REALIZING COMPUTABLY ENUMERABLE DEGREES IN SEPARATING CLASSES

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Abstract. We investigate what collections of c.e. Turing degrees can be realised as the collection of elements of a separating Π^0_1 class of c.e. degree. We show that for every c.e. degree $\mathbf{c}$, the collection $\{\mathbf{c}, \mathbf{0}'\}$ can be thus realized. We also rule out several attempts at constructing separating classes realizing a unique c.e. degree. For example, we show that there is no *super-maximal* pair: disjoint c.e. sets A and B whose separating class is infinite, but every separator of c.e. degree is a finite variant of either A or $\overline{B}$.

1. Introduction

This paper is concerned with (computably bounded) Π^0_1 classes. Of course we can consider these (up to Turing degree) as being a collection of infinite paths through a computable binary tree. They have deep connections with computability theory in general, as well as reverse mathematics, algorthmic randomness and many other areas. See, for example, Cenzer and Jockusch [1].

The meta-question we attack concerns realizing c.e. degrees as members of Π^0_1 classes. In this regard, we follow some earlier studies of Csima, Downey and Ng [2] and Downey and Melnikov [8], but in this case we will be looking at *separating classes*.

Recall that one of the fundamental theorems in this area is the Computably Enumerable Basis Theorem which says that each Π^0_1 class has a member of computably enumerable degree. Indeed, if α is the left- or right-most path of a Π^0_1 class P, then there is a c.e. set W_e such that $W_e \equiv_T \alpha$.

Definition 1 ([2])**.** We will say that a c.e. degree $\mathbf{w}$ is *realized* in a Π^0_1 class P iff there exists some $\beta \in P$ with $\deg_T(\beta) = \mathbf{w}$.

The fundamental question attacked in [2] was "What sets of c.e. degrees can be realized in a Π^0_1 class?" In [2], Csima, Downey and Ng give a surprising characterization of the sets of c.e. degrees that can be realized. They showed that the question is related to one of representing index sets.

For a set $\mathcal{W}$ of c.e. degrees, let

$$I(\mathcal{W}) = \{e : W_e \in \mathcal{W}\}$$

be the set of indices of c.e. sets whose degrees are in $\mathcal{W}$. Letting $\mathcal{R}$ be the collection of all c.e. degrees, for a Π^0_1 class P, let

$$\mathcal{W}[P] = \{\mathbf{w} \in \mathcal{R} : \mathbf{w} \text{ is realized in } P\}.$$

The second and third authors are partially supported by Marsden Fund of New Zealand. The first author is partially supported by NSF-DMS-1854136, a FRG grant from the NSF. We dedicate this paper to Hugh Woodin and Ted Slaman as part of their combined birthday celebrations.

A calculation shows:

Proposition 2 ([2]). *For any Π^0_1 class P, the index set $I(\mathcal{W}[P])$ is Σ^0_4.*

The first natural question is whether Proposition 2 reverses, that is, if every Σ^0_4 index set is realized as the collection of indices of c.e. sets whose degrees are realized in some Π^0_1 class. The answer is negative (see a discussion in [2]). To describe which sets can be realized, we observe that collections of degrees whose index sets are complicated can nonetheless have simple *representations*, in the following sense. For a set $S \subseteq \mathbb{N}$, let

$$\mathcal{D}(S) = \{\deg_T(W_e) \,:\, e \in S\}$$

be the collection of degrees of c.e. sets whose indices are in S. Since Turing equivalence is Σ^0_3, if S is Σ^0_4, then so is its closure under Turing equivalence, namely, $I(\mathcal{D}(S))$ (denoted by $G(S)$ in [2]), is also Σ^0_4. But it is possible for S to be very simple but $I(\mathcal{D}(S))$ to be complicated; for example, $I(\{\mathbf{0}'\})$ is Σ^0_4-complete, but of course $\{\mathbf{0}'\} = \mathcal{D}(S)$ where S is a singleton. For a complexity class Γ, we say that a set $\mathcal{W}$ of c.e. degrees is Γ-*representable* if $\mathcal{W} = \mathcal{D}(S)$ for some set S in the class Γ. This notion gives the following characterization:

Theorem 3 (Csima, Downey and Ng [2]). *The following are equivalent for a set $\mathcal{W}$ of c.e. degrees:*

(i) *$\mathcal{W}$ is Σ^0_3-representable;*
(ii) *$\mathcal{W}$ is computably representable;*
(iii) *$\mathcal{W} = \mathcal{W}[P]$ for some Π^0_1 class P.*

They also show that the class P in (iii) can be taken to be perfect. As mentioned, not every set of c.e. degrees whose index-set is Σ^0_4 is thus realized; in [2], the authors give an example of a Π^0_3-represented set of c.e. degrees which contains $\mathbf{0}$ but is not realized as the collection of c.e. degrees of elements of some Π^0_1 class.

Theorem 3 implies that, for example, the superlow c.e. degrees, the K-trivial c.e. degrees, and all upper cones are realizable. Lower cones were classified by Downey and Melnikov.

