Sacks Splitting Theorem Revisited

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Motivation

- One of the fundamental results of computability theory is Sacks' Splitting Theorem:

Theorem (Sacks, 1963)

If A is c.e. and $\varnothing <_T C \leq_T \varnothing'$ then there exists a c.e. splitting $A_1 \sqcup A_2 = A$ with $C \not\leq_T A_i$ for $i \in \{1, 2\}$.

- This fundamental result
  1. Showed that there were no minimal c.e. degrees,
  2. Ushered in one form of the infinite injury method (although it is not an infinite injury argument, but finite injury of "unbounded type".)
  3. Was the basis of huge technical progress on the c.e degrees.

For Example

Theorem (Robinson, 1971)

Everything c.e. If $C <_T A$ and C low, then $A = A_1 \sqcup A_2$ with $C \oplus A_1 |_T C \oplus A_2$. Hence every c.e. degree split over every lesser low c.e. degree.

Robinson's Theorem was very influential in that it showed how to use "lowness+c.e." a theme we follow to this day.

Theorem (Lachlan, 1975)

There exist $\mathbf{c} < a$ such that $\mathbf{a}$ does not split over $\mathbf{c}$.

Affected the architecture of computability theory thereafter. E.g. definability, decidability etc. Invented the $\mathbf{0}'''$ method to prove this result. Harrington improved Lachlan's Theorem to have $\mathbf{a} = 0'$.

Re-examining this

- Lots of questions can be asked about the 60 year old result.
- For example: "How unbounded is the finite injury?"
- In recent work, not talked about here this can be quantified:

Theorem (Ambos-Spies, Downey, Monath, Ng)

If A is c.e. then A can always be split into a pair of totally ω^2-c.a. c.e. sets. (Here totally ω^2 c.a. means that if $f \leq_T A_i$ is total, then it is ω^2-c.a. in the Downey-Greenberg classification.)

- Sacks' proof only gives ω^ω-c.a..
- Earlier Selwyn and I showed that this is tight:

Theorem (Downey and Ng, 2018)

There is a c.e. degree $\mathbf{a}$ such that if $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$ then $\mathbf{a}_i$ is not totally ω-c.a. for $i \in \{1, 2\}$.

- Lots of similar questions remain:
- For example:

Question

1. *Is the Ambos-Spies et. al. theorem valid if we also add cone avoidance?*
2. *What about adding lowness?*
3. *What can be said about the degrees which are joins of totally-ω-c.a. c.e. degrees? (This is a definable class.)*

Re-re-examining Sacks

- Here the question Guohua and I looked at:

Question

Is the natural analog for avoiding lower cones valid?

- The answer is no.

Theorem (Downey, Wu)

There are c.e. sets $B <_T A$ such that whenever $A_1 \sqcup A_2 = A$ is a c.e. splitting, then for some $i \in \{1, 2\}$, $A_i \leq_T B$.

- We remark that the degree analog is true because either $\mathbf{a}$ splits over $\mathbf{b}$, or $\mathbf{b}$ cups $\mathbf{a}_2$ to $\mathbf{a}$ for some $\mathbf{a}_2$ and we can then choose $\mathbf{b} < \mathbf{a}_1 < \mathbf{a}$ by Sacks's Density Theorem.
  (i.e. lower cone avoidance happens)

The Proof

- The proof is non-trivial, and uses the $\mathbf{0}'''$ method.
- We need $B \leq_T A$, say $\Xi^A = B$.
- Requirements $B \leq_T A$ and

$$R_e : W_e \sqcup V_e = A \to (\exists \Gamma_e(\Gamma_e^B = W_e) \vee \exists \Delta_e(\Delta_e^B = V_e)).$$

$$N_e : \Phi^B \neq A.$$

- We will define a rather complicated priority tree PT and there meet R_e at nodes τ, with outcomes $\infty <_L f$.
- The procedures Γ_e, Δ_e built via axioms as usual.
- We meet N_e at nodes σ.

