma)$; or we can use the relevant strategy to determine r_0 or r_1. Once the leaf selection sub-game ends, we are left with leaves t_0, r_0, r_1 of S_Γ such that $\Gamma_{t_0} \subseteq \Gamma_{r_0}$ and $\Gamma_{r_0} \subseteq \check{\Gamma}_{r_1}$ — either by choice of n, or since the relevant strategy produces such a leaf — and we then continue with the strategies $\mathfrak{S}_0$ and $\mathfrak{S}_1$. Again, if player 1 decreases the ordinal η_0 before the leaf selection sub-game has ended, then we abort this process, choose a new large n corresponding to the new m, and repeat. □

Proposition 6.3. *If Γ is a monotone class description of Σ-type, then $\mathbf{\Gamma}$ does not have the separation property.*

Proof. We show that player 1 has a winning strategy in $G_{\mathtt{sep}}(\Gamma,\Gamma;\Gamma,\Gamma)$. Again, this is done by induction on the length of the leftmost path. The base case is again when $o(\Gamma) = \omega_1$; this time, the labels of the outcome of the game are $(0,0;0,0)$, which is a win for player 1.

Suppose that $o(\Gamma) < \omega_1$ and that $\mathfrak{S}$ is a winning strategy for player 1 in the game $G_{\mathtt{sep}}(\Gamma_0,\Gamma_0;\Gamma_0,\Gamma_0)$. We use the notation t_0, t_1, r_0, r_1 as in the previous proof.

In the leaf selection sub-game that starts the game $G_{\mathtt{sep}}(\Gamma,\Gamma;\Gamma,\Gamma)$, if $o(\Gamma_0) > o(\Gamma)$ then player 1 starts with a pass (so $t_0 = t_1 = 0$). If $o(\Gamma_0) = o(\Gamma)$ then player 1 starts by following $\mathfrak{S}$, so it sets t_0 and t_1 both extending 0. As long as player 2 keeps both r_i extending the default outcome 0, player 1 either passes or follows $\mathfrak{S}$, depending on $o(\Gamma_0)$. Suppose that at some step, player 2 moves away from 0 on at least one of its trees, again, say by moving r_0. Player 1 now matches in both of their trees, the ordinal η_0 played by player 2 at the root as part of the choice of r_0. Just as the argument above, player 1 now can arrange for t_1 to emulate r_0, and t_0 to emulate $1 - t_1$, by choosing a new large n whenever η_0 decreases. □

Proposition 6.2 and Proposition 6.3, together with immediate implications, and the fact that every non-self-dual Borel Wadge class has a monotone description, gives the following:

Theorem 6.4. *Let $\boldsymbol{\Upsilon}$ be a non-self-dual Borel Wadge class. The following are equivalent:*

(1) *$\boldsymbol{\Upsilon}$ has the separation property.*
(2) *$\boldsymbol{\Upsilon}$ has the generalized separation property.*
(3) *Every monotone description of $\boldsymbol{\Upsilon}$ is of Π-type.*
(4) *Some monotone description of $\boldsymbol{\Upsilon}$ is of Π-type.*
(5) *Some / every monotone description of $\boldsymbol{\Upsilon}$ has the effective separation property.*
(6) *Some / every monotone description of $\boldsymbol{\Upsilon}$ has the effective generalized separation property.*

As a result, we see that the type of a monotone class description is invariant: if Γ, Λ are monotone and $\boldsymbol{\Lambda} = \boldsymbol{\Gamma}$ then Λ and Γ have the same type. We thus talk about the type of a class.

7. Characterising the reduction property

7.1. **Characterising reduction.** Armed with the game criterion for reduction, we can now characterise the Borel Wadge classes with the reduction property as those which have a description which is hereditarily Σ. First, we observe that a Borel Wadge class with the reduction property has to have Σ type: the reduction property for $\boldsymbol{\Gamma}$ easily implies the separation property for $\check{\boldsymbol{\Gamma}}$. Not every class of Σ-type has the reduction property though.

Example 7.1. The class $\mathrm{BiSep}(\boldsymbol{\Sigma}^0_1, \boldsymbol{\Sigma}^0_2, \{\varnothing\})$ is a Σ-type class that does not have the reduction property. Let Γ be the simplest description of this class (Fig. 3). The game $G_{\mathtt{red}}(\Gamma, \Gamma; \Gamma, \Gamma)$ starts with a leaf selection game on the subtree consisting of the root and its three children. Call the rightmost child "π" and the middle one "σ". To win, player 1 chooses the child π in both of their trees. If player 2 responds in kind, in the next leaf selection game, player 2 must move to outcome 0 on one of his trees; when he does so, player 1 moves to 0 on the opposite tree. If, on the other hand, player 2 chooses 0 or σ on one of his trees, player 1 will move to 0 on the opposite tree, forcing player 2 to move to the child 1 of σ (the choice of the child 0 of the root is terminal); player 1 then moves his other tree to 0.

Using our results of the first part, we can give a quick proof of a result that follows from work of van Wesep's [VW78]:

Proposition 7.2. *If $\boldsymbol{\Gamma}$ is a non-self-dual Borel Wadge class of Σ-type and is also closed under finite intersections, then it has the reduction property.*

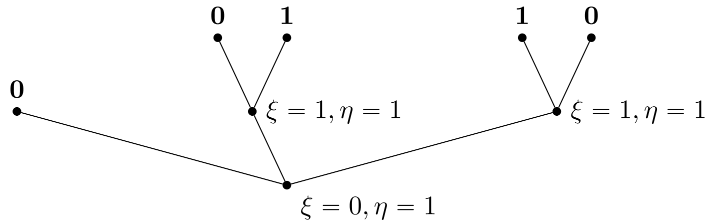

Figure 3. $\mathrm{BiSep}(\mathbf{\Sigma}^0_1, \mathbf{\Sigma}^0_2, \{\varnothing\})$

van Wesep showed, under AD, that if a non-self-dual $\mathbf{\Gamma}$ is closed under taking finite intersections, and $\check{\mathbf{\Gamma}}$ has the separation property, then $\mathbf{\Gamma}$ has the reduction property. The result for Borel Wadge classes follows from Borel determinacy.

Proof. Let $A_0, A_1 \in \mathbf{\Gamma}$. Since $\check{\mathbf{\Gamma}}$ has the generalized separation property, there are $G_0, G_1 \in \mathbf{\Gamma}$ such that $(G_0 \cap (A_0 \cup A_1), G_1 \cap (A_0 \cup A_1))$ reduces (A_0, A_1). Let $B_i = G_i \cap A_i$. $\square$

Definition 7.3. A class description Γ is *hereditarily* Σ*-type* if for every non-leaf $s \in T_\Gamma$, Γ_s has Σ-type.

What this means is that whenever the default outcome of some $s \in T_\Gamma$ is a leaf, then this leaf must be labelled 0. Unlike having Σ-type, being hereditarily Σ-type is not invariant for all descriptions of a given class (even restricting to acceptable or admissible descriptions). To see this, consider that $\mathrm{SU}_0(\Sigma_1, \Sigma_2, \Sigma_3, \dots) \equiv \mathrm{SU}_0(\Sigma_1, \Pi_1, \Sigma_2, \Pi_2, \dots) \equiv \mathrm{SU}_0(\Sigma_1, \Pi_2, \Pi_3, \Pi_4, \dots)$.

Definition 7.4. A class description Γ has the *effective reduction property* if $\mathbf{\Gamma}$ has the reduction property, uniformly: given any pair of Γ-names N^0 and N^1, we can compute a pair of Γ-names M^0 and M^1 which reduce (N^0, N^1).

The proof of Proposition 5.2 shows that if $\mathbf{\Gamma}$ has the reduction property, then there is some $z \in \Delta^1_1(y^\Gamma)$ such that after relativising to z, Γ has the effective reduction property. Our main result is:

Theorem 7.5. *The following are equivalent for a non-self-dual Borel Wadge class* $\mathbf{\Gamma}$*:*

(1) $\mathbf{\Gamma}$ *has a description which is hereditarily* Σ*-type.*
(2) $\mathbf{\Gamma}$ *has a description with the effective reduction property.*
(3) $\mathbf{\Gamma}$ *has the reduction property.*

Moreover, we will see that for any *admissible* Γ, if $\mathbf{\Gamma}$ has the reduction property, then there is some Λ with the effective reduction property such that $\mathbf{\Lambda} = \mathbf{\Gamma}$ and $y^\Lambda = y^\Gamma$.

One implication is easy, given Proposition 5.2.

Proposition 7.6. *If* Γ *is hereditarily* Σ*-type, then* Γ *has the effective reduction property.*

Proof. We describe a winning strategy for player 2 in the game $G_{\mathtt{red}}(\Gamma, \Gamma; \Gamma, \Gamma)$. In general, player 2 copies the moves of player 1, so that $s^j_1[k] = s^j_2[k]$ for $j = 0, 1$. The exception is when player 1 chooses (either within a leaf selection game, or outside it) two leaves of T_Γ that are labelled 1. If player 1 just selected the second such leaf, then on the corresponding tree, player 2 does not change their selection, and can continue taking the default outcome from their location to get to a 0-labelled leaf. If this is part of a leaf selection game, then player 2 will match player 1's move if and when she moves away from a 1-labelled leaf of T_Γ. $\square$

For the remaining implication (3) $\Longrightarrow$ (1), we analyse the *reducer* of a class $\mathbf{\Gamma}$. This will be the smallest class containing $\mathbf{\Gamma}$ that can reduce any pair of sets from $\mathbf{\Gamma}$ (the actual definition will be a bit different). It will turn out (as has been observed in [LSR88a]) that the reducer has the reduction property. Given an admissible class description Γ, we can easily describe the reducer of $\mathbf{\Gamma}$: we replace each 1-labelled default leaf by an appropriate $\mathbf{\Sigma}^0_{1+\xi}$. This is the "minimum action" required to turn Γ into a hereditarily Σ class.

