Computably Enumerable Degrees in Π^0_1 Classes

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Thanks

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References

- On the c.e. degrees realizable in Π^0_1 classes, with Csima and Ng
- On the c.e. degrees realizable in separating classes, (tentative title) with Greenberg, Turetsky, Wu
- Thin classes of separating sets, Reed Solomon, Contemporary Mathematics 425 (2007), 67-86.
- Cenzer-Remmel surveys, and preliminary version of book on Π^0_1 classes.

General Question

- (Computably bounded) Π^0_1 classes have become ever more important, and it would be nice to understand the kinds of members they can have.
- There are *lots* of older results implicitly with this theme. E.g. Low Basis Theorem.
- Starting point: Kreisel Basis Theorem Every Π^0_1 class has a member of c.e. degree.
- What else can be said about the c.e. members?

Motivation

- Well nothing I guess. The class could be empty. Not very sporting as an example.
- What nonzero c.e. degrees are realizable?
- Long ago it was shown that any nonzero c.e. (Δ^0_2, in fact) degree can be the only nonzero member of a Π^0_1 class. Proof e.g. use retraceable sets.
- In such a class, the c.e. degree of members are $\{\mathbf{0}, \mathbf{a}\}$.
- If we take separating classes representing PA degrees, then by Arslanov's Completeness Criterion, the only c.e. degree of a member will be $\{\mathbf{0}'\}$.
- Can we realize other singletons? What else is possible?

The Question

Question

What sets of c.e. degrees are realizable in a Π^0_1 class?

- For example, what singletons, what finite collections, what other collections of c.e. degrees. Also:

Question

What about c.e. degrees in more specialized Π^0_1 classes such as separating classes or thin classes?

Index sets

- The following definition is useful here:

Definition

We say that a real X is realizable in a Π^0_1 class C if $X \equiv_T \alpha$ for some $\alpha \in C$.

$$W[C] = \{e : W_e \text{ is realizable in } C\}.$$

$$\mathbf{W}[C] = \{\mathbf{a} : \exists X \in \mathbf{a} X \in W[C]\}.$$

Proposition (CDM)

$W[C]$ is Σ^0_4.

- The proof is to calculate the definition.
- Also use if $S \subset \mathbb{N}$, $G(S) = \{e : W_e \equiv_T W_j \wedge j \in S\}$. Also Σ^0_4.

Question

So what Σ^0_4 sets are realizable? What about if the class has restricted rank? What about if it is special? thin? separating?

If $\mathbf{0}$ is present

- This is the easiest case of all, and we will begin with this
- We sketch how to do $\{\mathbf{0}, \mathbf{a}\}$.
- We have $A = \cup_s A_s$. C is built in stages. At stage s we specify up to length s. Initially have the *spine* 0^n and then $0^n 1^{s-n}$ running off to the left. (board)
- If i enters A_{s+1} code with $0^i 1^{s-i}$ then work above $0^i 1^{s-i}$ similarly for $j > i$ (board). Can kill if you like things to the right, or not as desired.

Theorem (CDN)

*If S is a Σ^0_3 set of indices and an index for **0** is in S, then S can be realized. Furthermore, the rank of the Π^0_1 class can be 2.*

- For the proof it is easiest to begin with S Π^0_2. So $e \in S$ iff $\forall s \exists t R(e, s, t)$.
- The idea is to devote some specified part of the tree T with $P = [T]$ to coding e. For instance, choose the extensions of $0^{e+1}1$.
- Then in that cone, we need to code W_e each time e looks correct.
- That is, each time e is confirmed again by R, see what has happened to W_e (what has entered) below some boundary given inductively, and $\to \infty$, and code as in the singleton case.
- While we are away believing the Σ^0_2 outcome, extend by 0's (This is where we use **0**) (board).

- For the Σ^0_3 case, then $e \in S$ iff exists $j = j(e)$ where the Π^0_2 case happens.
- Have one construction like the above for each guess at j. Either one will succeed as its is believed infinitely often, or each will result in a finite collection of computable paths.

Not everything is possible

- The Π^0_3 set $\{e : W_e \text{ not computable}\}$ is not realizable by Jockusch and Soare (and in fact no downward dense set not containing an index for $\mathbf{0}$).

Theorem (CDN)

Also if $\mathbf{a}$ is c.e. you cannot realize $\mathbf{R} - \{\mathbf{a}\}$.

Theorem (CDN)

There is a Π^0_3 S with an index for $\mathbf{0}$ such that no Π^0_1 class has $W[P] = G(S)$.

- The proof uses straightforward diagonalization.
- For any c.e. set $L >_T \emptyset$, one can effectively obtain a c.e. set D and reduction Ψ such that $D = \Psi^L$ and $D > \emptyset$ and $D \not\geq_T L$.
- Iterate from a low$_2$ c.e. set L, we get computable increasing functions g and ℓ such that $L = W_{g(0)} >_T W_{g(1)} >_T W_{g(2)} >_T \cdots$, and $W_{g(n)} = \Phi^L_{\ell(n)}$ for all n.

