A NOTE ON COMPUTABLE DISTINGUISHING COLORINGS

NIKOLAY BAZHENOV, NOAM GREENBERG, ALEXANDER MELNIKOV, RUSSELL MILLER, AND KENG MENG NG

ABSTRACT. An α-coloring ξ of a structure $\mathscr{S}$ is *distinguishing* if there are no nontrivial automorphisms of $\mathscr{S}$ respecting ξ. In this note we prove several results illustrating that computing the distinguishing number of a structure can be very hard in general. In contrast, we show that every computable Boolean algebra has a $0''$-computable distinguishing 2-coloring. We also define the notion of a computabile distinguishing 2-coloring of a separable space; we apply the new definition to separable Banach spaces.

We study distinguishing numbers of computable structures and computable separable spaces. The distinguishing number of a structure is defined as follows. An α-*coloring*, where $\alpha \leq \omega$, of a structure $\mathscr{S}$ is a function from the domain of $\mathscr{S}$ into a set of size α. An α-coloring ξ is *distinguishing* if there are no nontrivial automorphisms of $\mathscr{S}$ respecting ξ: i.e. if f is a nontrivial automorphism of $\mathscr{S}$, then there is an element $a \in \mathscr{S}$ with $\xi(f(a)) \neq \xi(a)$. The *distinguishing number* of a countable structure $\mathscr{S}$ is the least $\alpha \leq \omega$ such that $\mathscr{S}$ has a distinguishing α-coloring. The idea is that the distinguishing number of a structure gives a new measure of the complexity of the structure. Distinguishing numbers have been extensively studied in combinatorics. Albertson and Collins [1] introduced the notion for finite graphs, and [16] studied distinguishing numbers of infinite graphs. For example, the random graph has a distinguishing 2-coloring [16]. For more results in combinatorics, the reader is referred to, e.g., [17, 23, 13].

How hard is it to decide whether the distinguishing number of a given algebraic structure is equal to 2? Also, how difficult is it to compute the α-coloring of a given structure? Miller, Solomon, and Steiner [20] initiated the study of computability-theoretic aspects of distinguishing colorings. They mainly restricted themselves to trees. The main purpose of this note is to extend the approach from [20] to arbitrary algebraic structures and also define the notion of a computable coloring for separable spaces. To do that, we apply the tools of computable structure theory [12, 2] and computable analysis [24]. To keep the note as brief as possible, we shall not give any detailed explanation or motivation here. We only note that the questions raised above can be viewed as computable classification problems. See surveys [14, 21, 10] for more about applications of computability to classification problems, and see [8, 9, 5, 19] for several recent results into this direction.

In Subsection 1.1 we prove that, for any computable ordinal α, there is a computable structure $\mathscr{S}$ with distinguishing number 2, which does not have $\mathbf{0}^{(\alpha)}$-computable distinguishing 2-colorings. In Subsection 1.2, we prove that the index set of structures having

---

Date: November 28, 2020.

Bazhenov was supported by RFBR, project number 20-31-70006. Greenberg was supported by Grant 17-VUW-090 from the Marsden Fund of New Zealand. Melnikov was supported by the Marsden Fund of New Zealand and the Rutherford Discovery Fellowship. Miller was partially supported by Grant # 581896 from the Simons Foundation and by the City University of New York PSC-CUNY Research Award Program. Ng was partially supported by the MOE grant RG23/19.

distinguishing number 2 is both Σ^1_1-hard and Π^1_1-hard. Since the results are fully relativizable, the results of the first two sections give a strong evidence that the distinguishing coloring problem does not have any tractable solution for countable structures. In contrast with this "anti-structure" result (in the sense of [14]), in Subsection 1.3 we prove that every computable Boolean algebra has a $0''$-computable distinguishing 2-coloring; we leave open whether $0''$ is sharp. In Section 2 we introduce the notion of a computable distinguishing 2-coloring for a separable space. We apply the new notion in Subsection 2.2 where we prove that every Banach space that has a strongly computable Schauder basis (to be defined) has a computable distinguishing 2-coloring. In each of these subsections we also state open problems, some of which seem to be rather challenging.

1. Coloring countable algebraic structures

1.1. **The complexity of distinguishing colorings.** Recall that a computable presentation of a countably infinite algebraic structure $\mathscr{A}$ is a structure $\mathscr{B} \cong \mathscr{A}$ upon the domain of natural numbers ω such that the operations and relations on $\mathscr{B}$ are (uniformly and Turing) computable. In this section, we show that for a computable structure $\mathscr{S}$, the complexity of an optimal distinguishing coloring cannot be bounded in the hyperarithmetical hierarchy.