Definition 7.7. Let Γ be a class description.

(a) We let $b(\Gamma)$ be the collection of 1-labelled leaves s of T_Γ such that either $s = \langle\rangle$ (when $o(\Gamma) = \omega_1$) or s is the leftmost child of its parent s^- on T_Γ.
(b) We let $R(\Gamma)$ be the class description obtained from Γ by attaching, to every $s \in b(\Gamma)$, two children $s\hat{}0$ and $s\hat{}1$, which are leaves of $T_{R(\Gamma)}$ labelled 0 and 1, respectively. We set $\xi_s^{R(\Gamma)} = \xi^\Gamma_{s^-}$ and $\eta_s^{R(\Gamma)} = 1$. If $s = \langle\rangle$ then $\xi_s^{R(\Gamma)} = 0$.

Note that even if Γ is efficient, $R(\Gamma)$ may fail to be efficient. The following is verified easily:

Lemma 7.8. Let Γ be a class description.

(a) $R(\Gamma)$ is hereditarily Σ-type.
(b) If $o(\Gamma) < \omega_1$ then $o(R(\Gamma)) = o(\Gamma)$.

Let $R(\mathbf{\Gamma})$ be the class described by $R(\Gamma)$.[3] Similarly, if $s \in T_\Gamma$, then we let $R(\mathbf{\Gamma})_s$ denote the class described by $R(\Gamma)_s$. Note that $R(\Gamma)_s = R(\Gamma_s)$ if $s \in T_\Gamma \backslash b(\Gamma)$, but when $s \in b(\Gamma)$ (and is not the root), $R(\Gamma)_s$ is the description of $\mathbf{\Sigma}^0_{1+\xi}$ where $\xi = o(\Gamma_s)$, whereas $R(\Gamma_s)$ is the description of $\mathbf{\Sigma}^0_1$.

By Proposition 7.6 and Lemma 7.8(a), $R(\mathbf{\Gamma})$ has the reduction property.

Example 7.9. Let Γ be the simplest description of $\check{D}_\eta(\mathbf{\Sigma}^0_{1+\alpha})$ (Fig. 2); it is admissible. Then $R(\mathbf{\Gamma})$ (see Fig. 4) equals $D_{\eta+1}(\mathbf{\Sigma}^0_{1+\alpha})$. Let Λ be the simple description of $D_{\eta+1}(\mathbf{\Sigma}^0_{1+\alpha})$. In $G_{\mathtt{cont}}(R(\Gamma), \Lambda)$, suppose that at a given round, the ordinal at the root played by player 1 is $\zeta \leqslant \eta$. If player 1 has already shifted to the outcome 1 of the leftmost child of the root, then player 2 matches the ordinal ζ; otherwise, player 2's ordinal is $\zeta + 1$. The other containment is easier.

Lemma 7.10. For all Γ, for all $s \in T_\Gamma$, $\mathbf{\Gamma}_s \subseteq R(\mathbf{\Gamma})_s$.

Proof. Let N be a Γ_s-name. An equivalent $R(\Gamma)_s$-name M is defined by setting $f_t^M = f_t^N$ and $\beta_t^M = \beta_t^N$ for all non-leaf $t \in T_{\Gamma_s}$; if $t \in b(\Gamma_s)$ then we set $f_t^M(\sigma) = t\hat{}1$ and $\beta_t^M(\sigma) = 0$ for all σ.[4] $\square$

[3] For now, this is abuse of notation; shortly we will see that restricted to admissible descriptions, the operation R induces a function on the described classes.

[4] Alternatively, a winning strategy for player 2 in the game $G_{\mathtt{cont}}(\Gamma_s, R(\Gamma)_s)$ has player 2 match the moves of player 1, except that if it is player 2's move and player 1 reached some $t \in b(\Gamma_s)$ then player 2 then chooses $t\hat{}1$. Similar steps need to be taken during a leaf selection sub-game.

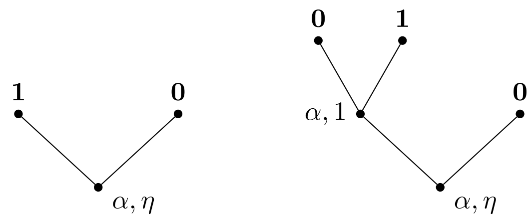

FIGURE 4. The admissible description of $\check{D}_\eta(\mathbf{\Sigma}^0_{1+\alpha})$ and its R.

Lemma 7.11. Let Γ and Λ be admissible class descriptions. Let $t \in T_\Gamma \backslash b(\Gamma)$ and $r \in T_\Lambda$. If $\mathbf{\Gamma}_t \subseteq \mathbf{\Lambda}_r$ then $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_r$.[5]

Proof. We prove the lemma by induction on the pair of ranks of t in T_Γ and r in T_Λ. We separate into a number of cases.

Case I: $\xi^\Gamma_t = \omega_1$. In this case, since $t \notin b(\Gamma)$, $R(\mathbf{\Gamma})_t = \mathbf{\Gamma}_t \subseteq \mathbf{\Lambda}_r \subseteq R(\mathbf{\Lambda})_r$, using Lemma 7.10 for the last containment.

Case II: $\xi^\Gamma_t > \xi^\Lambda_r$. By Corollary 3.7(a), there is some n such that $\mathbf{\Gamma}_t \subseteq \mathbf{\Lambda}_{r\hat{\ }n}$. By induction, $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_{r\hat{\ }n}$; and $R(\mathbf{\Lambda})_{r\hat{\ }n} \subseteq R(\mathbf{\Lambda})_r$.

Case III: $\xi^\Gamma_t < \xi^\Lambda_r$. For all n with $t\hat{\ }n \in T_\Gamma$, $\mathbf{\Gamma}_{t\hat{\ }n} \subseteq \mathbf{\Lambda}_r$. By induction $R(\mathbf{\Gamma})_{t\hat{\ }n} \subseteq R(\mathbf{\Lambda})_r$ for all non-default n, and for the default outcome n^* of t, if $t\hat{\ }n^* \notin b(\Gamma)$. If $t\hat{\ }n^* \in b(\Gamma)$ then $R(\mathbf{\Gamma})_{t\hat{\ }n^*} = \mathbf{\Sigma}^0_{1+\xi^\Gamma_t}$. Since Γ is efficient and $\xi^\Gamma_t < \omega_1$, we have $\varnothing, \mathcal{N} \in \mathbf{\Gamma}_t$, and so $\varnothing, \mathcal{N} \in \mathbf{\Lambda}_r$. Since $\xi^\Gamma_t < \xi^\Lambda_r$, $\mathbf{\Sigma}^0_{1+\xi^\Gamma_t} \subseteq \mathbf{\Lambda}_r \subseteq R(\mathbf{\Lambda})_r$ (Lemma 3.8 and again Lemma 7.10). Hence, for all n, $R(\mathbf{\Gamma})_{t\hat{\ }n} \subseteq R(\mathbf{\Lambda})_r$. By Corollary 3.7(b), $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_r$.

Case IV: $\xi^\Gamma_t = \xi^\Lambda_r < \omega_1$. Since in this case $R(\Gamma_t) = R(\Gamma)_t$ and $R(\Lambda_r) = R(\Lambda)_r$, we may simplify notation by assuming that $r = t = \langle\rangle$ are the roots of T_Γ and of T_Λ.

Let: $T_1 = S_\Gamma$, $T_2 = S_\Lambda$, $U_1 = S_{R(\Gamma)}$; and $U_2 = S_{R(\Lambda)}$. Below, we write $G_{\texttt{leaf}}(T_1, T_2)$ for $G_{\texttt{leaf}}(\Gamma, \Lambda)$ and $G_{\texttt{leaf}}(U_1, U_2)$ for $G_{\texttt{leaf}}(R(\Gamma), R(\Lambda))$.

By assumption, there is a strategy $\mathfrak{S}$ for player 2 in the game $G_{\texttt{leaf}}(T_1, T_2)$ that brings every play to an outcome (t, r) such that $\mathbf{\Gamma}_t \subseteq \mathbf{\Lambda}_r$. In fact, we may take $\mathfrak{S}$ to be prompt, in the sense of Remark 3.9. By Corollary 3.7(c), it suffices to show that there is a strategy for player 2 in the game $G_{\texttt{leaf}}(U_1, U_2)$ that brings every play to an outcome (t, r) such that $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_r$. By induction, for any pair of leaves $t \in T_1$ and $r \in T_2$, if $\mathbf{\Gamma}_t \subseteq \mathbf{\Lambda}_r$ and $t \notin b(\Gamma)$ then $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_r$.

Since Γ and Λ are admissible, we know that for all non-leaf $s \in T_i$, for all $n > 0$ such that $s\hat{\ }n \in T_i$, $s\hat{\ }n$ is a leaf of T_i, so the trees T_i have a very particular shape; other than the leaves, they only grow via the 0-outcome. Let q_i be the leftmost leaf of T_i; so the non-leaves of T_i are precisely the prefixes $s \prec q_i$. There are two possibilities for each i:

- If q_i is a 1-labelled leaf of the respective T_Γ or T_Λ, i.e., if $i = 1$ and $q_1 \in b(\Gamma)$, or $i = 2$ and $q_2 \in b(\Lambda)$, then $U_i = T_i \cup \{q_i\hat{\ }0, q_i\hat{\ }1\}$.
- Otherwise, $U_i = T_i$.

For the construction of the strategy, there are five sub-cases. In each, given a sequence of moves for player 1 in $G_{\texttt{leaf}}(U_1, U_2)$, we define an auxiliary play in the

[5] Note, not $R(\mathbf{\Gamma}_t) \subseteq R(\mathbf{\Lambda}_r)$, though these are close.

game $G_{\texttt{leaf}}(T_1, T_2)$. To keep things clear, we will refer to the players in $G_{\texttt{leaf}}(T_1, T_2)$ as player 3 and player 4. Given moves by player 1, we define a sequence of moves for player 3. We let player 4 follow $\mathfrak{S}$, and then explain how to use these moves to tell player 2 how to respond.

