Realizing c.e. degrees in $\Pi^0_1$ classes

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The question

- The Kriesel Basis Theorem says that a $\Pi_1^0$ class has a member of c.e. degree, e.g. the leftmost or rightmost path.

**Definition**

We say that a c.e. degree $a$ is **realized** in a class $C$ if there is a path $P \in C$ of degree $a$.

**Question**

*What sets of degrees are realized in some $C$?*

- For instance, it is possible to realize singletons: $0$, or more interestingly, $0'$ in e.g. the $\Pi_1^0$ class of Martin-Löf random reals.
We are interested in these sets of c.e. degrees up to Turing degree.

It follows that we are interested in index sets; sets of indices of c.e. sets closed under Turing equivalence.

**Definition**

We say that a set $S$ represents an index set $I$, iff

$I \overset{\text{def}}{=} G(S) = \{ e : (\exists j \in S) \ W_e \equiv_T W_j \}.$

Note that, for example, a singleton such as an index for $\emptyset'$ can represent $\{ e \mid W_e \text{ complete} \}$; a $\Sigma^0_4$ complete set, so representations of index sets can be much simpler.

In the same spirit:

**Definition**

If $P$ is a $\Pi^0_1$ class, then $W[P] = \{ e : W_e \text{ is realizable in } P \}.$
**Proposition**

If $P$ is a $\Pi^0_1$ class, then $W[P]$ is $\Sigma^0_4$.

**Proof**

$$e \in W[P] \iff (\exists j \exists k)(\forall n)(\exists m)(\exists s > n) \left[ \Phi_j^{W_e} \uparrow m[s] \downarrow \land W_e[s] \uparrow \varphi_j(m) \text{ is correct} \land \Phi_j^{W_e} \uparrow m[s] \text{ is extendible in } P[s] \land \Phi_k^{W_e} \uparrow m[s] \uparrow n = W_e \uparrow n \right].$$

Note that the predicate within the square brackets is $\Delta^0_2$. 
Specific Questions

1. Is every $\Sigma^0_4$ index set realizable in a $\Pi^0_1$ class?
2. If not, can we characterize the index sets $S$ which can be realized?
3. Is every upper and every lower cone of c.e. degrees realizable?
4. What about the c.e. degrees realizable in $\Pi^0_1$ classes of restricted rank? (Or type, such as separating classes.)

Hand in hand these go with analogous index set questions:

1. Can every $\Sigma^0_4$ index set representable as $\Sigma^0_3$ set?
2. If not, can we characterize the index sets $I$ which can be represented as $\Sigma^0_3$ sets, or by computable sets?
3. Is every upper and every lower cone of c.e. degrees representable?
Known results

- Mainly index set results: e.g. Yates 66, 69
- \( D \leq_T C \) are c.e. then if \( S \) is \( \Sigma^C_3 \), there is a computable collection \( \{ W_{f(k)} \mid k \in \mathbb{N} \} \) \( D \leq_T W_{f(k)} \leq_T C \) and \( e \in S \) is equivalent to \( W_{f(e)} \equiv_T C \).
- Yates \( C = \{ W_e \mid e \in C \} \) containing all finite sets is \( \Sigma^0_3 \) iff there is a computable collection \( \{ W_{f(d)} \mid d \in \mathbb{N} \} \) which equals \( C \).

**Theorem (Yates, 1966)**

The index of the lower cone \( \{ e \mid W_e \leq_T A \} \) is \( \Sigma^0_3 \) iff \( A \) is low\(_2\) or complete.
More recently Barmpalias and Lewis-Pye on ideals:

**Theorem**

For an ideal $I$ in the c.e. degrees:

1. If $I$ is uniformly superlow generated (That is there is a uniformly superlow collection of c.e. sets $\{W_e \mid e \in D\}$ such that $I = \{W_e \mid \exists F \subset D \land W_e \leq_T \bigoplus_{i \in F} W_i\}$) than it has a superlow upper bound in the c.e. degrees.

2. If $I$ is uniformly low generated then it has a low c.e. upper bound.

3. If $I$ is a $\Sigma^0_3$ generated proper ideal, then it has a low$_2$ c.e. upper bound. (also Downey-Hirschfeldt Book)

4. If $I$ is $\Sigma^0_4$ generated proper ideal, then it has an incomplete upper bound.

Our results sharpen some of these.
The main theorem

Theorem (Csima, Downey, Ng)

An index set $I$ is realizable in a $\Pi^0_1$ class iff $I$ has a $\Sigma^0_3$ representation.
Theorem (Csima, Downey, Ng)

Let $S \subseteq \omega$. The following are equivalent.

(i) $S$ has a computable representation, that is, $G(S) = G(R)$ for some computable set $R$.

(ii) There is a $\Pi^0_1$ class $P$ such that the set of c.e. degrees represented in $P$ has index set $G(S)$, that is, $W[P] = G(S)$.

(iii) There is a perfect $\Pi^0_1$ class $P$ such that $W[P] = G(S)$.

