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ON HYPER-TORRE ISOLS

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§1. Introduction. As Dekker [3] suggested, certain fragments of the isols can exhibit an arithmetic rather more resembling that of the natural numbers than the general isols do. One such natural fragment is Barback's "tame models" (cf. [2], [6] and [7]), whose roots go back to Nerode [8]. In this paper we study another variety of such fragments: the hyper-torre isols introduced by Ellentuck [4]. Let Y denote an infinite isol with $D(Y)$ the collection of all isols $A \leq f_\Lambda(Y)$ for some recursive and combinational unary function f. (Here, as usual, f_Λ is the Myhill-Nerode extension of f to the isols).

(1.1) DEFINITION [4]. An isol Y is called *hyper-torre* if Y is infinite, regressive and for all $m \geq 1$, recursive sets $\alpha \subset (\omega)^m$ and $A \in (D(Y))^m$ we have $A \in \alpha_\Lambda \cup \bar{\alpha}_\Lambda$.

The pretty fact discovered by Ellentuck concerning such Y's is

THEOREM (ELLENTUCK [4]). *If Y is hyper-torre then the universal theory of $(D(Y), +, \cdot)$ is the same as $(\omega, +, \cdot)$.*

Although Ellentuck directly constructed such isols, it was subsequently discovered that in 1976 Harrington (cf. [5, Chapter 20]) had constructed a hyper-torre isol. In a remarkable theorem Barback [1] (see [5, Chapter 20]) showed that Y is hyper-torre iff Y is infinite, regressive, and *heriditarily odd-even* (i.e. for all $A \leq Y$ either A is even or A is odd). Harrington constructed a heriditarily odd-even isol. Harrington's construction is a quite elaborate minimal-degree type tree construction, and Ellentuck's is a modification of this. Both constructions seem very different from the types of construction that lead to Π_1 or Σ_1 sets. It has since become a well-known open question in isol theory whether or not hyper-torre isols can exist in the co-simple isols. This appears as Question 9 in [5]. In this paper we solve McLaughlin's question affirmatively by proving

(1.2) THEOREM. *There exists a co-simple hyper-torre isol.*

In §2 we give the proof. In §3 we discuss some variations. Our construction uses a $\mathbf{0}''$-priority argument, and we refer to reader to Soare [9] for any unexplained notation or terminology, and for motivation for this technique. The heart of the

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paper is the discussion preceeding the formal construction. As usual all computations etc. are bounded by s at stage s.

We let $\{\varphi_e\}_{e\in\omega}$ be a standard enumeration of all one-to-one partial recursive functions, and we let λ denote the empty string.

§2. The proof. We shall build an r.e. set $A = \bigcup_s A_s$ in stages. At each stage s, we let $\{a_{i,s}: i \in \omega\}$ enumerate $\bar{A}_s$. The desired set will be $\bar{A} = \{a_i: i \in \omega\}$, where $a_i = \lim_s a_{i,s}$. We must build A co-retracable satisfying the requirements below:

N_e: $\lim_s a_{e,s} = a_e$ exists.

R_e: If $\operatorname{dom} \varphi_e \supset \bar{A}$ then $\varphi_e(\bar{A}) \cap 2\omega$ is either even or odd.

P_e: $\operatorname{card}(W_e) = \infty \to W_e \cap A \neq \varnothing$.

To ensure that A is co-retraceable we use a *dump construction* for A. Namely if $a_i = \mu z(z \in A_{s+1} - A_s)$ then $i < s$ and $A_{s+1} = A_s \cup \{a_j: i \leq j \leq s\}$. It is then easy to see that $\bar{A}$ is retraceable. The relevant retracing function g can be defined as follows. Let z be given. To define $g(z)$, go to stage z. If $z \notin \bar{A}_z$ define $g(z)$ arbitrarily. If $z \in \bar{A}_z$ then $z = a_{i,s}$ for some i. If $i = 0$ define $g(z) = z$, and otherwise define $g(z) = a_{i-1,s}$.

To meet the P_e we wait for an unrestrained element x to occur in $W_{e,s}$ (whilst $W_{e,s} \cap A_s = \varnothing$) and enumerate x into A_{s+1}, causing $W_{e,s+1} \cap A \neq \varnothing$. As we shall see, this requirement interacts quite strongly with the N_g and R_j of higher priority. As a consequence there will be several "guessed" versions of the restraint "$r(e,s)$" attempting to protect $a_{j,s}$ for $j \leq e$. The lim inf of the overall restraint $r(e,s)$ is finite, so we can eventually meet P_e.

