Alan Turing and Computation

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Plan

- In this first lecture I will talk about some of Turing's contributions to the theory and practice of computation.
- A couple of developments from this legacy.
- The a little break of 5 minutes.
- In lecture 2, I will look at how his ideas from computation have been applied in the theory of algorithmic randomness.
- Also in lecture 2 I will look at his anticipation of this theory with his work on normality.

The Scope of Turing's Work

- Turing worked famously on the Entscheidungsproblem
- How this had the key idea of stored program computers via universal machines....ACE
- Ideas in cryptography both breaking cryptosystems and making them for voice.
- Word problems in cancellation semigroups.
- Cryptography and Statistics.
- "Checking a Large Routine" symbolic model checking and program verification. His thesis was in this "Systems of logic based on ordinals" and looked at transfinite methods of verification.
- "Local Programming Methods and Conventions," programming methodology.
- "Rounding-off Errors in Matrix Processes" Ill-posed problems and "the other" theory of computation. (not discussed here, see the article by Lenore Blum)
- Intelligent Machinery and the Turing test
- Computer chess (before stored program computers)

Plan

- Clearly, I cannot discuss all of this here. For more refer to the archive http://www.turing.org.uk/sources/biblio.html and various upcoming Turing volumes.
- I will try to do some of these in some detail and use a broad sweep for others.
- I will begin with the birth of the digital computer, and the Turing Machine.

Born of logic

- Thanks to Moshe Vardi for this and the next quote (my highlighting).
- Cosma R. Shalizi, Santa Fe Institute.

> *If, in 1901, a talented and sympathetic outsider had been called upon (say by a granting agency) to survey the sciences an name a branch that would the the least fruitful in the century ahead, his choice might well have settled upon mathematical logic, and exceedingly recondite field whose practicioners could all have fit into a small auditorium. It had no practical applications, and not even that much mathematics to show for itself: its crown was an exceedingly obscure definition of cardinal numbers.*

More recently

- Martin Davis (1988) Influences of mathematical Logic on Computer Science.

  When I was a student, even the topologists regarded mathematical logicians as living in outer space. Today the connections between logic and computers are a matter of engineering practice at every level of computer organization.

- Read a somewhat dated but wonderful collection in the Bulletin of Symbolic Logic: On the Unusual Effectiveness of Logic in Computer Science (Halpern, Harper, Immerman, Kolaitis, and Vardi).
- Echoes Wigner's 1960 article "The unreasonable effectiveness of mathematics in the natural sciences," and Galileo's "The book of nature is writ in the language of mathematics."

Entscheidungsproblem

- The most famous mathematician of his generation, David Hilbert, famously asked for a decision procedure for number theory.
- This was born of 19th centure determinism which imagined the universe as a big machine whose path was completely determined.
- Of course this version is still open: is the universe mechanical; can the universe produce incomputability?
- Still others ask can the universe produce anything that is computable, or is everything random?
- This is too big for my brain and I will stick with questions in formal logic and number theory.
- Notice a weaker question is can everything that is true of some formal system be proved in such a system? (completeness)

A history of impossibility proofs

- You are taught at school that your can solve the quadratic $ax^2 + bx + c = 0$.
- Ingredients: numbers, $+, -$ $\times$, division, $\sqrt{}$, maybe cube roots, powers etc.
- Operations : Combine in sensible ways.
- Can we do the same for degree 3, the "cubic"

$$ax^3 + bx^2 + cx + d = 0?,$$

  what about degree 4, etc.
- This was one of the many questions handed to us by the Greeks.
- The answer is yes for degree 3 and degree 4.

The Sorry Tale

- For degree 3 this was first proven by Ferro (1500).
- Ferro left it to his son-in-law Nave and pupil Fiore.
- Fiore challenged Tartaglia (in 1535) who then re-discovered the solution with a few days to spare, leaving Foire in ignomy.
- Tartaglia also kept it secret, but told Cardano, who promised by his Christian faith to keep it secret, but....
- in 1545 Cardano published it in his great text Ars Magna
- Additionally Cardano published how to extend to degree 4, being discovered by a student Ferrari.

