Q-reducibility and Other Strong Reducibilities

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National University of Singapore, July, 2024
A Bevy of Reducibilities

- **(m-reducibility)** $A \leq_m B$ iff for computable $f$, $x \in A$ iff $f(x) \in B$.

- **(tt-reducibility)** $A \leq_{tt} B$ iff for some computable $f$, $B$ models the boolean expression $\sigma(f(x))$ where $x_i \in \sigma(f(x))$ is interpreted as $i$. For example, $f(4) = \sigma(4)$ might say $((x_1 \lor \neg x_3) \land x_5) \lor x_{17}$, so that $4 \in A$ iff either $(5 \in B$ and at least one of $(1 \in B$ or $(3 \notin B))$ or $(17 \in B)$ holds. Equivalently $A \leq_T B$ via $\Phi$ such that $\Phi^X$ is total for all oracles $X$.

- This is called **conjunctive tt-reducibility** iff $x \in A$ iff $D_{f(x)} \subseteq B$.

- **(wtt-reducibility)** $A \leq_T B$ via $\Phi$ with $\varphi(x) \leq f(x)$ for computable $f$. 
I will only talk about examples from “classical” computability theory, and not e.g. computable analysis or descriptive set theory.

There are, of course, examples in those areas.

One I could highlight comes from recent work of Day, D, and Westrick and then Kihara, in analysis.

The analog of Turing reducibility is something called parallelized Wiehrauch reducibility, and you can define versions of $m$- and $tt$-reducibility there.

In the boldface version, $tt$-reducibility gives a “computable explanation” of the Bourgain hierarchy in Baire classes (Kechris etc).

Classically, most reducibilities here tend to be like many one. So lots of open material to explore.
Why Bother?

- Well, why not? Theses to write, papers to write etc.
- There are, however, other very good reasons.
- **First**, reductions in actual mathematical situations tend to be strong. Computing the determinant of a matrix is an example of an \( m \)-reduction from non-singular matrices to \( \mathbb{R} \setminus \{0\} \). Most reductions in complexity theory are \( \leq_P^m \).
- More examples below.
- **Second**, even if you only care about \( \leq_T \), strong reductions give insight into it. This idea goes back to Post (1944).
- Finally you get into the **fine structure** of problems.
Some Examples from Mathematics

- If you are working in algorithmic randomness, a natural reduction for you is $\leq_{tt}$.
- For example, if $X \leq_{tt} Y$ then as $\Phi^Y = X$ is total, measures can be transferred.
- For example,

**Theorem (Demuth)**

*If* $X \leq_{tt} Y$ *and* $Y$ *is 1-random, then the (wtt-)degree of* $X$ *contains a 1-random.*

- The proof entails

$$\rho(\sigma) = \lambda(\bigcup\{[\tau] : \forall n < |\sigma| (\Phi^\tau(n) \downarrow = \sigma(n))\}).$$

- Much of the work of Slaman-Woodin and Reimann-Slaman uses $tt$-reductions.
Another “Randomness” Example

- This one uses \textit{wtt}-reducibility.
- Suppose that $\alpha$ is left-c.e. This is equivalent to it being a halting probability.

\textbf{Definition}
We say c.e. $A$ presents $\alpha$ if $\sum_{\sigma \in A} 2^{-|\sigma|} = \alpha$.

- (Easy) If $\alpha$ is left-c.e. then it has a computable presentation.

\textbf{Theorem (D and Laforte)}
\textit{There is a noncomputable (high) $\alpha$ with only computable presentations.}

- Question What can be said about the degrees of presentations of some $\alpha$?
If $A$ and $B$ present $\alpha$ so does $A \oplus B$.

Theorem (D and Laforte)

- If $A$ presents $\alpha$ then $A \leq_{wtt} \alpha$ with use identity.
- If $A$ presents $\alpha$ and $B \leq_{wtt} A$ then there is a presentation $\hat{B}$ of $\alpha$ with $\hat{B} \equiv_{wtt} B$. (This the wtt degrees of presentations form an ideal.)