Now we turn to the key requirements, the R_e. First we give the *basic module*.

The fundamental idea we use to meet R_e is what we call *binding*. Define

$$l(e,s) = \max\{x: (\forall y < x)(\varphi_{e,s}(a_{y,s})\downarrow)\}.$$

The process is quite simple for a single R_e. For the sake of R_e we shall wait till there occurs a stage s and least unbound numbers $a_{i,s}, a_{j,s}$ with $i < j < l(e,s)$ and such that $\varphi_e(a_{i,s}), \varphi_e(a_{j,s}) \in 2\omega$. We call such a stage *e-expansionary* since we know we have two new numbers to deal with. At this stage s we declare $a_{i,s}$ and $a_{j,s}$ as *bound* (together); we then promise that $a_{i,s} \in A$ iff $a_{j,s} \in A$ (or, in effect, we extend a partial recursive function h we are defining for the sake of R_e by setting $h(a_{i,s}) = a_{j,s}$). The reader should note that for a single R_e the effect is that if $\operatorname{card}(\varphi_e(\bar{A}) \cap 2\omega) = \infty$ then $\varphi_e(\bar{A}) \cap 2\omega$ is even. This follows since if $a_i \in A$ and $\varphi_e(a_i) \in 2\omega$ then a_i is bound to some unique a_j.

How does this strategy cohere with the other requirements? For the P_j and N_k cooperating with R_e the problem is this. Suppose P_j has lower priority than N_k, so that P_j does not wish to move $a_{k,s}$. Suppose P_j has been assigned restraint $r(j,s)$ and $a_{m,s} > r(j,s)$, so that $a_{m,s}$ is currently free to be used to P_j. Now perhaps R_e acts and binds $a_{n,s}$ to $a_{m,s}$, but the problem is that $n \leq k$. Now if we enumerate $z \leq a_{m,s}$ into $A_{s+1} - A_s$ we must fulfil our R_e commitments (perhaps $e < k$). Thus we must *also* enumerate $a_{n,s}$ into A too allowing P_j to injure N_k due to its interaction with R_e.

The solution to this dilemma is to simply reset the restraint and so ensure that if z enters $A_{t+1} - A_t$ for $t > s$ then $z = a_{p,s}$ for some $p > m$ such that, for all $q \geq p$, $a_{q,s}$ is not bound to some $a_{j,s}$ for $j \leq k$.

For the sake of the "α-module" below we shall introduce a little more terminology. In the "α-module", it is no longer possible to use pairs bound together. Rather we will use even finite collections of elements bound in a block. To facilitate this we refer to the least element x of a collection that is e-bound (in the pair situation above this refers to $a_{i,s}$) as the *lower boundary* of its *block*. Let $x = a_{i,s}$. If $a_{k,s}$ is the least element with $k > i$ such that $a_{k,s}$ is not in x's e-block then we refer to $a_{k-1,s}$ as the *upper boundary* of x's e-block. Note that we will try to ensure that there are an even number of y between these boundaries, that is, in x's e-block with $\varphi_e(y) \in 2\omega$. Also the upper boundary of a block is determined by the next block. We shall promise that if any element of an e-block enters, all of the e-block enters together.

(2.1) *The α-module and coherence of the R_e.* Now we discuss the α-module, that is, the modifications to the basic module to allow it to cohere with all the other R_j along the "true path" of the construction. Consider two versions R_e, R_f with $e < f$. The problem with the basic module is this.

Consider the situation which might occur where, for example, for all $j \leq s$ with $j \equiv 0, 1 \pmod 4$, $\varphi_{e,s}(a_{j,s}) \in 2\omega$, and with $j \equiv 2, 3$ we have $\varphi_f(a_{j,s}) \in 2\omega$. There may occur some n such that, if we bound e and f as in the basic module, we would have $a_{n,s}$ e-bound to $a_{n+4,s}$, $a_{n+5,s}$ to $a_{n+8,s}$, $a_{n+9,s}$ to $a_{n+12,s}$ etc.; and perhaps $a_{n+3,s}$ f-bound to $a_{n+6,s}$, $a_{n+7,s}$ to $a_{n+10,s}$, $a_{n+11,s}$ to $a_{n+12,s}$, etc. Then it is easy to see that these bindings have become *interlocked*, and if, for example, we enumerated $a_{n+14,s}$ into $A_{s+1} - A_s$ then this would cause $a_{n+11,s}$ and so (by the dump) $a_{n+12,s}$ and eventually $a_{n,s}$ to enter $A_{s+1} - A_s$.