Theorem 4 (Downey and Melnikov [8]). *The lower cone $[\mathbf{0}, \mathbf{c}]$ is realizable iff $\mathbf{c}$ is* low_2 *or $\mathbf{0}'$.*

In the present paper we will consider similar questions for the what are arguably the most important Π^0_1 classes in terms of applications in Reverse Mathematics; the *separating classes*. Recall that P is a separating class if there exist c.e. disjoint sets A, B such that $P = S(A, B) = \{Z \mid Z \supseteq A \wedge Z \cap B = \emptyset\}$. For example, if A represents the sentences provable in PA and B the sentences refutable, then $S(A, B)$ represents the complete extensions of PA. Also, as is well known, if we desire to show that a theorem of second order arithmetic is as strong as WKL_0, then it suffices to code in separating classes, in spite of the fact that Weak König's Lemma says that infinite binary trees have paths. That is, we don't have to look at all infinite binary trees, only separating classes. For example, it is known that the theorem "Every countable commutative ring with identity has a prime ideal" is equivalent to WKL_0, and this is proven by Friedman, Simpson and Smith [10] by coding a given separating class as the prime ideal structure of a commutative ring with identity, but their earlier claim ([9]) that, given a Π^0_1 class P there is a (computable) commutative ring with identity whose prime ideals are in 1-1 correspondence with the members of the class remains open. We refer the reader to e.g. Downey and Hirschfeldt [6] or Simpson [15]. The questions of what $\mathcal{W}[S(A,B)]$ can be for various A, B was asked in both [2] and [8].

1.1. **Our Results.** Whilst an exact characterization is elusive, we do prove some results to show how different the situation is for separating classes.

We do know that strange "separation sprectra" can occur. For example, Jockusch and Soare [11] showed that there two pairs of pairwise disjoint c.e. sets A_1, B_1, A_2, B_2 such that every (not necessarily c.e.) separator of A_1 and B_1 is Turing incomparible with every separator of A_2 and B_2. This result was extended by Downey, Jockusch and Stob [7] who showed that four such sets can be below a c.e. degree $\mathbf{a}$ iff $\mathbf{a}$ is array noncomputable. Moreover, Jockusch and Soare [12] extended their earlier result to construct the sets so that every separator of A_1, B_1 forms a minimal pair with each separator of A_2, B_2. This result was shown to be realizable below each *promptly* array noncomputable c.e. degree by Downey and Greenberg [5].

The basic result used in the arguments in [2] is that any c.e. singleton can be realized as a spectrum. Then, the authors use some computability theory approximation arguments for Σ^0_3-representable sets to get one direction of the characterization.

We do know of two singletons which can be realized. One is the trivial one $\mathbf{0}$ if we had a computable A and considered $S(A,\overline{A})$, but in the nontrivial situation where $\mathbb{N} - (A \sqcup B)$ is infinite, the only singleton we know of is the PA-one, namely $\mathbf{0}'$. The following question is open:

Question 5. *Is any other singleton* $\mathbf{a}$ *realizable as* $\mathcal{W}(S(A,B))$*?*

We conjecture that the answer is "no".

We can show that the answer is no in the case that $A \equiv_{wtt} B$, where $\leq_{wtt}$ denotes weak truth table reducibility. In fact, in §2, we prove the following "upward closure" result.

Theorem 6. *Suppose* $A \equiv_{wtt} B$ *are c.e. sets such that:*

- $A \cap B = \emptyset$*; and*
- $|\omega \setminus (A \cup B)| = \infty$.

Then for every $C \geq_T A$*, there is a separator of* A *and* B *of the same Turing degree as* C*.*

On the other hand, we really do need "$A \equiv_{wtt} B$" in the hypothesis, to force upward closure, as $A \leq_{wtt} B$ is not enough, as we see in the next result.

Theorem 7. *There are c.e. sets* $A \geq_{wtt} B$ *such that:*

- $A \cap B = \emptyset$*;*
- $|\omega \setminus (A \cup B)| = \infty$*; and*
- *No separator of* A *and* B *computes* $\emptyset'$.

One of the most natural ways to possibly get a singleton would be to have $A \equiv_T B$ and such that $S(A,B)$ was highly constrained in that members of c.e. degree X would be "close" to A and B. One place where this idea was used is where "c.e. degree" was replaced by "c.e." in, for example, Downey [3]. There (A,B) is called a *maximal pair* if whethever X is a c.e. set separating A and B then either $X - A$ or $X - B$ is finite (also see Muchnik [14]). Any simple c.e. set can be split into a maximal pair. They are quite useful in reverse mathematics as, for example, in [4]. Thus it would be very nice if there was a a stronger version of maximal pair with c.e. *set* replaced by set of c.e. *degree.*

Definition 8. Two c.e. sets A and B form a *super-maximal pair* if the following hold:

- $A \cap B = \emptyset$;
- $|\omega - (A \cup B)| = \infty$; and
- If X is of c.e. degree with $A \subseteq X$ and $B \subseteq \overline{X}$, then $X =^* A$ or $\overline{X} =^* B$.

Alas, no such pairs exist.