Abel

- Finally, in 1823, a young Norwegian mathematician, Abel proved that there is no recipe using the given ingredients for the degree 5 case, the quintic.
- (The paper was called "Memoir on algebraic purifications..." rather than "Memoir on algebraic equations..." due to a typsetting error.)
- (My favourite error in one of my own papers referred to a journal "Annals of Mathematical Logic" as "Animals of Mathematical Logic."It made me think of some of my colleagues!)
- Nobody believed him, for a long time. (There had been an earlier announcement by Ruffini, which contained "gaps".)

Galois

- Evariste Galois (1811-32) eventually gave a general methodology for deciding if a given degree n equation admits a solution with the ingredients described.
- This work laid the basis for group theory.
- Galois method is to associate a group with each equation, so that the equation is solvable in terms of the given ingredients (arithmetic operations and radicals) iff the group has a certain structure on its subgroups. This is one of the gems of m athematics.

Changing Ingredients

- It is not true to say that the quintic has no solution, just none with the given ingredients.
- We can add some new operations "elliptic functions" and show that there is a method of solving the general degree n equation.
- These operations are "mechanical" so there is a an algorithm for solving all such equations.

Hilbert

- David Hilbert, 1900, working from a background of 19th century determinism basically asked the question of whether mathematics could be finitely “mechanized”.

David Hilbert, 1912 — one of a group of portraits of professors which were sold as postcards in Göttingen

Hilbert

- David Hilbert, 1900, working from a background of 19th century determinism basically asked the question of whether mathematics could be finitely "mechanized".
- Can we create an algorithm, a machine, into which one feeds a statement about mathematics or at least in a reasonable "formal system" and from the other end a decision emerges: true or false.
- Or, for a given formal system, can we eventually produce proofs all the "truths" of that system.
- Hilbert also proposed that we should prove the consistency of various formal systems of mathematics.

Formal System

- Leibnitz's dream, the first order logic Frege 1879.
- It is not important what this is, save to say the type envisioned would be a bunch of axioms, saying things like
- for all numbers x, x+1 exists,
- for all numbers x and y $x + y = y + x$,
- and other "obvious truths."
- plus rules of inference, like "if whenever P is true then Q is true, and whenever Q is true then R is true; then whenever P is true R is true."
- induction.

Gödel

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Gödel

- Hilbert's dreams were forever shattered by a young mathematician, Kurt Gödel.
- He prove the two incompleteness theorems.
- The first incompleteness theorem says that any sufficiently rich formal system has statements
- expressible in the system
- true of the system, but
- cannot be proven in the system.
- Secondly no sufficiently rich formal system can prove its own consistency.
- The collective intuition of a generation of mathematicians was wrong.
- Of course, Tarski proved that some rich systems like Euclidean Geometry are decidable.

The confluence of ideas in 1936

- First Church, then Turing and Post proposed models for computation. In retrospect, it is clear that since the models are equivalent that Church first showed that that Entscheidungsproblem is undecidable (by lambda definable functions).
- Proposed his thesis that these modelled all effectively computable processes.
- Church λ-definable functions. Herbrand-Gödel general recursive functions (proved the same by Kleene in 1936).
- Post a Turing machine like model.
- Turing : Turing machine.

Why Turing?

- The arguments by Church for the acceptance of λ-definability as capturing were (i) by example (i) confluence (ii) step by step arguments echoing logical proof systems and (iii) failure of diagonalization.
- These were not accepted at the time. See e.g. Davis, Gandy 1995, Soare 2012, Kleene 1995.
- First and foremost Turing has a conceptual analysis giving what many regard as a proof of the thesis that TM's capture what is computable.
- This analysis is the fundamental contribution of Turing's paper.
- See "The Universal Turing Machine: A Half Century Survey" R. Herken (ed) Springer 1995 (2nd Ed).

Turing's analysis

- He considers an abstract human computor
- By limitations of sensory and mental apparatus we can (i) Fixed bound for the symbols. (ii) fixed bound for number of squares (iii) fixed bound to the number of actions at each step (iv) fixed bound on the movement. (v) fixed bound on the number of states.
- This justifies TM's
- Also examples like Bessel functions.
- Gandy, Soare (and others) argue that Turing proves any function calculable by an abstract human is computable by a TM.
- For instance, Turing's analysis was the only thing that convinced Gödel.