Given a graph G, we define a new graph $\mathrm{Double}(G)$: Every node v of G is replaced by two nodes $v[0]$ and $v[1]$. Nodes $v[i]$ and $w[j]$ are connected by an edge in $\mathrm{Double}(G)$ if and only if there is an edge from v to w inside G. It is not hard to prove:

Lemma 1.1. *Let G be a graph.*

(a) *For a node $v \in G$, the map*

$$f_v(x) = \begin{cases} v[1-i], & \text{if } x = v[i], \\ x, & \text{otherwise}, \end{cases}$$

is an automorphism of the structure $\mathrm{Double}(G)$. *Hence, if ξ is a distinguishing coloring of* $\mathrm{Double}(G)$, *then $\xi(v[i]) \neq \xi(v[1-i])$.*

(b) *Suppose that $\xi\colon \mathrm{Double}(G) \to \{0,1\}$ is a distinguishing 2-coloring of* $\mathrm{Double}(G)$. *Then for $i \in \{0,1\}$, the substructure $H_i \leq \mathrm{Double}(G)$ on the domain $\{x : \xi(x) = i\}$ is isomorphic to G.*

(c) *Suppose that $\xi\colon G \to \{1,2,\dots,n\}$ is a distinguishing n-coloring of G. Then the map*

$$\xi_{\mathrm{Double}}(v[i]) = \begin{cases} \xi(v), & \text{if } i = 0, \\ \xi(v) + n, & \text{if } i = 1, \end{cases}$$

is a distinguishing $2n$-coloring of $\mathrm{Double}(G)$.

Theorem 1.1. *For every computable ordinal α, there is a computable structure $\mathscr{S}$ with distinguishing number 2, which does not have $\mathbf{0}^{(\alpha)}$-computable distinguishing 2-colorings.*

Proof. The language of $\mathscr{S}$ contains unary predicates R_n, $n \in \omega$, and one binary predicate Q. For $e \in \omega$, the predicate R_e forms the e-th box inside $\mathscr{S}$. This box is intended to witness that the function $\varphi_e^{\mathbf{0}^{(\alpha)}}$ cannot be a distinguishing 2-coloring of $\mathscr{S}$. Beforehand, we add elements a_e, b_e, and c_e inside the e-th box.

Without loss of generality, one may assume that $\alpha > \omega$ and $\alpha = 2\beta + 1$. For convenience, we interpret 0 as "red" and 1 as "blue". By employing the technique of pairs of computable structures [3] (see, e.g., Theorem 3.1 of [4] for a similar argument), we build computable sequences $(\mathscr{A}_e)_{e\in\omega}$, $(\mathscr{B}_e)_{e\in\omega}$, and $(\mathscr{C}_e)_{e\in\omega}$ with the following properties:

(i) If some of the values $\varphi_e^{\mathbf{0}^{(\alpha)}}(a_e)$, $\varphi_e^{\mathbf{0}^{(\alpha)}}(b_e)$, or $\varphi_e^{\mathbf{0}^{(\alpha)}}(c_e)$ is undefined or some of these values does not belong to $\{0,1\}$, then $\mathscr{A}_e \cong \omega^\beta$, $\mathscr{B}_e \cong \omega^\beta \cdot 2$, and $\mathscr{C}_e \cong \omega^\beta \cdot 3$.

(ii) Assume otherwise. Then:
   (a) $\mathscr{A}_e$ is isomorphic to $\omega^\beta \cdot 4$ if $\varphi_e^{\mathbf{0}^{(\alpha)}}(a_e) = \varphi_e^{\mathbf{0}^{(\alpha)}}(b_e)$ or $\varphi_e^{\mathbf{0}^{(\alpha)}}(a_e) = \varphi_e^{\mathbf{0}^{(\alpha)}}(c_e) \neq \varphi_e^{\mathbf{0}^{(\alpha)}}(b_e)$. Otherwise, $\mathscr{A}_e \cong \omega^\beta$.
   (b) $\mathscr{B}_e$ is isomorphic to $\omega^\beta \cdot 4$ if $\varphi_e^{\mathbf{0}^{(\alpha)}}(a_e) = \varphi_e^{\mathbf{0}^{(\alpha)}}(b_e)$ or $\varphi_e^{\mathbf{0}^{(\alpha)}}(b_e) = \varphi_e^{\mathbf{0}^{(\alpha)}}(c_e) \neq \varphi_e^{\mathbf{0}^{(\alpha)}}(a_e)$. Otherwise, $\mathscr{B}_e \cong \omega^\beta \cdot 2$.
   (c) $\mathscr{C}_e$ is isomorphic to $\omega^\beta \cdot 4$ if $\varphi_e^{\mathbf{0}^{(\alpha)}}(a_e) \neq \varphi_e^{\mathbf{0}^{(\alpha)}}(b_e)$. Otherwise, $\mathscr{C}_e \cong \omega^\beta \cdot 3$.