Situations like this mean that R_f must be more careful with its bindings. The fundamental idea for the α-module is that, if R_f is guessing infinitary R_e behaviour, then R_f must attempt to bind within e-boundaries. R_f views this as follows. If $x < y$ are e-bound, then if z is f-bound to some q with $z < y$ then R_f has really bound all of x, z, y, q even if $x < z$.

Now if x lies within a previously defined f-block we have, in turn, really extended a previously set f-block. Suppose that the e-blocks (at stage s) appear as (x_1, y_1), $(x_2, y_2), \cdots, (x_n, y_n)$, where x_i (y_i) denotes the lower (upper) e-boundary. Thus we note that $y_1 = a_{j-1,s}$ if $x_2 = a_{j,s}$. Our task is to try to define our f-blocks so as not to interlock all of $\bar{A}$.

Now the reader should note that if for each i there are only an even number of z with $z = a_{k,s}$ some k, $x_i \leq z \leq y_i$ and $\varphi_{f,s}(z)\downarrow \in 2\omega$, there is no real problem. We can simply use the *same boundaries as e* to be the boundaries of the f-blocks. Then we keep happy whilst getting no interlocks since none of e's boundaries are crossed by f-bindings. The only remaining problem is if for some i, (x_i, y_i) contains only an odd number of $z = a_{k,s}$ with $\varphi_f(z) \in 2\omega$. The key rule is that we only ever bind *even numbers* of such z. Without loss, we may suppose that (x_1, y_1) is the least such e-block with an odd number. Our solution to this problem is this. We begin anew defining a new blocking predicated on the assumption that (x_1, y_1) is the *only* such *f-odd* e-block. Thus provided $(x_2, y_2), \ldots$ all turn out to be f-even *we do not f-block (x_1, y_1) at all but simply work on (x_j, y_j)* for $j \geq 2$. Obviously if this outcome is the correct one then again there are no problems; $\varphi_e(\bar{A})$ will be even plus a finite number. Finally, should we see some least (x_j, y_j) for $j \geq 2$ with (x_j, y_j) also f-odd, we shall cancel our previous f-blockings and define our new f-block as (x_1, y_j). Note ıat although our new f-block consists of j e-blocks the trade-off is that we have no ending commitment to (x_1, y_1) any more. (It is this outcome which necessitates ıanging our strategy from the basic module.) There is obviously no problem with n · 2 strategies since they will be inductively defined as above. Since each of the above determined by infinite recursive collections of Σ_1-events, the strategies can be, as sual, handled by a standard Π_2 guessing tree. We expect that readers familiar with ıch arguments will provide the details for themselves. However, for completeness, e give some formal details below.

Let $\Lambda = \{oe, o, e, w\}$ ordered in the manner given (i.e. $oe <_\Lambda o <_\Lambda e <_\Lambda w$). The terpretation here is that w means "wait", e means "even", o means "one odd the rest ven" and oe means "infinitely many changes from odd to even". The priority tree is $= \Lambda^{<\omega}$.

We refer to $\sigma, \tau \in T$ as *guesses* and let $\sigma \subseteq \tau$ mean that σ is an initial segment of τ. he priority ordering $\leq_L$ is the standard lexicographic ordering: thus $\sigma \leq_L \tau$ iff $\subseteq \tau$ or $\exists\gamma(\gamma\hat{}i \subseteq \sigma \;\&\; \gamma\hat{}j \subseteq \tau \;\&\; i <_\Lambda j)$. If $\sigma \in T$ and $\mathrm{lh}(\sigma) = e$ we devote σ to .tisfying R_e, and $\sigma\hat{}i$ for $i \in \Lambda$ are the outcomes of σ. We replace the idea of e-bound $\sigma\hat{}i$-bound for such σ. Of course σ encodes the guess as to the behaviour of the gher priority requirements (which act infinitely often). For $j \in \{oe, o, e\}$ we let $\sigma\hat{}j, i, s)$ and $y(\sigma\hat{}j, i, s)$ denote, respectively, the lower and upper boundaries of the ırrent ith block. For $\sigma\hat{}o$ we will also define a *critical block* $Q(\sigma, s)$ which will be the ıique pending odd block for which we are waiting for another odd block. We let