The Basic Module

- Drop the "e"
- One $N = N_j$ at a σ below $\tau\widehat{\ }\infty$ for $R = R_e$.
- The overall goal of N is to have $\ell(j, s) = \max\{z \mid \Phi_j^B \upharpoonright z = A \upharpoonright z[s]\} > y$, for some y and put y into A whilst preserving $B \upharpoonright \varphi_j(y)$.
- The obvious problem is that if we put y into A_{s+1} then assuming $\tau\widehat{\ }\infty \prec TP$, y will enter one of W or V.
- Now, depending on which we believe we are proving, $(\Gamma^B = W) \vee (\Delta^B = V)$, this would then entail putting something into B, i.e. something below $\gamma(y, s)$ or $\delta(y, s)$.
- On the other hand, if we are monitoring only Δ, say, and y enters W and not V, we would not care.

- σ We will either prove that W is computable (finite in the basic module) (the Σ^0_3 outcome) or if no σ does this, then τ will prove $\Delta^B = A$. (the Π^0_3 outcome)
- In the general construction we build $\Gamma^B_\sigma = W$.
- That is, we are "favouring" V at σ, in cooperation with τ.
- N picks a follower x with a trace $t_0 = \delta(x, s)$.
- The strategy runs in cycles. At each stage we will have a trace $t_n = \delta(x, s)$.
- The goal is to try to have
  1. Either $\delta(x, s) > \varphi_j(x, s)$ when $\ell(j, s) > x$, or
  2. Put something into A which meets N_j and went into W.
- In the first case if x entered V, we could still correct Δ^B using $\delta(x, s)$ without injuring $\Phi^B_j(x) \neq A(x)[s+1]$ as $\delta(x, s) > \varphi(x, s)$.
- Now it might be that neither occurs. Then
  1. Everything we use (i.e. the t_n's) to attack N will enter V and not W. (Thus W is computable (in fact empty).)
  2. $\varphi(x, s) \to \infty$ and hence $\Phi^B_j(x) \uparrow$. Note that Δ will be partial, but that's okay, as σ gives a proof that W is computable (or $\Gamma^B_\sigma = W$, more generally).

Cycle n

- We hit σ and see $\ell(j,s) > t_n (> x)$.
- Case 1. $t_n = \delta(x,s) > \varphi_j(x,s)$.
  Action Put x into $A_{s+1} - A_s$. This will meet N. At the next $\tau\widehat{\ }\infty$-stage, if x enters V put $\delta(x,s)$ into both A and B, and correct Δ.
- Case 2 Otherwise. Put t_n into A_{s+1}. Wait till the next $\tau\widehat{\ }\infty$-stage.
  1. If t_n enters W, then N is met, and we need to do nothing else. Note that Δ^B remains correct.
  2. If t_n enters V put t_n into B and $\xi(t_n) = t_n + 1$ (for example) into A. Pick a large fresh number $t_{n+1} = \delta(x,s')$. and enter cycle $n+1$

Analysis

- Notice that we keep $B \leq_T A$ by force.
- If we pick infinitely many t_n, then we can conclude
  1. σ adds an infinite computable set into B and A.
  2. Nothing we add to A enters W, so (basic module) $W = \varnothing$ (in general, $\Gamma_\sigma^B = W$).
  3. $\Phi_j^B(x) \uparrow$ so N is met.
- In all other cases we will succeed in meeting N after a finite number of cycles, and $\Delta^B = V$ is valid, since in the case we use x, if x enters V we correct $\Delta^B(x)$ at the next $\tau\widehat{\ }\infty$-stage.

Two τ's one σ

- Things become more complex when we consider $\tau_0\widehat{\ }\infty \preceq \tau_1\widehat{\ }\infty \preceq \sigma$, with N_j at σ as before, and R_i at τ_i, say.
- First we consider two in their primary phases, meaning believing Π^0_3 but being alert for Σ^0_3.
- It is not reasonable that τ_1 can drive $\delta_0(z)$ to infinity on general priority grounds (i.e. for any z), by priorities.
- But the converse is okay by general $\mathbf{0}'''$-grounds, and we could restart τ_1.
- Thus at σ x will (initially) have two traces $t^0_n = \delta_0(x,s)$ and $t^1_m = \delta_1(x,s) > \delta_0(x,s)$; and these can be chosen from e.g. separate columns of ω.
- The primary goal is to get
  1. Either have $\delta_0(y,s) > \varphi_j(y,s)$, (for some y) or
  2. get $\delta_0(x,s)$ entering W_0, not V_0, after enumeration into A.

