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THE MEMBERS OF THIN AND MINIMAL Π^0_1-CLASSES, THEIR RANKS AND TURING DEGREES

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ABSTRACT. We study the relationship among members of Π^0_1-classes, thin Π^0_1-classes, their Cantor-Bendixson ranks and their Turing degrees; in particular, we show that any nonzero Δ^0_2-degree contains a member of rank α for any computable ordinal α. Furthermore we observe that the degrees containing members of thin Π^0_1-classes are not closed under join.

1. INTRODUCTION

A (computably bounded) Π^0_1-class can be thought of as the collection of paths $[T]$ through a computable binary tree T. Their interest stems from the fact that they code many constructions in mathematics. For example, given any Π^0_1-class $\mathcal{C}$, there is a computable real closed field whose collection of orderings is precisely 1-1 correspondence with the members of $\mathcal{C}$, in fact, in a many-one degree preserving way (see [11]). Another classic example is the degrees of complete extensions of Peano Arithmetic are exactly those of certain kinds of Π^0_1-classes called separating classes (see [14]). As a consequence the study of Π^0_1-classes is an important topic in mathematical logic, of relevance to proof theory, reverse mathematics, computable algebra and analysis, model theory, and algorithmic randomness. The survey paper [1] provides excellent introductions to theory and applications of Π^0_1-classes.

The present paper is part of a long term programme which seeks to understand the relationships between the degrees of members of Π^0_1-classes, the Cantor-Bendixson ranks of Π^0_1-classes, and the position of a Π^0_1-class in the lattice of Π^0_1 classes. For example, if a countable Π^0_1-class $\mathcal{C}$ has rank 1, then every member of the class is computable from $\mathbf{0}''$, and if $\mathcal{C} = [T]$ where T is a tree having no dead ends, the degrees of members of $\mathcal{C}$ are further constrained to be computable from the halting problem. This result was generalized to all computable ordinals by Cenzer, Clote, Smith, Soare and Wainer [2]. At the other extreme, if a Π^0_1-class $\mathcal{A}$ has no computable members, then for any $\boldsymbol{x} \geq \mathbf{0}'$, $\mathcal{A}$ has a member X with X' of degree $\boldsymbol{x}$, the case with $\boldsymbol{x} = \mathbf{0}'$ being known as the *Low Basis Theorem* (all in Jockusch-Soare [9]).

The particular question motivating this paper is the following:

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Suppose that a degree $\boldsymbol{a}$ has a member of a Π^0_1-class rank r. Does it also have members of other ranks? If so, then what can other ranks be?

This question was first tackled by Cenzer and Smith [4] who proved that if $\boldsymbol{a} \leq \mathbf{0}'$ then $\boldsymbol{a}$ has a member of rank 1. On the other hand, Downey [7] proved that there are degrees $\boldsymbol{b}$ with $\mathbf{0} < \boldsymbol{b} \leq \mathbf{0}''$ all of whose members have a fixed rank $r < \omega$ for any r. In particular, there is such a $\boldsymbol{b}$ all of whose members have rank 1.

Cholak and Downey [5] generalized the Cenzer-Smith result. They showed that a computably enumerable and nonzero $\boldsymbol{a}$ contains a set X of *proper* rank α for any computable ordinal α. By proper rank α we mean that X lies on a countable Π^0_1-class of Cantor-Bendixson rank α, and not on one of any rank $< \alpha$. Furthermore, $\boldsymbol{a}$ contains a point Y which is not in any countable Π^0_1-class.

Our first result is to extend the Cholak-Downey result to all Δ^0_2-degrees.

Theorem 1.1. *Suppose that $\mathbf{0}' \geq \boldsymbol{a} \neq \mathbf{0}$, and that α is a computable ordinal. Then $\boldsymbol{a}$ contains a set X of proper rank α.*

There seems no natural extension of Theorem 1.1 beyond the Δ^0_2-degrees. The Downey example of a degree all of whose points have rank 1 is hyperimmune-free (i.e. computably dominated, meaning that all total functions $f \leq_T \boldsymbol{a}$ are dominated by a computable function). A possible guess would be that Theorem 1.1 might hold for all hyperimmune degrees, but it is not hard to see that a sufficiently (Cohen) generic degree will have all points unranked since this can be forced by finite information. This is also true of a sufficiently random degree.

Our remaining results concern what are called *thin* Π^0_1-classes. These classes are the analogues of maximal sets (or more properly hyperhypersimple sets) for the lattice of Π^0_1-classes. They are defined as classes $\mathcal{C}$ such that for any Π^0_1-subclass $\mathcal{D} \subset \mathcal{C}$, there is a clopen set $\mathcal{F}$ with $\mathcal{D} = \mathcal{F} \cap \mathcal{C}$. From Cholak, Coles, Downey and Herrmann [6], we know that this is equivalent to saying that the lattice of Π^0_1-subclasses of $\mathcal{C}$ forms a boolean algebra. Under duality, thin classes were first constructed by Martin and Pour-El [10] who constructed an essentially undecidable theory of propositions all of whose extensions were principal. In fact, they constructed a perfect thin Π^0_1-class. Perfect here means that there are no isolated points in the class. In passing, we remark that in [2] it is shown that these perfect thin classes form an orbit in the automorphism group of the lattice of Π^0_1-classes, and allow us to show that the "array noncomputable degrees" are invariant for this lattice in the same way that the high degrees are invariant via the maximal sets in the lattice of computably enumerable sets.