$$B(\sigma\hat{}j, i, s) = \{z \colon z = a_{k,s} \text{ for some } k \;\&\; x(\sigma\hat{}j, i, s) \leq z \leq y(\sigma\hat{}j, i, s)\}.$$

In the construction to follow we use the phrase "initialize σ". This means cancel all straints $r(\sigma, s)$ and declare as undefined all $x(\sigma, i, s)$, $y(\sigma, i, s)$, etc. Also define $c(\sigma, s)$ ıe current state of the control of the σ-module) to be w. Note that in the ınstruction to follow any parameter not specifically reset is simply extended to the ext stage without change.

If (for example) $x = x(\sigma, c, s)$ is defined and we enumerate x into A_{s+1} it then (of ıurse) becomes undefined. Moreover if $\sigma = \tau\hat{}o$ in the above and $x \in Q(\tau, s)$, then e reset $c(\tau, s + 1) = oe$ (from o).

We say P_e *requires attention at substage t of stage s* if $W_{e,s} \cap A_s = \varnothing$ and

$$\exists x(x \in W_{e,s} \;\&\; x > r(\sigma, s), \text{ where } \sigma = \sigma(t, s) \text{ and } e + 1 = \mathrm{lh}(\sigma)).$$

(t, s) is defined in the construction.)

Construction. Stage 0. Define $\sigma_0 = \lambda$ and initialize all $\sigma \in T$.

Stage $s + 1$. At stage $s + 1$ we proceed in substages $t \leq s + 1$.

Substage 0. Define $\sigma(0, s + 1) = \lambda$.

Substage $t + 1$, *part* 1. Let $\sigma = \sigma(t, s + 1)$ and $\mathrm{lh}(\sigma) = f$. Adopt the first case elow to pertain to σ, defining $m(\hat{\sigma}, s + 1) = \max\{r(\gamma, u), a_{g,s}, y(\gamma, h, u) \colon \gamma \leq_L \hat{\sigma} \;\&$ $= \mathrm{lh}(\hat{\sigma}) \;\&\; x(\gamma, h, s + 1) \leq \max\{r(\hat{\sigma}, u), a_{f,s}\} \;\&\; u \leq s + 1\}$.

Case 1. $\forall\tau \subsetneq \sigma(\tau\hat{}w \subseteq \sigma)$.

Subcase 1. $c(\sigma, s) = w$. See if there exist least $k > j > i > m(\sigma, s + 1)$ with $_{e,s}(a_{q,s})\downarrow$ and $\varphi_e(a_{q,s}) \in 2\omega$ for all $q \in \{i, j, k\}$. If so define $y(\sigma\hat{}e, 0, s + 1) = a_{k-1,s}$ ıd $x(\sigma\hat{}e, 0, s + 1) = \min\{a_{p,s} \colon m(\sigma, s + 1) < a_{p,s}\}$. Note that $p \leq i$ and we have efined $B(\sigma\hat{}e, 0, s + 1)$.

Now set $\sigma(t+1, s+1) = \sigma^\wedge e$ and if $t = s$ define $\sigma_{s+1} = \sigma^\wedge e$. Set $r(\sigma^\wedge e, s+1) = y(\sigma^\wedge e, 0, s+1)$ $(= m(\sigma^\wedge e, s+1))$ and $c(\sigma, s+1) = e$. Go to part 2.

If no k, j, i exist as above then define $\sigma(t+1, s+1) = \sigma^\wedge w$ and keep $c(\sigma, s+1) = w$. Now set $r(\sigma, s+1) = m(\sigma^\wedge w, s+1)$ and if $t = s$ define $\sigma_{s+1} = \sigma(t+1, s+1)$. Now go to part 2.