We write $(p[l])$ for the sequence of positions in the play of $G_{\texttt{leaf}}(U_1, U_2)$; we will let, as above, $t[l] = t^{p[l]}$, $c_s[l] = c_s^{p[l]}$ and $\eta_s[l] = \eta_s^{p[l]}$. We will let $(p'[k])$ denote the sequence of positions in the play of $G_{\texttt{leaf}}(T_1, T_2)$, and will let $t'[k] = t^{p'[k]}$, $c'_s[k] = c_s^{p'[k]}$ and $\eta'_s[k] = \eta_s^{p'[k]}$.

Sub-case IV(a): $U_1 = T_1$ and $U_2 = T_2$. In this case, the games are identical: for odd k we let $p'[k] = p[k]$; after player 4 responds with $\mathfrak{S}$, we let $p[k] = p'[k]$ for even k. Thus, the outcome (t, r) of the play in $G_{\texttt{leaf}}(U_1, U_2)$ is the same as the outcome of the play in $G_{\texttt{leaf}}(T_1, T_2)$. To show that this is a successful strategy, we need to show that $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_r$. By the assumption on $\mathfrak{S}$, we have $\mathbf{\Gamma}_t \subseteq \mathbf{\Gamma}_r$. The desired conclusion follows from the induction assumption if $t \notin b(\Gamma)$. However, no leaf of T_1 is in $b(\Gamma)$: since $U_1 = T_1$, $q_1 \notin b(\Gamma)$; no other leaf of T_1 can be in $b(\Gamma)$, as no other leaf of T_1 is the default child of its parent.

Sub-case IV(b): $U_1 = T_1$ and $U_2 \neq T_2$.

Again, since $U_1 = T_1$, player 3 can simply copy the positions played by player 1. In response, player 2 can copy the position played by player 4, unless the leaf $t'[k]$ played equals q_2, which is a leaf of T_2 but not of U_2. In this case we will set $t[k] = q_2\hat{\ }1$ (and this will necessitate setting $\eta_{q_2}[k] = 0$, since $q_2\hat{\ }1$ is not the default child of q_2 on U_2). More formally, for odd k we set $p'[k] = p[k]$; for even $k \geqslant 2$ such that $p'[k]$ is defined we set $p[k]$ to be $p'[k]$, except that we also set $c_{q_2}[k] = 1$ and $\eta_{q_2}[k] = 0$.

We need to check that this strategy is successful, but before that, we need to check that the auxiliary game does not terminate too quickly. We can imagine that there would be a problem. Suppose, for example, that $t'[1] = t[1] = q_1$ (so player 3's first move is a pass), and that the $\mathfrak{S}$-response to that is $t'[2] = q_2$. This is a pass for player 4, and this ends the game $G_{\texttt{leaf}}(T_1, T_2)$. However, the response with $t[2] = q_2\hat{\ }1$ is not a pass by player 2, and this means that the game $G_{\texttt{leaf}}(U_1, U_2)$ has not ended; now player 1 is free to make various moves, and we do not have $\mathfrak{S}$ to guide player 2's responses.

However, this imagined sequence of events does not actually happen. If it did, then the auxiliary play of $G_{\texttt{leaf}}(T_1, T_2)$ would end with the outcome (q_1, q_2). This is not possible because $\mathbf{\Gamma}_{q_1} \nsubseteq \mathbf{\Lambda}_{q_2}$: since $U_2 \neq T_2$, q_2 is a 1-labelled leaf of T_Λ, so $\mathbf{\Lambda}_{q_2} = \{\mathcal{N}\}$. On the other hand, since $U_1 = T_1$, q_1 is not a 1-labelled leaf of T_Γ (either it is not a leaf of T_Γ, or it is a 0-labelled leaf of T_Γ). Since Γ is efficient, this implies that $\mathbf{\Gamma}_{q_1} \neq \{\mathcal{N}\}$.

We also note that the only time that a pass played by player 4 can translate to a move by player 2 which is not a pass, is when $k = 2$, i.e., the first move by players 4 and 2, in which case we have $t'[2] = t'[0] = q_2$. At all other stages k, $t'[k-2] = t'[k]$ implies $t[k-2] = t[k]$: either $t'[k] \neq q_2$, in which case $t[k] = t[k-2] = t'[k]$; or $t'[k] = q_2$, in which case $t[k] = t[k-2] = q_2\hat{\ }1$. The only problem was that $t[0] = q_2\hat{\ }0$, since $p[0]$ is the default position.

Hence, the only time that the play of $G_{\texttt{leaf}}(T_1, T_2)$ may end prematurely is if player 1 (and so player 3) does not pass at their first move, player 4 passes, and then player 1 passes. In this case, player 2 can finish the game by passing.

Overall, we see that we can always carry both games to completion; if the outcome of $G_{\texttt{leaf}}(T_1, T_2)$ is (t, r), then the outcome of $G_{\texttt{leaf}}(U_1, U_2)$ is also (t, r), unless $r = q_2$, in which case the outcome is $(t, q_2\hat{\ }1)$. Since player 2 followed $\mathfrak{S}$, we have $\mathbf{\Gamma}_t \subseteq \mathbf{\Lambda}_r$. Since $U_1 = T_1$, no leaf of T_1 is in $b(\Gamma)$. Hence, by induction, $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_r$. If $r \neq q_2$ we are done. If $r = q_2$ then we need to show that $R(\mathbf{\Gamma})_t \subseteq R(\mathbf{\Lambda})_{q_2\hat{\ }1}$. However, as observed, $\mathbf{\Lambda}_{q_2} = \{\mathcal{N}\}$, and so $\mathbf{\Gamma}_t = \{\mathcal{N}\}$; since Γ is efficient, this means that t is a 1-labelled leaf of T_Γ. However, again, since $U_1 = T_1$, $t \notin b(\Gamma)$ (this means that t is not the default child of its parent). Hence t is also a leaf of $R(\Gamma)$, and so $R(\mathbf{\Gamma})_t = \{\mathcal{N}\}$ as well. Since the $R(\Lambda)$-label of $q_2\hat{\ }1$ is 1 (this was the whole point), we have $R(\mathbf{\Lambda})_{q_2\hat{\ }1} = \{\mathcal{N}\}$.

Sub-case IV(c): $U_1 \neq T_1$ and q_2 is not a leaf of T_Λ (in this case, $U_2 = T_2$).

Before going into the details, let us mention the main issues. Since $U_1 \neq T_1$, we need to translate player 1's moves on U_1 to moves for player 3 on T_1. It is not completely clear how to do this: what should $t'[k]$ be (for k odd) when $t[k] = q_1\hat{\ }0$ or $t[k] = q_1\hat{\ }1$? It seems that it should be q_1, but then, a move by player 1 from $q_1\hat{\ }0$ to $q_1\hat{\ }1$ is just a pass for player 3. Further, the label of q_1 in T_Γ is 1, whereas the label of $q_1\hat{\ }0$ in $R(\Gamma)$ is 0, so if the outcome of the $G_{\texttt{leaf}}(U_1, U_2)$ game is $(q_1\hat{\ }0, r)$ (and so, presumably, the outcome of $G_{\texttt{leaf}}(T_1, T_2)$ is (q_1, r)) it may be difficult to argue that $R(\mathbf{\Gamma})_{q_1\hat{\ }0} \subseteq R(\mathbf{\Lambda})_r$ based on the assumption $\mathbf{\Gamma}_{q_1} \subseteq \mathbf{\Lambda}_r$.

Thus, the moves for player 3 will not be in exact 1-1 correspondence with the moves for player 1. In the beginning, while player 1 plays leaves such as $q_1\hat{\ }0$ or $q_1\hat{\ }1$, or any leaf of $T_{R(\Gamma)}$, for that matter, player 2 can pass: since q_2 is not a leaf of Λ, $\mathbf{\Lambda}_{q_2}$, and so $R(\mathbf{\Lambda})_{q_2}$, contains both $\varnothing$ and $\mathcal{N}$, and so is an adequate response. Only once player 1 plays some $t = t[k]$ which is not a leaf of $T_{R(\Gamma)}$, do we copy this to be the first move $t'[1]$ of player 3. We let player 2 copy player 4's response, which is possible, since $U_2 = T_2$. Since t is not a leaf of $T_{R(\Gamma)}$, we have $t \notin b(\Gamma)$, and so copying the response gives player 2 a winning position. After this, if player 1 returns to playing leaves of $T_{R(\Gamma)}$, then player 2 can pass, as its current response is an adequate response to a class containing both $\varnothing$ and $\mathcal{N}$. Hence, we only need to define $t'[3]$ if player 1 eventually plays some $t[k']$ which is not a leaf of $T_{R(\Gamma)}$, and one different from $t'[1]$.

Now to the details. Since we will sometimes skip moves in $G_{\texttt{leaf}}(U_1, U_2)$, for every round k of the auxiliary game $G_{\texttt{leaf}}(T_1, T_2)$, we will define a corrsponding round $l(k)$ in the main game $G_{\texttt{leaf}}(U_1, U_2)$. The map $k \mapsto l(k)$ is strictly increasing and preserves parity (moves of player 3 correspond to moves of player 1, moves of player 4 correspond to moves of player 2), but may not be the identity: some rounds of the main game are not in the range of l. For every odd k we will have $l(k+1) = l(k) + 1$; but we may have $l(k+2) > l(k) + 2$.