The proof’s basic idea is that if $b$ enters $B$ at stage $s$, we put enough strings into $\hat{B}$ of an appropriate length $\langle b, n_s \rangle$ to account for the measure from the use of the reduction $\Gamma^A = B$.

- Counting quantifiers we see that the wtt-degrees of presentations of $\alpha$ forms a $\Sigma^0_3$ ideal in the c.e. wtt-degrees.

Theorem (D and Terwijn)

*Given any $\Sigma^0_3$ ideal in the wtt degrees, there is an $\alpha$ whose presentations are exactly that ideal.*
Another example.

(Metakides and Nerode) $V_\infty$ is the universal computable vector space over (e.g. $\mathbb{Q}$) with canonical basis $\{e_i : i \in \omega\}$. Every computable (c.e.) vector space $V$ is a quotient $V \cong_{comp} V_\infty/W$ for some computable (c.e.) $W \leq V_\infty$.

This structure attracted a lot of interest in the 1980’s and 1990’s.

**Theorem (D and Remmel)**

$B$ is the wtt-degree of a basis of c.e. $W \leq V_\infty$ iff $B \leq_{wtt} V$. 
Sometimes strong reducibilities code the possible combinatorics of the reductions.

**Theorem (Collins)**

*If* \( a \) *is a c.e. tt-degree, then there is a finitely presented group with word problem of degree* \( a \).

**Theorem (M. Ziegler)**

*Even for c.e. presented groups, this is not true for btt-degrees; which means that there is a constant bound on the size of the truth table.*
**Theorem (Overbeek)**

*Every c.e. m-degree is the degree of a word problem for a finitely presented semigroup.*

**Definition (Jockusch)**

A set $A$ is called **multiplicative** iff there is a computable $f(\cdot, \cdot)$ such that for all $x \in A$, $y \in A$ iff $f(x, y) \in A$ iff $f(y, x) \in A$.

- Examples include sets of words representing the same subgroup of a finitely presented group or cancellation semigroup.
Theorem (Jockusch)

- No simple non-hypersimple set is multiplicative.
- Recall that $A$ is hypersimple if $\overline{A}$ infinite and for all infinite c.e. setrong arrays $\{D_g(x) : x \in \omega\}$, $D_g(x) \subseteq A$ for some $x$. $A$ is simple if we restrict the arrays to singletons. (i.e. $|W| = \infty$ implies $W \cap A \neq \emptyset$.)
- and hence cannot be the $m$-degree of the word problem of a cancellation semigroup.
- It is easy to construct a simple non-hypersimple set.
A good example of using strong reducibilities in this way is $\leq_{\text{wtt}}$.

$\leq_{\text{wtt}}$ is distributive on c.e. degrees.

$A \leq B_1 \oplus B_2$ implies $A = A_1 \sqcup A_2$, with $A_i \leq_{\text{wtt}} B_i$.

Proof: $\Gamma^{B_1 \oplus B_2} = A$, and speed things up so that every stage is expansionary in that $\ell(s + 1) > \ell(s)$ where 

$$
\ell(s) = \max \{ x : \forall y \leq x (\Gamma^{B_1 \oplus B_2}(y) = A(y)[s]) \}.
$$

Then if $z$ enters $A_{s+1} = A_s$ some least thing enters $B_1$ or $B_2$, etc.

Sometimes we can transfer facts from the wtt degree to the T-degrees.
**Definition (Ladner and Sasso)**

A has **contiguous** Turing degree iff for all $B \equiv_T A$, $B \equiv_{wtt} A$.

**Definition (Ambos-Spies, Downey-Welch)**

A is called **strongly atomic** (anti-mitotic) if, whenever $A_1 \sqcup A_2 = A$ is a c.e. splitting $A_1, A_2$ form a minimal pair.

**Theorem (D-W)**

*If A is contiguous and strongly atomic, then there is a computable isomorphism from the $\Sigma^0_3$ atomless boolean algebra of c.e. splits of A to the complemented degrees below $\text{deg}_T(A)$.**
With a little work and results of Nies, this can be used to establish the undecidability of the c.e. wtt- and T-degrees.

