Some New Directions in Online Structure Theory

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Online

- Your job is to put objects $\{a_0, a_1, \dots\}$ into bins $\{b_j \mid j \in \omega\}$ of a certain maximum capacity and they are given one at a time. You must put a_i into some b_j at stage i before I give you a_{i+1} and try to minimize the number of bins used as you go along. ‘
- I give you a graph G one point at a time, giving the induced subgraph $\{a_0, a_1, \dots, a_i\}$ at step i and you must decide a colour before given a_{i+1}. Minimize the number of colours.
- You are in a vast graph and need to build, for example, an object incrementally, but don't have time to see more than a local neighbourhood.
- You are a triage nurse and patients arrive and you must order them in some priority ordering to be seen dynamically.
- You are a finitely branching tree T which I am giving you up to height n at step n and you must build a path of height n.

Online

- In all of the above you are in an online situation, though some are different than others.
- There are hundreds of algorithms for such problems both in the finite case, and in the case where e.g. a scheduler needs to works "forever".
- There are books with taxonomies of such algorithms.
- The goal is to give a theoretical basis for the above.
- One such basis has been talked about by Melnikov and Ng, and others.

Two approaches

- There seem two basic criteria needed for "online-ness".
- And there are two non-independent approaches based on them.
- The first is based on the punctuality of the online algorithm. We must do something immediately before the next item arrives. (Or leaves, this could be a Δ_2^0 process.) This has a growing and rich theory.
- The second is based on the uniformity of the operators against a hostile universe. This has almost no theory, yet!

Punctuality

- This especially enriches computable structure theory.
- Consider the proof that computable dense linear orderings without endpoints are computably categorical.
- Suppose that A and B are quick copies, could you build a quick isomorphism? No...
- No, but you could for the analogous problem is A and B are giant cliques.

Punctual structures

- The first model chosen focussed on the immediacy of decisions to be made.

Definition (Kalimullin, Melnikov, Ng)

A structure is punctual (fully primitive recursive) if it has domain $\mathbb{N}$ and all relations and functions are (uniformly) primitive recursive.

- You might say the use of primitive recursive seems quite arbitrary. What about automatic (Nerode, Khoussainov, Stephan, etc) or polynomial time (Cenzer, Remmel, Downey, others)....
- Primitive recursion is representative of the general total without delay set up, it is natural, and moreover no special coding or combinatorics/complexity classes etc are needed.
- The restricted Church-Turing thesis for primitive recursive functions says that a function is primitive recursive iff it can be described by an algorithm that uses only bounded loops.

Further logician's motivation

- A fundamental result in computable model theory is that a decidable theory has a decidable model.
- Almost any normal decidable theory is actually primitive recursively decidable.
- Almost any normal decision problem arising in algebra will be primitive recursive. For example, if a field has a splitting algorithm then that will naturally be primitive recursive.

Formalizing this

- Tempting: If T has a primitive recursive decision procedure then it has a punctually decidable model.
- But what do we mean by this? As observed in work by Cenzer and Remmel, for example, we need primitive recursive Skolem function

Theorem (Bazhenov,D, Kalimullin, M)

If T has a primitive recursive decision procedure then it has a punctually decidable model.

- Proof Henkin. Contrast with:

Theorem (KMN)

There is a punctually 1-decidable theory with no punctually 1-decidable model.

- Several known results lift.

Theorem (BDKM)

Suppose that T is a complete theory with primitive recursive decision procedure. TFAE

1. *T has a punctually decidable prime model*
2. *T has a prime model and an increasing uniformly primitive recursive sequence of principal types with quick witnesses.*

- D, Harrison-Trainor, Greenberg and Turetsky observed that a complete primitive recursive theory has a punctual model that omits a given primitive recursive non-principle type.
- Question Develop punctual pure model theory.

Natural Punctual Structures

Theorem (KMN)

The following structures have punctual presentations iff they have computable ones.

1. *Linear orderings (Gregorieff)*
2. *Boolean algebras*
3. *Equivalence structures*
4. *Torsion-free abelian groups.*
5. *Abelian p-groups*
6. *Locally finite graphs.*

Theorem (KMN)

The following have computable structures which have no punctual presentation.