We will always have, for even k, $p[l(k)] = p'[k]$: player 4's response by $\mathfrak{S}$ at round k is copied over to be a move by player 2 at round $l(k)$. For odd k we will have $p'[k] = p[l(k)] \upharpoonright T_1$; that is, for all non-leaf $s \in T_1$, $c'_s[k] = c_s[l(k)]$ and $\eta'_s[k] = \eta_s[l(k)]$. In other words, player 1's move at round $l(k)$ is copied over to player 3's move at round k, except that we do not copy $c_{q_1}[l(k)]$ and $\eta_{q_1}[l(k)]$, since q_1 is a leaf of T_1.

We start with $l(-1) = -1$ and $l(0) = 0$. Note that since all positions at these stages are the default positions, we indeed have $p[0] = p'[0]$ and $p'[-1] = p[-1] \upharpoonright T_1$.

Let $k \geqslant 1$ be odd, and suppose that $l(k-2)$ and $l(k-1) = l(k-2)+1$ have been defined; we have also described the moves for player 2 in $G_{\texttt{leaf}}(U_1, U_2)$ up to and including round $l(k-1)$. If the main game has not yet terminated, then player 1 plays $p[l(k-1)+1]$. We then let player 2 pass, and let player 1 keep playing, until we encounter some odd $m \geqslant l(k-1)+1$ at which one of the following holds:

- player 1 passes at m; or
- $t[m]$ is not a leaf of $T_{R(\Gamma)}$, and $t[m] \neq t[l(k-2)]$.

When such an m is encountered, if the play does not end at round m:

- If player 1 passes at m, then player 2 passes at $m+1$ and halts the play.
- Otherwise, we set $l(k) = m$.

In the latter case, as promised, we set $p'[k] = p[m] \upharpoonright T_1$. We then let player 2 respond according to $\mathfrak{S}$, set $l(k+1) = m+1$, and $p[m+1] = p'[k+1]$.

Before we verify that this strategy for player 2 is successful, we quickly check that the various plays can be performed as described. That is:

(1) for odd k, the move $p'[k]$ is legal for player 3.
(2) for even k, the move $p[l(k)]$ is legal for player 2.
(3) The auxiliary play does not terminate prematurely.

For (1), for odd $k \geqslant 1$, we see that $p'[k] \leqslant p'[k-2]$ because $p'[k] = p[l(k)] \upharpoonright T_1$, $p'[k-2] = p[l(k-2)] \upharpoonright T_1$, and $p[l(k)] \leqslant p[l(k-2)]$, since player 1 plays legally. Similarly, for (2), for even $k \geqslant 2$, $p[l(k)] = p'[k]$, $p[l(k-2)] = p'[k-2]$ and $p'[k] \leqslant p'[k-2]$ since player 4 plays legally; but since player 2 is instructed to pass in the rounds between $l(k-2)$ and $l(k)$, we have $p[l(k)-2] = p[l(k-2)]$, so $p[l(k)] \leqslant p[l(k)-2]$. For (3), we show that $t'[k] \neq t'[k-2]$. This is because $t[l(k)]$ is not a leaf of $T_{R(\Gamma)}$, in particular, $t[l(k)]$ does not extend q_1, and so setting $p'[k] = p[l(k)] \upharpoonright T_1$ results in $t'[k] = t[l(k)]$. For $k = 1$, we have $t'[1] \neq q_1 = t'[-1]$. For $k \geqslant 3$, since in the search for $m = l(k)$ we required $t[m] \neq t[l(k-2)]$, we have $t'[k] = t[l(k)] \neq t[l(k-2)] = t'[k-2]$. Hence, no move by player 3 is a pass, so the play of $G_{\mathtt{leaf}}(T_1, T_2)$ does not terminate.

Now we check that the strategy is successful. Let (u, r) be the outcome of the play of $G_{\mathtt{leaf}}(U_1, U_2)$; we show that $R(\mathbf{\Gamma})_u \subseteq R(\mathbf{\Lambda})_r$.

First, suppose that $l(1)$ is undefined. This means that player 1 only chooses leaves of $T_{R(\Gamma)}$ until he passes; player 2 only passes. So u is a leaf of $T_{R(\Gamma)}$ and $r = q_2$; as discussed, $\varnothing, \mathcal{N} \in \mathbf{\Lambda}_{q_2} \subseteq R(\mathbf{\Lambda})_{q_2}$ (using Lemma 7.10), and $R(\mathbf{\Gamma})_u$ is either $\{\varnothing\}$ or $\{\mathcal{N}\}$.

Otherwise, let k be the greatest number such that $l(k)$ is defined; $k \geqslant 2$ is even. This means that after round $l(k)$, player 2 only passes; so $r = t[l(k)] = t'[k]$. However, both $u = t[l(k-1)]$ and $u \neq t[l(k-1)]$ are possible, since player 1 is allowed to move about after playing $t[l(k-1)]$, and only then pass.

Suppose first that $u = t[l(k-1)]$. By our instructions, $t[l(k-1)]$ is not a leaf of $T_{R(\Gamma)}$. As discussed, $t'[k-1] = t[l(k-1)]$ is a leaf of T_1, different from q_1. Hence, $u \notin b(\Gamma)$. The auxiliary $G_{\mathtt{leaf}}(T_1, T_2)$ play does not end, but since $\mathfrak{S}$ is assumed to be prompt (see Remark 3.9 again), $\mathbf{\Gamma}_{t'[k-1]} \subseteq \mathbf{\Lambda}_{t'[k]}$. That is, $\mathbf{\Gamma}_u \subseteq \mathbf{\Lambda}_r$. Since $u \notin b(\Gamma)$, by induction, $R(\mathbf{\Gamma})_u \subseteq R(\mathbf{\Lambda})_r$, as required.

Next, suppose that $u \neq t[l(k-1)]$. Then it must be that u is a leaf of $T_{R(\Gamma)}$ (otherwise, we would have defined $l(k+1)$, contradicting the maximality of k). As in the first case, $\mathbf{\Gamma}_{t'[k-1]} \subseteq \mathbf{\Lambda}_r$. Since $t'[k-1] = t[l(k-1)]$ is not a leaf of $T_{R(\Gamma)}$, and Γ is efficient, we have $\varnothing, \mathcal{N} \in \mathbf{\Gamma}_{t'[k-1]}$; so $\varnothing, \mathcal{N} \in \mathbf{\Lambda}_r$. By Lemma 7.10, $\varnothing, \mathcal{N} \in R(\mathbf{\Lambda})_r$. Since u is a leaf of $T_{R(\Gamma)}$, $R(\mathbf{\Gamma})_u = \{\varnothing\}$ or $R(\mathbf{\Gamma})_u = \{\mathcal{N}\}$. In either case, $R(\mathbf{\Gamma})_u \subset R(\mathbf{\Lambda})_r$, as required.

Sub-case IV(d): $U_1 \neq T_1$ and $U_2 \neq T_2$ (so $q_1 \in b(\Gamma)$ and $q_2 \in b(\Lambda)$). This is a little more complicated than the previous sub-case. In this case, player 2 cannot just rest until player 1 plays a non-leaf; the default outcome for player 2 is $q_2\hat{}0$, which is not an adequate response to a 1-labelled leaf of $T_{R(\Gamma)}$.

Now it would seem that this should not be a problem. When player 1 plays $q_1\hat{}j$, then player 2 can play $q_2\hat{}j$, and otherwise, we will play an auxiliary $G_{\mathtt{leaf}}(T_1, T_2)$ game as above. However, player 1 can thwart us by first moving away from q_1, then coming back to $q_1\hat{}0$, and later moving to $q_1\hat{}1$. If player 2 responded by moving away from q_2 when player 1 moved away from q_1, starting the auxiliary game, then there is no guarantee that player 2 can later return to q_2 when player 1 returns to q_1, and so player 2 cannot respond to the move from $q_1\hat{}0$ to $q_1\hat{}1$ by moving from $q_2\hat{}0$ to $q_2\hat{}1$.

The solution is for us to delay the start of the auxiliary game, and to "spend" the move from $q_2\hat{}0$ to $q_2\hat{}1$ *first*, even if player 1 moves to other leaves of $T_{R(\Gamma)}$. In the beginning, while player 1 plays 0-labelled leaves of $T_{R(\Gamma)}$, whether $q_1\hat{}0$ or others (which are 0-labelled leaves of T_Γ), player 2 has no problems with just passing. If then player 1 plays a 1-labelled leaf, regardless of whether it is $q_1\hat{}1$ or not, player 2 can respond by moving to $q_2\hat{}1$. Suppose that then, player 1 returns to $q_1\hat{}0$. Player 3 has an advantage over player 1: the latter spent already two moves elsewhere, moving away from q_1 and then back, but player 3 has not.

Let w be the parent of q_1 on T_Γ, and suppose that all of player 1's moves so far are children of w. Thus, after two moves, $\eta_w[m] + 2 \leqslant \eta_w^\Gamma$ (with m odd). Since Γ is efficient, there is some child $w\hat{}n$ of w with $\varnothing \in \mathbf{\Gamma}_{w\hat{}n}$; player 3 can choose such $w\hat{}n$ and still have an advantage over player 1, by setting $\eta'_w = \eta_w[m] + 1$. If player 1 then moves from $q_1\hat{}0$ to $q_1\hat{}1$, then this extra ordinal now allows player 3 to move again (say to q_1 itself), now matching η_w and η'_w.

A slight complication is if player 1 chooses a leaf of T_Γ that is not a sibling of q_1 (a child of w). That is, player 1 can cycle between 0 and 1-labelled leaves without decreasing any single ordinal more than once. The ordinal book-keeping gets complicated.

But in this case, we can use "heavy artillery". This move of player 1 allows us to move away from $s\hat{}0$ for some $s \prec w$. This position of s implies $o(\Gamma_{s\hat{}0}) = o(\Gamma)$. Since Γ is admissible, by Lemma 4.7, there is a child $s\hat{}n$ of s such that $\Gamma_{s\hat{}0} < \Gamma_{s\hat{}n}$. Player 3 can choose such a child $s\hat{}n$ that adequately mimics player 1's move, and never needs to move to any extension of $s\hat{}0$ ever again.