Inside the structure $\mathscr{S}$, we put the graphs Double($\mathscr{A}_e$), Double($\mathscr{B}_e$), and Double($\mathscr{C}_e$) into the e-th box. We add an edge between a_e and every element of Double($\mathscr{A}_e$). We note that by Lemma 1.1, the structure Double($\mathscr{A}_e$) has a distinguishing 2-coloring. Treat b_e and Double($\mathscr{B}_e$), c_e and Double($\mathscr{C}_e$) in a similar way.

It is not difficult to show that the structure $\mathscr{S}$ is not rigid. Moreover, there is a distinguishing 2-coloring of $\mathscr{S}$: For every e, the e-th box contains at most two isomorphic "double graphs" — e.g., suppose that $\mathscr{A}_e \cong \mathscr{B}_e \not\cong \mathscr{C}_e$. Then the desired coloring can be defined as follows. Choose arbitrary distinguishing 2-colorings of Double($\mathscr{A}_e$), Double($\mathscr{B}_e$), and Double($\mathscr{C}_e$). After that, the node a_e is colored blue, b_e is colored red, and for c_e one can assign any color.

Aiming for a contradiction, assume that $\varphi_e^{\mathbf{0}^{(\alpha)}}$ is a distinguishing 2-coloring of $\mathscr{S}$. Then there are at least two nodes from the set $\{a_e, b_e, c_e\}$ such that $\varphi_e^{\mathbf{0}^{(\alpha)}}$ colors them in the same color; say, $\varphi_e^{\mathbf{0}^{(\alpha)}}(b_e) = \varphi_e^{\mathbf{0}^{(\alpha)}}(c_e) = 0$.

If $\varphi_e^{\mathbf{0}^{(\alpha)}}(a_e) = 0$, then by Lemma 1.1.(b), each of the graphs Double($\mathscr{A}_e$) and Double($\mathscr{B}_e$) can be decomposed in a red copy of $\omega^\beta \cdot 4$ and a blue copy of $\omega^\beta \cdot 4$. Note that both a_e and b_e are colored red. Thus, one can recover a color preserving automorphism of $\mathscr{S}$, which maps Double($\mathscr{A}_e$) onto Double($\mathscr{B}_e$).

If $\varphi_e^{\mathbf{0}^{(\alpha)}}(a_e) = 1$, then a similar argument shows that there is a color preserving automorphism mapping Double($\mathscr{B}_e$) onto Double($\mathscr{C}_e$). We obtained a contradiction, hence, no $\mathbf{0}^{(\alpha)}$-computable function can be a distinguishing 2-coloring for the structure $\mathscr{S}$. □

1.2. **Index sets.**

Theorem 1.2. *The index set of computable structures with distinguishing number* 2 *is both* Σ^1_1*-hard and* Π^1_1*-hard.*

Proof. First, we establish the Σ^1_1-hardness. In order to do this, we prove the following:

Lemma 1.2. *Let* $\mathscr{L}$ *be a computable linear order such that every block of* $\mathscr{L}$ *is infinite. Then there is a* $\mathbf{0}'$*-computable distinguishing* 2*-coloring of* $\mathscr{L}$.

Proof. We write $a \prec b$ if b is an immediate successor of a inside $\mathscr{L}$. We define our coloring as follows. At stage 0, pick the element 0 of $\mathscr{L}$, and find three elements $a \prec b \prec c$ such that $0 \in \{a,b,c\}$ (using the $\mathbf{0}'$-oracle to find successors). Both a and c are colored red, and b is colored blue.

At stage $s > 0$, use the oracle to find $s+3$ immediate successors $n_0 \prec n_1 \prec \dots \prec n_{s+1} \prec n_{s+2}$ such that $s \in \{n_0, n_1, \dots, n_{s+2}\}$. If any of these elements is already colored, then simply color all of these $s+3$ elements red, except those which are already colored. If none of these elements is already colored, then color n_0 and n_{s+2} red, and color the intervening elements $n_1, \ldots, n_{s+1}$ blue, creating a sequence of exactly $s+1$ consecutive blue elements in $\mathscr{L}$. This completes stage s.