1. *Torsion abelian groups.*
2. *(undirected) Graphs*
3. *Archimedean ordered abelian groups*

Theorem (Bazhenov, Harrison-Trainor, KMN)

$\{e \mid M_e$ *has a punctual (automatic, poly) presentation*$\}$ *is* Σ^1_1 *complete.*

The Operator Approach

- A criticism of the work above is that there is another aspect of online algorithms, reflected in practice.
- Consider online colouring of a graph with the simple monotone model.
- The online algorithm A acts on G_{s+1} to (irrevocably) colour $v = s + 1$.
- You could include delay where it sees $f(s+1)$ many new points, where f would be primitive recursive, before making its decision, but we'll stick with the simple version.
- The crucial insight is that A must act *uniformly* on any sequence $G_0, \ldots, G_{s+1}, \ldots$. The offline algorithm can be considered as a sequence of algorithms $\hat{A}_s$ acting on G_s for each s.

- The key observation: whilst there are only a primitive recursive number of graphs of size s, there is no reason that the limit graph the opponent builds is even remotely primitive recursive.
- There are $2^{\aleph_0}$ many possible graphs.
- We are thinking of the algorithm acting on objects represented as paths in a computable tree. An operator.

- We could argue that any countable structure could be considered, where A_n is some kind of n-bounded fragment of the open diagram.
- But, really, in practice online structures are given in "layers". For instance, the n-th step in colouring is to consider an induced subgraph on n vertices, not, for instance, taking some enumeration of the vertices and edges and giving only part of the picture.
- Moreover functions make everything problematical. (How many iterations should we allow?)
- We want a theory which reflects practice.

Definitions

Definition

A class $\mathcal{C}$ of relational structures is called inductive if $A \in \mathcal{C}$ implies A has a filtration $A = \cup_s A_s$ where each A_s is finite and has universe $\{1, \ldots, s\}$ and for all $s' > s$ the substructure induced by $\{1, \ldots, s\}$ in $A_{s'}$ is A_s. (Similarly g-filtration for a computable g with universe $\{1, \ldots, g(n)\}$, etc.) n ($g(n)$) is the height of A_n.

Definition

A *representation* of a class $\mathcal{C}$ of structures is a surjective function $F : \omega^{<\omega} \to \mathcal{C}^{<\omega}$, which acts computably in the sense that $F(\sigma) = C_n$ for $|\sigma| = n$ and $|C_n| = n$, and if $\sigma \preceq \tau$ then $F(\sigma)$ is an induced substructure of $F(\tau)$. (Later this might be partial, and objects might have several names.)

Online

Definition

A on-line problem is a triple (I, S, s) where I is the space inputs viewed as strings in a finite or infinite computable alphabet, S is the space of outputs (solutions) viewed as strings in (perhaps, some other) alphabet, and $s : I \to S^{<\omega}$ is a function which maps I to the set of admissible solutions of σ of S.

Intuitively, to solve a problem (I, S, s) we need to find an online computable function f which, on input i, chooses an admissible solution from the finite set $s(i)$.

Definition

A solution to an online problem (I, S, s) is a function $f : I \to S$ with the properties:

(O1) f is computable without delay (to be clarified);

(O2) $f(\sigma) \in s(\sigma)$ for every $\sigma \in I$;

(O3) $f(\sigma)$ uses only σ in its computation.

Remark: the difference

- The key difference from the punctual view of online-ness of viewed earlier, there is now no reason that the structures we are dealing with, even in the punctual case, need to be primitive recursive structures.
- Suppose that the representation is 2^ω. Then there will be uncountably many structures represented as paths through the tree. It is the algorithm acting on the paths which is uniform, and the primitive recursiveness would be relative to the path.

Finite=Infinite

Proposition

- *Suppose that A acts in an online fashion uniformly on all finite strings. Then A acts uniformly online on all computable paths through the representing space.*
- *Suppose that the algorithm A is total and acts uniformly online on all computable paths. Then A acts uniformly on all paths.*

Proof.

(i) If A fails on some computable path α it must fail on some finite initial segment. (ii) Computable paths are dense. $\square$

More precisely

- Suppose f is a solution to an online problem (I, S, s).
- The space of inputs carries a natural totally disconnected topology, and the completion of I forms the space of paths or infinite words in the language of I.
- The solution f induces a solution for the completion of the initial problem (I, S, s), in the sense that f can be uniquely extended to a functional $\overline{f} : [I] \to [S]$ between completions.
- Then $\overline{f}$ is a primitive recursive ibT operator (which means that its oracle use is bounded by the identity) with the property that, for every $n, f(p \upharpoonright n) \in s(p \upharpoonright n)$.
- In this case we say that $\overline{f}$ is a solution to the completion of (I, S, s).