Let us give the details. As in the previous sub-case, we match rounds k of the auxiliary game with rounds $l(k)$ of the main game. However, as discussed, we will not always have $p'[k] = p[l(k)] \upharpoonright T_1$ for odd k. To make sure that the moves we make are legal, we inductively ensure the following, for all odd $k \geqslant 1$:

(i) If $t[l(k)]$ is a leaf of $T_{R(\Gamma)}$ then $R(\Gamma)_{t[l(k)]} \subseteq \Gamma_{t'[k]}$. Otherwise, $\Gamma_{t[l(k)]} \subseteq \Gamma_{t'[k]}$.
(ii) For all $s \preccurlyeq w$, $\eta'_s[k] \geqslant \eta_s[l(k)]$.
(iii) If $t'[k] = s\hat{}n$ (for any $s \preccurlyeq w$) then for all $r \prec s$, $\eta_r[l(k)] = \eta_r^\Gamma$.
(iv) If $t'[k] = s\hat{}n$ where $s \prec w$, then $\Gamma_{s\hat{}0} < \Gamma_{t'[k]}$.
(v) If $t'[k]$ is a leaf of T_Γ and $c_{q_1}[l(k)] = q_1\hat{}0$ then $\eta'_w[k] > \eta_w[l(k)]$.

We run the plays as follows. We start with $l(-1) = -1$ and $l(0) = 0$. Suppose that $k \geqslant 1$ is odd and that $l(k-1)$ has been defined. We search for an odd round $m \geqslant l(k-1) + 1$ such that one of the following holds:

- player 1 passes at round m.
- $t[m]$ is not a leaf of $T_{R(\Gamma)}$, and $\Gamma_{t[m]} \nsubseteq \Gamma_{t'[k-2]}$.
- $t[m]$ is a leaf of $T_{R(\Gamma)}$, $k \geqslant 3$, and $R(\Gamma)_{t[m]} \nsubseteq \Gamma_{t'[k-2]}$.
- $k = 1$, $t[m]$ is a 0-labelled leaf of $T_{R(\Gamma)}$, and $t[m-2]$ is a 1-labelled leaf of $T_{R(\Gamma)}$.

If $k \geqslant 3$, then until we find such m, player 2 just passes. However, if $k = 1$, then for any even round $n < m$, if $t[n-1]$ is a 1-labelled leaf of $T_{R(\Gamma)}$ then we set $c_{q_2}[n] = q_2\hat{\ }1$ (so $\eta_{q_2}[n] = 0$; and $\eta_s[n] = \eta_s^\Gamma$ for all $s \prec q_2$). Otherwise, player 2 passes.

If player 1 passes at round m (and this does not end the play), then we let player 2 pass at round $m+1$ and end the play. Otherwise, we set $l(k) = m$. We then define $p'[k]$. Let s be the longest $s \preccurlyeq w$ such that $\eta_s[m] < \eta_s^\Gamma$. [There is such an s since otherwise, $t[n] \succ q_1$ for all odd $n \leqslant m$, implying that $k = 1$ and contradicting the choice of m.]

(1) If $s \prec w$, we set $\eta'_s[k] = \eta_s[m]$ and $c'_s[k] = t'[k]$ to be some non-default child of s such that $\Gamma_{c_s[m]} \subseteq \Gamma_{t'[k]}$ and $\Gamma_{s\hat{\ }0} < \Gamma_{t'[k]}$. (For all other $r \preccurlyeq w$, we leave $\eta'_r[k] = \eta'_r[k-2]$ so $c'_r[k] = c'_r[k-2]$.)
(2) If $s = w$ and either $c_{q_1}[m] = q_1\hat{\ }1$ or $t[m]$ is not a leaf of $T_{R(\Gamma)}$, then we set $p'[k] = p[m] \restriction T_1$.
(3) If $s = w$, $c_{q_1}[m] = q_1\hat{\ }0$, and $t[m]$ is a leaf of $T_{R(\Gamma)}$, then we set $\eta'_w[k] = \eta_w[m] + 1$ and $c'_w[k] = t'[k]$ to be some child of w such that $R(\Gamma)_{t[m]} \subseteq \Gamma_{t'[k]}$. For all $s \prec w$ we leave $\eta'_s[k] = \eta^\Gamma$ (and so $c'_s[k] = s\hat{\ }0$).

We let $p'[k+1]$ be player 4's response by $\mathfrak{S}$, $l(k+1) = m+1$, and $p[m+1] = p'[k+1]$, extended with $c_{q_2}[m+1] = q_2\hat{\ }1$ (and $\eta_{q_2}[m+1] = 0$).

Let us verify that this is all legal. First, we note that the described moves for player 2 while we are waiting to define $l(1)$ are all legal: as long as player 1 plays 0-labelled leaves of $T_{R(\Gamma)}$, player 2 passes; if player 1 then switches to 1-labelled leaves of $T_{R(\Gamma)}$, then player 2 changes the position once and then passes; then, if player 1 returns to a 0-labelled leaf of $T_{R(\Gamma)}$, or to a non-leaf of $T_{R(\Gamma)}$, then that stage is $l(1)$.

Suppose that $k \geqslant 1$ is odd, and everything has been verified up to round $k-1$. We check that it is possible and legal to define $p'[k]$ as we did, and that (i)–(v) hold at k. We consider which of the three cases of defining $t'[k]$ applies. For simplicity of notation, for all odd m, let $\Theta_m = R(\Gamma)_{t[m]}$ if $t[m]$ is a leaf of $T_{R(\Gamma)}$, and $\Theta_m = \Gamma_{t[m]}$ otherwise.

Suppose that (1) applied. We claim that $\eta_s[l(k)] < \eta'_s[k-2]$, so setting $\eta'_s[k] = \eta_s[l(k)]$ allows us to redefine $c'_s[k]$ as we like. If $\eta'_s[k-2] = \eta_s^\Gamma$ then this is by the choice of s. Otherwise, $k-2 \geqslant 1$, and by (iii) (and (ii)) at $k-2$, $t'[k-2]$ is some child of s (necessarily non-default, as $s \prec w$). By (i), $\Theta_{l(k-2)} \subseteq \Gamma_{t'[k-2]}$. By the choice of $l(k)$, $\Theta_{l(k)} \nsubseteq \Gamma_{t'[k-2]}$, so $t[l(k)] \neq t[l(k-2)]$. By (iv), $\Gamma_{s\hat{\ }0} < \Gamma_{t'[k-2]}$, so it must be that $t[l(k)]$ as well is a child of s. This implies that $c_s[l(k)] \neq c_s[l(k-2)]$, whence $\eta_s[l(k)] < \eta_s[l(k-2)]$. By (ii), $\eta_s[l(k-2)] \leqslant \eta'_s[k-2]$.

Next, we check that a child $c'_s[k]$ as described in (1) indeed exists. If $\Gamma_{s\hat{\ }0} < \Gamma_{c_s[l(k)]}$ then we can choose $c'_s[k] = c_s[l(k)]$. Otherwise, $\Gamma_{c_s[l(k)]} \subseteq \Gamma_{s\hat{\ }0}$ or $\Gamma_{c_s[l(k)]} \subseteq \check{\Gamma}_{s\hat{\ }0}$. As discussed above, since $s \prec w$, we can choose $c'_s[k]$ be some child of s such that $\Gamma_{s\hat{\ }0} < \Gamma_{c'_s[k]}$.

(i) holds at k: if $t[l(k)]$ is a leaf of $T_{R(\Gamma)}$ then $R(\mathbf{\Gamma})_{t[l(k)]} = \{\varnothing\}$ or $= \{\mathcal{N}\}$; but $\Gamma_{s\hat{}0} < \Gamma_{t'[k]}$ so $\varnothing, \mathcal{N} \in \Gamma_{t'[k]}$. Otherwise, as $c_s[l(k)] \preccurlyeq t[l(k)]$, we have $\Gamma_{t[l(k)]} \subseteq \Gamma_{c_s[l(k)]}$; and we ensured that $\Gamma_{c_s[l(k)]} \subseteq \Gamma_{t'[k]}$.

(ii), (iii) and (iv) hold at k by design. (v) holds vacuously.

Suppose that (2) applied at round k. We check that $p'[k] \leqslant p'[k-2]$.

Suppose that $k \geqslant 3$ and that (2) applied at round $k-2$ as well. Then $p'[k-2] = p[l(k-2)] \upharpoonright T_1$, $p'[k] = p[l(k)] \upharpoonright T_1$, and $p[l(k)] \leqslant p[l(k-2)]$, so $p'[k] \leqslant p'[k-2]$.

Suppose that $k \geqslant 3$ and that (3) applied at round $k-2$. Then $\eta'_w[k-2] > \eta_w[l(k-2)]$; since $\eta_w[l(k-2)] \geqslant \eta_w[l(k)]$ we have $\eta'_w[k] < \eta'_w[k-2]$, as required.

If $k = 1$ then any S_Γ-position is legal for player 3 at round k.

(i) holds at k if $t'[k] = t[l(k)]$. Otherwise, $t[l(k)] \succ q_1$, so $t'[k] = q_1$ and by assumption, $t[l(k)] = q_1\hat{}1$. Hence $R(\mathbf{\Gamma})_{t[l(k)]} = \{\mathcal{N}\}$ and $\mathbf{\Gamma}_{t'[k]} = \{\mathcal{N}\}$ as well (recall that $q_1 \in b(\Gamma)$). So (i) holds in this case as well. (ii) and (iii) hold at k by design; (iv) and (v) hold vacuously (if $t[l(k)]$ is not a leaf of $T_{R(\Gamma)}$ then by (i), $t'[k]$ is not a leaf of T_Γ.)