- If this never occurs, then as in the basic module,
  1. $\delta_0(x,s) \to \infty$, $\varphi_j(x,s) \to \infty$ and W_0 is empty,
  2. A computable set is enumerated into A,
  3. And, by the way we nest δ_0 inside of δ_1, this also drives $\delta_1(x,s) \to \infty$.
- So we have been enumerating $\delta_0(x,s) < \delta_1(x,s)$ which can be both taken as t_n into A at σ-stages.
- We might as well assume that $\delta_0(x,s) \not> \varphi_j(x,s)$ as this case is easy (more or less).
- We hit τ_0 at an expansion stage.
- Since this all looks like the basic module unless t_n enters W_0, we explore what to do when t_n enters W_0.

If $t_n = \delta_0(x,s)$ enters W_0, then currently we have no obligations to Δ_0^B. So we could play $\tau_0\widehat{\ }\infty$ and move to τ_1.

1. If t_n entered W_1, then we are lucky and have met N, and need do nothing more.
2. The universe is cruel, and of course t_n entered V_1.
   Thus we want to correct $\Delta_1^B = V_1$, and would change $B \upharpoonright \delta_1^B(x,s)$ into A at this stage s_1.
   To make sure that $\Xi^A = B$ is satisfied, we would also have to put (e.g.) $t_n^0 + 1 < t_n^1$ into A at s_1. Potentially this could later change V_0.
3. In the second case above at the next $\tau\widehat{\ }\infty$-stage s_2, we would see if $t_n^0 + 1$ entered W_0 or V_0.
4. If V_0, then we would need to correct $\Delta_0{}^{\text{`}B}(x,s)$, again and pretend the fact that "t_n^0 entered W_0 at s_1" never happened but could correct $\Gamma_\sigma^B(t_n^0)$.
   Now we'd be back in the basic module thinking that $\delta(x,s) \to \infty$.
5. If W_0, we discuss next page.

- At s_2 $t_n^0 + 1$ also entered W_0. Now, we are in a bit of a quandary.
  1. The B-change at s_1 allows us to correct $\Gamma^B \upharpoonright t_0 + 1$, with no further work.
  2. The fact that we changed $B \upharpoonright \delta_1^B(x, s)$ at s_1, means no further work is needed for Δ_1^B at the next $\tau_2\widehat{\ }\infty$-stage.
  3. But we can't now continue to keep moving $\delta_0(x, s)$ for $s > s_2$, since τ_0 has fulfilled its obligations.
     Thus the plan is to detach τ_0 from x, until τ_1 looks like it fulfils its obligations.
- To wit: We would now choose a $t_{n,1}^0 = \delta_0(t_n^0, s_2)$ large and bigger than $\delta_0(x, s_2) = \delta_0(x, s)$ and make this more or less $t_{n+1}^1 = \delta_1(x, s_2)$. (Assuming this is also a $\tau_2\widehat{\ }\infty$-stage).
- Again we only attack N at σ at σ-stages where $\ell(j, s) >$ all current traces.
- If we ever see $\delta(t_{n,1}^0, s_2) > \varphi_j(t_{n,1}^0, s)$ we can win by enumeration of $t_{n,1}^0$ into A (as in the basic module, with the role of x taken by $t_{n,1}^0$) and correct the Δ^B's.
- Assuming not, we continue until the next W_0 change at a $\tau_0\widehat{\ }\infty$ stage, and then work as above with the new numbers.

1. If also a W_1 change then we are done.
2. If a V_1 change then we use the two step process to first correct $\Delta_1^B(x,s)$ and then at the next $\tau_0\widehat{\ }\infty$ stage, see if another W_0 enumeration occurred. In this case we detach again and if not we continue.

- The only other possibility is that at some stage t, we see $\varphi_j(t_n^0,t) < \delta_1(t_n^0,t)$.
- Now the problem is that inevitably $\delta_0(t_n^0, s_2+1) = \delta_0(t_n^0,t) = t_n^0[s_2+1] < \varphi_j(t_n^0,s)$. That is, we can correct Δ_1 if t_n^0 entered V, but not Δ_0.
- It is now that we add $t_n^0 = \delta_0(x,t)$ into A.
  1. At the next $\tau_0\widehat{\ }\infty$ stage we see if t_n^0 enters W_0.
  2. If it does, then we can put $\delta_1(t_n^0,t)$ into A and B, meeting N and allowing for correction, where necessary, at the next $\tau_i\widehat{\ }\infty$-stages.
  3. If it enters V_0 then we would correct Δ_0 by putting t_n^0+1 into A and t_0^n into B, and go back to the primary sequence picking t_{n+1}^0.