Next in Cenzer, Downey, Jockusch and Shore [3], it was shown that it is possible to construct a *minimal* class $\mathcal{M}$; a thin class of rank 1 with a unique rank one point. An equivalent formulation of minimality is that $\mathcal{M}$ is infinite and every Π^0_1-subclass is either finite or co-finite. The Cenzer-Smith phenomenon is not true for members of thin classes, that is, Cenzer, Downey, et al. [3] showed that not every computably enumerable degree has members of thin classes: there is a c.e. $\boldsymbol{a}$ such that if $X \in \boldsymbol{a}$, then X is not thin. They also showed that for any computable α there is a set X of proper rank α and which lies on a thin class of rank α.

One of the goals of this project was to examine the relationship between thinness, ranks, and degrees. Our initial guess was that ranks and thinness should behave in a complex way. However, to our surprise, in the Δ^0_2-case, we found that thinness has nothing to do with rank.

Theorem 1.2.

(i) *For any $\boldsymbol{a}$ with $\mathbf{0} < \boldsymbol{a} \leq \mathbf{0}'$, and for any computable ordinals $\alpha, \beta > 0$, if $\boldsymbol{a}$ has a point X in a thin class $\mathcal{C}$ with which X has rank α, then $\boldsymbol{a}$ also contains a point Y of proper rank β, and Y also lies on a thin class of rank β.*

(ii) *Moreover, $\boldsymbol{a}$ also has a point Z which does not lie on any countable class, but lies in a perfect thin Π^0_1-class.*

The proof of this result builds on the work for Theorem 1.1, plus a method of applying "thin pressure" on the opponent when we need to make sure of thinness. This proof is given in Section 3.

In Section 4, we examine the thin analogues of Downey's result on the existence of degrees all of whose points had the same rank. In particular, we will show that there are degrees all of whose members are minimal, and there are degrees all of whose members are thin and lie on no countable Π^0_1-class.

At the end of the paper, we observe that there are two c.e. degrees $\boldsymbol{a}$ and $\boldsymbol{b}$, each of which contains a member of a thin Π^0_1-class, but their join $\boldsymbol{a} \vee \boldsymbol{b}$ does not contain such a member.

This paper forms part of a pair of papers investigating the degrees of members of thin classes. In a sequel [8], we will prove that the c.e. degrees which are not thin, that is, contain no thin members, are dense in the c.e. degrees. This complements a result from Cenzer, Downey, Jockusch and Shore [3] who showed that the c.e. degrees containing minimal points were dense in the c.e. degrees.

During the investigation into thinness and minimality, a natural connection between the underlying trees was observed. This observation not only makes our presentation simpler, but we realized that it made some old constructions conceptually clearer. It was in this way, we realized the possibility to solve the open problem left in Cholak and Downey [5] by proving Theorem 1.1.

2. Preliminaries

For concreteness we remind the reader of some definitions. Recall that a tree $T \subseteq 2^{<\omega}$ is a set of strings closed downwards. For a finite string $\sigma \in 2^{<\omega}$, we use $|\sigma|$ to denote its length. We say that $\sigma \in T$ is a *leaf* or a *dead end* in T, if for all $\tau \in T$, $\sigma \not\prec \tau$. We say that $X \in 2^\omega$ is a path through T if for all $\sigma \in 2^{<\omega}$, $\sigma \prec X$ implies $\sigma \in T$. The set of paths through T is denoted by $[T]$. Since we often approximate an infinite tree T by a sequence of finite trees $\{T_s : s \in \omega\}$, we use $[T_s]$ to denote the set of maximal branches of T_s without counting those known to be dead ends in T_s. For each $\sigma \in T$, we use $T[\![\sigma]\!]$ to denote the set $\{X \in [T] : \sigma \prec X\}$, which form the basic clopen sets of the usual tree topology. A Π^0_1-class can be thought of as $[T]$ for some computable T. We recall the notion of the Cantor-Bendixson derivative and rank:

Definition 2.1. The *Cantor-Bendixson derivative* of a Π^0_1-class P is defined by

$$D(P) = \{X : X \text{ is in the closure of } (P \setminus \{X\})\},$$

namely the set of nonisolated points of P. This process can be iterated to ordinals by defining $D^0(P) = P$, $D^{\alpha+1}(P) = D(D^\alpha(P))$ and $D^\lambda(P) = \bigcap_{\alpha<\lambda} D^\alpha(P)$ for limit ordinal λ. A member X of a Π^0_1-class P is *ranked* in P, if $X \notin D^\alpha(P)$ for some ordinal α, otherwise it is called *unranked* in P. The *rank* of X *in* P, written

$|X|_P$, is the least α such that $X \notin D^{\alpha+1}(P)$; and the rank of X, written $|X|$, is the minimum $|X|_P$ for all Π^0_1-class P in which X is ranked.