Clearly this construction gives a $\mathbf{0}'$-computable 2-coloring of $\mathscr{L}$. Moreover, within $\mathscr{L}$, every block contains at least one sequence of finitely many blue elements with red at each end, created at the first stage at which the construction encountered an element of this block. However, no two of these sequences have the same length, since no two were created at the same stage. Therefore, no automorphism can map any block to a different block and still respect the coloring. Furthermore, a similar argument shows that if an automorphism respects the coloring, then every ζ-block must stay fixed. Therefore, the only automorphism respecting the coloring is the identity. □

Let S be a Π^1_1 set. There exists a computable total function f such that, for all n, $f(n)$ is the index of a computable linear order $\mathscr{L}_n$ with

$$\mathscr{L}_n \cong \begin{cases} \text{some } \alpha < \omega_1^{CK}, & \text{if } n \in S, \\ \omega_1^{CK} \cdot (1+\eta), & \text{if } n \not\in S. \end{cases} \tag{1}$$

Any ordinal α has distinguishing number 1. By Lemma 1.2, the Harrison order $\omega_1^{CK} \cdot (1+\eta)$ has distinguishing number 2. Therefore, our index set is Σ^1_1-hard.

Now we show that the index set is Π^1_1-hard. For the sequence $(\mathscr{L}_n)_{n\in\omega}$ from (1), define $\mathscr{M}_n = \mathrm{Double}(\mathscr{L}_n)$. If $n \in S$, then $\mathscr{L}_n$ is isomorphic to an ordinal. By Lemma 1.1, the structure $\mathscr{M}_n$ has distinguishing number 2.

Suppose that $n \notin S$. Let $\mathscr{M} = \mathscr{M}_n$, and let $\xi \colon \mathscr{M} \to \{0,1\}$ be a distinguishing 2-coloring. Consider the substructure $\mathscr{A}$ of $\mathscr{M}$ on the domain $\{a : \xi(a) = 0\}$. By Lemma 1.1.(b), the graph $\mathscr{A}$ is isomorphic to the Harrison order. Fix a nontrivial automorphism g of $\mathscr{A}$. Define $D(v[i]) := v[1-i]$. Then the map

$$g^*(x) = \begin{cases} g(x), & \text{if } x \in \mathscr{A}, \\ D(g(D(x))), & \text{if } x \notin \mathscr{A}, \end{cases}$$

is a nontrivial automorphism of $\mathscr{M}$, which respects ξ. Therefore, $\mathscr{M}$ has no distinguishing 2-colorings, and our index set is Π^1_1-hard. □

Problem. What is the exact (optimal) complexity of the 2-coloring problem in the analytic hierachy?

1.3. **Boolean algebras.** In contrast to Theorem 1.1, computable structures from some familiar classes admit distinguishing colorings of fairly low arithmetical complexity. We illustrate this by the following:

Theorem 1.3. *Any computable Boolean algebra $\mathscr{B}$ has a $\mathbf{0}''$-computable distinguishing 2-coloring.*

Proof. Without loss of generality, we assume that $\mathscr{B}$ is infinite. We build a generating tree T for $\mathscr{B}$ such that T is $\mathbf{0}''$-computable (a detailed exposition of the generating trees technique can be found in [15]). The desired coloring ξ is defined as follows:

(a) If a node from T has precisely two children a, b, and both a and b are atoms of $\mathscr{B}$, then color a red and b blue.
(b) All other nodes from T are colored red. All elements from $\mathscr{B} \setminus T$ are colored blue.

Let f be an automorphism of $\mathscr{B}$, which respects our coloring. If a is a generator (i.e., an element of T), then $f(a)$ is also a generator. The construction of T will ensure that $f(a)$ equals a. Since every element of $\mathscr{B}$ is a finite sum of generators, we deduce that f is the identity map.

We build T as a subtree of $\omega^{<\omega}$. For $a \in \mathscr{B}$, let $\mathrm{cr}(a) = \mathrm{card}(\{x : x \leq_{\mathscr{B}} a\})$.