Theorem 9. *Super maximal pairs do not exist.*

We believe that this result is of independent interest aside from our interest in degrees of members of separating classes. The proof is surprisingly difficult and requires *three* levels of nonuniformity, in the same way that the Lachlan Nondiamond Theorem (Lachlan [13]) needs one level of nonuniformity. The only other example where exactly 3 levels are needed occurs in an unpublished manuscript of Slaman where he shows that there is a c.e. degree $\mathbf{a} \neq \mathbf{0}$ which is not the top of a diamond lattice in the Turing degrees. Thus this proof is of some technical interest.

Giving up on *one* degree, we ask whether *two* degrees are possible. This time the answer is yes, provided that one is $\mathbf{0}'$. One easy way to see this is to take A a complete c.e. set with $\overline{A}$ introreducible. Then $S(A, \emptyset)$ has spectrum $\mathbf{0}', \mathbf{0}$. The next result shows that $\mathbf{0}$ can be replaced by any c.e. degree.

Theorem 10. *For every c.e. degree* $\mathbf{c}$*, the separating spectrum* $\{\mathbf{c}, \mathbf{0}'\}$ *is possible.*

We remark that we are unaware of any other definite spectrum which can be realized, even for the two degree case.

2. Wtt-results

In this section we prove the comparibility results about weak truth table reducibility for A and B. The first shows that we can have no upward closure whilst having comparibility.

Theorem 11. *There are c.e. sets* $A \geq_{wtt} B$ *such that:*

- $A \cap B = \emptyset$*;*
- $|\omega \setminus (A \cup B)| = \infty$*; and*
- *No separator of* A *and* B *computes* $\emptyset'$*.*

Proof. We construct such sets. To achieve $A \geq_{wtt} B$, we promise to never enumerate an element into B unless we simultaneously enumerate a smaller element into A.

We build an auxiliary c.e. set D and meet the following requirements:

N_k: $(\exists x > k)[x \notin (A \cup B)]$.

R_e: For any separator $Z \in S(A, B)$, $\Phi_e^Z \neq D$.

Clearly this will suffice.

Strategy for N_k*:* Wait for a stage $s > k$. By construction, $s \notin (A_s \cup B_s)$. Restrain $(A \cup B)\restriction_{s+1}$.

Strategy for R_e*:* We fix a restraint r to be the stage at which the strategy was last initialized, such that the strategy will not be permitted to enumerate elements below r into $A \cup B$. The strategy repeats the following loop:

(1) Claim a large n not yet claimed by any strategy.
(2) At stage s, search for a $\sigma \in 2^s$ such that:
   - $\Phi_e^\sigma \upharpoonright_{n+1} [s] = D_s \upharpoonright_{n+1}$; and
   - For all $x < |\sigma|$, $x \in A_s \to \sigma(x) = 1$, and $x \in B_s \to \sigma(x) = 0$.
(3) Having found such a σ, if there is an $m < |\sigma|$ such that $m \geq r$, $\sigma(m) = 1$ and $m \notin A_s$, fix the least such. For all $x \in [m, |\sigma|)$, if $\sigma(x) = 1$, then enumerate x into A_{s+1}, and if $\sigma(x) = 0$, then enumerate x into B_{s+1}.
(4) Regardless of whether a desired m exists, enumerate n into D and return to Step (1).

Construction: Arrange the strategies into a priority ordering. At stage s, run the first s strategies, in order of priority. Whenever a strategy acts, initialize all lower priority strategies.

Verification: By construction, we never enumerate a number into B unless we simultaneously enumerate a smaller number into A, and so $B \leq_{wtt} A$. Also, we never enumerate a number into $(A \cup B)[s+1]$ unless that number is smaller than s.

Claim 11.1. *Suppose the R_e-strategy is only initialized finitely many times. Then it only enumerates finitely many numbers into D.*

Proof. Let r be the final restraint imposed on the strategy. Towards a contradiction, suppose there are infinitely many stages at which the strategy enumerates an element into D, and list those which occur after the final time the strategy was initialized as $s_0 < s_1 < \dots$. Fix n_i the element enumerated at stage s_i, σ_i the witnessing σ, and let m_i be the selected m, if it exists, and otherwise let $m_i = s_i$. By well-ordering properties, there are infinitely many i such that for all $j > i$, $m_i \leq m_j$. Fix such an i.

Fix a $j > i$. By construction, we have $n_i < n_j$. So it cannot be that σ_j extends σ_i, as that would give $\Phi_e^{\sigma_j}(n_i) = \Phi_e^{\sigma_i}(n_i) = 0$, but $n_i \in D_{s_i+1} \subseteq D_{s_j}$, contrary to our choice of σ_j. So there must be some $y < |\sigma_i|$ with $\sigma_i(y) \neq \sigma_j(y)$. By construction, $y < m_i$.

If $y \geq r$, we claim that $\sigma_i(y) = 0$. For if not, then $\sigma_{i+1}(y) = 0$ implies $y \notin A_{s_{i+1}} \supseteq A_{s_i}$, and so y contradicts our choice of m_i.