Recall also the definitions of minimal and thin Π^0_1-classes, which together with other conventions were given in Cenzer, Downey, et al. [3].

Definition 2.2.

(1) A Π^0_1-class P is said to be *minimal* if for every Π^0_1-subclass Q of P, either Q is finite or $P \setminus Q$ is finite.

(2) A Π^0_1-class P is said to be *thin* if every Π^0_1-subclass Q of P is relatively clopen in P, that is, there is a clopen set U such that $Q = U \cap P$.

Also notice that since we work on binary trees, compactness implies the following fact: Q is clopen in P, if and only if there are finitely many basic clopen sets $O_1, \dots, O_n$ such that $Q = P \cap (O_1 \cup \dots \cup O_n)$.

3. Δ^0_2-Degrees

Before we prove the results relating thin and minimal members of Δ^0_2-degrees, we present a proof of a result of Cenzer and Smith [4]. The purpose of this proof is to observe the natural connection mentioned in the introduction. This connection will be crucial in the later proofs and can be thought of as a game between trees.

Theorem 3.1 (Cenzer and Smith)**.** *Suppose that A is Δ^0_2. Then $A \equiv_T B$ for some rank-one B.*

Proof. Fix a computable tree T of which A is a path. In this proof, T can be the full binary tree, however, with further modifications in mind it is helpful to think in terms of an arbitrary computable tree. We may further assume that A is non-computable, hence not an isolated path on T. Fix a computable enumeration $\langle T_s : s \in \omega \rangle$ of T as $T_s = \{\sigma \in T : |\sigma| \leq s\}$. Also fix a computable enumeration $\langle A_s : s \in \omega \rangle$ of the Δ^0_2-set A such that $A_s \in T_s$. Since A is not isolated on T, we may further assume that $A_s\hat{\ }\langle k \rangle \in T_s$ for $k = 0, 1$. Our goal is to build another computable tree M and a set B such that B is a rank-one member of the Π^0_1-class $[M]$.

At each stage s, we also enumerate a (partial) function $f_s : T_s \to M_s$. We shall verify that each of f_s satisfies the conditions below, and will call such f_s *partial isomorphisms*:

(a) The domain and range of f_s are subtrees of T_s and M_s respectively.

(b) Any $\sigma \in [T_s]$ extends some leaf σ' of domf_s. [Intuitively, the leaves of the domain of f_s form a maximal antichain in T_s (expect the dead ends).]

(c) f_s is an isomorphism (of trees) from its domain to its range, that is, for all σ and σ' in the domain of f_s, $\sigma \subseteq \sigma'$ if and only if $f_s(\sigma) \subseteq f_s(\sigma')$.

(d) A_s as a binary string is in domf_s.

(e) $f_s \subseteq f_{s+1}$ and the function $f = \bigcup_s f_s$ is partially computable.

For each $\sigma \in \text{dom} f_s$, we also associate the basic clopen set $O = T_s[\![\sigma]\!]$ of T_s with the basic clopen set $X = M[\![f_s(\sigma)]\!]$ of M_s which happens to be a single path. This one-one correspondence $O \mapsto X$ will play an important role later. Let B_s be $f_s(A_s)$. Since $\lim_s A_s = A$ and by (d), $\lim_s B_s$ exists. Let B be the limit of B_s. Then B is Δ^0_2. Because f is partially computable and total on A, $A \equiv_T B$ follows from $f[A] = B$ and $f^{-1}[B] = A$.

The idea is to mimic the "topological structure" of T_s but only "near" A_s, whereas at the parts "far away" from A_s, we reduce the richness of T into a single isolated path on M.

Stage 0. Let M_0 be the empty set (as the root) and $f_0 = \{\langle \emptyset, \emptyset \rangle\}$.

Stage $s+1$, suppose that at the end of stage s, we have defined M_s and a partial isomorphism $f_s : T_s \to M_s$ satisfying the above conditions (a) - (d), moreover, $f_{s'} \subseteq f_s$ for all $s' \leq s$.

