T-DEGREES, JUMP CLASSES, AND STRONG REDUCIBILITIES

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ABSTRACT. It is shown that there exist r.e. degrees other than 0 and 0' which have a greatest r.e. 1-degree. This solves an old question of Rogers and Jockusch. We call such degrees 1-*topped*. We show that there exist incomplete 1-topped degrees above any low r.e. degree, but also show that no nonzero low degree is 1-topped. It then follows by known results that all incomplete 1-topped degrees are low$_2$ but not low. We also construct cappable nonzero 1-topped r.e. degrees and examine the relationships between 1-topped r.e. degrees and high r.e. degrees.

Finally, we give an analysis of the "local" relationships of r.e. sets under various strong reducibilities. In particular, we analyze the structure of r.e. wtt- and tt-degrees within a single r.e. T-degree. We show, for instance, that there is an r.e. degree which contains a greatest r.e. wtt-degree and a least r.e. tt-degree yet does not consist of a single r.e. wtt-degree. This depends on a new construction of a nonzero r.e. T-degree with a least tt-degree, which proves to have several further applications.

1. Introduction. This paper concerns itself with relationships between Turing (T-) reducibility and several other reducibilities—many-one (m-), one-one (1-), truth table (tt-), bounded truth table (btt-), and weak truth table (wtt-)— for r.e. sets and various other Δ_2^0 sets. In particular in §2 we are interested in the ordering of 1-degrees within a single r.e. T-degree. In §3 we study connections between the (local) ordering of r.e. 1-degrees in an r.e. T-degree a and the (global) properties possessed by a within **R**, the r.e. T-degrees. In §4 we analyze (local) relationships between various reducibilities and by this we are mainly concerned with *interactions* of the various orderings of r-degrees (for various r) in a single r.e. T-degree. Thus we continue our analysis along the lines of [**Dg1,2, Jo1, 2, 3, LS, and DS**]. For background information we refer to Rogers [**Rg**] or Odifreddi's excellent survey [**Od1**].

It is well known that 0' and 0 are r.e.-degrees containing greatest r.e. 1-degrees. It is an old question of Rogers and Jockusch (cf. [**Jo1**, Problem 14–14; **Rg**, §4]) whether these are the only such degrees.

Let r_1 and r_2 be reducibilities. We shall say an r.e. r_1-degree a is r_2- *topped* if there is an r.e. set A of r_1-degree a such that for all r.e. sets B of r_1-degree a we have $B \leqslant_{r_2} A$. In this case A is obviously unique up to r_2-equivalence and is called the r_2-*top* of a. Thus an r.e. T-degree a is tt-topped by an r.e. set A if A has T-degree a

and for all r.e. sets $B \leqslant_T A$, $B \leqslant_{tt} A$. The question above may be rephrased in our terminology as asking if there exist nontrivial 1-topped r.e. (T-) degrees. Odifreddi [**Od1**, Problem 12] noted that it was even unknown whether there exist nontrivial tt-topped r.e. degrees.

In §2 we solve these questions affirmatively by constructing a nontrivial 1-topped r.e. degree. The proof makes essential use of the fact that we are dealing with *r.e.* 1-degrees—or *r.e.* tt-degrees—within a single r.e. degree. If we drop "r.e." and simply deal with all tt-degrees of degree a, our result then fails. In fact, Jockusch [**Jo2**] showed that no hyperimmune degree (and hence no nonzero degree $\leqslant 0'$) contains a greatest tt-degree. However, as we shall see in §4, it is possible for an r.e. degree a to contain a *least* tt-degree (amongst *all* sets of degree a). With this in mind we define an r.e. r_1-degree a to be r_2-*bottomed* if there is an r.e. set A of r_1-degree a such that for all *r.e.* sets B of r_1-degree a we have $A \leqslant_{r_2} B$, and we define an a to be *strongly r_2-bottomed* (by A) if for *all* sets B of r_1-degree a we have $A \leqslant_{r_2} B$.