If a is a sum of $n \geq 2$ atoms of $\mathscr{B}$, then its *fishbone* is the following tuple of elements. For $n = 2$, put $\mathrm{fb}(a) := (b, c)$, where $b <_{\mathbb{N}} c$ are the atoms below a. For $n \geq 3$, define $\mathrm{fb}(a) := (b, a \setminus b, \mathrm{fb}(a \setminus b))$, where b is the $\leq_{\mathbb{N}}$-least atom below a. Every fishbone is associated with a finite tree in a natural way: For $n = 2$, its root a has two children b and c. For $n \geq 3$, the root a has two children b and $a \setminus b$; and $a \setminus b$ serves as the root of an adjoined tree corresponding to $\mathrm{fb}(a \setminus b)$. It is clear that there is no nontrivial automorphism of a fishbone tree, which respects the coloring described above.

At stage 0, we put the element $1^{\mathscr{B}}$ as the root of T. Choose an element $a \notin \{0^{\mathscr{B}}, 1^{\mathscr{B}}\}$. One may assume that $2^2 \leq \mathrm{cr}(a) \leq \mathrm{cr}(\overline{a})$. Here $\overline{a}$ denotes the complement of a. We add a and $\overline{a}$ as the children of $1^{\mathscr{B}}$.

For $k \geq 2$, define $r(k) = 2 + 3 + \cdots + (k+1)$. Using the $\mathbf{0}''$-oracle, check whether a is a sum of finitely many atoms. If a is a sum of n atoms, then adjoin the fishbone tree corresponding to $\mathrm{fb}(a)$ under $1^{\mathscr{B}}$. Split $\overline{a}$ into three parts b_0, b_1, b_2 such that for each i, we have $\mathrm{cr}(b_i) \geq 2^{r(4)}$. Put all b_i as children of $\overline{a}$.

If $\mathrm{cr}(a) = \omega$, then split a into 2 parts (they will be children of a) and $\overline{a}$ into 3 parts (children of $\overline{a}$). We require that for each of these parts u, we have $\mathrm{cr}(u) \geq 2^{r(4)}$. In both cases, it is clear that the elements a and $\overline{a}$ are not automorphic as elements of the tree.

At stage $s > 0$, we look at each element a of the tree such that a does not have children and a is not an atom. Note that each of the siblings of a also has these properties. Let v be the parent of a, and let $b_0, b_1, \dots, b_k$ be all children of v. The previous stage ensured that for each i, $\mathrm{cr}(b_i) \geq 2^{r(k+2)}$, and at least one of b_i satisfies $\mathrm{cr}(b_i) = \omega$.

If b_i is a sum of m atoms, then we split b_i into $(i+2)$ parts $c_0, \dots, c_{i+1}$ (the children of b_i) such that $\mathrm{cr}(c_j) = 2^{j+2}$ for $j \leq i$, and add the fishbone trees corresponding to all c_j under b_i. If $\mathrm{cr}(b_i) = \omega$, then we split b_i into $(k+3)$ parts $d_0, d_1, \dots, d_{k+2}$ such that $\mathrm{cr}(d_j) \geq 2^{r(k+4)}$ for each j. This kind of procedure ensures that the elements b_i, $i \leq k$, of the tree are pairwise not automorphic. Moreover, all c_j are pairwise not automorphic.

Consider the coloring ξ of $\mathscr{B}$ discussed in the beginning. Let f be an automorphism of $\mathscr{B}$ respecting ξ. Then $f \upharpoonright T$ is an automorphism of T. A nontrivial automorphism of T can only switch a pair of atoms a and b described in the condition (a) above. This implies that the coloring ξ is distinguishing. □

We do not know if the estimate $0''$ is sharp. Note that if $\mathscr{B}$ is an atomless Boolean algebra, then the proof of Theorem 1.3 produces a computable 2-coloring of $\mathscr{B}$.

Problem. Does every computable Boolean algebra possess a computable distinguishing coloring?

2. Coloring separable spaces

2.1. **The definition of a computable 2-coloring for a separable space.** Let $\mathscr{M} = (M, d)$ be a metric space and κ be a cardinal. We follow [6] and define the *distinguishing number* of $\mathscr{M}$ as the least cardinal κ such that $\mathscr{M}$ has a distinguishing κ-coloring, up to surjective isometry. For simplicity, in this section we focus on the case when $\kappa = 2$.