Lifting things

- We can see what lefts from the previous punctual setting.
- It is possible to define online categoricity without too much difficulty.

Definition

A structure G is online categorical if there is an online computable f which, on input α and β and arbitrary representations of G outputs an isomorphism from α to β.

- Notice that we are using (representations of) partial maps in that they only need to work on copies of the structure.

Proposition

A relational structure is online categorical if, and only if, it is totally automorphically trivial.

- Here this means that for all $\overline{x}, y, z$, y and z are in to same orbit over $\overline{x}$.
- We don't know what happens if we add functions; and this is tricky to define anyway.

- I should prove something.
- If G not automorphically trivial, let $\overline{x}$ be shortest (length n) such that for some z is not in the same automorphism orbit as y over $\overline{x}$.
- To make α, β, copy $\overline{x}$ into both and calculate $f : \alpha \restriction n \to \beta \restriction n$.
- If we identify these with $\overline{x}$, then f induces a permutation o$\beta \restriction n$.
- As n is least, any permutation of $\overline{x}$ can be extended to an automorphism of the whole structure.
- Adjoin z to α and find y' playing the role of y over $f(\alpha \restriction n)$.
- Then necessarily $f(z) = f(y')$, because f has already shown its computation on the first n bits.
- However, by the choice of z and y', f cannot be extended to an isomorphism

- We comment that the theme the theory tends to be smoother (and surely involves definability and forcing) than the corresponding punctual theory as the punctual theory involves no uniformity.
- This is akin to computable structure theory vs uniform computable structure theory.
- Question: Also connections between uniform computable structure theory and (type-2) computable analysis?

Graphs; incremental computation

- There is a notion of incremental computation due to Milterson et. al. and we can show that this aligns to an online version of Weihrauch reduction.
- We can also have ratio preserving Weihrauch reductions.
- The performance ratio of a minimization problem (e.g. coloring here) is
$$\frac{|\{f_\chi(G \restriction n)\}|}{|\{\chi(G \restriction n\}|}.$$
- E.g. Famously First Fit Bin Packs with Performance ratio 2. (currently 1.7)
- One example as above is colouring. E.g. a graph of pathwidth k can be online coloured by $3k-2$ many colours reduces to chain covering of interval orderings. (Kierstead and Trotter)
- G is an interval graph of width k, if it can be represented as intervals I_x for each $x \in V$ with $I_x \cap I_y \neq \emptyset$ if $xy \in E$, and the cutwidth is $\leq k+1$. Also called bounded pathwidth

Computable analysis

- So $f : 2^\omega \to 2^\omega$ is online computable if for all α, $f(\alpha \upharpoonright n) = f(\alpha) \upharpoonright n$.
- There are obvious extensions of this. For a fixed function g, f is g-online computable if $f(\alpha \upharpoonright g(n)) = f(\alpha) \upharpoonright n$. An obvious case is when $g(n) = n + k$, which would be online with delay k.
- E.g. $+$ on the reals (below) is online computable with constant 2.

Representations

- Suppose we want to look at computable relationships between a totally disconnected space like 2^ω, and, for example, $\mathbb{R}$. Topological considerations rule out "computable" injective functions from $\mathbb{R}$ to 2^ω, since we have see such functions must be continuous. In the finite case, no such topological considerations occur.
- So let X be our (topological) space, with $\mathbb{R}$ as a canonical model. As above we could define a representation of a space X as a partial function $\delta : \omega^\omega \to X$, so that elements $x \in X$ have *δ-names* p_x (strictly a set $\{p_x \mid \delta(p_x) = x\}$). Note that x can have many names p_x; consider the case of names being Cauchy sequences and the space being the reals.

Proposition

If f is online computable on $[a, b] \subseteq [0, 1]$ then $\int_a^b f(x)dx$ is online Lipschitz computable with constant 2.