- Gandy (1995):

> *What Turing did, by his analysis of the processes and limitations of calculations of human beings, was to clear away, with a single stroke of his broom, this dependency on contenporary experience, and produce a characterization-within clearly percieved limits- which will stand for all time..... What Turing also did was to show that calculation can be broken down into the iteration (controlled by a "program") of extremely simple concrete operations; so concrete that they can easily be described in terms of (physical) mechanisms. (The operations of λ-calculus are much more abstract.)*

The Universal Machine

- The other major contribution was the notion of a universal machine, a compiler.
- Turing has the first universal machine. The idea that there could be a single machine which interpreted programs to emulate any other machine.
- Church's ingeneous solution did not use the "halting problem" encoded.

Prehistory and posthistory

- Babbage said of his Analytical Engine (not a stored program machine) "it could do anything except compose country dances." (quoted in Huskey and Huskey 1980, p 300)
- Actually now computers do compose country dances.
- The idea that a computer could be universal was a long time penetrating.
- Howard Aitken (1956)

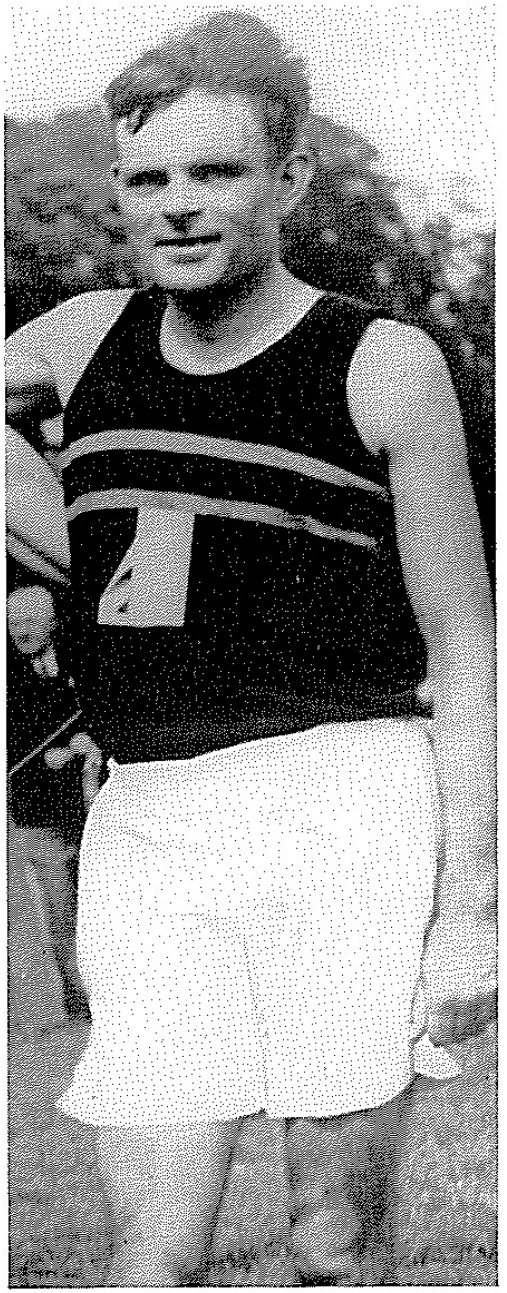
> *If it should turn out that the basic logics of a machine designed for numerical solution of differential equations coincide with the logics of a machine intended to make bills for a department store, I would regard this as the most amazing coincidence that I have ever encountered.*

- Read more on this in Martin Davis' or Herken's books.

The birth of computers

- Turing was aware of the possibilities of using stored program machines.
- The war intervened and Turing famously was involved in Hut 8. Identified as one of the key codebreakers.
- Bletchley park had 10,000 members during the war, but it was widely regarded that the people in Hut 8 and, in particular Tutte, Foss, and notably Tuuring were key players.
- For example, the Bombe modified from Polish ideas, kind of running the Enigma machine backwards
- Incidentally, I learned only recently that the codes used in the war were actually secure, but brought down by humans, who provided the ingresses for the cracking.