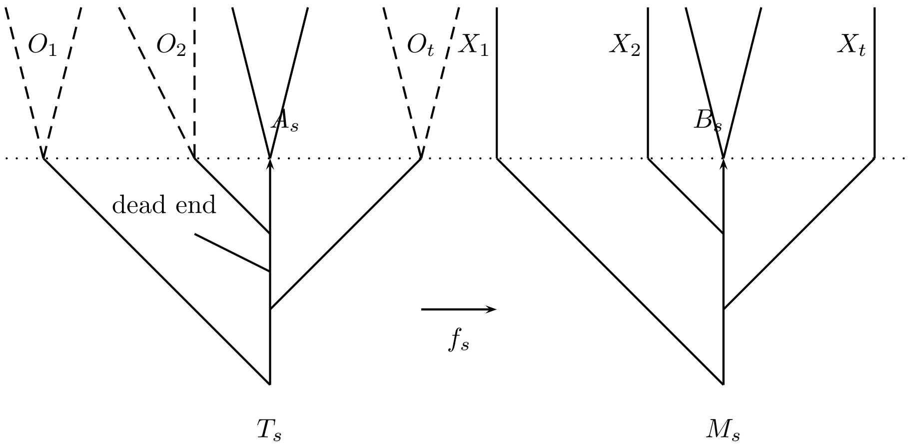

FIGURE 1. picture of f_s

The intuitive picture is as follows. Within the domain of f_s (below the dotted line in Figure 1), T_s and M_s are the same modulo finitely many dead ends. Extending a terminal node σ of domain of f_s, T_s may has an open neighborhood $O = T_s[\![\sigma]\!]$ whereas M_s merely has an isolated path X extending $f_s(\sigma)$. The isomorphism building will be focused along A_s.

Now back to the construction. At stage $s+1$, first calculate A_{s+1}. By speeding up the enumeration if necessary, we may assume that A_{s+1} extends some leaf σ of domf_s. By assumption, $f_s(\sigma)$ will be a leaf τ of the subtree ranf_s of M_s. Let τ^* be the (unique) node extending τ which is a leaf in M_s. We extend f_s to f_{s+1} as follows. First we need to mimic the "branching points" between σ and A_{s+1} on top of τ^* on M: Let η be the binary string that $\sigma\hat{}\eta = A_{s+1}$. Let $\gamma_0 = \sigma$. Suppose we have defined $\gamma_k \subset A_{s+1}$. If $\gamma_k\hat{}\langle i \rangle \prec A_{s+1}$ and $\gamma_k\hat{}\langle 1-i \rangle$ has an extension on T_s of length s, define $f_{s+1}(\gamma_k\hat{}\langle 1-i \rangle) = \tau^*\hat{}(\eta \restriction k)\hat{}\langle 1-i \rangle$, then associate the open neighborhood $T_{s+1}[\![\gamma_k\hat{}\langle 1-i \rangle]\!]$ with the string $M_{s+1}[\![\tau^*\hat{}(\eta \restriction k)\hat{}\langle 1-i \rangle]\!]$. This process stops when $\gamma_k = A_{s+1}$. Let σ_0 and σ_1 be the first two incompatible extensions of σ on T_{s+1} and σ_0 is to the left of σ_1. Define $f_{s+1}(\sigma_i) = \tau^*\hat{}\eta\hat{}\langle i \rangle$ (for $i = 0, 1$). This extends the definition of $f_{s+1}(\gamma\hat{}\langle i \rangle)$ for all γ from σ to A_{s+1} and $i = 0, 1$.

For other leaves β of ranf_s, let β^* be the unique leaf of M_s such that $\beta \prec \beta^*$. If $f_s^{-1}(\beta)$ is a dead end in T_{s+1}, then terminate β^* on M_{s+1}; otherwise, extend β^* naturally as follows: If the last digit of β^* is k, let $\beta^*\hat{}\langle k \rangle \in M_{s+1}$.

To make M computable, if a string is incompatible with any leaves of M_{s+1}, declare it not belong to M. This ends the construction.

It is easy to see from the construction that (a) to (e) hold at the end of stage $s+1$. By (d) f is total on A and $A \equiv_T B$ as argued above. Because A is Δ^0_2, B is also Δ^0_2. Let p be any infinite path on M which is incompatible with B, say the branching point is τ, i.e., both p and B extend τ, $\tau\hat{\ }\langle k\rangle \in p$ and $\tau\hat{\ }\langle 1-k\rangle \in B$ for some $k \in \{0,1\}$. By Δ^0_2-ness of A, there is a stage s such that for all $t > s$, $f_t(A_t)$ and p are incompatible, thus all paths extending $p \restriction (|\tau|+1)$ do not have any splits added after stage s, thus isolated. Hence B is rank-one. $\square$

We now can state the observation in terms of the partial isomorphism f_s. Let $\sigma \in T$ be a string in the domain of f_s, and $\tau \in M$ be its image $f_s(\sigma)$. What matters is to make the following correspondence: The clopen set $T[\![\sigma]\!]$ on T is empty iff the corresponding clopen set $M[\![\tau]\!]$ on M is empty. What exactly are in the clopen sets do not matter.

We now establish one connection between Δ^0_2-degrees containing members of thin class and the ones containing minimal members. This is a "thin analogue" of the Cenzer-Smith Theorem.

Theorem 3.2. *Suppose that A is Δ^0_2 and in a thin class. Then $A \equiv_T B$ for some minimal B.*

Proof. We modify our proof of Theorem 3.1, with a new feature to make M witness the minimality of B under the assumption that T witnessing the thinness of A.

We suppose that $A \in [T]$ with $[T]$ thin. We build a minimal class M by stages.