- Represent a function $f : X \to Y$ is an exactly analogous manner to 2^ω, but taking into account non-uniqueness of representation. That is, $f(x) = y$ is represented (realized) by some $F : \omega^\omega \to \omega^\omega$ taking each p_x to some p_y. (The first is a δ_X-name and the second δ_Y, but will will suppress this explicitness in the pursuit of clarity.)
- Let f, g be as above. Then $f \leq_W g$, Weihrauch reducibility, is defined to mean that there are computable A and B defined now on ω^ω, such that for any p_x, and any representation (realizer) G of g,

$$A(p_x, G(B(p_x))$$

realizes f (i.e. is a name for $f(x)$). (Henceforth, we will suppress the coding when the context is clear; particularly in the case that we are dealing with a metric space.

Ratio Preserving Reductions

- For online applications, ratio preserving Weihrauch Reductions.

Definition

Let f, g be functions on 2^ω. Then f is called *ratio preserving online reducible to g*, $f \leq_O^r g$, if there are (type II) online computable functions A and B with and a constant d, such that for all n,

$$f(\alpha \upharpoonright n) = A(\alpha \upharpoonright n, g(B(\alpha \upharpoonright n)),$$

and the ratio of $c(f(\alpha \upharpoonright n))$ to $c(f_{\text{off}}(\alpha \upharpoonright n))$ is at most d times the ration of $c(g(B(\alpha \upharpoonright n)))$ to $c(g_{\text{off}}(B(\alpha \upharpoonright n)))$.

Fact

If $f \leq_O^r g$ then , for some $d > 0$, $\dfrac{c(f \upharpoonright n)}{c(f_{off} \upharpoonright n)} \leq d \, \dfrac{c(g \upharpoonright n)}{c(g_{off} \upharpoonright n)}$.

- A classical reduction is a ratio preserving Weihrauch reduction from colouring interval graphs to chain cover for interval orderings.

- Lots other applications of this setting.
- E.g. EX-learning, Distributed computing, Büchi automata, etc.
- The idea is to somehow tie these together.
- Here are two examples, one from Tree Decompositions, and one from proof theory:

- Graphs of bounded treewidth are usually solved by tree automata.
- But if we present a graph by a root to leaf online representation, we call a promise, then the apparatus of Courcelle's Theorem on MS_2 theory of graphs of bounded treewidth applies.

Theorem (D and Long Qian)

Given a formula $\varphi(X)$ which is first order on graphs and X only occurs positively or negatively in $\varphi(X)$, then the online problem corresponding to $\varphi(X)$ has an online algorithm (the greedy algorithm) which has constant competitive ratio for graphs of bounded degree.

- Principal tool: Gaifman's Locality Lemma, and greed.
- Associated with any structure is a Gaifman graph where more or less edges correspond to relations, in the sense that x and y are joined if they occur in some tuple of the structure A together. The Locality Theorem says every first order formula is equivalent to a boolean combination of "basic local sentences" of a small radius.
- Obviously there are lots of interesting questions open here. For example, is there any analog of this result for some set of φ where there is no promise. Maybe in bounded pathwidth?

Δ^0_2 Processes

- Imagine you are in a situation where the data you are dealing with is so large that you cannot see it all. At each stage s your goal is to build a solution f to some problem.
- Imagine you are in a situation where the data you are dealing with is so large that you cannot see it all. At each stage s your goal is to build a solution f to some problem.
- Or Online Bitartite Matching.

Definition

- A *limiting online algorithm* on 2^ω is a computable function A such that for each s, $A(\alpha \upharpoonright s)$ computes a string $\{f_A(n,s) \mid n \leq s\}$ such that $\lim_s f_A(n,s)$ exists for each n.
- As usual we would have $A(\alpha \upharpoonright g(s))$ for the g-online version.

Reductions

- We can then compare combinatorial problems by how fast their limits converge.
- We say that algorithm $A \leq_{O,lim} B$ if there is an online Weihrauch reduction reducing A to B such that $f_B(n,s) = f_B(n,t)$ for all $t \geq s$ implies $f_A(n,s) = f_A(n,t)$ for all $t \geq s$.
- This gives a fine grained measure of the complexity of combinatorial problems.
- Finitary Reverse Mathematics

Example

- A binary tree T of height n is called separating if for each $j \leq n-1$, for any node σ on T of height j, and $i \in \{0,1\}$, if $\sigma * i$ does not have an extension in T of height n, then for all τ of length j, neither does $\tau * i$.
- Let X_0 denote the space whose paths are separating trees, and X the paths of trees.

Proposition

There is a 2^{n+1}-limiting online reduction which finds limiting online paths in X from those in X_0.