So if $y \geq r$, $y \notin A_{s_i}$. Higher priority strategies will never act after stage s_i, and lower priority strategies were initialized and so have restraints greater than y, so neither can enumerate y into A. By choice of i, $y < m_i \leq m_k$ for all $k \in [i, j]$, and so the action of this strategy cannot enumerate y into A before stage s_j. So $y \notin A_{s_j}$, and so if $y \geq r$, then y is a viable candidate for m_j. This contradicts $m_i < m_j$. It follows that $y < r$, and so $\sigma_i \upharpoonright_r \neq \sigma_j \upharpoonright_r$, for all $j > i$.

But there are only finitely many strings in 2^r, and so there can be only finitely many i such that for all $j > i$, $m_i \leq m_j$, contrary to our earlier observation. The claim follows. □

It is now a simple induction to show that each strategy is only initialized finitely many times. Clearly each N_k-strategy ensures its requirement. Also each R_e-strategy must eventually wait forever at Step (2), and so there can be no separator computing D. This completes the proof. □

The second result shows that if there is a realizable singleton which is not $\mathbf{0}$ or $\mathbf{0}'$, then we cannot use non-adaptive reductions.

Theorem 12. *Suppose $A \equiv_{wtt} B$ are c.e. sets such that:*

- $A \cap B = \emptyset$; *and*
- $|\omega \setminus (A \cup B)| = \infty$.

Then for every $C \geq_T A$*, there is a separator of* A *and* B *of the same Turing degree as* C*.*

Proof. Fix wtt-operators Γ and Δ with $\Gamma^A = B$ and $\Delta^B = A$, and let f be a computable function bounding the use of both Γ and Δ. We assume that f is monotonic and $f(x) > x$ for all x. We split the argument into two cases, depending on whether or not there are infinitely many x with $(x, f(x)] \subseteq A \cup B$.

Case 1. Suppose there are only finitely many x with $(x, f(x)] \subseteq A \cup B$.

Fix k such that there are no such $x \geq k$. Define the computable sequence: $m_0 = k$; $m_{i+1} = f(m_i)$. Then we define a separator Z such that if $n \notin C$, then Z agrees with A on $(m_n, m_{n+1}]$, and if $n \in C$, then Z agrees with $\overline{B}$ on $(m_n, m_{n+1}]$. As C computes A (and thus B), $Z \leq_T C$.

To establish $C \leq_T Z$, note that Z cannot agree with both A and $\overline{B}$ on any $(m_n, m_{n+1}]$, as $(m_n, m_{n+1}] \not\subseteq A \cup B$. Thus $n \in C$ iff Z agrees with $\overline{B}$ on $(m_n, m_{n+1}]$ iff Z differs from A on $(m_n, m_{n+1}]$. So Z can determine whether $n \in C$ by waiting for a stage s such that it agrees with either A_s or $\overline{B}_s$ on $(m_n, m_{n+1}]$.

Case 2. Suppose there are infinitely many x with $(x, f(x)] \subseteq A \cup B$.

Define the following sequence:

- $m_0 = -1$;
- m_{n+1} is the least $x > m_n$ such that:
  - $(m_n, x] \not\subseteq A \cup B$; and
  - $(x, f(x)] \subseteq A \cup B$.

By assumption, m_n exists for all n. We define a separator Z such that if $n \notin C$, then Z agrees with A on $(m_n, m_{n+1}]$, and if $n \in C$, then Z agrees with $\overline{B}$ on $(m_n, m_{n+1}]$.

First observe that since C computes A (and thus B), C computes $(m_n)_{n \in \omega}$. Thus C computes Z.

Next, we argue that $(m_n)_{n \in \omega}$ is computable from Z. Suppose we have determined m_i for $i \leq n$. Then let s be a stage such that the following hold:

- For every $i < n$, Z on $(m_i, m_{i+1}]$ agrees with either A_s or $\overline{B}_s$;
- For every $i < n$, $(m_i, f(m_i)] \subseteq A_s \cup B_s$; and
- There is some $x > m_n$ such that:
  - $\Gamma^{A_s} \upharpoonright_{x+1} = B \upharpoonright_{x+1}$ and $\Delta^{B_s} \upharpoonright_{x+1} = A \upharpoonright_{x+1}$;
  - Z on $(m_n, x]$ agrees with either A_s or $\overline{B}_s$;
  - $(m_n, x] \not\subseteq A_s \cup B_s$; and
  - $(x, f(x)] \subseteq A_s \cup B_s$.

Fix y the least such x.

Claim 12.1. $y = m_{n+1}$.

Proof. Suppose not. If $m_{n+1} > y$, then by minimality of m_{n+1}, it must be that $(m_n, y] \subseteq A \cup B$. By choice of y, this means that some element $z \leq y$ is enumerated into $A \cup B$ after stage s.

If $m_{n+1} < y$, then by minimality of y, it must be that $(m_{n+1}, f(m_{n+1})] \not\subseteq A_s \cup B_s$. By monotonicity, $f(m_{n+1}) \leq f(y)$, so there must be some element $z \leq f(y)$ enumerated into $A \cup B$ after stage s. By choice of y, $z \leq y$.