- We define $n \notin S$ iff whenever $n = g(e)$, then there are some i, j for which
  (I) $\Phi_i^{W_{g(e)}}$ is total and is in $[T_e]$, and
  (II) $\Phi_j(\Phi_i^{W_{g(e)}})$ is total and equals $W_{g(e)}$.
- S is Π^0_3, using the low-2-ness of L.
- No P such that $W[P] = G(S)$.
- To see that S can be made to contain $\emptyset$, note that $S \cup \tilde{S}$ is also good for any Π^0_3 set $\tilde{S}$ where $G(\tilde{S}) \cap G(W_{g(n)}) = \emptyset$ for every n.

- The low-2-ness is needed in some sense.

Theorem

If S is a Σ^0_4 index set containing an index for a computable set, and S_H is a computable sequence of high c.e. degrees with $deg(S) \subseteq S_H \cup \{\mathbf{0}\}$, then there is a rank 2 P with $W[P] = S$.

Singletons and the special case

- It is easy to get **0** as a singleton, take a finite class. Again not sporting.

Theorem (CDN)

There is a Π^0_1 class which is perfect, and every member is either computable or has minimal degree.

- Compare with

Theorem (Groszek-Slaman)

There is a special Π^0_1 class with only members of c.e degree or minimal degree.

Nonzero Singletons

Theorem (CDN)

For any Σ^0_3 set S there is a Π^0_1 class with $W[P] = S$.

- We begin discussing singletons.

Lemma (CDN)

For any c.e. degree $\mathbf{a}$ *we can (effectively from an index from a member of* $\mathbf{a}$*) find a* Π^0_1 *perfect class with* $W[P] = \mathbf{a}$.

The separating case

- Separating classes are defined as $C = \{Z$ such that $Z \supseteq A$ and $Z \cap B = \emptyset$ where A and B are disjoint c.e. sets. Write $S(A, B)$ for such C.
- They are naturally of interest wrt e.g. WKL_0 and the like.
- Not everything is possible. For example.

Theorem (Downey, Jockusch, Stob)

Suppose that $A \cup B$ has array computable degree (that is, there is a $f \leq_{wtt} \emptyset'$ such that for all $g \leq_T A \cup B$, for almost all x, $g(x) < f(x)$.) Then $\mathbf{0}' \in W[S(A, B)]$.

Some known results

Theorem (Jockusch-Soare)

There exist A_1, A_2, B_1, B_2 with every $S(A_1, B_1)$ forming a minimal pair with $S(A_2, B_2)$. (D-Greenberg have shown this is possible below and promptly anc degree)

Theorem (Solomon)

There is pair A, B with $S(A, B)$ having every member $Z' \equiv_T \emptyset' \oplus Z$, hence each c.e. member low.

Theorem (Solomon)

There is pair A, B with $S(A, B)$ such that if Z and X are members, either $Z =^* X$ or they are Turing incomparable.

- Results still in formation here. Some of the questions still open are embarrassing.

Question

Is **a** possible for a nonzero **a**?

- We tried to do this using a *supermaximal pair*. A maximal pair is a A, B such that for all disjoint $\hat{A}, \hat{B}$ containing A, B respectively and suitably disjoint, $\hat{A} - A, \hat{B} - B$ are finite. (Downey thesis, used in coding e.g. ideals in computable rings.)
- This pair is supermaximal if the same is true for $\hat{A}, \hat{B}$ of c.e. *degree*.

Theorem (DGTW)

There are no supermaximal pairs. That is, there is always a separating set of c.e. degree Z, with $Z - A$ and $Z - B$ both infinite.

- The proof is kind of interesting, and nonuniform.

Theorem (DGTW)

There are c.e. $A \equiv_T B$ such that they have low degree and compute every separating Z of c.e. degree.

- If we have **0** we can easily realize upper cones.

Theorem (DGTW)

For any c.e. $\mathbf{a}$ *we can realize* $\{\mathbf{0}\} \cup \{\mathbf{d} : \mathbf{d} \geq \mathbf{a}\}$.

- Let A be c.e. and consider the graph of the modulus function of A, $\hat{A}$.. Then the complement is uniformly introreducible, and hence any infinite subset (and hence any separator Z) can compute A, and a simple argument shows that coding is possible.

- Without **0** this is more difficult but can be done.

Theorem (DGTW)

For any c.e. $\mathbf{a}$ *there is a* A, B *with* $A \equiv_T B \in \mathbf{a}$ *and the c.e. members* $S(A, B)$ *having degrees exactly* $\mathbf{R} \cap \{\mathbf{d} : \mathbf{d} \geq \mathbf{a}\}$

Thin

- The case where C is thin (meaning that if $\hat{C}$ is a Π^0_1 subclass of C then $\hat{C} = C \cap M$ for some clopen M) is still being explored.
- Even what possible c.e. degrees can be realized as members is nontrivial.

Theorem (D, Cenzer, Jockusch, Sore)

There is a c.e. **a** *which cannot be realized in a thin class.*

- Proof (Board)

Theorem (DCJS)

However, realizable c.e. degrees are dense.

Theorem (D, Wu, Yang)

If $\mathbf{b}$ *is c.e.a* $\mathbf{0}'$ *then there is a c.e. degree* $\mathbf{a}$ *which cannot be realized and* $\mathbf{a}' = \mathbf{b}$*.*

Theorem (DWY)

There is a $\mathbf{a} < \mathbf{b}$ *with all of* $[\mathbf{a}, \mathbf{b}] \cap \mathbf{R}$ *thin.*

Thank you