- Also unexplored is the situation for online algorithms acting on Δ^0_2 inputs. E.g. modelling users in a network. So-called dynamic graphs.

New directions

- Many counterexamples come from not knowing Skolem functions in an online situation.
- Imagine you are in a maze and navigating. Unless you were in Harry Potter, you are not going to get new egresses from the current location appearing. In many situations in e.g. online graphs you would expect at least to know your immediate surroundings, without a global picture.
- This idea leads to a new class of online structures with algorithmic parameterizations:

Definition

1. A locally strongly online graph is an given by a filtration (G_s, N_s) where $N_s = N(G_s)$, the neighbours of G_s in $\lim_s G_s = G$.
2. A strongly online graph G is a filtration (G_s, H_s) where $H_{s+1} = N(G_s \cup H_s \cup \{v_{s+1}\})$.

- This could be re-prhrased as adding certain function symbols to the langauge.

- You can think of G_s as the blue part of the graph and $H_s - G_s$ (or $N_s - G_s$) as the red part of the graph. The online algorithm is running on the blue part of the graph only.
- In, for example, navigation we can see a space around us but only process the algorithm when we traverse it. We could also specify N_k instead of N for the k-neighbourhood of G_s.
- This accords with old work on "highly recursive" graphs. (Kierstead, Bean, Schmerl, etc)

Theorem (Askes)

If G is a strongly online graph, then G can be strongly online coloured in $2\chi(G)$ many colours.

We compare this with

Theorem (Schmerl 1980)

If G is a highly computable, k-colourable graph then, G is computably $(2k-1)$-colourable.

Theorem (Askes)

1. *For all k there is a strongly online k-colourable graph that cannot be strongly $(2k-1)$-online coloured.*
2. *There is a locally strongly online tree T which cannot be finitely coloured online.*

Pathwidth

Recall that G has pathwidth d if every vertex x can be represented as an interval I_x and such that if if $xy \in E$, then $I_x \cap I_y \neq \emptyset$. The width is the max cutqwidth of the decomposition, and this must be $\leq d + 1$. The cuts determine a set of bags of size $\leq k + 1$ in an order.

Theorem (Askes)

1. *Every strongly online graph, G, with strongly online pathwidth k can be strongly online coloured in $2k + 1$ colours.*
2. *There is a strongly online graph with pathwidth k that cannot be strongly online coloured in $2k$ colours.*

Can online path decompositions be built?

Theorem (Askes)

For all $n > 0$, there is a strongly online graph of pathwidth 2 but whose strongly online pathwidth is $\geq n$

A sample proof

We will diagonalize against all possible φ_e, online algorithms attempting to show that the pathwidth is $\leq n$. We present $n(2n+1)$ many vertices as the centres of paths which are ever growing. The blue vertices are the presented ones which are the centre. We wait till φ_e declares bags (intervals I_x) for vertices in these paths. As G has $n(2n+1)$ many vertices into bags of size $\leq n$, there must be $2n+1$ bags containing a vertex not in the other $2n+1$ many bags.

We now join up the ends of the lines (with vertices) to force the bags to grow.

The next pictures might help.