To wit: Nies showed that the theory of an effectively dense boolean algebra interprets arithmetic. In the case of the wtt-degrees use an exact pair theorem for ideals, and in the T-degree case build a specific set where it does.

Similar ideas were used by Ambos-Spies and Soare to show that the c.e. degrees have infinitely many 1-types.

You construct for each \( n \), a contiguous degree \( a_n \) which is the join of exactly \( n \) contiguous degrees which do not bound minimal pairs.
Q-DEGREES

**Definition (Tennenbaum)**

\[ A \leq_Q B \text{ iff there is a computable } f \text{ such that } x \in A \text{ iff } W_f(x) \subseteq B. \]

- First came to prominence via Post’s programme.
- Post suggested that if \( A \) has a “very thin” complement it would not be T-complete.
- Inspired by the fact that if \( A \) is simple then \( A \) is not \( m \)-complete.
- And hypersimple sets are not \( tt \)-complete.
On the c.e. sets this really is a reducibility, since $W_f(x)$ can be taken as finite. (Odifreddi attributes this observation to someone called Soloviev.)

Moreover, we can always have a unique $m(x, s)$ such that, at any stage $s$, if $m(x, s)$ enters $B_{s+1} - B_s$, then either $x \in A_{s+1}$ or a new $m(x, s + 1) \not\in B_{s+1}$ is picked, and such that $\lim_s m(x, s)$ exists.

Kind looks like a “fickle” $m$-reduction, and we say $x$ can enter $A$ only if $m(x, s)$ enters $B$.

On the c.e. sets $A \leq_Q B$ means $A \leq_T B$.

Henceforth, everything is c.e., so $\leq_Q$ is a strong reducibility.
Post’s Problem was whether there is an intermediate c.e. Turing degree, or are all c.e. problems simply the halting problem in disguise.

**Theorem (Friedberg, Muchnik)**

There are incomparable c.e. Turing degrees.

- Friedberg and Muchnik solved this not using Post’s programme.
- Why would Post think a thinness property of $\overline{A}$ guarantee incompleteness?
- $\phi' = \{\langle x, y \rangle : \varphi_x(y) \downarrow\}$ so there are lots of elements of $\phi'[s]$ waiting to halt.
- So if $\phi' \leq_m A$, and $A$ is c.e. then this must also be true of $\overline{A}$. So simple sets solve Post’s problem for $\leq_m$.
- Similar argument for hypersimple (in fact hypersimple c.e. sets are not even wtt-cuppable (Downey-Jockusch)).
Post’s programme, in its original form fails.

**Theorem (Yates)**

There is a $T$-complete maximal set. Here co-infinite $A$ is maximal if for all c.e. $W \supset A$, either $W =^* A$ or $W =^* \omega$.

Soare in a BSL announcement “conjectured” that every c.e. noncomputable set is automorphic to a complete set.

**Theorem (Cholak, D, Stob)**

No property of the lattice of supersets of $A$ alone can guarantee Turing incompleteness.
However:

**Theorem (Harrington and Soare)**

*There is a property $P$ such that there are c.e. sets satisfying $P(A)$, and for any c.e. set $A$ with $P(A)$, $A$ cannot be complete.*

- $P(A)$ is a 4 quantifier statement obtained from analysing how the “automorphism machinery” fails when you try to prove all c.e. noncomputable sets are automorphic to complete ones.
Marchenkov (1976) gave a resurrection of Post’s programme in the original thinness spirit.

**Definition (Jockusch)**

A is called **semirecursive** if there is a computable function $f(x, y)$ such that for all $x, y$, $f(x, y) \in \{x, y\}$ and

$$x \in A \lor y \in A \iff f(x, y) \in A.$$
**Lemma**

If c.e. $B$ is semirecursive and c.e. $A \leq_T B$, then $A \leq_Q B$.

