Effectively Categorical Torsion Free Abelian Groups

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Our Concern

- A computable structure $\mathcal{A}$ is computably categorical iff for all $\mathcal{B} \cong \mathcal{A}$, $\mathcal{A} \cong_{\text{computable}} \mathcal{B}$.
- There is a longstanding program to understand the relationship between $\cong, \cong_{\text{comp}}$, classical structure of $\mathcal{A}$ and logical structure of $\mathcal{A}$ in terms of definability.
- These all also have "higher up" versions, like Δ^0_α categocity, definability etc.

Theorem (Goncharov, 1975)

If $\mathcal{A}$ is $2-$decidable, then $\mathcal{A}$ is computable cat iff it is relatively computably cat iff it has an effective naming, that is a c.e. Scott family of existential formulae with parameters $\overline{c}$, such that for all $\overline{a}, \overline{b}$ if they satisfy the same ϕ, then they are automorphic.

More Recent Metatheorems

Theorem (Downey, Kach, Lempp, Turetsky)

If $\mathcal{A}$ is 1-decidable and it is computably cat, then it is relatively Δ^0_2 cat, as it has a Σ_2 Scott family.

Theorem (Downey, Kach, Lempp, Lewis, Montalbán, Turetsky)

For each $\alpha < \omega_1^{CK}$ there is a computably cat $\mathcal{A}$ which is not relatively Δ^0_α cat.

Categoricty questions for abelian groups

- When we specialize to specific structures within which it is hard to code graphs questions become more complex. You actually have to do some algebra!
- This is not too hard if you have torsion, and in particular p-groups.
- Khisamiev (and independently Ash-Knight-Oates later) gave a characterization of computable abelian groups in terms of generalized limitwise monotonic functions
- These have proven useful in lots of areas, $\aleph_1$ categorical theories, equivalence relations, linear orderings, etc.

Theorem (Goncharov, Smith)

A computable p-group is computably categorical iff it can be written in one of the following forms.

1. $(\mathbb{Z}(p^\infty))^\ell \oplus G$ *for* $\ell \in \omega \cup \{\infty\}$ *and* G *finite;*
2. $(\mathbb{Z}(p^\infty))^n \oplus (\mathbb{Z}_{p^k})^\infty \oplus G$ *where* G *is finite, and* $n, k \in \omega$.

- Calvert-Cenzer-Harizanov-Morozov A p-group is computably categorical iff it is uniformly computably categorical (and hence has a simple algebraic structure by results of Goncharov-Smith)

Torsion-Free Abelian Groups

- Here we will study torsion-free abelian groups. That is, they have no elements z with z^n trivial.
- Some kind of good behaviour.

Theorem (Khisamiev)

Every Π^0_{n+1} presentable torsion-free abelian group is isomorphic to one which is Δ^0_n-presentable.

- In general the isomorphism problem is very complex:

Theorem (Downey and Montalbán)

The isomorphism problem for torsion-abelian groups is Σ^1_1 complete.

- The point is that this result means that the cannot be reasonable invariants for the isomorphism problem.

Better algebraic classes

- The idea is to look at algebraically more tractable classes; this is what is done classically anyway.
- Recall that if G is a torsion-free then G embeds into $\oplus_{i \in F}(\mathbb{Q}, +)$. The cardinality of the least such F is called the (Prüfer) rank of G.
- Khisamiev proved that there is an effective embedding.

Rank One Groups

- The only groups we understand well are the rank one groups (and certain mild generalizations) If $g \in G$, define $t(g) = (a_1, a_1, \dots)$ where $a_i \in \{\infty\} \cup \omega$ and represents the maximum number of times p_i divides g. Say that $t(g) = t(h)$ if they are $=^*$, meaning that they must be ∞ in the same places, but otherwise are finitely often different. Thus we can write $t(G)$.
- For example, a divisible group would have $(\infty, \infty, \dots)$ as its type.

Theorem (Baer)

For rank 1 torsion-free abelian groups, $G \cong H$ iff they have the same type.

- One corollary is that if we consider $T(G) = \{\langle x, y\rangle \mid x \leq t(G)_y\}$, then G is computably presentable iff $T(G)$ is c.e.. (Mal'tsev)

Two Corollaries

- G is a computably categorical torsion-free abelian group iff it has finite rank.

Definition

A structure $\mathcal{A}$ has a degree iff $\min\{\deg(\mathcal{B}) \mid \mathcal{B} \cong \mathcal{A}\}$ exists.

- Strictly speaking, we would mean the isomorphism type here.
- Can define jump degree by replacing $\deg(\mathcal{B})$ by $\deg(\mathcal{B})'$.
- (Coles, Downey and Slaman) Every torsion free abelian group of finite rank has first jump degree.
- (Anderson, Kach, Melnikov, Solomon) For each computable α and $\mathbf{a} > \mathbf{0}^{\alpha}$ there is a torsion-free abelian group with proper α-th jump degree $\mathbf{a}$.

The infinite rank case

- It could be hoped that if G has infinite rank, then $G \cong \bigoplus_{i \in \omega} H_i$ with H_i of rank one.
- Alas, this is not true, however, there is a class of groups for which this is true, called completely decomposable for which this does happen.
- What about categoricity for such groups?
- We cannot hope for computable categoricity, but can hope for things "higher up".

The homogeneous case

- If $G \cong \bigoplus H$ for a fixed H then G is called homogeneous

Theorem (Downey and Melnikov)

Homogeneous computable torsion free abelian groups are Δ^0_3 categorical.

- The proof relies on a new notion of independence called S-independence generalizing a notion of Fuchs to sets of primes.
- Q, a set of primes, is S-independent (in G) iff for all $p \in Q$ and $b_1, \ldots, b_k \in G$,

$$p \mid \sum_{i=1}^{k} m_i b_i \text{ implies } p \mid m_i \text{ for all } i.$$

- This bound is tight.

But when can it be Δ^0_2 categorical?

- Recall that a set S is called semilow if $\{e \mid W_e \cap S \neq \emptyset\} \leq \emptyset'$.
- Semilow sets allow for a certain kind of local guessing, and aroze in (i) automorphisms of the lattice of computably enumerable sets (Soare) and in (ii) computational complexity as non-speedable ones. (Soare, Blum-Marques, etc.)

Theorem (Downey and Melnikov)

G is Δ^0_2 categorical iff the type of H consists of only 0's and ∞'s and the position of the 0's is semilow.

- The proof is tricky and splits into 5 cases depending on "settling times".
- We remark that this is one of the very few known examples of when Δ^0_2 categoricity of structures has been classified.

The general completely decomposable case

Theorem (Downey and Melnikov)

A completely decomposable G is Δ^0_5 categorical. The bound is tight.

The proof uses methods from the homogeneous case, plus some new ideas. The sharpness is a coding argument. For sharpness we use copies of $\oplus_{i\in\omega}\mathbb{Z} \oplus \oplus_{i\in\omega}\mathbb{Q}^{(p)} \oplus \oplus_{i\in\omega}\mathbb{Q}^{(q)}$., where $p \neq q$ primes and $\mathbb{Q}^{(r)}$ denotes the additive group of the localization of $\mathbb{Z}$ by r. Then a relation θ on this group which is decidable in one copy and very bad in another.
With some extra work we can also prove the following. We don't know if the bound is sharp here.

Corollary (Downey and Melnikov)

The index set of completely decomposable groups is Σ^0_7.

References

- Computable completely decomposable groups, (DM) submitted
- Effectively Categorical Abelian Groups, (DM) to apepar J. Algebra.

Thank You