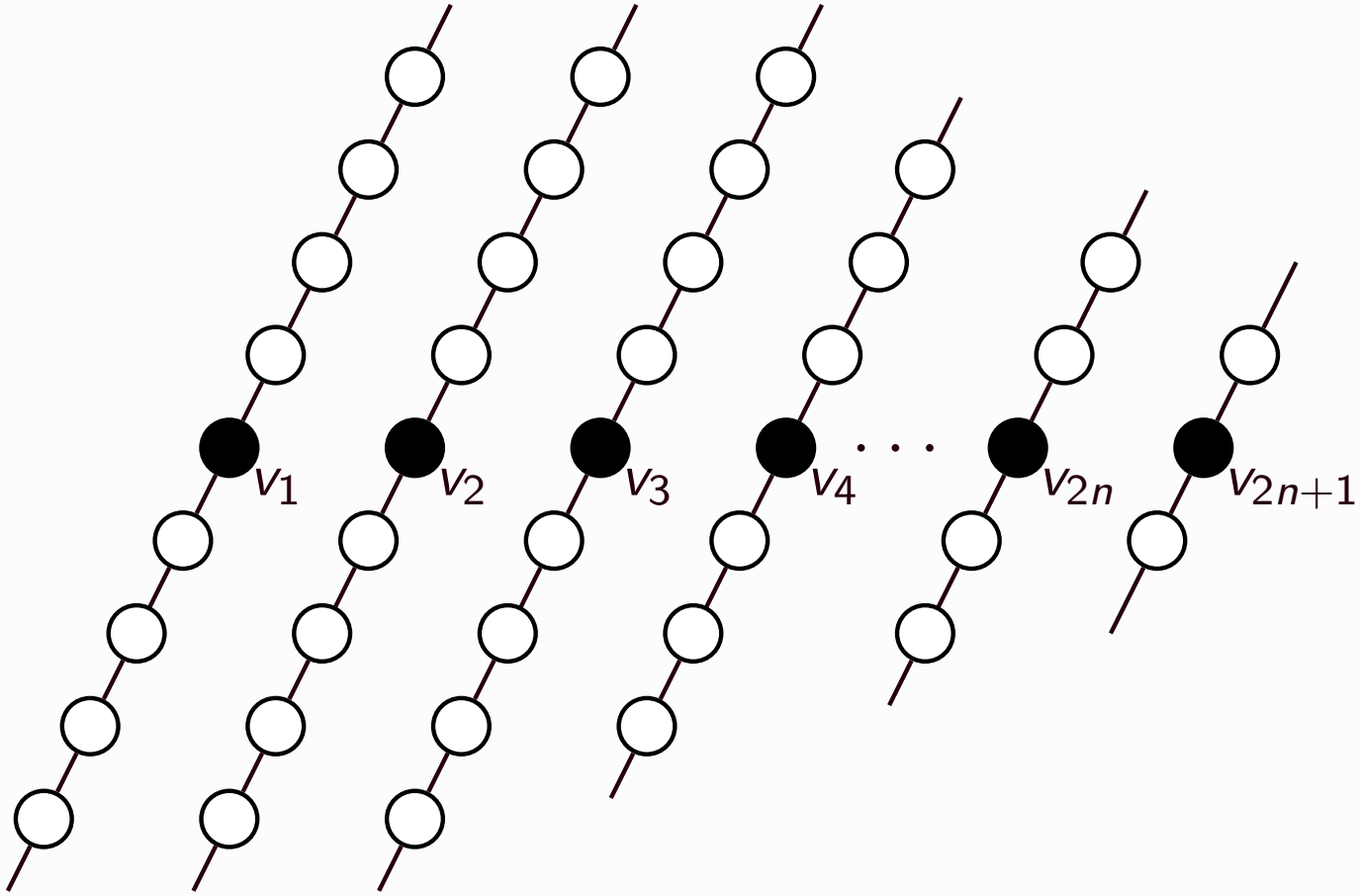

Figure: The initial $n+1$ paths and the presented vertices (in black)