In §3 we analyze the R-global properties possessed by 1-topped r.e. degrees. In particular we analyze how 1-topped r.e. degrees behave under the jump operator. We show that no nonrecursive semilow r.e. set is an m-top and no nonrecursive low r.e. set is a tt-top. From a known result of Jockusch [**Jo3**] it follows that all nontrivial tt-topped degrees are low$_2$ but not low. We also construct incomplete 1-topped r.e. degrees above each low r.e. degree. In particular, this means that there exist incomplete promptly simple 1-topped r.e. degrees (by e.g. [**AJSS**]). We remark that many characterizations of promptly simple degrees appear in [**AJSS**], and in particular promptly simple degrees are exactly the noncappable r.e. degrees (i.e. those nonzero r.e. degrees that are not part of a minimal pair of r.e. degrees). To complete §3 we show that there exist cappable 1-topped r.e. degrees, and examine the relationship between high r.e. degrees and 1-topped r.e. degrees.

In §4 our focus shifts to the interrelationships between various reducibilities at a more "local" level (rather than the "global" aspects of §3). The main theme of this section is the structure of the tt-degrees within a single T- (or wtt-) degree. We start with a direct proof of Kobzev's [**Ko3**] result that there exist nonzero r.e. T-degrees with strong tt-bottoms. Our construction is much more flexible than Kobzev's indirect method (using "η-maximal semirecursive sets") and allows us to solve a question of Odifreddi [personal communication] by constructing a strong r.e. tt-bottom which is not of minimal tt-degree.

Strongly tt-bottomed r.e. degrees may or may not be (strongly) contiguous, where (following [**Do2**]) an r.e. degree is called *contiguous* (resp. *strongly contiguous*) if it contains a single r.e. wtt-degree (resp. single wtt-degree). These and other results enable us to show that there are

(1) low$_2$ r.e. degrees that are 1-topped but not tt-bottomed,

(2) strongly tt-bottomed and strongly contiguous r.e. degrees,

(3) strongly tt-bottomed and not wtt-topped r.e. degrees, and

(4) wtt-topped strongly tt-bottomed noncontiguous r.e. degrees.

Many of these results are to some extent the best possible by various results from Lachlan [**La2**], Cohen [**Co**], and Kobzev [**Ko1, 2, 3**].

Two other results from §4 are the following. First, there exists an r.e. T-degree that contains infinitely many r.e. wtt-degrees, the structure of which is a lattice with least and greatest members. Second, we generalize the Friedberg and Rogers [**FR**] result that hypersimple r.e. sets are not wtt-complete by showing that hypersimple r.e. sets are *not wtt-cuppable*. That is, if H is hypersimple and r.e., and D is any r.e. set with $D \oplus H \equiv_{wtt} \varnothing'$ then $D \equiv_{wtt} \varnothing'$. In particular each r.e. T-degree contains r.e. wtt-degrees that are not wtt-cuppable and no contiguous r.e. degree is wtt-cuppable. (This last result is due to Ambos-Spies, Jockusch, Soare, and Shore in [**AJSS**].) Finally, we show that $0'_T$ fails to be wtt-bottomed in a very strong way. Namely, we show that if A is any r.e. set with $A \leqslant_{wtt} B$ for all r.e. sets B of T-degree $0'$ then $A \equiv_T \varnothing$.

Our notation is for the most part standard and will follow the rules that Φ_e and Γ_e will denote T-functionals, $\hat{\Gamma}_e$ will denote a wtt-functional with use γ_e, and in general γ_e will denote a partial recursive function. W_e will denote the eth r.e. set. The arguments of all computations, etc. will be bounded at any stage s by s. We denote by α_e the eth tt-condition, so that $B \leqslant_{tt} A$ iff there is a recursive function f such that for all x, $x \in B$ iff $A \vDash \alpha_{f(x)}$. When convenient we suppose that use functions and $\mathrm{dom}\,\alpha_x$ are both monotone in x. We denote by $u(-)$ the use function of any computation $(-)$. We let dtt and btt denote, respectively, disjunctive truth table and bounded truth table reducibility. Here and elsewhere we refer to Odifreddi [**Od1**].

Finally, most of our constructions use "tree of strategies" constructions, presented in the manner of Stob [**St**]. It would be helpful if the reader were familiar with this or Soare [**So3** or **So4**]. We do abuse notation slightly by sometimes using W_e for different roles in the same construction. However, we shall only do so when the meaning will be clear from the context.

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2. A 1-topped r.e. degree. The goal of this section is to give the basic construction of a 1-topped r.e. degree.