Recall that a *computable presentation* of a Polish metric space $\mathscr{M}$ is a countable metric space $X = (\omega, d)$ such that $d(i, j)$ is a real uniformly computable in i, j, and the completion $\overline{X} \cong \mathscr{M}$, where $\cong$ stands for isometric isomorphism[1]. A Cauchy name of a point $a \in \mathscr{M}$ is a sequence $(i_n)_{n\in\omega}$ in X such that $d(a, i_n) < 2^{-n}$. A point is computable if it has

[1]For a finite metric space we also allow the domain to be an initial segment of ω. We allow $d(i, j) = 0$ in $X = (\omega, d)$; a standard trick can be used to remove repetitions.

a computable Cauchy name. Let X, Y be computable presentations of Polish metric spaces. A map $F\colon \overline{X} \to \overline{Y}$ is *computable* if there is a Turing functional Φ such that, for each x in the domain of F and for every Cauchy name χ for x, $(\Phi^{\chi}(n))_{n\in\omega}$ is a Cauchy name for $F(x)$. Note that we do not require the Cauchy names to be necessarily computable. It is well-known that computability of F implies that F is continuous; e.g., [24].

The first, naive attempt to define computable 2-coloring says that the coloring function $\xi : \mathcal{M} \to \{0,1\}$ is computable. In particular, if $\xi : \mathcal{M} \to \{0,1\}$ is computable then then $\xi^{-1}(0) \sqcup \xi^{-1}(1)$ must be a partition of $\mathcal{M}$ into its clopen components. For a connected $\mathcal{M}$, this is vacuously impossible unless one of the $\xi^{-1}(i)$ is empty. Even for spaces which are not connected, the condition "$\xi^{-1}(i)$ is clopen" seems too strong.

We therefore abandon this idea. Instead, we put computability-theoretic conditions on the sets $\xi^{-1}(i)$, as follows. Let X be a computable presentation of $\mathcal{M}$. Recall that an open subset L of $\mathcal{M}$ is Σ^0_1 or c.e. (with respect to X) if there is a a computably enumerable set W such that $L = \bigcup_{\langle i,r\rangle\in W} B_r(i)$, where $B_r(i) = \{x : d(i,x) < r\}$ is the basic open ball centered in i and with radius $r \in \mathbb{Q}$. A closed set C is Π^0_1 if $\mathcal{M} \setminus C$ is Σ^0_1. A closed set C is *computable* if it is Π^0_1 and additionally it contains a sequence $(x_i)_{i\in\omega}$ of uniformly computable points such that its completion is equal to C.

We return to colorings of a metric space $\mathcal{M}$. Most common separable metric and normed spaces have a natural computable presentation. We fix such a computable structure X on $\mathcal{M}$.

Definition 1. A distinguishing 2-coloring $\xi : \mathcal{M} \to \{0,1\}$ is *computable* if $\xi^{-1}(0)$ is a computable closed subset of $\mathcal{M}$.

2.2. **Coloring separable Banach spaces.** We test the new notion to separable Banach spaces. For the theory of computable Banach spaces, see [22]. For a Banach space, on top of computability of the metric induced by the norm, we also require that the standard operations of addition and scalar multiplication are computable with respect to this metric induced by the norm. This extra assumption about computability of the operations cannot be dropped even for the space $C[0,1]$; see [18]. However, every computable Banach space is still a computable Polish space, thus Definition 1 does not have to be adjusted.

Let B be a Banach space. A sequence $(e_n)_{n\in\omega}$ of elements of B is a *Schauder basis* of B if for any element $v \in B$, there is a unique sequence of scalars $(\alpha_n)_{n\in\omega}$ such that $v = \sum_{n=0}^{\infty} \alpha_n e_n$. In a computable Banach space B, we say that a Schauder basis $(e_n)_{n\in\omega}$ of B is *strongly computable* if the sequence $(e_n)_{n\in\omega}$ is uniformly computable and furthermore there is a computable procedure $B \to R^{\omega}$ which, on input $x \in B$, outputs a sequence of real numbers (α_i) such that $x = \sum_i \alpha_i x_i$. (Here R^{ω} is the standard computable presentation of the ω -dimensional Hilbert space.)

Theorem 2.1. *Let B be a real computable Banach space with a strongly computable Schauder basis. Then B has a computable distinguishing 2-coloring (as a metric space).*

Proof. Let $(e_i)_{i\in\omega}$ be a strongly computable Schauder basis of B. We color the following set of elements from B red:

$$U = \{0\} \cup \{e_i, (i+2)e_i : i \in \omega\};$$

and all other elements are colored blue.