Analysis

- If for any cycle we never get to a W_0 change, then cycle i, based on $t^0_{n,i}$ gives a proof that W_0 is computable, and $\varphi_j(t^0_m, s)$ is unbounded for some m. (Outcome of kind (g, i).)
- If there are infinitely many complete cycles resulting in a V_0 change, we get a proof that $\varphi_j(x, s)$ is unbounded, and W_0 is B-computable. (Outcome (g, u).)
- Otherwise we will win N with finite effect, and Δ_0 will be correct.
- Notice that on the assumption that $\Delta^B = V_0$ we only need concern Δ^B_1 with $W_1 \cap W_0$ and $V_1 \cap V_0$, since $A = (W_0 \cap W_1) \sqcup (W_0 \cap V_1) \sqcup (V_0 \cap V_1) \sqcup (V_0 \cap W_1)$. And $V_0 = (V_0 \cap V_1) \sqcup (V_0 \cap W_1)$ meaning that $(V_0 \cap V_1) \leq_T V_0$ and $(V_0 \cap W_1) \leq_T V_0$. So, it only when things enter W_0 we even need to monitor Δ_1.
- If either of the first two outcomes occur then $W_0 \leq_T B$, via Γ^B_σ and a version of τ_1 guessing this outcome will be below some kind of outcome of σ like $\sigma\widehat{\ }g$. It will accordingly only care about numbers entering V_0 for its primary Δ_1.

The other configurations

- Now we have $\tau_0\hat{\ }\infty \preceq \sigma\hat{\ }(\tau_0, g) \prec \tau_2$.
- This τ_2 mother "knows" that $W_0 \leq_T B$ is proven at σ and an infinite stream of $\delta_0(z, s)$'s will be entering B and A. First suppose $g = (g, i)$, say.
- It only issues axioms claiming $V_1 \cap V_0 \leq_T B$ via some $\Delta^B_{\tau_2}$.
- Some σ' extending $\tau_2\hat{\ }\infty$ has a follower x' with trace $\delta_2(x', s) > x_\sigma$.
- On realization via σ'-correct computations, bigger than $\delta_1(x', s)$, we can put $\delta_1(x', s)$ into A instead of $t^0_{m,i}$ as the case might be.
- Thus we can put $\delta_1(x', s)$ into A.
- Since this will enter V_0, by τ_0's assumption, Δ_1 will be correct. If it enters W_0 we meet σ.
- Entering V_0 means that τ_0 will put (more or less) it into B to correct V_0, in which case we can move $\delta(x', s)$ to the current $t^0_{m,i}$
- On reaching τ_2 we can correct Δ^B_1 if necessary.

- Now consider a version of τ_2 below $g = (g, u)$, so actually lots of numbers enter W_0, but later we put correctors into B and the primary t_n^0 sequence is resurrected.
- This would happen if we had $\tau\hat{\ }\infty \prec \hat{\tau}_2\hat{\ }\infty \prec \sigma\hat{\ }(g, u) \prec \tau_2\hat{\ }\infty$ where $\hat{\tau}_2$ is the original τ_2 mother, guessing Π^0_3.
- The only difference is there are infinitely many W_0 then V_0 changes, and the inner cycle slowly goes to ∞.
- If τ_2 guesses this, when we hit τ_2 we would correct Γ_1, Δ_2 as appropriate since τ_1 will use τ_2's numbers.

One τ, two σ's

- Now suppose that we have $\tau\hat{\ }\infty \prec \sigma_1 \prec \sigma_2$.
- In the case that σ_2 extends $\sigma_1\hat{\ }f$ no problem; finite injury.
- Thus suppose that σ_2 extends $\sigma_1\hat{\ }(\tau, g)$ for one of the infinitary outcomes of σ_1 giving the Σ^0_3 outcome for τ.
- Thus σ_2 expects an infinite stream of t^0_n of some type to enter V_0.
- Hence it should have no obligations to τ if this really is the case, but maybe it's not. This version of N at σ_2 believes that $W_0 \leq_T B$ via $\Gamma^B_{\sigma_1}$.
- The idea is that numbers associated with σ_1 will be shared by σ_2 in their uses $\Delta = \Delta_\tau$.