Aged 5



After a successful race. May, 1950

The Enigma Machine, employed by the Germans to encrypt classified and sensitive messages during World War II. (HultonArchive/Getty Images)

John von Neumann, Princeton, 1932

The Birth of Computers

- Turing learnt of the possibilities for large scale computers through the work of Tommy Flowers on the Colossus machine. A several tonne valve machine, the first large scale computer. (NB This is not what you read in texts, but now known after declassified documents.)
- McCulloch and Pitt used Turing ideas to show the control mechanism for a TM could be simulated by a finite collection of gates with delays. (1943)
- Von Neumann knew of Turing's ideas and with two other co-authors prosed a practical architecture for stored program machines. He uses the McCulloch and Pitt ideas. (1945) EDIAC.
- Later ENVAC.
- Turing proposed ACE (automated computing engine), Architecture very influential.
- However, first stored program computer in Manchester, in lab run by his lifetime friend Max Newmann.
- Turing wrote the (first) programming manual.

A Basic Undecidable Question

- Using the fact that all Turing machines can be enumerated we can use a beautiful argument of Cantor about differing sizes of infinite sets(!) to show that there is no algorithm to decide to following question.
- INPUT Turing machine number x and an input y.
  QUESTION Does the machine x halt on input y.

- (Proof. Suppose that we could decide this algorithmically. We can then use the decision procedure to construct a machine M that halts on input n if T_n does not halt on input n, and our machine M does *not* halt if machine T_n does halt on input n. Then M would be some machine T_m, but then $T_m(m)$ halts if and only if $M(m)$ halts iff $T_m(m)$ does not halt....)
- We code this problem into others.

Example-Conway's Theorem

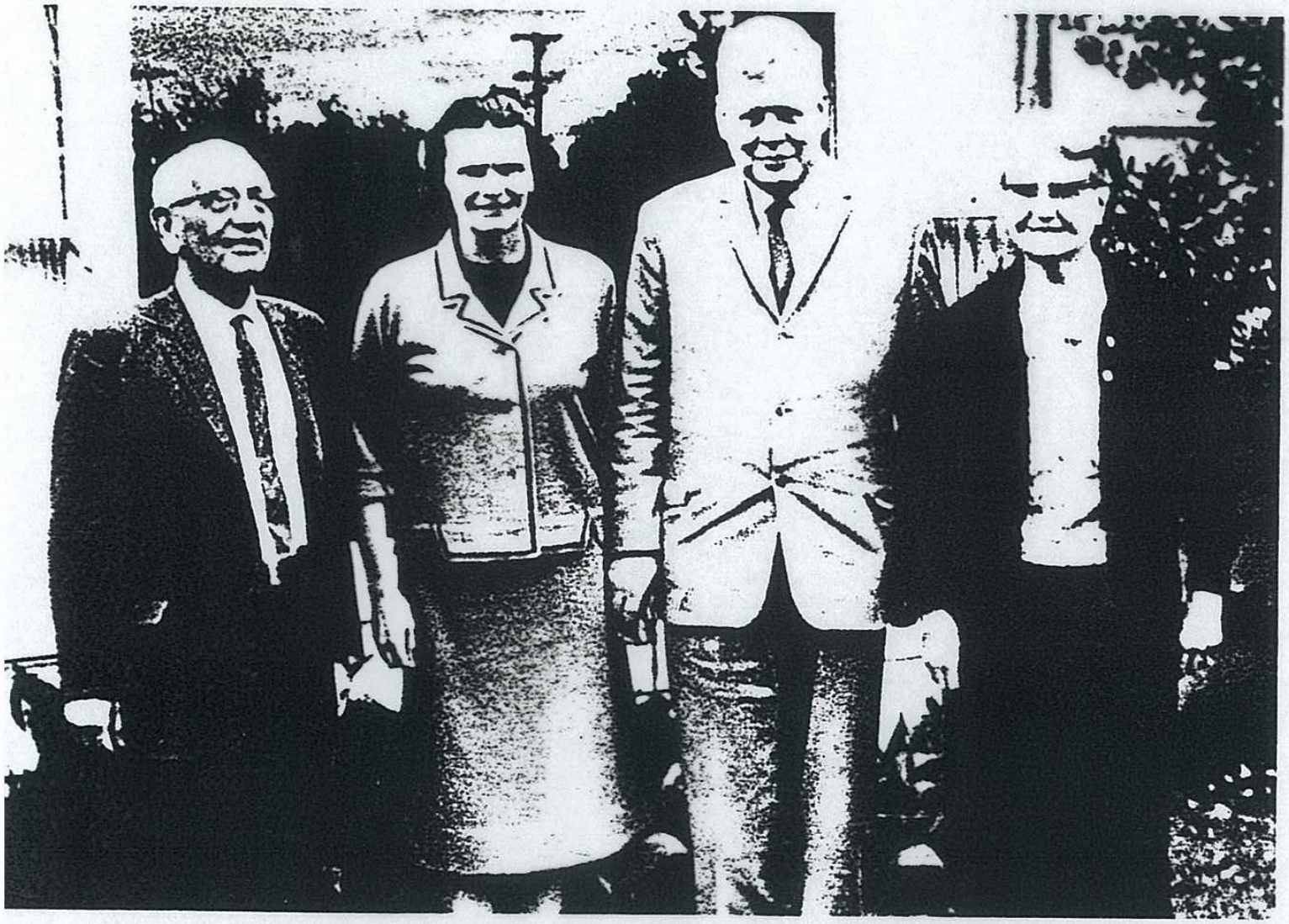
- Collatz-type functions. $f(x) = \frac{x}{2}$ if x is even, and $f(x) = 3x + 1$ if x odd.
- e.g. $f(3) = 10$ $f(f(3)) = 5$, get the sequence, 3,10,5,16,8,4,2,1
- Do you always get to 1? (Still open)
- General type of question : e.g $g(x) = 1/2x$ if x divisible by 4, $g(x) = 5x - 1$ if x has remainder 1 when divided by 4, etc.
- John Conway (1980's) showed that there is no general algorithm to decide
  INPUT A system like the above, and a number x.
  QUESTION Does x get back to 1?