In either case, we see that there is an element $z \leq y$ enumerated into $A \cup B$ after stage s. By choice of s and correctness of Δ, if $z \in A$, then there must be a $w \leq f(z)$ such that w is enumerated into B after stage s, and by monotonicity and choice of s and y, $w \leq y$. If $z \in B$, then symmetric reasoning shows there is a $w \leq y$ enumerated into A after stage s. Without loss of generality, assume that z is the least element enumerated into $A \cup B$ after stage s

If there is an $i < n$ such that $z \in (m_i, m_{i+1}]$, then by choice of s and correctness of Γ and Δ, $w \in (m_i, m_{i+1}]$. But Z agrees with either A_s or $\overline{B}_s$ on $(m_i, m_{i+1}]$ by choice of s, so there cannot be such z and w: e.g. if Z agrees with A_s, then $z \in A \setminus A_s$ contradicts Z being a separator. So it must be that $z, w \in (m_n, x]$. But again, Z agrees with either A_s or $\overline{B}_s$ on $(m_n, x]$, so this is a contradiction. □

Having computed $(m_n)_{n\in\omega}$, Z can determine whether $n \in C$ by waiting for a stage s such that Z on $(m_n, m_{n+1}]$ agrees with either A_s or $\overline{B}_s$ (in fact, the stage s used to determine m_{n+1} suffices). □

3. No Super-Maximal Pairs

Recall that super-maximnal pairs are c.e. sets A, B where all X separating of c.e. degree must be finite variants of either A or B.

Theorem 13. *Super maximal pairs do not exist.*

Proof. Let A and B be two c.e. sets satisfying $A \cap B = \emptyset$ and $|\omega - (A \cup B)| = \infty$. We will build a separating set X of c.e. degree such that $X \neq^* A$ and $\overline{X} \neq^* B$. This construction is necessarily non-uniform, however, so we may make up to three attempts. If the first attempt fails, it will be because our set has either $X =^* A$ or $\overline{X} =^* B$. If the second attempt fails, it will fail in the opposite manner. Then the third attempt will succeed.

The First Attempt. We build a computable sequence $(X_{1,s})_{s\in\omega}$ approximating X_1. We begin with $X_{1,0} = \emptyset$. We also define the auxiliary sets $Y^n_{1,0} = \emptyset$ for all n.

At stage $s+1$, we first define a sequence $x^1_{-1,s+1} < x^1_{0,s+1} < \cdots < x^1_{k,s+1} = s$:

- $x^1_{-1,s+1} = -1$.
- Given $x^1_{n,s+1} < s$, if $x^1_{n+1,s}$ is undefined or $x^1_{n+1,s} \leq x^1_{n,s+1}$, we let $x^1_{n,s+1} = s$.
- Given $x^1_{2i,s+1} < x^1_{2i+1,s} < s$, if there is a $y \in (x^1_{2i,s+1}, x^1_{2i+1,s}]$ satisfying:
  - $y \in X_{1,s} - (A_{s+1} \cup B_{s+1})$; or
  - $y \notin A_{s+1} \cup B_{s+1}$, and there is some $z < y$ with $z \in X_{1,s} \cap B_{s+1}$ or with $z \in \overline{X}_{1,s} \cap A_{s+1}$; or
  - $y \notin A_{s+1} \cup B_{s+1}$ and $y = s$,

  then we let $x^1_{2i+1,s+1} = x^1_{2i+1,s}$. Otherwise, we let $x^1_{2i+1,s+1} = s$.
- Given $x^1_{2i+1,s+1} < x^1_{2i+2,s} < s$, if there is a $y \in (x^1_{2i+1,s+1}, x^1_{2i+2,s}]$ satisfying:
  - $y \in \overline{X}_{1,s} - (A_{s+1} \cup B_{s+1})$; or
  - $y \notin A_{s+1} \cup B_{s+1}$, and there is some $z < y$ with $z \in X_{1,s} \cap B_{s+1}$ or with $z \in \overline{X}_{1,s} \cap A_{s+1}$; or
  - $y \notin A_{s+1} \cup B_{s+1}$ and $y = s$,

  then we let $x^1_{2i+2,s+1} = x^1_{2i+2,s}$. Otherwise, we let $x^1_{2i+2,s+1} = s$.

Note that the sequence we define is strictly increasing. It terminates when it achieves s.

On each interval $(x^1_{2i,s+1}, x^1_{2i+1,s+1}]$, we will attempt to ensure that $X \neq^* A$ by putting whatever elements we can into X_1. On each interval $(x^1_{2i+1,s+1}, x^1_{2i+2,s+2}]$, we will do the opposite. Of course, this is restricted by our need to make X_1 a separator of c.e. degree. Accordingly, we make the following definition

Definition 14. We say that y is *permitted at stage* $s+1$ (for X_1) if $y \notin A_{s+1} \cup B_{s+1}$, and $y = s$ or there is some $z < y$ with $z \in X_{1,s} \cap B_{s+1}$ or with $z \in \overline{X}_{1,s} \cap A_{s+1}$.