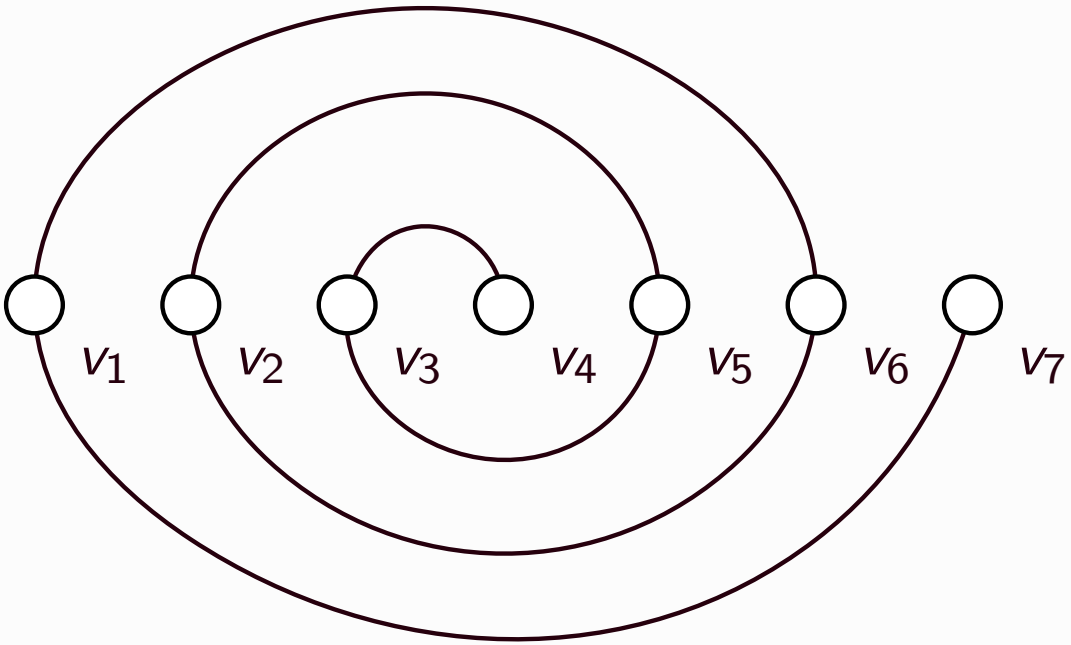

Figure: The connections between vertices (paths) for $n = 3$

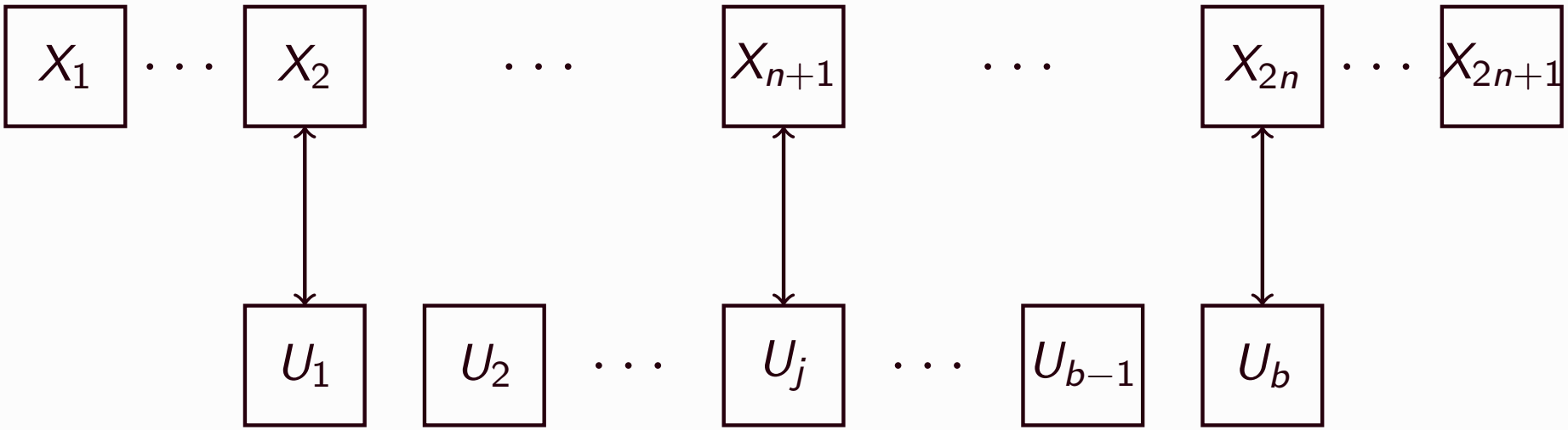

Figure: The bags in $\mathcal{Q}$ and the bags corrosponing to the path v_i—v_{2n+2-i}.

Thus X_{n+1} appears between U_1 and U_b. Hence there must be some U_j such that $U_j = X_{n+1}$. Therfore X_{n+1} contains some u_j. Then by letting i vary we can see that X_{n+1} must contain $n+1$ vertices (one from each path along with the vertex v_{n+1}). Therefore G_e satisfies R_e.

A model theoretical framework

- Speculation: We can have a constant ratio approximation scheme for first order properties if the Gaifman rank of the formula is below the online neighbourhood distance.
- Current model theory needs enriching to fit into this framework. For example, structures which seem to be reasonably well behaved online (e.g. good approximation ratios) seem to have a good shape according to some pseudo-metric on them. How to include this in model theory, whose inspiration was more about fields and the like.
- Further finitization of computability theory.

References

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- Foundations of Online Structure Theory, II, `https://arxiv.org/abs/2007.07401` final version in Logical Methods in Computer Science, 2021.
- M. Askes "Adversarial and Online Adversarial Games" MSc Thesis, Victoria University, in preparation.
- Lots of papers on Melnikov's and Ng's home pages.

Thank You