Suppose that (3) applied at round k. We check that $\eta'_w[k] < \eta'_w[k-2]$. Suppose first that $k = 1$. The choice of $l(1)$, and the fact that (1) does not hold at $k = 1$, imply that $\eta_w[l(1)] + 2 \leqslant \eta^\Gamma_w$: player 1 had to make at least two changes, and neither of them is the change from $q_1\hat{}0$ to $q_1\hat{}1$ (recall that $\eta^{R(\Gamma)}_{q_1} = 1$). Hence, the choice $\eta'_w[1] = \eta_w[l(1)] + 1$ is legal, and $\eta'_w[1] < \eta^\Gamma_1$. This allows us to choose $c'_w[1]$ as we like.

Suppose that $k \geqslant 3$. Then $c_{q_1}[l(k-2)] = q_1\hat{}0$. By the choice of $l(k)$, $R(\Gamma)_{t[l(k)]} \nsubseteq \Gamma_{t'[k-2]}$; this means that $t'[k-2]$ is a leaf of T_Γ. Hence, (v) applies at $k-2$, so $\eta'_w[k-2] > \eta_w[l(k-2)]$. Now (i) at $k-2$ implies that $t[l(k)] \neq t[l(k-2)]$, so $\eta_w[l(k-2)] > \eta_w[l(k)]$. Hence, $\eta'_w[k] < \eta'_w[k-2]$.

Because Γ is efficient, we can choose $c'_w[k] = t'[k]$ with the desired property $R(\Gamma)_{t[l(k)]} \subseteq \Gamma_{t'[k]}$.

Hence, $p'[k]$ is well-defined and is a legal move for player 1.

(i), (ii), (iii), and (v) hold at k by our definitions. (iv) holds vacuously. This concludes the verification that $p'[k]$ is legal and that (i)–(v) hold at k.

We verified that $t'[k] \neq t'[k-2]$ in some of the situations above, but the argument holds in general. By choice of $l(k)$, $\Theta_{l(k)} \nsubseteq \Gamma_{t'[k-2]}$; (i) now implies that $t'[k] \neq t'[k-2]$. So the move $t'[k]$ is not a pass for player 3, and the auxiliary game does not end prematurely.

We verify that the described strategy is successful. Let (u, r) be the outcome of the play of the main game $G_{\mathtt{leaf}}(U_1, U_2)$. If $l(1)$ is never defined then u is $q_1\hat{}j$ for some $j \in \{0, 1\}$, and we ensured that in this case, $r = q_2\hat{}j$, so $R(\mathbf{\Gamma})_u = R(\mathbf{\Lambda})_r$.

Suppose that $l(1)$ is defined; let k be the greatest such that $l(k)$ is defined; $k \geqslant 2$ is even. As in the previous sub-case, $r = t[l(k)]$. Let m be the last stage at which player 1 makes a move. So $u = t[l(m)]$. The maximality of k ensures that $\mathbf{\Theta}_m \subseteq \mathbf{\Gamma}_{t'[k-1]}$; promptness of the strategy $\mathfrak{S}$ ensures that $\mathbf{\Gamma}_{t'[k-1]} \subset \mathbf{\Lambda}_{t'[k]}$. By the definition of our strategy, $r = t'[k]$ if $t'[k] \neq q_2$, and $r = q_2\hat{}1$ if $t'[k] = q_1$. In the latter case, $\mathbf{\Lambda}_{t'[k]} = R(\mathbf{\Lambda})_r = \{\mathcal{N}\}$.

If u is a leaf of $T_{R(\Gamma)}$, then $\mathbf{\Theta}_u = R(\mathbf{\Gamma})_r$, and the string of containments just discussed shows that $R(\mathbf{\Gamma})_u \subseteq \mathbf{\Gamma}_r$, so Lemma 7.10 shows that $R(\mathbf{\Gamma})_u \subseteq R(\mathbf{\Lambda})_r$. Otherwise, $\mathbf{\Theta}_m = \mathbf{\Gamma}_u$ so we get $\mathbf{\Gamma}_u \subseteq \mathbf{\Lambda}_r$. In this case $u \notin b(\Gamma)$, so the indutive hypothesis applies, and we get $R(\mathbf{\Gamma})_u \subseteq \mathbf{\Gamma}_r$ as required.

Sub-case IV(e): $U_1 \neq T_1$ and q_2 is a 0-labelled leaf of T_Λ (so $U_2 = T_2$). This case is almost identical to the previous one, with one difference: at the beginning, if player 1 moves from 0-labelled leaves to a 1-labelled leaf, player 2 cannot respond with $q_2{}^\frown 1$. Instead, we start the auxiliary game, and choose $t'[1] = q_1$ (as it is a 1-labelled leaf of T_Γ). Note that this is a pass for player 3, while the corresponding move was not a pass for player 1. Nonetheless, this is not a problem, because in this case player 4 cannot pass, as $\mathcal{N} \notin \mathbf{\Lambda}_{q_2}$. So the auxiliary play does not end prematurely. The fact that this is a pass for player 3 means that no ordinal was spent, so the ordinal advantage over player 1 is the same as in the previous sub-case. □

Definition 7.12. For a Borel Wadge class $\mathbf{\Upsilon}$, let $\mathcal{C}(\mathbf{\Upsilon})$ be the collection of all non-self-dual Borel Wadge classes $\mathbf{\Theta}$ of Σ-type such $\mathbf{\Upsilon} \subseteq \mathbf{\Theta}$, and for some $\mathbf{\Theta}^0, \mathbf{\Theta}^1 \in \{\mathbf{\Theta}, \check{\mathbf{\Theta}}\}$, the pair $(\mathbf{\Theta}^0, \mathbf{\Theta}^1)$ reduces $(\mathbf{\Upsilon}, \mathbf{\Upsilon})$.

If $\mathbf{\Upsilon} \subseteq \mathbf{\Theta}$ and $\mathbf{\Theta}$ has the reduction property, then $\mathbf{\Theta} \in \mathcal{C}(\mathbf{\Upsilon})$. In particular, for all Γ, $R(\mathbf{\Gamma}) \in \mathcal{C}(\mathbf{\Gamma})$. If $\mathbf{\Gamma}$ has the reduction property, then $\mathbf{\Gamma}$ is the $\subseteq$-least element of $\mathcal{C}(\mathbf{\Gamma})$.

Lemma 7.13. Suppose that $(\mathbf{\Lambda}^0, \mathbf{\Lambda}^1)$ reduces $(\mathbf{\Gamma}^0, \mathbf{\Gamma}^1)$, and that $\varnothing \in \mathbf{\Gamma}^1$. Then $\mathbf{\Gamma}^0 \subseteq \mathbf{\Lambda}^0$.

Proof. Let $A \in \mathbf{\Gamma}^0$. The only pair that reduces $(A, \varnothing)$ is $(A, \varnothing)$ itself, so $A \in \mathbf{\Lambda}^0$. □

Corollary 7.14. *Suppose that $\mathbf{\Gamma} \neq \{\mathcal{N}\}$, $\mathbf{\Theta}$ has Σ-type, and that for some $\mathbf{\Theta}^0, \mathbf{\Theta}^1 \in \{\mathbf{\Theta}, \check{\mathbf{\Theta}}\}$, the pair $(\mathbf{\Theta}^0, \mathbf{\Theta}^1)$ reduces $(\mathbf{\Gamma}, \mathbf{\Gamma})$. Then $\mathbf{\Gamma} \subseteq \mathbf{\Theta}$ (so $\mathbf{\Theta} \in \mathcal{C}(\mathbf{\Gamma})$).*

Proof. By Lemma 7.13, $\mathbf{\Gamma} \subseteq \mathbf{\Theta}^0$ and $\mathbf{\Gamma} \subseteq \mathbf{\Theta}^1$.

By the semi-linear-ordering principle, we need to exclude the case that $\mathbf{\Gamma}$ has Π-type and $\mathbf{\Theta} = \check{\mathbf{\Gamma}}$. In this case, $\mathbf{\Theta}^0 = \mathbf{\Theta}^1 = \mathbf{\Gamma}$, but then $\mathbf{\Gamma}$ has the reduction property, which is impossible. □

The following proposition, together with Theorem 4.11, then finishes the proof of Theorem 7.5.

Proposition 7.15. *If Γ is admissible, then $R(\mathbf{\Gamma})$ is the $\subseteq$-least element of $\mathcal{C}(\mathbf{\Gamma})$.*

Proof. By Proposition 4.13, it suffices to show that for all monotone admissible Λ with $\mathbf{\Lambda} \in \mathcal{C}(\mathbf{\Gamma})$ we have $R(\mathbf{\Gamma}) \subseteq \mathbf{\Lambda}$. By Proposition 4.13 (and Lemma 7.11), we may assume that Γ is both admissible and monotone. [Monotony is not fundamental to the proof; it only makes notation a little cleaner.]

In fact, we show:

($*$): For all $t \in T_\Gamma$ such that $t \notin b(\Gamma)$ or $t = \langle\rangle$, for all monotone admissible Λ with $\mathbf{\Lambda} \in \mathcal{C}(\mathbf{\Gamma}_t)$, we have $R(\mathbf{\Gamma})_t \subseteq \mathbf{\Lambda}$.

We prove this by induction on the rank of t in T_Γ. For a fixed t, we prove $(*)_{\Gamma,t}$ by induction on the complexity of Λ.

Case I: $\xi^\Gamma_t = \omega_1$. There are two sub-cases:

- If $\mathbf{\Gamma}_t = \{\varnothing\}$ then $R(\mathbf{\Gamma})_t = \{\varnothing\}$ and $\varnothing \in \mathbf{\Lambda}$ since Λ has Σ-type.
- If $\mathbf{\Gamma}_t = \{\mathcal{N}\}$ then $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Lambda}$; the assumption $t \notin b(\Gamma)$ or $t = \langle\rangle$ implies that $R(\mathbf{\Gamma})_t = \{\mathcal{N}\}$ or $R(\mathbf{\Gamma})_t = \mathbf{\Sigma}^0_1$.