To make M minimal we must meet the requirements:

$$R_e : U_e \subseteq M \text{ and } [U_e] \text{ is infinite imply } [U_e] =^* [M].$$

Here, U_e denotes the e-th primitive computable tree.

Observe that we may assume $B \in [U_e]$, otherwise, say τ is an initial segment of B which is not in U_e. Then by construction, any clopen set $M[\![\sigma]\!]$ with σ and τ incompatible is eventually finite, consequently $[U_e]$ would be finite.

Since B is the only rank one point on M, any X in $[M]\setminus[U_e]$ must be isolated; i.e., like those X's in Figure 1. To satisfy R_e, we will use the neighborhood O associated with X to "press" T. To do this, we build subtrees V_e of T corresponding to U_e. At stage s, $V_{e,s}$ is always a subtree of T_s. Whilst $U_{e,s} \subseteq M_s$, we do the following. Suppose we see some $\tau \in [M_s] \setminus [U_{e,s}]$ (there could be more than one such τ. Find the associated neighborhoods O of τ and terminate all nodes in $O \cap V_{e,s}$. We refer this action of as "putting thin pressure on T". This completes the construction.

To see that this works, suppose for the sake of contradiction that M is not minimal. Then for some e, $U_e \subseteq M$, $[U_e]$ is infinite and $[M]\setminus[U_e]$ is also infinite. We argue that the corresponding Π^0_1-class $[V_e]$ is not clopen in $[T]$ as follows. Suppose for the sake of contradiction that $[V_e]$ is clopen. Then $[V_e] = [T]\cap(O_1 \cup \cdots \cup O_n)$ for some $n \in \omega$, where $O_i = T[\![\sigma_i]\!]$ and the σ's are pairwise incompatible. Observe that $A \in [V_e]$ since $B \in [U_e]$ (here we used the fact that $[U_e]$ is infinite). Without loss of generality, we may assume that $A \in O_1$. Let $\tau_1 = f(\sigma_1)$. Since $[M]\setminus[U_e]$ is infinite, there exists some Y extending τ_1 such that $Y \in [M] \setminus [U_e]$. At the moment when Y leaves U_e, by construction, we will terminated all nodes in the corresponding neighborhood O on V_e. However $[T]\cap O \neq \emptyset$ (otherwise $Y \notin [M]$ by construction). Thus $[V_e]\cap O = \emptyset \neq [T]\cap O$, contradiction. It follows that M is minimal. $\square$

Observe that by the partial isomorphism f_s, or more precisely by the one-one correspondence between the open neighborhoods O on T and X on M, we are able to transfer the thinness to minimality for free. This observation will be used again in the proof of Theorem 3.7.

We now working on the opposite direction to find a member of arbitrarily higher rank in a given degrees. Previous work has been done by Cholak and Downey [5], where they established the following result for computably enumerable degrees.

Theorem 3.3 (Cholak and Downey)**.** *For every computable ordinal $\alpha \neq 0$, and c.e. degree $\boldsymbol{d} \neq 0$ there is a c.e. set of rank α and degree $\boldsymbol{d}$.*

In [5], they left the open problem that whether the same conclusion holds for Δ^0_2-degrees. We now give a positive answer.

Theorem 3.4. *Suppose that A noncomputable and Δ^0_2 then for any computable ordinal $\alpha \neq 0$ there is a set $B \equiv_T A$ of proper rank α.*

Before the proof, we need to fix some terminology as preparations. First fix an ordinal notation a of α. Computably enumerate the set of notations $\{b : b <_{\mathcal{O}} a\}$, which is known to be c.e. (see for example, [13]), and using transfinite induction on a, one can show that there is a computable way to produce a Π^0_1-class having a unique member of rank $|a|_{\mathcal{O}} = \alpha$. We will call it an *$a$-tree*. For each finite string σ, we define an a-tree *on top of* σ naturally. Note that although it looks like our construction is notation dependent, it actually is not. As we shall see, we are going to successfully diagonalize every computable tree which is not a super tree of an a-tree, any surviving member must be of rank at least $|a|_{\mathcal{O}}$, in other words, it cannot be of rank β for any $\beta < |a|_{\mathcal{O}}$.

The new issue now is the Δ^0_2-permitting, here we use the incomputability of A in a crucial way.

Proof. By the proof of Theorem 3.1, we may assume that A lies on a rank one tree T which has no dead ends. What we will do is in some sense the reverse of the above proof. We simultaneously define a computable tree M and the partial isomorphism f as before, except we build a suitable b-tree on top of every leaf of $\operatorname{ran} f$. Here the notations b are taken for the sole purpose that the limit of $|b|$ is the desired α (more details below). This will artificially increase the M-rank of B. Fix a computable enumeration of primitive computable trees $\langle U_e : e \in \omega \rangle$. To make the rank proper, we try to diagonalize every U_e which looks "less fluffy" than the tree we are building, in particular, the trees whose members are of lesser rank. To be more precise, we have the following requirements for B:

R_e : If $B \in [U_e]$ then the rank of B on U_e is larger than or equal to α.

Stage 0. Let M_0 be the empty set (as the root) and $f_0 = \{\langle \emptyset, \emptyset \rangle\}$.