We now define $X_{1,s+1}$ as follows:

- If $y \in A_{s+1}$, then $y \in X_{1,s+1}$.
- If $y \in B_{s+1}$, then $y \notin X_{1,s+1}$.
- If $y \in (x^1_{2i,s+1}, x^1_{2i+1,s+1}]$ and is permitted at stage $s+1$, then $y \in X_{1,s+1}$.
- If $y \in (x^1_{2i+1,s+1}, x^1_{2i+2,s+1}]$ and is permitted at stage $s+1$, then $y \notin X_{1,s+1}$.
- If none of the above apply, then $x \in X_{1,s+1} \iff x \in X_{1,s}$.

This completes the first construction.

Claim 14.1. *$(X_{1,s})_{s\in\omega}$ converges (in a Δ^0_2 fashion) to a set X_1, and X_1 is a separator of A and B of c.e. degree.*

Proof. Observe first that if $x \in X_{1,s} \triangle X_{1,s+1}$, then one of the following must hold:

(1) $x \in X_{1,s} \cap B_{s+1}$;
(2) $x \in \overline{X}_{1,s} \cap A_{s+1}$;
(3) For some $z < x$, one of (1) or (2) holds; or
(4) $x = s$.

By induction on x, each of these can occur only finitely many times for each x, and so the limit X_1 exists.

Now let $W = \{x : \exists s\, x \in X_{1,s} \cap B_{s+1} \vee x \in \overline{X}_{1,s} \cap A_{s+1}\}$ with the obvious c.e. approximation $(W_s)_{s\in\omega}$. If $x \in W_{s+1} - W_s$, then $x \in X_{1,s} \triangle X_{1,s+1}$. Conversely, if $x \in X_{1,s} \triangle X_{1,s+1}$ for some $s > x$, then there is a $z \leq x$ with $z \in W_{s+1} - W_s$. Thus $X_1 \equiv_T W$, and so X_1 is of c.e. degree.

That $A \subseteq X$ and $X \cap B = \emptyset$ is immediate from our definition of $X_{1,s+1}$. □

Now, for each n, we consider the sequence $(x^1_{n,s})_{s\in\omega}$. Note that not every $x^1_{n,s}$ will be defined. However, if $x^1_{n,s}$ is defined for almost every s, we can consider whether the sequence has a limit.

Claim 14.2. *Suppose $x^1_n = \lim_s x^1_{n,s}$ exists with $x^1_n < \infty$ (and, implicitly, $x^1_{n,s}$ is defined for almost every s). Then for every $j < n$, $x^1_j = \lim_s x^1_{j,s}$ exists, and $x^1_j < x^1_n$. Further, if $n > -1$, then there is a $y \in (x^1_{n-1}, x^1_n]$ with $y \notin A \cup B$, and $y \in X_1$ if and only if n is odd.*

Proof. By construction, for $j < n$ and every s at which $x^1_{n,s}$ is defined, $x^1_{j,s}$ is also defined. So $x^1_{j,s}$ is defined for almost every s. By construction, $x^1_{j,s}$ is nondecreasing in s and bounded by x^1_n, so $x^1_j < x^1_n$ exists.

For s with $x^1_{n,s+1} = x^1_n < s$, there is some y witnessing that $x^1_{n,s+1} \neq s$. This y is in $X_{1,s+1}$ if and only if n is odd by construction. So by pigeon hole, there is some y in $(x^1_{n-1}, x^1_n]$ with this property for infinitely many s, and thus for almost every s (as the approximation to X_1 converges). Thus $y \in X_1$ if and only if n is odd. □

So if $x^1_n < \infty$ exists for every n, then X_1 is as desired. So instead assume ℓ_1 is the greatest n such that $x^1_n < \infty$ exists. Let $k_1 = \ell_1 + 1$, and WLOG assume k_1 is odd. Observe that $x^1_{k_1,s+1}$ is defined for every $s > x^1_{\ell_1}$.

Claim 14.3. *For every s with $x^1_{\ell_1,s} = x^1_{\ell_1}$ and $x^1_{k_1,s}$ defined, and for every $y \in (x^1_{\ell_1}, x^1_{k_1,s}] \cap X_{1,s}$, $y \in A \cup B$.*

Proof. Fix $t \geq s$ such that $x^1_{k_1,t+1} \neq x^1_{k_1,t} = x^1_{k_1,s}$. Then, by definition of $x^1_{k_1,t+1}$, every such y must be in $A_{t+1} \cup B_{t+1} \cup \overline{X}_{1,t}$. But, by induction on $s' \in (s,t]$, if $y \notin B$, then $y \in X_{1,s'}$. □

It follows that for any $y > x^1_{\ell_1}$, $y \in X_1 \iff y \in A$.

The Second Attempt. We make a second attempt based on the knowledge of how the first attempt failed. First, we perform a speedup of the enumerations of A and B and of the previous construction such that the following all hold:

- For each $n < k_1$ and every s, $x^1_{n,s} = x^1_n$.
- For every s, $x^1_{k_1,s} > s$ is defined.
- For every $y \in (x^1_{\ell_1}, s]$, $y \in A_s \cup \overline{X}_{1,s}$.

We now build X_2 in this new timeline. We simply repeat the construction of X_1, except that we refer to the sequence we build at each stage as $(x^2_{n,s+1})$, to avoid confusion, and we always begin with $x^2_{-1,s+1} = x^1_{\ell_1}$. Analogues of Claims 14.1 and 14.2 for X_2 proceed as the originals. Note that for every n and s with $x^2_{n,s}$ defined, $x^2_{n,s} < x^1_{k_1,s}$.