(2.1) THEOREM. *There exists an r.e. degree* $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ *such that* $\mathbf{a}$ *is* 1-*topped.*

PROOF. We shall build $A = \bigcup_s A_s$ together with an auxiliary set $B = \bigcup_s B_s$ in stages, to satisfy the requirements:

$$P_e : \overline{A} \neq W_e,$$
$$Q_e : \neg(B \leqslant_1 A \text{ via } \gamma_e),$$
$$R_e : \Phi_e(A) = W_e \text{ implies } W_e \leqslant_1 A.$$

Here, for the Q_e we suppose $\{\gamma_e\}_{e \in \omega}$ is a recursive list of all 1-1 partial recursive functions, and for the R_e we regard (Φ_e, W_e) as a list of all pairs consisting of a T-functional (Φ_e) and an r.e. set (W_e). Our priority ranking here is $R_0, P_0, Q_0, R_1, \ldots$.

We shall satisfy the P_e in the usual way: we appoint followers x, wait till x appears in W_e and then enumerate x into A when this occurs. The basic idea for the satisfaction of the Q_e requirements is also quite simple. We pick a follower x

targeted for B and wait till $\gamma_{e,s}(x)\downarrow$. We then enumerate x into B iff $\gamma_{e,s}(x) \notin A_s$. If $\gamma_{e,s}(x) \notin A_s$ we restrain A in the sense that we would not allow $\gamma_{e,s}(x)$ to be enumerated into A. (Strictly speaking we attempt to restrain by cancellation and use the fact that s bounds all current computations.)

Our problems occur with the satisfaction of the R_e requirements in conjunction with the P_i and Q_i requirements. First we need to describe the method of satisfaction of the R_e. Let $l(e,s) = \max\{x : \forall y < x(\Phi_{e,s}(A_s; y) = W_{e,s}(y))\}$.

CONVENTION. Here we regard $\Phi_{e,s}(A_s)$ as controlling $W_{e,s}$ and do not allow $W_{e,s}$ to change until allowed to by $\Phi_{e,s}(A_s)$. Explicitly, this means that once $l(e,s) > y$ then for all $t > s$, $W_{e,s}(y) = W_{e,t}(y)$ unless some number z enters $A_t - A_s$ with $z < u(\Phi_{e,s}(A_s; y))$. We shall adopt this convention here and for our other constructions. This convention is important since it allows us to control $W_{e,s}$ by controlling A_s.

Let $ml(e,s)$ denote $\max\{l(e,t) : t < s\}$. We say a stage is e-expansionary if $l(e,s) > ml(e,s)$. For a single R_e requirement, roughly speaking, our idea is as follows. We intend to give each $y \in \omega$ a trace $f(e,y)$ and ensure that $y \in W_e$ iff $f(e,y) \in A$. Thus at e-expansionary stages, we give each $y < l(e,s)$ not already possessing a trace, a trace $f(e,y)$ which will be a large fresh number. Now we promise that if ever $y \in W_{e,s}$ then $f(e,y) \in A_{s+1}$. (The reader should note that the convention above on $W_{e,s}$ will mean, in a sense, we will cause such $y \in W_{e,s}$ since such y can enter only when we change A.)

By choosing followers of P_e correctly, this will not interfere with the satisfaction of the P_e requirements. However the combination of this P_e with the R_e may affect the Q_e. When x enters A for the sake of some P_j, this allows A to change and so W_e to change and hence perhaps force many traces into A, because of our commitments to R_k for $k \leq e$. Now in the situation to satisfy the Q_e we wait till $\gamma_{e,s}(x)\downarrow$, and act according to whether or not $\gamma_{e,s}(x) \in A_s$. It is however possible that $\gamma_{e,s}(x)\downarrow$ but $\gamma_{e,s}(x) = f(i,y)$ for some $i \leq e$, and $\gamma_{e,s}(x) \notin A_s$. Hence although we set $B_{s+1} = B_s \cup \{x\}$, perhaps at some later stage t, $A_t[u(\Phi_{i,t}(A_t; y))]$ changes, allowing y to enter W_i and now forcing $f(i,y) = \gamma_e(x)$ into A. This problem is acute due to the infinitary nature of the R_e requirements.

Our solution is to be more shrewd in our choice of stages at which to attack the Q_e. Thus we wait for a stage s where s is i-expansionary (for each "appropriate" $i \leq e$) and for all $z < u(\Phi_{i,s}(A_s; y))$, $z \in W_{i,s}$ iff $f(i,z) \in A_s$. We then cancel all lower priority followers and traces for R_k and P_k for $k > e$ and know that with priority e we have satisfied Q_e since $A[u(\Phi_{i,s}(A_s; y))]$ is now fixed.