Stage $s+1$. Suppose that at the end of stage s, we have defined a partial isomorphism $f_s : T_s \to M_s$ satisfying the conditions (a) - (d) in the proof of Theorem 3.1. For bookkeeping purpose, for each leaf τ of $\operatorname{ran} f_s$ and $\tau \neq B_s$, we assign a number $k = k(\tau)$ to τ, indicating τ is the k-th split off the target set B. In the case of $a = 3 \cdot 5^e$, we build $\Phi_e(k)$-tree on top of τ; otherwise, i.e. $a = 2^b$ for some notation b, we simply build a b-tree on top of τ regardless which k it is. The construction below is described for the case when α is a limit ordinal, because the successor case is similar and simpler.

Now at stage $s+1$, first calculate A_{s+1}. By speeding up the enumeration if necessary, we may assume that A_{s+1} extends some leaf σ of $\mathrm{dom} f_s$. By assumption, $f_s(\sigma)$ will be a leaf τ of the subtree $\mathrm{ran} f_s$.

Check if there is any $e < s+1$ such that the requirement R_e has not been satisfied yet, $\tau \in U_{e,s}$, and $U_{e,s}$ is not a superset of the $k(\tau)$-tree (say, V) on top of τ, and $e > k(\tau)$. (The last condition ensures that R_e will not touch first e many subtrees off B, otherwise B might become isolated.) We refer to this as R_e *requires attention at stage s on top of τ*. If no R_e requires attention, just extend f_s to f_{s+1} naturally as in the proof of Theorem 3.1. Otherwise, choose the least e such that R_e requires attention on top of τ, choose the (canonically) least $\tau' \in V \setminus U_e$ with $|\tau'| = s$, define $f_{s+1}(\sigma_i) = \tau' \hat{}\langle i \rangle$ (refer as *reroute to τ'*), terminate all other nodes of length s on V (to ensure that the range of f_{s+1} is still a maximal antichain on M_{s+1}). For the remaining terminal nodes η other than τ on $\mathrm{ran}(f_s)$, recalculate its $k(\eta)$, build $k(\eta)$-tree on top of η. This ends the construction.

We pause to explain on how permitting was actually done. We are making B as the unique rank α path on M. If some primitive computable tree U_e witnesses that B is of lesser rank, U_e must miss many paths on M. Whenever we see U_e missed a path, we try to "reroute" B through the missing path. However, this action requires A's permission. Since in our construction, B is like a shadow passively following A, if A does not extend some node $\sigma \in \mathrm{dom} f_s$, then B cannot reroute to extend $f_s(\sigma)$, even though U_e misses many nodes above $f_s(\sigma)$. Also note, even if A temporarily gives us the permission, A could take it back since A is a Δ^0_2 set. These are the difficulties of permitting. What saves us is the abundance of permissions. If U_e is really of lesser rank, it must miss nodes almost everywhere. Since A is non-computable, it should offer us plenty of opportunities to satisfy R_e.

The set B is again defined the limit of $B_s = f_s(A_s)$. As in the proof of Theorem 3.1, B is Δ^0_2 and $A \equiv B$.

To see the rank of B is at most α, notice that any node τ off B on M, the number $k(\tau)$ will eventually be fixed and $|k(\tau)|_{\mathcal{O}} < \alpha$ and rank B is at most α.

To see the rank of B is at least α, we argue that all requirements R_e are satisfied by induction on e. Suppose that e is the least index such that B is on U_e and of rank $< \alpha$. For each $e' < e$, the requirement $R_{e'}$ will be satisfied in either one of the following two ways: (1) there is some $m_{e'}$ such that $B \restriction m_{e'} \in U_{e'}$ but $B \restriction (m_{e'}+1) \notin U_{e'}$, in other words, our rerouting effort is successful, we refer to this case as *$R_{e'}$ is satisfied in a Σ-way*; or (2) any other ways than (1) to satisfy $R_{e'}$, in fact, we will see this only means B is of rank $\geq \alpha$ on U_e. Let $m = \max\{m_{e'}+1 : e' < e \text{ and } R_{e'} \text{ is satisfied in } \Sigma\text{-way}\}$. Let s_0 be a stage such that for all $t > s_0$ $A \restriction m = A_t \restriction m$. To see R_e is satisfied, let us assume that B is on U_e and U_e witnesses the rank of B is $< \alpha$, thus there is some $\tau_0 \in U_e$ such that $\tau_0 \prec B$ and for all $\tau \in M$ which extend τ_0 but are not initial segments of B, the subtree $\{\eta \in U_e : \tau \prec \eta\}$ is not a super tree of the $k(\tau)$-tree which we are building. We want to derive that A is computable by showing B is computable. Without loss of generality we may further assume that τ_0 has length $> m$. Let s_1 be a stage $> s_0$ such that for all $t > s_1$ $B_t \supseteq \tau_0$. We show that starting from τ_0, we can computably figure out more and more bits of B as follows. Suppose that we have found $\tau \prec B$, we wanted to know whether $\tau \hat{}\langle i \rangle \prec B$ for $i = 0, 1$.