In the remaining cases, $\xi^\Gamma_t < \omega_1$, so $R(\Gamma)_t = R(\Gamma_t)$; to save ink, we assume that $t = \langle\rangle$.

Case II: $o(\Gamma) > o(\Lambda)$. By Proposition 5.2, there are n and m such that for some $\Theta_0 \in \{\Lambda_n, \check{\Lambda}_n\}$ and $\Theta_1 \in \{\Lambda_m, \check{\Lambda}_m\}$, the pair $(\mathbf{\Theta}_0, \mathbf{\Theta}_1)$ reduces $(\mathbf{\Gamma}, \mathbf{\Gamma})$. Since Λ is monotone, for $k = \max\{n, m\}$ we have $\mathbf{\Theta}_0, \mathbf{\Theta}_1 \in \{\mathbf{\Lambda}_k, \check{\mathbf{\Lambda}}_k\}$; without loss of generality, Λ_k has Σ-type. By Corollary 7.14, $\mathbf{\Lambda}_k \in \mathcal{C}(\mathbf{\Gamma})$ (and this is why we defined $\mathcal{C}(\mathbf{\Gamma})$ the way we did). By induction, $R(\mathbf{\Gamma}) \subseteq \mathbf{\Lambda}_k$, and $\mathbf{\Lambda}_k \subseteq \mathbf{\Lambda}$. [If $\check{\Lambda}_k$ has Σ-type then we use $\check{\mathbf{\Lambda}}_k \subseteq \mathbf{\Lambda}$, since Λ is monotone.]

Case III: $o(\Gamma) < o(\Lambda)$. By Corollary 3.7(b), it suffices to show that for all n, $R(\mathbf{\Gamma})_n \subseteq \mathbf{\Lambda}$. For all n, since $\mathbf{\Gamma}_n \subseteq \mathbf{\Gamma}$, $\mathbf{\Lambda} \in \mathcal{C}(\mathbf{\Gamma}_n)$. If $n \notin b(\Gamma)$ then by induction, $R(\mathbf{\Gamma})_n \subseteq \mathbf{\Lambda}$. If $n \in b(\Gamma)$ then $n = 0$. In this case, $R(\mathbf{\Gamma})_0 = \mathbf{\Sigma}^0_{1+o(\Gamma)}$. However, for any $n > 0$, $o(\mathbf{\Gamma}_n) > o(\Gamma)$, showing that $\mathbf{\Sigma}^0_{1+o(\Gamma)} \subseteq \mathbf{\Gamma}_n$ (Lemma 3.8); as $\mathbf{\Gamma}_n \subseteq R(\mathbf{\Gamma}_n) = R(\mathbf{\Gamma})_n$ (Lemma 7.10, and $n \notin b(\Gamma)$) and $R(\mathbf{\Gamma})_n \subseteq \mathbf{\Lambda}$; so $\mathbf{\Sigma}^0_{1+o(\Gamma)} \subseteq \mathbf{\Lambda}$.

In the remaining cases, let $\xi = o(\Gamma) = o(\Lambda) < \omega_1$. For all n, the classes $R(\mathbf{\Gamma})_n$ have Σ-type, and so they are all $\subseteq$-comparable. Note that by Lemma 7.11, if $n > 0$ then $R(\mathbf{\Gamma})_n \subseteq R(\mathbf{\Gamma})_{n+2}$, however equality may hold even if $\Gamma_n < \Gamma_{n+2}$. Further, it is not clear what the relationship is between $R(\mathbf{\Gamma})_n$ and $R(\mathbf{\Gamma})_{n+1}$ when $\Gamma_{n+1} = \check{\Gamma}_n$. Also, it is possible that $R(\mathbf{\Gamma})_0$ is larger than each $R(\mathbf{\Gamma})_n$ for $n > 0$.

Case IV: For all n, $o(\Gamma_n) = \omega_1$. Since Γ is monotone, either $\mathbf{\Gamma} = D_{\eta^\Gamma}(\mathbf{\Sigma}^0_{1+\xi})$ or $\mathbf{\Gamma} = \check{D}_{\eta^\Gamma}(\mathbf{\Sigma}^0_{1+\xi})$. If the former, then $R(\Gamma) = \Gamma$, so $R(\mathbf{\Gamma}) \subseteq \mathbf{\Lambda}$. If the latter, then by Example 7.9, $R(\mathbf{\Gamma}) = D_{\eta^\Gamma+1}(\mathbf{\Sigma}^0_{1+\xi})$. By Example 5.3 (and the semi-linear-ordering principle), $R(\mathbf{\Gamma})$ is the $\subseteq$-least element of $\mathcal{C}(\mathbf{\Gamma})$.

Suppose that case IV does not apply. Then there is some n such that $R(\mathbf{\Gamma})_0 \subseteq R(\mathbf{\Gamma})_n$. For otherwise, since $\Gamma_0 \subseteq \Gamma_2$, by Lemma 7.11 we must have $0 \in b(\Gamma)$. So $o(\Gamma_0) = \omega_1$ and $R(\mathbf{\Gamma})_0 = \mathbf{\Sigma}^0_{1+\xi}$. Let $n > 0$. Since $o(\Gamma_n) > \xi$, and by assumption, $\mathbf{\Sigma}^0_{1+\xi} \nsubseteq \mathbf{\Gamma}_n$ (as $\mathbf{\Gamma}_n \subseteq R(\mathbf{\Gamma})_n$ (Lemma 7.10)), by Proposition 2.4 (or Lemma 3.8), $o(\Gamma_n) = \omega_1$; so case IV applies.

In the remaining cases, let $\mathbf{\Upsilon} = \bigcup_{n \geqslant 0} R(\mathbf{\Gamma})_n$. Note that if $n > 0$ and $o(\Gamma_n) = \omega_1$ then $R(\Gamma)_n = \Gamma_n$. Since case IV does not hold,

$$\mathbf{\Upsilon} = \bigcup \{R(\mathbf{\Gamma})_n \;:\; n > 0 \;\&\; o(\Gamma_n) < \omega_1\}\,.$$

Further, we observe that

$$\mathbf{\Upsilon} \subseteq \bigcup_m \mathbf{\Lambda}_m.$$

To see this, let $n > 0$. Since $\mathbf{\Gamma}_n \subseteq \mathbf{\Gamma}$, $\mathbf{\Lambda} \in \mathcal{C}(\mathbf{\Gamma}_n)$, so by induction, $R(\mathbf{\Gamma})_n \subseteq \mathbf{\Lambda}$. Since $o(R(\Gamma)_n) = o(\Gamma_n) > \xi$ and $o(\Lambda) = \xi$, by Corollary 3.7, there is some m such that $R(\mathbf{\Gamma})_n \subseteq \mathbf{\Lambda}_m$.

Case V: For some n, $\mathbf{\Upsilon} = R(\mathbf{\Gamma})_n$. We may assume that $n > 0$. In this case $o(R(\Gamma)_n) = o(\Gamma_n) > \xi$. Since $o(R(\Gamma)) = \xi$, by Corollary 3.7(b), $R(\mathbf{\Gamma}) = R(\mathbf{\Gamma})_n$; and we just cheked that $\mathbf{\Upsilon} \subseteq \mathbf{\Lambda}$.

Case VI: For some n, $\mathbf{\Upsilon} \subseteq \mathbf{\Lambda}_n$. Since Λ is monotone, we may assume $n > 0$. Since Λ is admissible, $o(\Lambda_n) > \xi$. By Corollary 3.7(b), $R(\mathbf{\Gamma}) \subseteq \mathbf{\Lambda}_n$, and $\mathbf{\Lambda}_n \subseteq \mathbf{\Lambda}$.

Case VII: $\eta^\Lambda > \eta^\Gamma$. In this case, $R(\mathbf{\Gamma}) \subseteq \mathbf{\Lambda}$ follows from Proposition 4.8.

Case VIII: None of the above. We claim that

$$\boldsymbol{\Upsilon} = \bigcup_n \boldsymbol{\Lambda}_n = \bigcup_n \boldsymbol{\Gamma}_n$$

and that (Γ_n) and (Λ_n) do not settle to be a dual pair: for all n there is some m such that $\Gamma_n < \Gamma_m$ and $\Lambda_n < \Lambda_m$. For the first equality, observe that $\boldsymbol{\Upsilon}$ is the union of the Σ-classes $R(\boldsymbol{\Gamma})_n$, and that case VI does not apply. If the second fails then there is some m^* with $\bigcup_n \boldsymbol{\Gamma}_n \subseteq R(\boldsymbol{\Gamma})_{m*}$ (as by Lemma 7.10, $\bigcup_n \boldsymbol{\Gamma}_n \subseteq \boldsymbol{\Upsilon}$); we may assume that $m^* > 0$ and $o(\Gamma_{m*}) < \omega_1$. This implies that $R(\Gamma)_{m*} = R(\Gamma_{m*})$ has the reduction property. By induction, for all n, $R(\boldsymbol{\Gamma})_n \subseteq R(\boldsymbol{\Gamma})_m$; so case V applies.

Also observe that $\eta^\Lambda = \eta^\Gamma$. Otherwise, since case VII does not apply, $\eta^\Lambda < \eta^\Gamma$. Then $\bigcup \boldsymbol{\Lambda}_n \subseteq \bigcup \boldsymbol{\Gamma}_n$ and $\boldsymbol{\Gamma}$ being admissible would impliy $\boldsymbol{\Lambda} < \boldsymbol{\Gamma}$ (Proposition 4.8), contradicting $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$.

Fix $\Theta, \Upsilon \in \{\Lambda, \check{\Lambda}\}$ such that $(\boldsymbol{\Theta}, \boldsymbol{\Upsilon})$ reduces $(\boldsymbol{\Gamma}, \boldsymbol{\Gamma})$.