Wang Tiles

- INPUT a set of square coloured tiles of the same size. Only same colour borders next to one another.
  QUESTION Can an initial configuration be extended to colour the plane?
- Wang in the 60's showed that there is no algorithm to decide this.

Hilbert's 10th problem

- INPUT A polynomial P in variables $x_1, ..., x_n$
  QUESTION Is there a positive solution to the equation $P = 0$?
- Matijasevich, after Julia Robinson in the 70's showed there is no algorithm to decide such questions.
- But there is now a polynomial whose only positive rational zeroes are the primes!



This shows myself, Julia Robinson, Raphael Robinson, and my wife.

$$
\begin{aligned}
Q(a,\ldots,z) = (k+2)\{&1-[wz+h+j-q]^2\\
&-[(gk+2g+k+1)(h+j)+h-z]^2\\
&-[2n+p+q+z-e]^2-[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2\\
&-[e^3(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2\\
&-[16r^2y^4(a^2-1)+1-u^2]^2\\
&-[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2\\
&-[n+1+v-y]^2\\
&-[(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2\\
&-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2\\
&-[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2\\
&-[z+pl(a-p)+t(2ap-p^2-1)-pm]^2\}.
\end{aligned}
$$

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Recent Examples

- Recently it was shown by Braverman and Yampolsky (STOC, 2007) that Julia sets can be noncomputable, any halting problem being codable. (Also Blum-Smale-Shub, but that's another story.)
- Julia set: $z \mapsto z^2 + \alpha z$, where $\alpha = e^{2\pi i\theta}$.
- Nabutovsky and Weinberger (Geometrica Dedicata, 2003) showed that basins of attraction in differential geometry faithfully emulated certain computations. Refer to Soare Bull. Symbolic Logic.
- Remark: Earlier and ignored work by Lee Rubel on universal PDE's.