Now, as usual, we do not know which $i \leq e$ are appropriate (that is, have $l(i,s) \to \infty$) and this necessitates some sort of nested strategies procedure. We choose to use Lachlan's tree of strategies approach. This method equips Q_e with *guesses* as to the action of higher priority requirements. The crucial definition is that of "σ-correct computation" which, as in (say) a high minimal pair, lies at the heart of the construction. For more on tree arguments the reader is referred to Soare [So2, and So3, Chapter XIV].

We now give the formal details of the argument. Let $T = 2^{<\omega}$. R_e is identified with those $\sigma \in 2^{<\omega}$ with $\mathrm{lh}(\sigma) = e + 1$ where $\mathrm{lh}(\sigma)$ denotes the length of σ.

Members of $2^{<\omega}$ are referred to as *guesses*. We let $\sigma \subseteq \tau$ denote σ is an initial segment of τ and $\sigma \leqslant_L \tau$ the usual left tree ordering: $\sigma \leqslant_L \tau$ iff $\sigma \subseteq \tau$ or $\exists \gamma (\gamma^\wedge 0 \subseteq \sigma \,\&\, \gamma^\wedge 1 \subseteq \tau)$. We remark that $\sigma \leqslant_L \tau$ should be read as σ is *stronger* than τ (for $\sigma \neq \tau$).

Instead of $f(e, y)$ above, we shall build guessed versions $f(\sigma^\wedge 0, y)$ for each $\sigma^\wedge 0$ with $\text{lh}(\sigma) = e$. A guess $\sigma^\wedge 0$ is guessing that $\Phi_e(A) = W_e$ plus whatever σ encodes. In fact $f(\sigma^\wedge 0, \cdot)$ will be partial recursive but have cofinite domain in the second variable for each "truly correct" guess $\sigma^\wedge 0$. Also dom f is finite at each stage s. It is important to note that dom $f(\sigma^\wedge 0, -)$ is only extended at $\sigma^\wedge 0$-stages (which we define later). The key definition is

(2.2) DEFINITION. We say a computation $\Phi_{e,s}(A_s; x) = W_{e,s}(x)$ is *σ-correct at stage s* where $\text{lh}(\sigma) = e + 1$ and $\sigma = \tau^\wedge 0$ if $l(e, s) > x$ and for all $\rho^\wedge 0 \leqslant_L \sigma$ and all z if

(i) $f(\rho^\wedge 0, z)$ is defined, and

(ii) $f(\rho^\wedge 0, z) \leqslant u(\Phi_{e,s}(A_s; x))$,

then

(iii) $z \in W_{i,s}$ iff $f(\rho^\wedge 0, z) \in A_s$ for $i = \text{lh}(\rho)$.

The reader should note that the above must hold for all $\rho^\wedge 0 \leqslant_L \sigma$ not just $\rho^\wedge 0 \subseteq \sigma$. Roughly speaking, (2.2) says that σ does not believe any computation until all higher priority R_i action that might affect the computation has been completed.

(2.3) We define the notions $l(\sigma, s)$ and *σ-stage* by induction on $\text{lh}(\sigma)$ and s.

(i) Every stage s is a $\varnothing$-stage.

(ii) If s is a τ-stage with $\text{lh}(\tau) = e$ then we define

$$l(\tau^\wedge 0, s) = \max\{ x : \forall y < x (\Phi_{e,s}(A_s; y) = W_{e,s}(y)$$

$$\text{and these computations are } \tau^\wedge 0\text{-correct at stage } s)\}.$$

Then if $l(\tau^\wedge 0, s) > \max\{l(\tau^\wedge 0, t) : t$ is a $\tau^\wedge 0$-stage and $t < s\}$, we say s is a $\tau^\wedge 0$-stage. Otherwise s is a $\tau^\wedge 1$-stage.

Now let σ_s denote the unique guess of length s such that s is a σ_s-stage. Following the ideas of Stob [St], followers of P_e and Q_e are given guesses σ with $\text{lh}(\sigma) = e + 1$. We shall say that P_e *requires attention* at stage $s + 1$ if $W_{e,s} \cap A_s = \varnothing$ and one of the following options holds.

(2.4) P_e has no follower.

(2.5) P_e has a follower x with $x \in W_{e,s}$.

We say that Q_e *requires attention* at stage $s + 1$ if Q_e is not currently *declared satisfied*, and one of the following options holds.

(2.6) Q_e has no follower with guess $\subseteq \sigma_s$.