(iv) There is a computable function $g$ such that for every $n$,

$$n \in G(S) \iff W_{g(n)}^{W_n} \text{ is cofinite.}$$

(v) There is a truth-table functional $R$ such that for every $n$,

$$n \in G(S) \iff \exists a \forall b \exists c \ R^{W_n}(n, a, b, c).$$

Examples include e.g. the $K$-trivial sets and all upper cones.
Theorem

For any c.e. degree \(a\), we can find (effectively in an index of a member for \(a\)) a perfect \(\Pi^0_1\) class \(P\) such that \(W[P] = \{a\}\).

Proof Let \(C \in a\) be c.e., and fix a 1-1 enumeration \(\{C_s\}_{s \in \mathbb{N}}\) of \(C\). We build a perfect \(\Pi^0_1\) class \(P\) satisfying the requirements

\[
\mathcal{R}_e : \text{If } Z_e(x) := \lim_{s} Z_e(x, s) \text{ exists for every } x, Z_e \in P, \Phi^Z_e \text{ is total and is a modulus for the approximation } Z_e(x, s), \text{ then } Z_e \equiv_T C.
\]

Recall that a function \(f\) is a modulus for the computable approximation \(Z(x, s)\) if for every \(x\) and every \(t > f(x)\), \(Z(x, t) = Z(x, f(x))\). (This is because if a path has c.e. degree, \(P = \Xi^W \land W = \Psi^P\), then if \(W\) changes inducing a change in \(P\) this “can’t go back”. It is perhaps easiest to think of this.)
The key idea is that $Z_e$ must behave itself because we can force it off the tree. (Diagram) Second we can keep a coding location for e.g. “$x \in C$?” in the form of some split $\sigma * 0, \sigma * 1$ where $\sigma * i \prec Z_e[s]$. This is done by pruning the tree and forcing a change if necessary. The usual priorities to make the tree perfect, etc.
Realizing Index Sets

**Theorem (Csima, Downey, Ng)**

For any non-empty $\Sigma^0_3$ set $S$, there is a perfect $\Pi^0_1$ class $P$, such that $W[P] = G(S)$. Furthermore if $S$ contains an index for $\emptyset$, then we can make $P$ rank 2.

The proof comes from “setting aside cones of the tree for the guesses”. To wit: $x \in I$ iff $\exists e \forall s \exists t R(x, e, s, t)$.

In the part of the tree we devote to $\langle x, e \rangle$, we will pursue the strategy above (essentially) each time the $\Pi^0_2$ predicate fires, and whilst the $\Sigma^0_2$-outcome looks correct, either directly extend (in the rank 2 case with 0) or pursue a singleton strategy for a known member of $S$. 

Can all $\Sigma_4^0$ sets be realized?

- Jockusch and Soare: The $\Pi_3^0$ index set $\{e : W_e \text{ is not computable}\}$.
- Using a direct diagonalization argument:

**Theorem (Csima, Downey, Ng)**

There is a $\Pi_3^0$ set $S$ containing an index for $\emptyset$, such that there is no $\Pi_1^0$ class $P$ with $W[P] = G(S)$.

- This also follows from some later results.
If \( A \) is a high set then by Martin 1966, there is an \( \overline{A} \)-computable function \( \Gamma^\overline{A} \), such that which dominates every computable function, meaning that if \( f \) is computable, then for almost all \( x \), \( \Gamma^\overline{A}(x) > f(x) \).

Thus, if \( A \) is high then some \( \Gamma = \Phi_j \) satisfying this will exist. We will call such a \( j \) a domination index for \( A \).

**Theorem (Csima, Downey, Ng)**

Consider a computable collection \( C \) of pairs of the form \( \langle e, j(e) \rangle \), where \( j \) is a computable function such that each \( W_e \) has high degree, and \( j(e) \) is a domination index for \( W_e \). Let \( S \) be any \( \Sigma^0_4 \) subset of \( C \), and fix an index \( e_0 \) for \( \emptyset \). Then there is a rank 2 \( \Pi^0_1 \) class \( P \) with such that \( W[P] = G(S \cup \{e_0\}) \).

*In particular, if \( S \) is \( \Sigma^0_4 \)-complete, then \( S \cup \{e_0\} \) is an example of a realizable set which is not \( \Sigma^0_3 \).*
Proof idea

- $e \in S$ iff $\exists p \forall m \exists s \forall t R(e, p, m, s, t)$. For a fixed $p$, should the $\Pi^0_3$ condition $\forall m \exists s \forall t R(e, p, m, s, t)$ hold, we need to code $W_e$.
- We have parts devoted to each pair $\langle e, p \rangle$.
- $m$ points at a coding location for $m$ which can move.
- And in fact we’d like it to move and kick each time the $\Pi^0_2$ predicate fires for a fixed $\langle e, p, m \rangle$, kicking and dumping.
- The idea is that if for one such $m$ this fires infinitely often, this generates a computable path. Hence if the $\Pi^4_0$ outcome is correct $W_e$ won’t be present.
- If not then the coding location $c_{\langle e, p \rangle}(m, s)$ will eventually settle.
- Uniform highness does the rest.
The Characterization, again

**Theorem (Csima, Downey, Ng)**

Let $S \subseteq \omega$. The following are equivalent.