Proof. Let $\Phi^B = A$, and accelerate things so that every stage is expansionary. Suppose $x \not\in A_s$. Consider $D_s = \{z : z \leq \varphi(x, s) \land z \not\in B_s\} = \{b_1, \ldots, b_{n(s)}\}$. Consider the pairs of elements of $D_s$. Compute $f(b_1, b_2) = b_{i_1}$. Note if either $b_1$ or $b_2$ enter $B - B_s$, then $b_{i_1}$ does. Now consider $f(b_{i_1}, b_3) = b_{i_2}$. The if any of $b_1, b_2, b_3 \in B - B_s$, then so is $b_{i_2}$, etc. So we can compute an elements $m(x, s) = b_{i_{n(s)}},$ where enters $B - B_s$ if $B$ changes on the use $\varphi(x, s)$ after stage $s$. This is a $Q$-reduction.

**Corollary (Marchenkov)**

If a semirecursive set is $T$-complete, it is also $Q$-complete.
Unfortunately, no (hyperhypersimple) maximal set is $Q$-complete. (Soloviev, Gill and Morris)

No (hhs-) maximal set is semirecursive. (Martin)

**Definition (Malcev)**

An equivalence relation $\equiv$ is called **positive** if $x \equiv y$ is c.e.. $A$ is called **closed** if $x \equiv y$ implies $x \in A$ iff $y \in A$.

- Now we can re-do the lattice of c.e. sets replacing $=$ by $\equiv$ for some positive $\equiv$.
- $A \equiv$-finite means that it has only a finite number of cells.

**Lemma (Marchenkov)**

If $A$ is $\equiv$-maximal (even hhs), then $A$ is not $Q$-complete.

- Proof. For a contradiction assume it is, so that the halting problem is $\leq_{Q} M$. Then we use the Recursion Theorem to build an infinite collection of sets in $\overline{K}$ and these will point at a weak array hitting $\overline{M}$.
**Theorem (Degtev)**

There is a positive \( \equiv \) and an \( \equiv \)-maximal semirecursive c.e. set.

Proof. First to make \( M \) semirecursive use a “dump” construction \( a \) enters \( M - M_s \) then also put \( a + 1, \ldots, s \) into \( M \). Then use an \( e \)-state construction a al Friedberg, but note that for Friedberg if we have \( m_{i,s}, \ldots, m_{j,s} \) in \( \overline{M}_s \) and want to make \( m_{i+1,s+1} = m_{j,s} \) we need to put the middle elements into \( M \), whereas here we simply declare them \( \equiv \) to \( m_{i,s} \).
Dobritsa noticed that $Q$-reducibility was relevant to combinatorial group theory.

For all c.e. $X$ there is a finitely generated c.e. presented group with word problem the same $Q$-degree as $X$.

**Definition (W Scott)**

$G$ is existentially closed (algebraically closed) if all $W_i(x_j, g_k) = e$ and $W(x_j, g_k) \neq e$ have solutions in $G$ or there is no solution in any group extension of $G$. This is the group version of being algebraically closed.

**Theorem (Belegradek)**

Suppose that $H_1, H_2$ are finitely generated c.e. groups, with $H_2$ nontrivial. Then $H_1$ is embeddable in any EC group that $H_2$ is embeddable iff $H_1 \leq Q H_2$. Consequently, $H_1$ is embeddable into every AC group iff $H_1$ has a solvable word problem.
Earlier, Macintyre proved the only if for $\leq_T$ and Belegradek proved that the iff is not true for $\leq_T$.

No a.c. group can have a c.e. presentation, which is vaguely strange to me as the construction looks like a Henkin construction.

The proof is by C. Miller, reported in Macintyre (1972). It relies on the fact that EC groups are simple, and if it has c.e. presentation it would then have a solvable word problem and thus so too would finitely generated subgroups, contradicting a result of Miller.

I am not sure whether people have investigated how complicated EC extensions are. ?One jump?
- For non-c.e. sets, $\leq_Q$ is not so well behaved. For example $\{e : \varphi_e \text{ total}\} \leq_Q \emptyset'$.

- If we define $A^Q = \{e : W_e \subseteq A\}$ then $A^Q \equiv_Q A$, and hence $\emptyset' \equiv [\emptyset]_Q$.