(2.7) Q_e has a follower x with guess $\sigma \subseteq \sigma_s$ and the following conditions are satisfied.

(i) $\gamma_{e,s}(x)\downarrow$.

(ii) For all $\tau^\wedge 0 \subseteq \sigma$, $l(\tau^\wedge 0, s) > \gamma_{e,s}(x)$.

At each stage s we take as the requirement to require attention the one of highest priority according to the above definitions and the given scheme.

CONSTRUCTION, STAGE $s + 1$.

Step 1 (CANCELLATION). For all $\tau \not\leqslant_L \sigma_s$, cancel all $f(\tau, x)$ and followers with guess τ, and declare as unsatisfied any Q_e that is currently satisfied by a number with guess τ.

Step 2 (TRACE ASSIGNMENT). For each $\tau^\wedge 0 \subseteq \sigma_s$ and for each $x < l(\tau^\wedge 0, s)$ if $f(\tau^\wedge 0, x)$ has not as yet been defined at any stage $t \leqslant s$, find a large fresh number $y(x) > s$ and define $f(\tau^\wedge 0, x) = y(x)$.

Step 3 (ATTEND P_e, Q_e). Find the requirement R of highest priority to require attention. Then $R = P_e$ or Q_e for some e.

Substep 1. First cancel all followers with guess τ and traces $f(\tau, z)$ for all τ with $\sigma \subseteq \tau$ and $\sigma \neq \tau$ where $\text{lh}(\sigma) = e + 1$ and $\sigma \subseteq \sigma_s$.

Substep 2. Adopt the appropriate case below.

Case 1. $R = P_e$ and (2.4) holds. Appoint a large fresh number x as a follower of P_e with guess σ where $\sigma \subseteq \sigma_s$ and $\text{lh}(\sigma) = e + 1$.

Case 2. $R = P_e$ and (2.5) holds. Enumerate x into A_{s+1}. P_e is now met (forever). Declare those Q_j for $j \geqslant e$ as unsatisfied.

Case 3. $R = Q_e$ and (2.6) holds. Appoint a large fresh number as a follower of Q_e.

Case 4. $R = Q_e$ and (2.7) holds. Set $B_{s+1} = B_s \cup \{x\}$ if $\gamma_{e,s}(x) \notin A_s$. Otherwise set $B_{s+1} = B_s$. Declare Q_e is currently satisfied via x.

Step 4 (ATTENDING R_e). For each $\gamma^\wedge 0$ and x with $f(\gamma^\wedge 0, x)$ defined and $x \in W_{e,s}$ where $e = \text{lh}(\gamma)$,

(2.8) If $f(\gamma^\wedge 0, x) \notin A_s$ enumerate $f(\gamma^\wedge 0, x)$ into A_{s+1}.

Please note that we do not ask that $\gamma^\wedge 0 \subseteq \sigma_s$ here.

END OF CONSTRUCTION.

VERIFICATION. Let β denote the leftmost path. Namely $\beta \in 2^\omega$ with β defined by induction on substrings: $\varnothing \subseteq \beta$ and $\sigma \subseteq \beta$ implies $\sigma^\wedge 0 \subseteq \beta$ iff $\exists^\infty s$ (s is a $\sigma^\wedge 0$-stage), otherwise $\sigma^\wedge 1 \subseteq \beta$.

We first verify that all the P_j for $j \leqslant e$ receive attention finitely often and Q_j for $j \leqslant e$ receive attention finitely often at σ stages where $\sigma \subseteq \beta$ and $\text{lh}(\sigma) = e + 1$, and all are eventually met. Fix $\sigma \subseteq \beta$ with $\text{lh}(\sigma) = e + 1$. Let s_0 be a σ-stage such that for all $s > s_0$

(i) $\sigma \leqslant_L \sigma_s$.

(ii) s is a σ-stage implies P_j and Q_j for $j < e$ do not receive attention at stage s.

(iii) For all z, τ such that $\tau^\wedge 0 \leqslant_L \sigma$, $\tau^\wedge 0 \nsubseteq \sigma$, and $f(\tau^\wedge 0, z)$ defined, $f(\tau^\wedge 0, z) \in A$ iff $f(\tau^\wedge 0, z) \in A_{s_0}$. Note that (iii) is possible since $f(\tau^\wedge 0, z)$ are only appointed at $\tau^\wedge 0$-stages, and since there are only finitely many $\tau^\wedge 0$-stages, only finitely many $f(\tau^\wedge 0, z)$ are ever defined.