Again, if $x^2_n < \infty$ exists for every n, then X_2 is our desired separator. So instead assume ℓ_2 is the greatest n such that $x^2_n < \infty$ exists, and let $k_2 = \ell_2 + 1$. Again, $x^2_{k_2,s+1}$ is defined for every $s > x^2_{\ell_2}$.

Claim 14.4. *k_2 is even.*

Proof. Suppose not. Fix an $s > x^2_{\ell_2} > x^1_{\ell_1}$ with $s \notin A \cup B$.

Fix t least with $x^2_{k_2,t+1} \geq s$. We claim that s is permitted for X_2 at stage $t+1$. This is immediate if $s = t$, so suppose not. Then $x^2_{k_2,t} < s < t$, so there is some $y \in (x^2_{\ell_2}, x^2_{k_2,t}] \cap X_{2,t}$ witnessing that $x^2_{k_2,t} < t-1$. As this y does not suffice for $t+1$, it must be that $y \in A_{t+1} \cup B_{t+1}$. In second case, y witnesses that s is permitted for X_2 at stage $t+1$. In the first case, since $y \notin A_t$, $y \in \overline{X}_{1,t}$. So y witnesses that s is permitted for X_1 at some point in the time period between stages t and $t+1$ of our speedup. But as $x^1_{\ell_1} < s < x^1_{k_1,s} \leq x^1_{k_1,t}$, a definition $s \in X_1$ would be made. This contradicts Claim 14.3.

So s is permitted for X_2 at stage $t+1$, and so $s \in X_{2,t+1}$. But then s will forever witness that $x^2_{k_2,s'}$ does not need to change, contrary to choose of k_2. □

Analogously to Claim 14.3, we obtain the following.

Claim 14.5. *For every s with $x^2_{\ell_2,s} = x^2_{\ell_2}$ and $x^2_{k_2,s}$ defined, and for every $y \in (x^2_{\ell_2}, x^2_{k_2,s}] \cap \overline{X}_{2,s}$, $y \in A \cup B$.*

It follows that for any $y > x^2_{\ell_2}$, $y \in X_2 \iff y \notin B$.

The Final Attempt. We make our final attempt based on the knowledge of how the first two attempts failed. Again, we begin with a speedup of our previous speedup such that the following all hold:

- For each $n < k_2$ and every s, $x^2_{n,s} = x^2_n$.
- For every s, $x^2_{k_2,s} > s$ is defined.
- For every $y \in (x^2_{\ell_2}, s]$, $y \in X_{2,s} \cup B_s$.

Again, we always begin with $x^3_{-1,s+1} = x^2_{\ell_2}$. We again prove analogues of Claims 14.1 and 14.2. Define k_3 to be the least such that $x^3_{k_3} < \infty$ does not exist. We show two versions of Claim 14.4, one showing that k_3 cannot be odd by arguing to a contradiction with Claim 14.3, and another showing that k_3 cannot be even by arguing to a contradiction with Claim 14.5. It follows that there is no such k_3, and so X_3 is as desired.

□

4. Two degrees

In this section we examine degree spectra containing two degrees.

Theorem 15. *For every c.e. degree* $\mathbf{c}$*, the separating spectrum* $\{\mathbf{c}, \mathbf{0}'\}$ *is possible.*

Proof. Fix $C \in \mathbf{c}$. We build disjoint c.e. sets A and a B with $|\omega \setminus (A \cup B)| = \infty$ and meeting the following requirements, for all e and n:

R_e: If $\Phi_e^{W_e} = Z$, $A \subseteq Z$, and $|\overline{Z} \setminus B| = \infty$, then W_e computes K.

P_n: There is at most one i with $\langle n, i\rangle \in B$, and such i, if it exists, is bounded by $n^2 + 1$. Further, $n \in C \iff \exists i\, [\langle n, i\rangle \in B]$.

First we argue that meeting these requirements suffices.

By the P_n, $B \equiv_T C$. For one direction, $n \in C \iff (\exists i \leq n^2 + 1)[\langle n, i\rangle \in B]$, which is bounded quantification. For the other, suppose we wish to determine whether $\langle n, j\rangle$ is in B. We first check whether $n \in C$; if not, $\langle n, j\rangle \notin B$. If so, we enumerate B until we see some $\langle n, i\rangle \in B$. Then $\langle n, j\rangle \in B \iff i = j$.

Now, suppose Z is a separator of c.e. degree. If $|\bar{Z} \setminus B| = \infty$, then by the appropriate R_e, Z is of degree $\mathbf{0}'$. If instead $\bar{Z} =^* B$, then Z is of degree $\mathbf{c}$. In particular, A itself is a separator of degree $\mathbf{0}'$, and $\overline{B}$ is a separator of degree $\mathbf{c}$.

We assume that for all i, $\langle n, i\rangle \geq n^3$.