Claim: B extends $\tau \hat{}\langle i \rangle$ if and only if there is a stage $t > s_1$ such that for every $\mu \in \mathrm{ran} f_t$ extending $\tau \hat{}\langle 1-i \rangle$, R_e requires attention on top of μ.

Proof of Claim: If B extends $\tau\hat{}\langle i\rangle$, then there is a stage $v > s_1$ after which no node extending $\tau\hat{}\langle 1-i\rangle$ can enter the range of f. For each $\mu \in \mathrm{ran} f_v$, by choice of τ_0, the subtree $\{\eta \in U_e : \mu \prec \eta\}$ is going to have lesser rank than $k(\mu)$, thus it will require attention at some stage v_μ and remain so forever. Thus $t = \max\{v_\mu : \mu \in \mathrm{ran} f_v\}$ is the stage that we wanted. On the other hand, suppose that there is a stage $t > s_1$ such that for every $\mu \in \mathrm{ran} f_t$ extending $\tau\hat{}\langle 1-i\rangle$, R_e requires attention on top of μ. Let $v > t$ be the first stage (if exists) such that for some $\mu \in \mathrm{ran}(f_t)$, $\tau\hat{}\langle 1-i\rangle \prec \mu \prec B_v$. By construction, we will reroute B to (temporarily) satisfy one of $R_{e'}$ ($e' \leq e$) at stage v and delete all other incompatible nodes extending μ. This deleting ensures that whenever B returns to μ it would satisfy $R_{e'}$ again. By induction hypothesis, this will not happen. Thus B will not return to μ permanently. But there are only finitely many such μ in $\mathrm{ran} f_v$. We can rule them out one by one. Thus B must extend $\tau\hat{}\langle i\rangle$. This finishes the proof of Claim.

It follows from Claim that B is computable. Since $A \equiv_T B$, A would be computable, contradiction. Therefore R_e is satisfied. This establishes the theorem. □

Corollary 3.5. *For every computable ordinal $\alpha \neq 0$, and Δ^0_2 degree $\boldsymbol{d} \neq \boldsymbol{0}$ there is a Δ^0_2 set of rank α and degree $\boldsymbol{d}$.*

Corollary 3.6. *For every Δ^0_2 degree $\boldsymbol{d} \neq \boldsymbol{0}$ there is a Δ^0_2 set of degree $\boldsymbol{d}$ which does not lie on any countable class.*

Proof. This is a corollary of the proof, rather than the corollary of the statement. Instead of diagonalizing all primitive computable trees of lesser rank, we diagonalize all imperfect ones. □

Theorem 3.7 (Stronger version of Theorem 3.4)**.** *Suppose that A is minimal, noncomputable and Δ^0_2 then for any ordinal notation a for a computable ordinal α there is a real $B \equiv_T A$ of proper rank α and thin of rank α.*

Proof. We modify the proof of Theorem 3.4 with the new thinness requirements:

$$R_e : [U_e] \subseteq [M] \text{ implies } [U_e] \text{ is clopen in } [M].$$

Here, U_e denotes the e-th primitive computable tree.

First, the informal ideas: We carry on the construction exactly as in the proof of Theorem 3.4 except adding the features to satisfy the thinness requirement R_e. As observed before, the thinness requirement is basically satisfied by rerouting part of M through some missing part of U_e. If the missing part of U_e is on top of some leave τ of $\mathrm{ran} f$, where we build a rank $k(\tau)$ tree, then we can do the rerouting without any problems, because we have complete freedom in $M[\![\tau]\!]$. The trouble is the missing part could belong to $\mathrm{ran} f$, thus we cannot move as we wish. To solve this problem, we also build auxiliary trees $V_e \subset T$, so that we can use V_e to pressure T whenever our thinness of M is under threat by U_e. This is what we have remarked after the proof of Theorem 3.2, but with the roles of minimality and thinness reversed.

To be more precise, during the construction of the partial isomorphism f_s, we also have the one-one correspondence between the isolated points X in T and the open neighborhoods O in M. At stage s, we say that V_e requires attention, if there is a (shortest) $\tau \in M_s \setminus U_{e,s}$. Since we have taken care the case when τ extending a leave in $\mathrm{ran} f_s$, we may assume that $\tau \subset f_s(\sigma)$ for some $\sigma \in T_s$, list them all out by

$\sigma_1, \ldots, \sigma_n$. If V_e does not require attention, we just copy T by setting $V_{e,s} = T_s$. Suppose some V_e requires attention, we simply terminate $V_{e,s} \cap T_s[\![\sigma_i]\!]$ on V_e for all $1 \leq i \leq n$.