Subcase 2. $c(\sigma, s) \neq w$. We claim $c(\sigma, s) = e$ in this case. (The reader should verify this as an easy induction on the construction.) There will be a greatest block $B(\sigma^\wedge e, g, s)$ defined at stage s. Let $y(\sigma^\wedge e, g, s) = a_{p,s}$. See if there exist least $k > j > i > p$ such that $\varphi_{e,s}(a_{q,s})\downarrow$ and $\varphi_e(a_{q,s}) \in 2\omega$ for $q \in \{i, j, k\}$. If so, define

$$x(\sigma^\wedge e, g+1, s+1) = a_{p+1,s} \quad \text{and} \quad y(\sigma^\wedge e, g+1, s+1) = a_{k-1,s}.$$

Now set $\sigma(t+1, s+1) = \sigma^\wedge e$. ($t = s$ is impossible here. To have $c(\sigma, s) = e$ necessitates a previous visit to σ.) Now go to part 2.

If no such k, j, i exist as above, then define $\sigma(t+1, s+1) = \sigma^\wedge w$ and go to part 2. (Note that here $c(\sigma, s+1) = e$ and $r(\sigma^\wedge q, s+1)$ remain the same for all q.)

Case 2. $\exists \hat{\tau} \subsetneq \sigma(\hat{\tau}^\wedge w \not\subseteq \sigma)$. Let τ denote the longest such $\hat{\tau}$ and let $\tau^\wedge q \subseteq \sigma$. Note that $q \in \{oe, o, e\}$. Adopt the first case below to pertain.

Subcase 1. $c(\sigma, s+1) = w$. See if there exists a least block $B(\tau^\wedge q, i, s+1)$ with $l(f, s+1) > y = y(\tau^\wedge q, i, s+1)$ and $x(\tau^\wedge q, i, s+1) > m(\sigma, s+1)$. If none exists define $\sigma(t+1, s+1) = \sigma^\wedge w$ and $r(\sigma^\wedge w, s+1) = m(\sigma^\wedge w, s+1)$. If $t = s$ define $\sigma_{s+1} = \sigma^\wedge w$. Go to part 2.

If one exists, let $d = \text{card}\{z\colon \varphi_f(z) \in 2\omega \,\&\, z \in \bar{A}_s \,\&\, x \leq z \leq y\}$. If d is even, define $x(\sigma^\wedge e, 0, s+1) = x$ and $y(\sigma^\wedge e, 0, s+1) = y$, set $\sigma(t+1, s+1) = \sigma^\wedge e$ and $c(\sigma, s+1) = e$ and $r(\sigma^\wedge e, s+1) = m(\sigma^\wedge e, s+1)$, and go to part 2. If $t = s$, define $\sigma_{s+1} = \sigma^\wedge e$.

If d is odd define $x(\sigma^\wedge o, 0, s+1) = x$; $y(\sigma^\wedge o, 0, s+1) = y$ and $r(\sigma^\wedge o, s+1) = m(\sigma^\wedge o, s+1)$. Set $\sigma(t+1, s+1) = \sigma^\wedge o$, $Q(\sigma^\wedge o, s+1) = B(\sigma^\wedge o, 0, s+1)$ and $C(\sigma, s+1) = 0$. If $t = s$ define $\sigma_{s+1} = \sigma^\wedge o$. Go to part 2.

Subcase 2. $c(\sigma, s+1) = e$. Let $B(\sigma^\wedge e, i, s)$ be the largest currently defined $\sigma^\wedge e$-block. Then $y(\sigma^\wedge e, i, s) = y(\tau^\wedge q, j, s+1)$ for some j. If

$$l(f, s+1) > y = y(\tau^\wedge q, j+1, s+1)$$

(and this is defined), let

$$d = \text{card}\{z\colon \varphi_f(z) \in 2\omega \,\&\, z \in \bar{A}_s \,\&\, x(\tau^\wedge q, j+1, s+1) \leq z \leq y\}.$$

If d is even, define $x(\sigma^\wedge e, i+1, s+1) = x$ and $y(\sigma^\wedge e, i+1, s+1) = y$ and set $\sigma(t+1, s+1) = \sigma^\wedge e$. If d is odd, define

$$x(\sigma^\wedge o, k, s+1) = x(\sigma^\wedge e, k, s+1), \qquad y(\sigma^\wedge o, k, s+1) = y(\sigma^\wedge e, k, s+1)$$

for all $k \leq i$, and $c(\sigma, s+1) = o$. Also define $x(\sigma^\wedge o, i+1, s+1) = x$ and $y(\sigma^\wedge o, i+1, s+1) = y$ and $Q(\sigma, s+1) = B(\sigma^\wedge o, i+1, s+1)$. Go to part 2.