- When we visit σ_2 we give it some follower x', and we will give this the current t_n^0 for the current x (or $t_{i,m}^0$) at σ, for its $\delta(x', s)$. Note that if this is on TP then this use will be driven to ∞ by σ_1, but that's okay.
- We don't believe that the computation at σ_2 is correct unless $\ell(\sigma_2, s) > x'$ via σ_2-correct computations. (After all, an infinite stream is entering B at σ_1.)
- Put x' into A.
- At the next $\tau\hat{}\infty$ stage s_1 after s, At the next $\tau\hat{}\infty$-stage t see which or W or V x' enters.
  1. If W, then σ_1 is met.
  2. If V then put the current $t_n^0[s] = \delta(x)[s]$ into both A and B.
  3. At the next $\tau\hat{}\infty$-stage if t_n^0 enters W make $\Gamma_{\sigma_1}^B(t_n^0) = 1$ else $\Delta^B(t_n^0) = 1$, and in either case pick a new t_{n+1}^0.

- Notice that since $t_n[s] = \delta(x')[s]$ also we have fulfilled our obligations to τ, so that B comprehends the entry of x' into V.
- Notice that we have met σ' since $t_n[s]$ was bigger than the use of $\Phi_{\sigma'}(x')$.
- There are no other splitting requirements around so we are in good shape.

- We consider next the situation where σ' extends an version of τ_1 which is guessing that $\delta_0(x)\uparrow$ at σ, say.
- This version of τ_1, say τ_1' believes that the stream from A above enters V_0, essentially.
- It deals with $(V_0 \cap V_1) \sqcup (V_0 \cap W_1)$ trying to prove that $(V_0 \cap V_1) \leq_T B$, or $(V_0 \cap W_1) \leq_T B$ will be proven at some σ' extending τ_1'.
- Thus, as above, x' will be chosen, but now it will have a τ_1' trace $t_n^{[}s_0]'$.
- The stream $t_m[s]$ for τ_0 will be used as traces for $t_n[s_0]$.
- When we see σ' realized we also ask that the length $\ell(\sigma', s) > t_n[s_0]', x'$ and the computation has seen the traces $t_m[s]$ clear the use.

- Now we put $t_n[s_0]'$ into A.
- At the next τ_0 stage if this has entered W_0 we are done.
- If it entered V_0, we will put $t_m[s]$ into A and B.
- At the next τ_0-stage we update Δ_0 and Γ_0. and define t^0_{m+1}.
- When we hit τ'_1, we would see if $t'_n[s_0]$ entered V_1 or W_1.
- If W_1, then σ' is met.
- If V_1, we need to update $\delta'_1(x')$
- To do this we need to add $t_n[s_0]'$ into B.
- We'd wait till the σ length of agreement (note NOT σ' we have not yet visited there again) is bigger than $t_n[s_0]'$ again.
- Now we add this to A and wait till the next τ_0-stage, and see if this enters W_0. If so we are done.
- if V_0, then add it to B and $t^0_m[t]$ to A, etc
- There is some noise from the fact that we don't play this outcome of σ each time etc.

- More specifically, suppose that $\sigma\hat{\ }(\delta(x)\uparrow) \prec \tau_1' \prec \sigma'$.
- So we can only access σ' when $\delta(x)\uparrow$ look correct.
- We have done the above for $\delta_1'(x')$, viz put it into A instead of $\delta_0(x)$, hit τ_0 noted that it went V_1, corrected $\Delta_0^B(\delta_1'(x'))$ by putting e.g. $\delta_1'(x')+1$ into B and A at τ.
- Then we can't access τ_1' again until again we play $\sigma\hat{\ }(\delta(x)\uparrow)$ again. At such a stage we will hit τ_1' and note perhaps that $\delta_1'(x')$ entered $V_1 \cap V_0$.
- τ_1' can now correct Δ_1' by putting $\delta_1'(x')+1$ into A and B. Note that this might enter W_0, but that can be corrected by Γ_σ^B.
- Now the cycle repeats.

- The rest is putting this on a a priority tree and using induction.
- This argument uses "capricious destruction" and is something that a young computability theorist should know.

Thank You