Strategy for R_e: We construct a c.e. operator V_e, with the intention that $V_e^{W_e} = \overline{K}$ if $\Phi_e^{W_e}$ is as described. For each m, if $m \notin [K \cup V_e^{W_e}][s]$, we search for an x and a γ satisfying the following:

- For all $n \leq \max\{e, m\}$ and $i < n^2 + 1$, $x > \langle n, i\rangle$.
- $x \notin B_s$;
- For all $y \leq x$, $\Phi_e^{W_e \restriction \gamma}(y)[s]\downarrow$; and
- $\Phi_e^{W_e \restriction \gamma}(x)[s] = 0$.

If these exist, we fix the least such x and γ (by standard use assumptions, we can minimize these both simultaneously), and we define $m \in V_e^{W_e \restriction \gamma}[s]$. At every subsequent stage $t > s$, if $W_e \restriction_\gamma [t] = W_e \restriction_\gamma [s]$, then x is *blocked from* B at stage t.

If instead $m \in [K \cap V_e^{W_e}][s]$, we fix the x used in the definition of $m \in V_e^{W_e}[s]$. If $x \notin B_s$ (as we will later argue must be the case), we enumerate x into A.

Otherwise, do nothing for m.

Strategy for P_n: When n is enumerated into C, fix the least i such that $\langle n, i\rangle \notin A_s$ and $\langle n, i\rangle$ is not blocked from B at stage s by any of the R_e. We will later argue that $i < n^2 + 1$. Enumerate $\langle n, i\rangle$ into B.

Construction: Simply run all of the above strategies simultaneously. We take no care to ensure that different R-strategies are choosing different witnesses x.

Verification: By construction, we never enumerate an element into $A \cup B$, and so A and B are disjoint. Next, we show that our claim in the P_n-strategy holds.

Claim 15.1. *At any stage s, for every n,*

$$\#\{i < n^2 + 1 : \langle n, i\rangle \in A_s \vee \langle n, i\rangle \text{ is blocked from } B \text{ at stage } s\} \leq n^2.$$

Proof. The only way for $x = \langle n, i\rangle$ to be blocked or enumerated into A is for it to be selected by some R_e-strategy on behalf of some m. By our choice of such x in the R_e-strategy, it must be that $e, m < n$. Further, no pair (e, m) can contribute more than one x in this fashion: the R_e-strategy blocks at most one x at a time on behalf of each m, and if the strategy has enumerated an x on behalf of m, then $m \in C_s$ and so the strategy will not block or enumerate another element on behalf of m.

The claim follows. □

Thus our P_n-strategy meets its requirement.

Claim 15.2. *For all k,*

$$|(A \cup B) \cap k^3| \leq O(k^2).$$

It follows that $|\omega \setminus (A \cup B)|$ is infinite.

Proof. By construction, since $\langle k, 0\rangle \geq k^3$, the only strategies which can enumerate elements into $A \cap k^3$ are R_e-strategies with $e < k$, and then only on behalf of some $m < k$. As in the previous claim, each pair (e, m) can contribute at most one such enumeration, and so $|A \cap k^3| \leq k^2$.

Also, the only strategies which can enumerate elements into $B \cap k^3$ are P_n-strategies with $n < k$, and each necessarily enumerates at most one element. So $|B \cap k^3| \leq k$.

The claim follows. □

Claim 15.3. *Each R_e-strategy meets its requirement.*

Proof. Fix m. Suppose first that $m \notin K$. Fix the least $x \notin (Z \cup B)$ such that $x > \langle n, i\rangle$, for all $n \leq \max\{e, m\}$ and $i < n^2 + 1$. Fix γ least such that $\Phi_e^{W_e \upharpoonright_\gamma} \supseteq Z\upharpoonright_{x+1}$, and fix s_0 such that $W_e \upharpoonright_\gamma = W_{e,s_0} \upharpoonright_\gamma$. Then at any stage $s > s_0$ with $m \notin V_e^{W_e}[s]$, x and γ will be chosen for the new definition of $m \in V_e^{W_e \upharpoonright_\gamma}[s]$. By choice of s_0, this ensures $m \in V_e^{W_e}$.

If instead $m \in K$, then fix s_0 least with $m \in K_s$. By construction, we will never define $m \in V_e^{W_e}[s]$ for any $s \geq s_0$. So suppose $m \in V_e^{W_e}[s_0]$ because of our action at some stage $t < s_0$. Fix the witnessing x and γ. Then necessarily, $W_{e,r} \upharpoonright_\gamma = W_{e,s_0} \upharpoonright_\gamma$ for every $r \in [t, s_0]$, and so x was blocked from B at every such stage. Also, $x \notin B_t$. By construction, $x \notin B_{s_0}$. So x will be enumerated into A at stage s_0.

By assumption, $\Phi_e^{W_e \upharpoonright_\gamma}(x)[t] = \Phi_e^{W_e \upharpoonright_\gamma}(x)[s_0] = 0$. So if $\Phi_e^{W_e} = Z$ contains A, and in particular contains x, it must be that $W_{e,s_0} \upharpoonright_\gamma \neq W_e \upharpoonright_\gamma$. Since no future stage s will define $m \in V_e^{W_e}[s]$, it follows that $m \notin V_e^{W_e}$. □

This completes the proof. □

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