- This is also true for general, non-c.e. groups. To remedy this Ziegler defined Ziegler reducibility, where $A \leq_Z B$ means $A \leq_Q B$ and $A \leq_e B$, where $\leq_e$ is enumeration reducibility.

- That is $A \leq_e B$ means there is a $W$ such that $x \in A$ iff $\exists D(\langle x, D \rangle \in W$ and $D \subseteq B$.

- $\leq_Z$ is strictly stronger than $\leq_T$ on arbitrary sets and is useful for things like AC groups applied outside of finitely generated c.e. presented groups.

- Largely unexplored to my knowledge.
\[ \leq_e \text{ is a weak reducibility.} \]
\[ \text{But has had applications in computable algebra.} \]
\[ \text{For example, in Linda Jean Richter’s PhD Thesis she used it to construct models without degree (i.e. in their isomorphism type).} \]

**Theorem (Richter)**

Let \( T \) be a theory of a computable language \( L \). Suppose there is a recursive antichain \( W\{V_i : i \in \omega\} \) of finite \( L \)-structures and a method of combining, for all \( S \subseteq \omega \), the collection \( \{V_i : i \in S\} \) into a structure \( V_S \) with the following properties:

1. \( V_S \) is a countable model of \( T \).
2. \( V_S \) is enumeration reducible to \( S \).
3. \( V_i \) embeds into \( V_S \) iff \( i \in S \).

Then there is a set \( A \) such that the isomorphism type of \( V_A \) has no (least Turing) degree.
Richter used this on Abelian groups, to construct one with no degree of its isomorphism type.

The only other method I know is one she used for linear orderings using minimal pairs.

Echoes Joe Miller’s proof that there are continuous functions with no degree.
I remark that Belegradek-Macintye results on $Q$- and EC groups probably inspired by the relativised Higman Embedding theorem.

**Theorem (Higman)**

A finitely generated group is embeddable into a finitely presented group iff it has a c.e. presentation.

**Theorem (C. F. Miller, Higman-Scott)**

Given two finitely generated groups, $H_1, H_2$, then $H_1$ is embeddable into a group finitely presented over $H_2$, iff $H_1$ is enumeration reducible to $H_2$.

Belegradek (1996) has an extension to certain algebras.
STRUCTURE OF $\leq_Q$ DEGREES OF C.E. SETS

- Call this $R_Q$.
- Upper-semilattice with join induced by $\oplus$.
- Some things are inherited from $\leq_T$, such as there are minimal pairs.

THEOREM (AMBOS-SPIES AND FISCHER)

$R_Q$ is not a lattice and not distributive.
- Not distributive: make $B \leq A_1 \oplus A_2$, so that $B \not\equiv_Q B_1 \oplus B_2$ with uses $m_1(x, s)$, $m_2(x, s)$. Roughly bait and switch. Not a lattice, using a variation of Jockusch’s non-inf construction, for example.
**Theorem (D, Laforte, Nies)**

$R_Q$ is a dense USL, has nonbranching degrees, and has an undecidable first order theory.

- The proofs are genuinely complicated.
- One interesting feature of $R_Q$ is that it has join irreducible elements.
Define the dump set as follows, given a computable enumeration \( f(\omega) = A \). Let \( \overline{D}_s = \{d_{0,s}, \ldots, d_{n,s}, \ldots\} \), and let \( D_{s+1} = D_s \cup \{d_{f(s),s}, \ldots, d_{s,s}\} \).

Then \( D \equiv_T A \) and \( D \) is semirecursive (and hypersimple).

**Theorem (D, Laforte, Nies)**

\( D(A) \) has maximal \( Q \)-degree in the \( T \)-degree of \( A \). The \( Q \)-degree of \( A \) is join irreducible.

Proof. It is maximal as it is semirecursive. The second part follows from the following more general result.