We claim that the set U is closed. Indeed, suppose that a sequence $(v_n)_{n\in\omega}$ of distinct elements from U converges to some $w \in B$. For $m \in \omega$, consider the projection operator $P_m\colon \sum_{k=0}^{\infty} \alpha_k e_k \mapsto \sum_{k=0}^{m} \alpha_k e_k$. For almost all v_n, we have $P_m(v_n) = 0$. Since the operator

P_m is continuous, we obtain $P_m(w) = 0$. Therefore, we deduce that $w = 0 \in U$. Furthermore, we claim that U is a computable closed set. To see why, fix the computable map $s : B \to R^\omega$ witnessing strong computability of the Schauder basis $(e_n)_{n\in\omega}$. Then $B \setminus U$ is equal to the pre-image under s of the effectively open set

$$\Big\{\bar{v} = (v_i) \in R^\omega : \big(\exists i \neq j\; v_i \neq 0 \,\&\, v_j \neq 0\big) \vee \big(\exists i\; v_i \notin \{1, i+2\}\big)\Big\}.$$

It follows that U is Π^0_1. Since the sequence $(e_n)_{n\in\omega}$ is uniformly computable by assumption and the operations on B are computable, we conclude that the set $U = \{0\} \cup \{e_i, (i+2)e_i : i \in \omega\}$ is a computable closed set.

Now we show that our 2-coloring is distinguishing. Suppose that F is a surjective isometry of B, which respects the coloring. Since B is a real normed space, the Mazur–Ulam Theorem implies that there is a linear map $L\colon B \to B$ such that $F(x) = L(x) + F(0)$ for all x.

Since F respects the coloring, we have $F(0) \in U$. Assume that $F(0) = me_i$ for some $i \in \omega$ and $m > 0$. For every $j \in \omega$, we have

$$F(e_j) = L(e_j) + F(0) = L(e_j) + me_i \in U;$$

hence, $L(e_j) = q_j e_{r_j} - me_i$ for some $q_j, r_j \in \omega$. Since F is a bijection, one can choose j such that $r_j \neq i$. On the other hand, we obtain

$$F((j+2)e_j) = (j+2)L(e_j) + me_i = (j+2)q_j e_{r_j} - m(j+1)e_i \notin U,$$

which contradicts the coloring preservation. Therefore, we deduce that $F(0) = 0$, and F itself is a linear map.

By employing the linearity of F, it is not hard to show that $F(e_i) = e_i$ for all $i \in \omega$. Let x be an arbitrary element from B. Consider its decomposition $x = \sum_{n=0}^{\infty} \alpha_n e_n$. Then we have

$$d\Big(x, \sum_{n=0}^{k} \alpha_n e_n\Big) = d\Big(F(x), F\Big(\sum_{n=0}^{k} \alpha_n e_n\Big)\Big) = d\Big(F(x), \sum_{n=0}^{k} \alpha_n e_n\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

Therefore, $F(x) = x$, and F is the identity map. □

Many common spaces such as $C[0,1]$ and ℓ_n satisfy Theorem 2.1. Nonetheless, not every Banach space possesses a Schauder basis; see, e.g., [11]. In fact, the existence of such a space had been an open problem for quite some time. On the other hand, we conjecture that there is a computable Banach space that admits a Schauder basis but has no computable presentation with a strongly computable Schauder basis. Note that Bosserhoff [7] constructed a computable Banach space with a Schauder basis, which does not have a computable Schauder basis.

Problem. Is there a Banach space with no distinguishing 2-coloring or at least with no computable 2-coloring.

More generally, the problem below is wide open.

Problem. Investigate the computability-theoretic content of the coloring problem for Polish metric and separable Banach spaces.

Finally, we wonder whether Definition 1 can be naturally extended to the case of more than 2 colors.

REFERENCES

[1] M. O. Albertson and K. L. Collins. Symmetry breaking in graphs. *Electron. J. Comb.*, 3(1):R18, 1996.

[2] C. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[3] C. J. Ash and J. F. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. Elsevier Science B.V., Amsterdam, 2000.

[4] N. Bazhenov. Autostability spectra for decidable structures. *Math. Struct. Comput. Sci.*, 28(3):392–411, 2018.

[5] Nikolay Bazhenov, Matthew Harrison-Trainor, Iskander Sh. Kalimullin, Alexander G. Melnikov, and Keng Meng Ng. Automatic and polynomial-time algebraic structures. *J. Symb. Log.*, 84(4):1630–1669, 2019.

[6] A. Bonato, C. Laflamme, M. Pawliuk, and N. Sauer. Distinguishing number of Urysohn metric spaces. Preprint. arXiv:1811.06023.