(i) $S$ has a computable representation, that is, $G(S) = G(R)$ for some co-
computable set $R$.

(ii) There is a $\Pi^0_1$ class $P$ such that the set of c.e. degrees represented in
$P$ has index set $G(S)$, that is, $W[P] = G(S)$.

(iii) There is a perfect $\Pi^0_1$ class $P$ such that $W[P] = G(S)$.

(iv) There is a computable function $g$ such that for every $n$,

$$n \in G(S) \iff W^{W_n}_g(n) \text{ is cofinite.}$$

(v) There is a truth-table functional $R$ such that for every $n$,

$$n \in G(S) \iff \exists a \forall b \exists c \ R^{W_n}(n, a, b, c).$$

The main idea in the proof is to show that realizable c.e. degrees are
essentially the leftmost or rightmost paths in some cone.
If $S$ and $T$ have computable representations then so do $G(S) \cap G(T)$ and $G(S) \cup G(T)$.

The following classes of c.e. degrees have a computable representation and can be realized in some (perfect) $\Pi^0_1$ class:

- The set of all superlow c.e. degrees.
- The set of all $K$-trivial c.e. degrees.
- Any $\Sigma^0_4$ subset of a computable collection of high indices with an index for $0$.
- Any uppercone of c.e. degrees, that is, the set $\{b \mid b \text{ is c.e. and } b \geq a\}$ for any c.e. degree $a$.
- Any lowercone of c.e. degrees below a low$_2$ c.e. degree, that is, the set $\{b \mid b \text{ is c.e. and } b \leq a\}$ for any low$_2$ c.e. degree $a$. 

 CDN conjectured that if $a$ is incomplete the cone below $a$ is realizable iff $a$ is low$_2$.

 Showing some cones cannot be realized uses:

**Theorem (CDN)**

Let $C$ be a uniformly computable collection of c.e. sets $\{A_i \mid i \in \omega\}$ which are independent in that for any finite set $F$, if $i \not\in F$, then $A_i \not\leq_T \bigoplus_{j \in F} A_j$. Let $S$ be a $\Sigma^0_4$ set. Then there is a c.e. $B$ such that for all $i \in \omega$, $i \in S$ iff $A_i \leq_T B$.

 This strengthens Barmpalias and Lewis-Pye when restricted to ideals: $B$ selects $S$. 

Corollary (CDN)

Not every lowercone of c.e. degrees has a computable representation.

Proof Apply the above to a uniformly low independent set of \( \{ A_i \mid i \in \omega \} \), and \( S \) \( \Sigma_4^0 \)-complete set of indices. If \( \{ e \mid W_e \leq_T B \} \) had a \( \Sigma_3^0 \) representation we could reduce the complexity of \( S \).

In fact:

Theorem (Downey and Melnikov)

The cone below \( a \) is realizable iff \( 0 =' \) or \( a \) is low\(_2\).
Recall that $P$ is a separating class if there exist c.e. disjoint sets $A$, $B$ such that $P = S(A, B) = \{ Z \mid Z \supseteq A \land Z \cap B = \emptyset \}$.

For example, if $A$ represents the sentences provable in PA and $B$ the sentences refutable, then $S(A, B)$ represents the complete extensions of PA.

Used extensively in Reverse Maths.

You can realize a singleton via ML-randoms or PA-degrees.

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Any other singleton-Open?

If $A$ and $B$ have $A \oplus B$ of array computable degree, then Downey-Jockusch-Stob show that $S(A, B)$ contains a Turing complete element.
Theorem (Cholak, Downey, Greenberg, Turetsky)

Suppose \( A \equiv_{wtt} B \) are c.e. sets such that:

- \( A \cap B = \emptyset \); and
- \( |\omega \setminus (A \cup B)| = \infty \).

Then for every \( C \geq_T A \), there is a separator of \( A \) and \( B \) of the same Turing degree as \( C \).

Theorem (Cholak, Downey, Greenberg, and Turetsky)

There are c.e. sets \( A \geq_{wtt} B \) such that:

- \( A \cap B = \emptyset \);
- \( |\omega \setminus (A \cup B)| = \infty \); and
- No separator of \( A \) and \( B \) computes \( \emptyset' \).

Theorem (Cholak, Downey, Greenberg, and Turetsky)

For every c.e. degree \( c \), the separating spectrum \( \{ c, 0' \} \) is possible.
- On the c.e. degrees realizable in \( \Pi^0_1 \) classes, Csima, Downey and Ng, JSL in press.
- On realisation of index sets in \( \Pi^0_1 \)-classes, Downey and Melnikov, *Algebra i Logika* Vol. 58 (2019), 659-663 (Russian).
- Realizing computably enumerable degrees in separating classes, Cholak, Downey, Greenberg, and Turetsky), accepted in ”Higher Recursion Theory and Set Theory”, (James Cummings, Andrew Marks, Yue Yang and Liang Yu, eds) volume in celebration of Ted Slaman and Hugh Woodin, Lecture Notes Series Institute for Mathematical Sciences. National University of Singapore, Singapore.
Thank You