To verify the strategy works. Assume that $[U_e]$ witnesses the non-thinness of M, then $[U_e]$ is a subclass of $[M]$ and there are two sequences of clopen sets of $[M]$ $\langle O_i : i \in \omega \rangle$ and $\langle C_j : j \in \omega \rangle$ such that $[U_e] \cap [O_i] \neq [M] \cap [O_i]$ and $[U_e] \cap [C_j] = [M] \cap [C_j]$. By construction, we may assume that B is in the closure of $\bigcup_i O_i$ and also in the closure of $\bigcup_i C_i$. Then by the one-one correspondence, the isolated path X_i corresponding to $[O_i]$ belongs to $[T] \setminus [V_e]$ and the isolated path Y_j corresponding to $[C_j]$ belongs to $[T] \cap [V_e]$, contradict the minimality of T. □

Corollary 3.8. *For every Δ^0_2 degree $\boldsymbol{d} \neq \mathbf{0}$ there is a Δ^0_2 set of degree $\boldsymbol{d}$ which does not lie on any countable class but lies on a perfect thin Π^0_1-class.*

4. Beyond Δ^0_2-Degrees

From the proof of Theorems 3.1 and 3.4, it is clear that the Δ^0_2-ness plays a crucial role. We now formally demonstrate that the Δ^0_2-ness condition is necessary.

In what follows, we will make heavy use of the properties of hyperimmune-free sets. We remind the readers some of the properties that we are going to use:

- For any hyperimmune-free set A, if $B \leq_T A$ then $B \leq_{tt} A$.
- Given a computable tree T and a tt-reduction Φ, since tt-reduction is always total, Φ will turn any infinite path $\lambda \in [T]$ into an infinite path $\Phi(\lambda) \in [\Phi[T]]$.
- If A is ranked on T, then $\Phi(A)$ is ranked on $\Phi(T)$.
- Hyperimmune-free basis Theorem: Every infinite computable tree has a hyperimmune-free infinite path. It was first shown by Jockusch and Soare [9].

Theorem 4.1. *There is a Δ^0_3 thin degree $\boldsymbol{a}$ containing only unranked members.*

Proof. Let P be a thin perfect Π^0_1-class, whose existence was first shown in [10]. Modify the construction of Martin and Pour-El class and add the requirements which make sure that any path $A \in [T]$, A is not on a ranked computable tree. Call the resulting tree U.

Apply hyperimmune-free basis theorem to U, we get a real A which is a member of a thin class and is unranked. Since $0''$ is sufficient to carry out the proof of hyperimmune-free basis theorem, A is Δ^0_3. Furthermore, by construction, A is not computable. If $B \equiv_T A$ then $B \equiv_{tt} A$ by hyperimmune-freeness, and hence if B lies on a tree H, and H witnesses that B has a rank, then the tt-reduction would induce a tree G which gives a rank to A, a contradiction. □

Theorem 4.2. *There is a Δ^0_3 degree of only minimal points.*

Sketch. This proof is the only one we will not give in detail. In Downey [7], Downey constructed a degree $\boldsymbol{a}$ each point of which has rank 1. The construction is a full approximation proof which constructs a noncomputable set hyperimmune-free degree of rank 1, below $\mathbf{0}''$. The proof we need is to simply add thinness to this construction. Thinness is a finite injury requirement relative to the construction, and simply results in the deletion of certain parts of the construction. Adding such requirements is purely routine, and the whole construction would give no additional insight.

This construction now gives a minimal point A on a tree T, and T has rank 1. We claim that if $B \equiv_T A$ then B is minimal. Certainly it has rank 1. Let $\Phi^A = B$ and $\Delta^V = A$ be the relevant tt-reductions, and let U be the tree containing B induced by Φ. (i.e. Φ^T.) Suppose that R is a Π^0_1 subclass of $[U]$ which is neither finite nor co-finite in $[U]$. Then we can use Δ to pull this back to T, so that Δ^U would be a Π^0_1 subclass of T which was neither finite nor co-finite in T, and this is a contradiction. □

We end the paper with the following fact that the degrees of members of thin classes do not closed under join.

Theorem 4.3. *There are c.e. degrees $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, both of which contains members of thin classes, and their join $\boldsymbol{b}_1 \vee \boldsymbol{b}_2$ does not contain any member of any thin class.*

Proof. In Cenzer et al [3], they have established the existence of an c.e. degree $\boldsymbol{a}$ which does not contain any member of thin class; they also shown that the degrees containing thin members are dense in c.e. degrees. Now take $\boldsymbol{a}$ which does not contain any member of thin class, by Sacks Splitting Theorem [12], $\boldsymbol{a} = \boldsymbol{a}_1 \vee \boldsymbol{a}_2$ and $\boldsymbol{a}_1, \boldsymbol{a}_2 < \boldsymbol{a}$. By the density of thin c.e. degrees, there are $\boldsymbol{b}_1, \boldsymbol{b}_2$ both contains member of thin classes and $\boldsymbol{a}_1 < \boldsymbol{b}_1 < \boldsymbol{a}$ and $\boldsymbol{a}_2 < \boldsymbol{b}_2 < \boldsymbol{a}$. Hence $\boldsymbol{b}_1 \vee \boldsymbol{b}_2 = \boldsymbol{a}$. The result follows. □

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