[7] V. Bosserhoff. On the effective existence of Schauder bases. *J. UCS*, 15(6):1145–1161, 2009.

[8] R. Downey and A. Montalbán. The isomorphism problem for torsion-free abelian groups is analytic complete. *J. Algebra*, 320(6):2291–2300, 2008.

[9] Rod Downey, Alexander G. Melnikov, and Keng Meng Ng. Iterated effective embeddings of abelian p-groups. *Internat. J. Algebra Comput.*, 24(7):1055–1084, 2014.

[10] Rodney G. Downey and Alexander G. Melnikov. Computable analysis and classification problems. In Marcella Anselmo, Gianluca Della Vedova, Florin Manea, and Arno Pauly, editors, *Beyond the Horizon of Computability - 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 - July 3, 2020, Proceedings*, volume 12098 of *Lecture Notes in Computer Science*, pages 100–111. Springer, 2020.

[11] Per Enflo. A counterexample to the approximation problem in Banach spaces. *Acta Math.*, 130:309–317, 1973.

[12] Y. Ershov and S. Goncharov. *Constructive models*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 2000.

[13] E. Estaji, W. Imrich, R. Kalinowski, M. Pilśnyak, and T. Tucker. Distinguishing Cartesian products of countable graphs. *Discuss. Math., Graph Theory*, 37(1):155–164, 2017.

[14] S. Goncharov and J. Knight. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.

[15] S. S. Goncharov. *Countable Boolean Algebras and Decidability*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1997.

[16] W. Imrich, S. Klavžar, and V. Trofimov. Distinguishing infinite graphs. *Electron. J. Comb.*, 14(1):R36, 2007.

[17] C. Laflamme, L. Nguyen Van Thé, and N. Sauer. Distinguishing number of countable homogeneous relational structures. *Electron. J. Comb.*, 17(1):R20, 2010.

[18] Alexander G. Melnikov and Keng Meng Ng. Computable structures and operations on the space of continuous functions. *Fund. Math.*, 233(2):101–141, 2016.

[19] Alexander G. Melnikov and Keng Meng Ng. Computable torsion abelian groups. *Adv. Math.*, 325:864–907, 2018.

[20] R. Miller, R. Solomon, and R. M. Steiner. On the effectiveness of symmetry breaking. In A. Beckmann, E. Csuhaj-Varjú, and K. Meer, editors, *Language, Life, Limits*, volume 8493 of *Lect. Notes Comput. Sci.*, pages 314–323. Springer, Cham, 2014.

[21] Antonio Montalbán. Computability theoretic classifications for classes of structures. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 79–101. Kyung Moon Sa, Seoul, 2014.

[22] Marian B. Pour-El and J. Ian Richards. *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989.

[23] S. M. Smith, T. Tucker, and M. E. Watkins. Distinguishability of infinite groups and graphs. *Electron. J. Comb.*, 19(2):P27, 2012.

[24] Klaus Weihrauch. *Computable analysis*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2000. An introduction.

SOBOLEV INSTITUTE OF MATHEMATICS, 4 AKAD. KOPTYUG AVE., NOVOSIBIRSK, 630090, RUSSIA

KAZAKH-BRITISH TECHNICAL UNIVERSITY, 59 TOLE BI ST., ALMATY, 050000, KAZAKHSTAN
E-mail address: bazhenov@math.nsc.ru

SCHOOL OF MATHEMATICS AND STATISTICS, VICTORIA UNIVERSITY OF WELLINGTON, WELLINGTON, NEW ZEALAND
E-mail address: greenberg@msor.vuw.ac.nz

MASSEY UNIVERSITY AUCKLAND, PRIVATE BAG 102904, NORTH SHORE, AUCKLAND 0745, NEW ZEALAND
E-mail address: alexander.g.melnikov@gmail.com

DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE — C.U.N.Y., 65-30 KISSENA BLVD., QUEENS, NEW YORK 11367, U.S.A.

PHD PROGRAMS IN MATHEMATICS & COMPUTER SCIENCE, C.U.N.Y. GRADUATE CENTER, 365 FIFTH AVENUE, NEW YORK, NEW YORK 10016, U.S.A
E-mail address: Russell.Miller@qc.cuny.edu

DIVISION OF MATHEMATICAL SCIENCES, SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES, NANYANG TECHNOLOGICAL UNIVERSITY, SINGAPORE 637371, SINGAPORE
E-mail address: kmng@ntu.edu.sg