**Theorem (D, Laforte, Nies)**

Suppose that semirecursive \( D \leq_Q A \oplus B \). Then either \( D \leq_Q A \) or \( D \leq_Q B \).
Proof: First try to make $D \leq_Q B$. Suppose that $D \leq A \oplus B$ via $m$, and $f$ witness the semirecursiveness of $D$. Given some $x$ find some $y > x$ which will enter $D$ if $x$ enters $D$, and $m(y, s)$ is targeted at $B$. Then let $x$ point at $m(y, s)$. If this purported $Q$-reduction fails, then it can only be that some least $x \notin D$ yet $m(y, s) \to \infty$ and all of the $m(y, s)$ relevant to $x$ enter $B$. Choose some $d \notin A$. Then for $y > x$, if $y$ points at $B$, as above we will map $y$ to $d$. Else it must point at $A$.

**Corollary (Ambos-Spies and Fischer)**

The $Q$-degree of $\emptyset'$ is join irreducible.
Some recent results

- When you have join irreducible elements you can have what are called Ahmad Pairs. $A, B$ such that $A \nleq B$, and such that for all $X \leq A$, either $A \leq X$ or $X \leq B$.

**Theorem (Ben-Sahar, D, Soskova)**

Neither $R_Q$ nor $R_{sQ}$ below have these pairs.

**Definition (Omanadze)**

$A \leq_{sQ} B$ iff $A \leq_Q B$ via some $Q$-procedure $\Gamma$ for which there is a computable function $f$ with $\gamma(x) < f(x)$.

- $\leq_{sQ}$ implies $\leq_{wtt}$ on the c.e. sets.

**Theorem (Omanadze)**

$R_{sQ}$ is a dense USL, which is not a lattice. It has join irreducible elements.

- Question How do $\leq_Q$ and $\leq_{sQ}$ interact?
**Definition (BDS)**
We say that a $Q$-degree $[A]$ is *sQ*-contiguous iff for all $B \equiv_Q A$, $B \equiv_{sQ} A$.

**Theorem (BDS)**
*Every c.e. Turing degree contains a sQ-contiguous Q-degree.*

- There are even stranger $Q$-degrees.

**Definition (Batyryshin)**
We say that a $Q$-degree $[A]$ is *sQ*-singular iff for all $B \equiv_Q A$, $B \equiv_m A$.

**Theorem (Batyryshin)**
*There exist non-zero sQ-singular Q-degrees.*

**Theorem (BDS)**
*They cannot have low Turing degree. Can be high.*
- We had hoped that since $\leq_{sQ}$ refines $\leq_{wtt}$ it would also be distributive.

**Lemma (BDS)**

$R_{sQ}$ is not distributive.

- The problem is that permitting does not work here. e.g.

**Theorem (BDS)**

There is a non-computable c.e. set $D$ such that, for all $A \leq_{Q} D$, $A$ is not simple.

- The proof uses a tree of strategies argument.
- Also $\leq_{sQ}$ and $\leq_{Q}$ interact badly.

**Theorem (BDS)**

There exist $A \equiv_{Q} B$ such that the $sQ$-degrees of $A$ and $B$ form a minimal pair.

**Theorem (D, Laforte, Nies)**

There exist $A \equiv_{T} B$ such that the $Q$-degrees of $A$ and $B$ form a minimal pair.
Theorem (BDS)
No Ahmed triples.

Theorem
No initial segment of the c.e. sQ-degrees is a lattice. (This should be compared with the c.e. wtt-degrees.)
But the sQ-degrees seem more stable within the wtt degrees:

Theorem (BDS)

- Every c.e. wtt degree contains a maximal c.e. sQ-degree.
- If $A \leq_{wtt} B$ are c.e., then there exists $D \equiv_{wtt} A$ with $D \leq_{sQ} A, B$. Hence if $A \equiv_{wtt} B$ then if the infimum of $A$ and $B$ in the Q-degrees or sQ degrees, should it exist, will have the same wtt-degree as $A$. 
Work is still ongoing.

For example, can $\Sigma^0_3$ ideals have exact pairs?

What is the complexity of the theories?

Do $Q$-degrees have applications in algebra/analysis outside of the ones so far. What about other finitely generated structures?

Lattice embeddings: 1-3-1 surely, but what about $S_7$?

$sQ$ degrees outside of $R_{sQ}$. Remains a reducibility.

Thank You.