If $l(f, s+1) > y$ define $\sigma(t+1, s+1) = \sigma^\wedge w$.

Subcase 3. $c(\sigma, s+1) = o$. Let $B(\sigma^\wedge o, i, s)$ denote the largest currently defined $\sigma^\wedge o$-block. Then $y(\sigma^\wedge o, i, s) = y(\tau^\wedge q, j, s+1)$ for some j. See if $l(f, s+1) > y = y(\tau^\wedge q, j+1, s+1)$. If not, define $\sigma(t+1, s+1) = \sigma^\wedge w$ and go to part 2.

If so, let $d = \text{card}\{z\colon \varphi_f(z) \in 2\omega \,\&\, x(\tau^\wedge q, j+1, s+1) \leq z \leq y \,\&\, z \in \bar{A}_s\}$.

Subsubcase 1. d is even. Define $y(\sigma\hat{\ }o, i+1, s+1) = y$ and $x(\sigma\hat{\ }o, i+1, s+1) = x$. Set $\sigma(t+1, s+1) = \sigma\hat{\ }o$ and go to part 2.

Subsubcase 2. d is odd. Now $Q(\sigma, s+1)$ is defined and

$$Q(\sigma, s+1) = B(\sigma\hat{\ }o, k, s+1)$$

for some $k \leq i$. Define

$$x(\sigma\hat{\ }e, i+1, s+1) = x(\sigma\hat{\ }oe, i+1, s+1) = x(\sigma\hat{\ }o, k, s+1),$$

and $y(\sigma\hat{\ }e, i+1, s+1) = y(\sigma\hat{\ }oe, i+1, s+1) = y$. For all $m \leq i$ define

$$x(\sigma\hat{\ }oe, m, s+1) = x(\sigma\hat{\ }e, m, s+1) = x(\sigma\hat{\ }o, m, s+1)$$

and similarly y. *Note that under this identification we give $x(\sigma\hat{\ }e, n, s+1)$ the same priority as $x(\sigma\hat{\ }oe, n, s+1)$, and so we do not cancel $x(\sigma\hat{\ }e, n, s+1)$ henceforth unless we also cancel $x(\sigma\hat{\ }oe, n, s+1)$.* Set $r(\sigma\hat{\ }oe, s+1) = m(\sigma\hat{\ }oe, s+1)$. Now define $\sigma\hat{\ }oe = \sigma(t+1, s+1)$ and $c(\sigma, s+1) = e$.

Part 2. Having computed $\sigma = \sigma(t+1, s+1)$, see if $P_{\mathrm{lh}(\sigma)-1}$ requires attention. Let $f = \mathrm{lh}(\sigma) - 1$. If not, go to substage $t+1$ unless $t = s$. If $t = s$ go to part 3. If P_f requires attention via z, say, let $z = a_{i,s}$. For any $\tau\hat{\ }q \leq_{\mathrm{L}} \sigma$ and $q \in \{o, e, oe\}$, if $a_{i,s} \leq y(\tau\hat{\ }q, n, s+1)$ declare $a_{i,s}$ as bound to $x(\tau\hat{\ }q, n, s+1)$. Find the least such x. Then (by induction) $x = a_{g,s}$ for some g. We set $A_{s+1} = A_s \cup \{a_{g+p,s}: p \leq s\}$. Now set $\sigma_{s+1} = \sigma(t+1, s+1)$. Go to part 3.

Part 3. Initialize all $\gamma \not\leq_{\mathrm{L}} \sigma_{s+1}$.

End of construction.