Claim 7.15.1. If $0 \in b(\Gamma)$ then 0 is not a leaf of T_Λ.

Proof. Suppose that $0 \in b(\Gamma)$ and that 0 is a leaf of T_Λ. We show how player 1 wins the game $G_{\texttt{red}}(\Gamma, \Gamma; \Theta, \Upsilon)$, contradicting the assumption that $(\boldsymbol{\Theta}, \boldsymbol{\Upsilon})$ reduces $(\boldsymbol{\Gamma}, \boldsymbol{\Gamma})$.

Let T_0 and T_1 be the two copies of S_Γ used by player 1 in the game $G_{\texttt{leaf}}(\Gamma, \Gamma; \Theta, \Upsilon)$; let S_0 and S_1 be the two copies of $S_\Lambda = S_\Theta = S_\Upsilon$ used by player 2 in that game.

We show that there is a move $p[1]$ for player 1 in which he does not move on T_0, that forces player 2 to move on S_0 (or it is an easy win for player 1). This depends on the labels of the leaf 0 in the classes Θ and Υ.

- If $\Theta = \Upsilon$, so the labels are either $(0, 0)$ or $(1, 1)$: player 1 can pass, since the quadruples $(1, 1; 0, 0)$ and $(1, 1; 1, 1)$ are winning for player 1.
- $\Theta = \check{\Upsilon}$: by exchanging Θ and Υ (as $(\boldsymbol{\Upsilon}, \boldsymbol{\Theta})$ also reduces $(\boldsymbol{\Gamma}, \boldsymbol{\Gamma})$), we may assume that the label of 0 on Θ is 0. On T_1, player 1 moves to some outcome m such that $\varnothing \in \boldsymbol{\Gamma}_m$ (and sets the η-ordinal at the root of T_1 to 0); the tree T_0 remains in default position. Now player 1 passes until player 2 moves on S_0. If this never happens, then player 1 can ensure that the outcome is $(1, 0; 0, *)$, which is winning for player 1.

After player 2 moves on S_0, player 1 now moves on T_0 and matches the η-ordinal at the root; if the current outcome on S_0 is n, player 1 can choose an outcome m such that $\boldsymbol{\Theta}_n < \boldsymbol{\Gamma}_m$. Further, if $p[1]$ is a pass, then player 1 also moves on T_1 to some outcome k such that $\varnothing \in \boldsymbol{\Gamma}_k$. At the end of the play of $G_{\texttt{leaf}}(\Gamma, \Gamma; \Theta, \Upsilon)$, we obtain an outcome $(m, a; n, k)$ with $\boldsymbol{\Theta}_n < \boldsymbol{\Gamma}_m$ and $\varnothing \in \boldsymbol{\Gamma}_k$. By Lemma 7.13, $(\boldsymbol{\Theta}_n, \boldsymbol{\Upsilon}_a)$ does not reduce $(\boldsymbol{\Gamma}_m, \boldsymbol{\Gamma}_k)$, so player 1 has a winning strategy in the corresponding reduction game. $\square$

Claim 7.15.2. $(\boldsymbol{\Theta}_0, \boldsymbol{\Upsilon}_0)$ reduces $(\boldsymbol{\Gamma}_0, \boldsymbol{\Gamma}_0)$.

Proof. To devise a strategy for player 2 in $G_{\texttt{red}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$, we play an auxiliary play of $G_{\texttt{red}}(\Gamma, \Gamma; \Theta, \Upsilon)$. Call the players in the auxiliary play, players 3 and 4. Let $\mathfrak{S}$ be a winning strategy for player 4 in the auxiliary game. As above, let T_0, T_1 and S_0, S_1 denote the trees for players 3 and 4, respectively, in the game $G_{\texttt{leaf}}(\Gamma, \Gamma; \Theta, \Upsilon)$. There are three cases.

If $o(\Gamma_0) = o(\Lambda_0) = \xi$*:* the game $G_{\mathtt{red}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$ starts with $G_{\mathtt{leaf}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$, where the corresponding trees are the restrictions of the trees T_i and S_i to extensions of 0. Player 3 copies player 1's moves. We argue that player 4 also only plays extensions of 0 on both S_0 and S_1, so player 2 can copy player 4's moves. Otherwise, suppose that at some round, player 4 moves away from 0, say on S_0. Then player 3 can abandon copying player 1, rather, player 3 can behave as in the proof of the previous claim: on T_1, player 3 moves to an outcome k with $\varnothing \in \mathbf{\Gamma}_k$; on T_0, player 3 reacts to a choice $n \in S_0$ by some $m \in T_0$ with $\mathbf{\Theta}_n < \mathbf{\Gamma}_m$. This gives player 3 a winning position in $G_{\mathtt{red}}(\Gamma, \Gamma; \Theta, \Upsilon)$, defeating $\mathfrak{S}$.

Hence, $G_{\mathtt{leaf}}(\Gamma, \Gamma; \Theta, \Upsilon)$ ends with leaves all extending 0, the same leaves being therefore the outcome of $G_{\mathtt{leaf}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$; in the rest of $G_{\mathtt{red}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$, player 2 can continue following $\mathfrak{S}$.

If $o(\Gamma_0) > \xi$*:* in $G_{\mathtt{leaf}}(\Gamma, \Gamma; \Theta, \Upsilon)$, player 3 only passes. Again, player 4 cannot move away from 0 on either S_0 or S_1, or he exposes himself to defeat. Hence, the auxiliary leaf selection game ends with leaves extending 0. These leaves can be chosen by player 2 as the result of their first moves in $G_{\mathtt{red}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$. Player 2 can then follow $\mathfrak{S}$.

If $o(\Gamma_0) = \xi$ *and* $o(\Lambda_0) > \xi$*:* in $G_{\mathtt{red}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$, player 2 is instructed to wait, while player 1 chooses some leaves t_0, t_1 of S_{Γ_0}; we identify these with the leaves of S_Γ extending 0. Then, we start the auxiliary play of $G_{\mathtt{leaf}}(T_0, T_1; S_0, S_1)$. In that play, player 3 first chooses (t_0, t_1) (setting all of their η-ordinals above 0 to 0) and then passes. As in the other cases, player 4 must respond with extensions of 0. Say the outcome for player 4 of the auxiliary game is a pair of leaves (s_0, s_1) on S_Λ. In the main game $G_{\mathtt{red}}(\Gamma_0, \Gamma_0; \Theta_0, \Upsilon_0)$, since $o(\Gamma_{t_0}), o(\Gamma_{t_1}) > \xi$, player 1 is instructed to wait, and player 2 can walk up to (s_0, s_1), and then follow $\mathfrak{S}$. □

As a result:

Claim 7.15.3. $R(\mathbf{\Gamma})_0 \subseteq \mathbf{\Lambda}_0$.

Proof. Suppose that $0 \notin b(\Gamma)$. So $\varnothing \in \mathbf{\Gamma}_0$. Since Λ_0 has Σ-type, Claim 7.15.2 and Corollary 7.14 imply that $\mathbf{\Lambda}_0 \in \mathcal{C}(\mathbf{\Gamma}_0)$. Since $0 \notin b(\Gamma)$, the claim follows from the induction hypothesis $(*)_{\Gamma,0}$.

Suppose that $0 \in b(\Gamma)$. Then $R(\mathbf{\Gamma})_0 = \mathbf{\Sigma}^0_{1+\xi}$. By Claim 7.15.1, 0 is not a leaf of T_Λ, so by Lemma 3.8 (as Λ_0 has Σ-type), $\mathbf{\Sigma}^0_{1+\xi} \subseteq \mathbf{\Lambda}_0$. □

Putting it all together, we see that $\bigcup_n R(\mathbf{\Gamma})_n \subseteq \bigcup_m \mathbf{\Lambda}_m$, that $R(\mathbf{\Gamma})_0 \subseteq \mathbf{\Lambda}_0$, and that $o(R(\Gamma)) = o(\Lambda)$ and $\eta^{R(\Gamma)} = \eta^\Lambda$. By Proposition 4.9, $R(\mathbf{\Gamma}) \subseteq \mathbf{\Lambda}$, as required. □

References

[DDW] Adam R. Day, Rod G. Downey, and Linda B. Westrick. Three topological reducibilities for discontinuous functions. To appear.

[DGHTTa] Adam R. Day, Noam Greenberg, Matthew Harrison-Trainor, and Daniel Turetsky. An effective classification of Borel Wadge classes. Submitted.

[DGHTTb] Adam R. Day, Noam Greenberg, Matthew Harrison-Trainor, and Daniel Turetsky. Iterated priority arguments in descriptive set theory. Submitted.

[DM] Adam R. Day and Andrew S. Marks. The decomposability conjecture. In preparation.

[GQT] Noam Greenberg, Renrui Qi, and Daniel Turetsky. Extending Selivanov's fine hierarchy. In preparation.

[Lou83] Alain Louveau. Some results in the Wadge hierarchy of Borel sets. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 28–55. Springer, Berlin, 1983.

[LSR88a] Alain Louveau and Jean Saint Raymond. Les propriétés de réduction et de norme pour les classes de Boréliens. *Fund. Math.*, 131(3):223–243, 1988.

[LSR88b] Alain Louveau and Jean Saint Raymond. The strength of Borel Wadge determinacy. In *Cabal Seminar 81–85*, volume 1333 of *Lecture Notes in Math.*, pages 1–30. Springer, Berlin, 1988.

[VW78] Robert Van Wesep. Wadge degrees and descriptive set theory. In *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, volume 689 of *Lecture Notes in Math.*, pages 151–170. Springer, Berlin, 1978.

[Wad84] William W. Wadge. *Reducibility and determinateness on the Baire space*. PhD thesis, University of California, Berkeley, 1984.

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