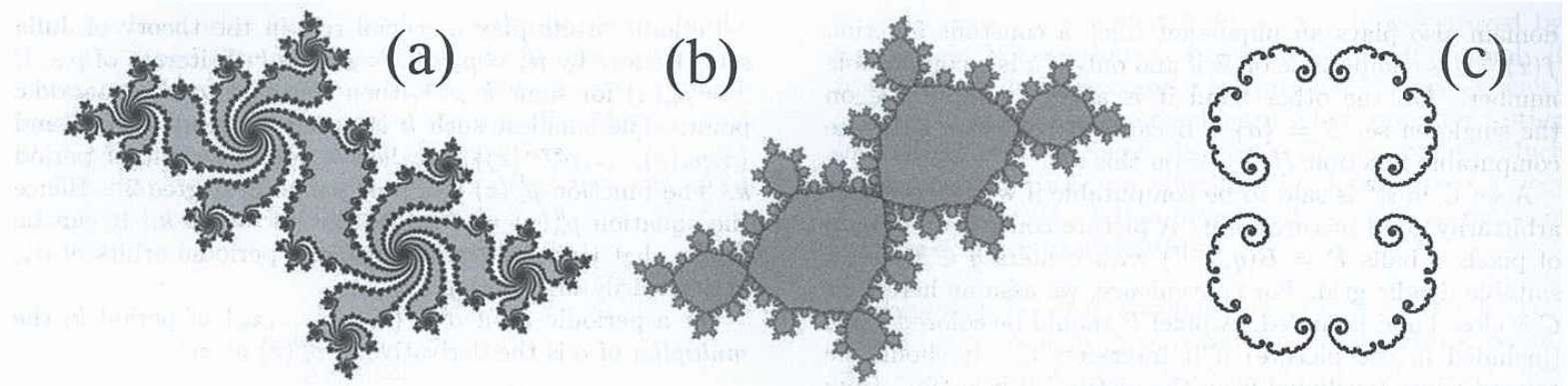

Figure 1: Examples of quadratic Julia sets J_p (black), and filled Julia sets K_p (gray); orbits that originate at white points escape to ∞; note that on picture (c) $K_p = J_p$, since K_p has empty interior

- The answer is inescapable: these diverse mathematical objects, tiles, Conway sequences, and polynomials can be used to simulate computations.

Using computation to show No Invariants

- Much of mathematics is concerned with classification of structures (groups rings, DE's etc) by invariants.
- Bases for vector spaces, Ulm invariants for abelian groups.
- How can we show that no invariants are possible?
- A computability theorists's view.

Analytic=Σ^1_1

- The halting problem is called Σ^0_1 in that $\varphi_x(y)$ halts iff

$$\exists t \in \mathbb{N} \varphi_x(y)$$

  halts in t steps. (And $\varphi_x(y)$ halts in t steps is computable.) This is arithmetic, where the quantifier searches over $\mathbb{N}$.
- Almost all problems in normal mathematics are analytic.
- A is analytic or Σ^1_1 iff deciding $x \in A$ entails asking if there is a function f from $\mathbb{N}$ to $\mathbb{N}$ such that some computable relation holds for all $f(n)$.
- E.g. isomorphism is typically in Σ^1_1.

Σ^1_1 completeness

- Many problems in Σ^1_1 are much easier. E.g. isomorphism for finitely presented groups is Σ^0_3. (Is there a matching of generators for which every equation in the first holds in the second?)
- If some problem is shown to be Σ^1_1 complete, then no simpler set of invariants is possible.
- E.g. (Downey and Montalbán) the problem of deciding if two finitely presented groups have $H_i(G) \cong H_i(\hat{G}$ for $i \leq 3$ is Σ^1_1 complete.
- Uses the result that the isomorphism problem for computable torsion free Abelian groups is Σ^1_1 complete. (DM)

But does it matter?

- Most problems in real life seem to be tractable.
- For example: why do sat solvers work so well on real problems?
- For example: word problems are generically decidable.
- For example: big hardware is routinely verified.
- Your challenge: explain this.

Other work of Turing

- Lots of technical work in logic.
- Proofs of equivalence of the models. (JSL papers)
- Undecidability of the word problem for cancellation semigroups.
- Proposed methods for symbolic verification of programs. This has grown into modern model checking.
- Proposed methods of logically constructing programs.
- First computer chess program (1950). See the webcast of Kasparov's talk in Manchester, Turing 100 conference.

Machine Intelligence

- Famous unpublished paper on this from a sabbatical at Cambridge.
- His boss thought it was a "schoolboy paper". Now it is regarded as a classic.
- Later famously posed the Turing Test.
- Often mis-quoted as saying machine intelligence by the end of the 20th century. Actual quote (from a radio discussion with Max Newmann) "at least 100 years."
- Emphasized optimization as a key strategy for artificial intelligence, and realized in his chess program.

Some Other Things Left Out

- "Rounding-off Errors in Matrix Processes" Ill-posed problems and "the other" theory of computation.
- He was the first to properly study complexity of matrix algorithms like determinant computations when dividing by near zero quantities.
- This was centered in numerical analysis
- Morphogenesis: How do leopards get their spots?
- Suggests a simple mechanism based on partial differential equations.
- diffusion/reaction equations.
- Basically stable, but under peturbation creates a feedback loop.

Thank You