Verification (*sketch*). Let β denote the true path of the construction. Thus $\beta \in [T]$ is defined inductively via $\lambda \subseteq \beta$. If $\tau \subseteq \beta$ then $\tau\hat{\ }i \subseteq \beta$ iff there are infinitely many $\tau\hat{\ }i$-stages (that is, when $\tau\hat{\ }i = \sigma(t+1, s+1)$) and only finitely many $\tau\hat{\ }j$-stages for $\tau\hat{\ }j <_{\mathrm{L}} \tau\hat{\ }i$. To see that β exists, first note that P_j can receive attention at most once. Thus β exists since once $\mathrm{lh}(\sigma_{s+1}) > z$, we can have $\mathrm{lh}(\sigma_{t+1}) \leq z$ only if P_j receives attention for some $j \leq z - 1$. To see that P_j is met it suffices to argue that for $j = \mathrm{lh}(\tau)$ if $\sigma = \tau\hat{\ }n \subseteq \beta$ then $\lim_s r(\gamma, s)$ exists for $\gamma \leq_{\mathrm{L}} \tau\hat{\ }n$.

For an induction, find a stage s_0 such that for all $s > s_0$

a) $\gamma \leq_{\mathrm{L}} \sigma$ & $\gamma \not\subseteq \beta \rightarrow s$ is not a γ-stage,

b) $\forall j \leq \mathrm{lh}(\sigma)$($P_j$ does not receive attention at stage s), and

c) $(\forall j)(r(\gamma, s) = r(\gamma, s_0))$.

Note that a), b) and c) ensure that for all $\gamma \leq_{\mathrm{L}} \sigma$ if $\gamma \neq \sigma$ then $r(\gamma, s) = r(\gamma, s_0)$. Restraints are only reset at γ-stages for those $\gamma \leq_{\mathrm{L}} \sigma$ with $\gamma \not\subseteq \sigma$. If $\sigma = \tau\hat{\ }w$ then $r(\sigma, s_0) = r(\sigma)$. Let $R = \max\{a_{\mathrm{lh}(\sigma)}, r(\gamma, s_0): \gamma \leq_{\mathrm{L}} \sigma\}$. If $\sigma \neq \tau\hat{\ }o$, compute a σ-stage $s > s_0$ where $x(\sigma, n, s) > R$. Then $r(\sigma, s) = r(\sigma)$, since this will be specifically protected in the construction.

It is implicit here that in the construction, until we see such an $x(\sigma, n, s)$, the P_j of lower priority than σ are restrained from enumerating $x(\sigma, i, s)$. It follows that $\lim_s r(\sigma, s) = r(\sigma)$ exists and similarly all the N_e are met.

Finally, the R_e are met for exactly the reasons discussed before the formal construction. If $\sigma\hat{\ }e$ or $\sigma\hat{\ }oe$ are on the true path then almost all elements are in even blocks. If $\sigma\hat{\ }o \subseteq \beta$, then one block $\lim_s Q(\sigma, s) = Q(\sigma)$ is odd and all those exceeding it are even. The formal verification of this provides absolutely no further insight, and we leave it to the reader.

§3. Variations and comments. One can add permitting to the above construction as follows. For the sake of P_e one defines *followers*. For example, use $x = x(\sigma, i, s)$ to indicate x is the current ith follower at guess σ, where $e = \mathrm{lh}(\sigma)$. A follower is realized when $\hat{x} \in W_{e,s}$ for some $x(\sigma, i-1, s) < \hat{x} < x(\sigma, i, s)$. Whilst $A_s \cap W_{e,s} = \varnothing$, when $x(\sigma, i, s)$ becomes realized we initialize all $\gamma \leq_{\mathrm{L}} \sigma$ and define a new follower $x(\sigma, i+1, s)$ in such a way that it does not violate $x(\sigma, i, s)$'s block and ensure that $\gamma \leq_{\mathrm{L}} \sigma$ cannot violate $x(\sigma, i+1, s)$ by restraint. To meet P_e we enumerate $y \in W_{e,s}$ as before; if $W_{e,s} \cap A_s = \varnothing$, y is unrestrained (as before), and additionally ask that $y > \hat{x}$ for some realized x with $\hat{x}$ permitted we will enumerate as before (defining σ_s to be σ). With this modification we get

(3.1) THEOREM. *Let C be any r.e. nonrecursive set. Then there exists a co-simple hyper-torre isol $A \leq_{\mathrm{T}} C$.*

Here, of course, we use the fact that retraceable isols have well-defined degrees (see [5, Chapter 5]).

I do not know if all nonzero r.e. degrees contain Π_1 hyper-torre isols. Indeed it is not even clear if $\mathbf{0}'$ contains such an isol. It seems to me that a new construction would be needed to get this. I conjecture that not all r.e. nonzero degrees contain such isols.

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