Now if P_e has no follower at stage s_0 it will get one with guess σ. If P_e already has a follower it must have guess $\tau \leqslant_L \sigma_s$. In either case such a follower is now uncancellable, and so after stage s_0 (2.4) cannot pertain to P_e. Hence P_e can require attention at most once more after stage s_0 (namely when (2.5) holds) and thereafter is met.

We now turn to the activity of the Q_e. Let $s_1 > s_0$ be a σ-stage such that also

$$\forall s > s_1 \ (P_e \text{ does not receive attention at stage } s).$$

Let $s_2 > s_1$ be a σ-stage. Now at stage $s_2 + 1$, Q_e must have an uncancellable follower x with guess σ. Certainly Q_e is met and will never again receive attention at a σ-stage if $\gamma_e(x)\uparrow$. Similarly, if $\gamma_e(x)\downarrow$ and $\gamma_e(x) \in A_s$ with $x \notin B_s$ for some stage $s \geqslant s_2$, then Q_e is clearly thereafter forever met. (Nothing will induce us to add x to B.) Hence the only situation we need worry about is exactly the one discussed in the introduction, that all the mechanics of the construction were set up for. Thus let $s_3 \geqslant s_2$ be the least σ-stage with $\gamma_{e,s_3}(x)\downarrow$ and $\gamma_{e,s_3}(x) \notin A_{s_3}$. Now, as $\sigma \subseteq \beta$ for each $\tau\,^\wedge 0 \subseteq \sigma$ we know that $l(\tau\,^\wedge 0, s) \to \infty$ for σ-stages s. Hence there is a σ-stage $s_4 \geqslant s_3$ such that $l(\tau\,^\wedge 0, s_4) > \gamma_e(x)$ for each such τ. Now at stage $s_4 + 1$, Q_e will receive attention via x. Thus we will cancel all lower priority followers and traces $f(\rho, z)$ for all ρ with $\sigma \leqslant_L \rho$ and $\sigma \neq \rho$ (Step 3, Substep 1).

Also, since $l(\tau\,^\wedge 0, s) > \gamma_e(x)$ for each $\tau\,^\wedge 0 \leqslant_L \sigma$, we know that for all $z \leqslant \gamma_e(x)$,

$$z \in W_{i,s_4} \quad \text{iff} \quad f(\tau\,^\wedge 0, z) \in A_{s_4} \quad \text{where } i = \mathrm{lh}(\tau)$$

by definition of $l(\tau\,^\wedge 0, s)$ and σ-correctness. Thus by cancellation and choice of s_4, we see that $\forall s > s_4$ ($l(\tau\,^\wedge 0, s) > \gamma_e(x)$ and $l(i, s) > \gamma_e(x)$). The reader should realize here that we are explicitly appealing to our convention concerning the way numbers may enter W_i. Thus we have achieved our desired aim that $\gamma_e(x) \in A$ iff $\gamma_e(x) \in A_{s_4}$. Hence we now win the requirement Q_e in Substep 2 by ensuring $x \notin B_{s_4+1}$ iff $\gamma_e(x) \in A_{s_4}$ and so $x \notin B$ iff $\gamma_e(x) \in A$.

Finally, we argue that all the R_e are met. Let $\gamma \subseteq \beta$ with $\mathrm{lh}(\gamma) = e + 1$. Suppose $\Phi_e(A) = W_e$. Since $\Phi_e(A) = W_e$ we know that the use function of any computation is bounded. Therefore $\gamma = \tau\,^\wedge 0$. If s_0 is the stage defined at the beginning of the verification, we see that no trace $f(\tau\,^\wedge 0, z)$ (for any z) defined after stage s_0 is ever cancelled. Then the activity in Step 4 (2.8) specifically ensures that $z \in W_e$ iff $f(\tau\,^\wedge 0, z) \in A$. As $l(\tau\,^\wedge 0, s) \to \infty$ for $\tau\,^\wedge 0$-stages, we see that for almost all z, $f(\tau\,^\wedge 0, z)$ is defined after stage s_0 and so $W_e \leqslant_1 A$. $\square$

There are several corollaries or easy modifications to the above. Using infinitary Sacks restraints with guessing as above, we may avoid cones: viz

(2.9) COROLLARY. *Let* c *be any nonzero* Δ^0_2 *degree. Then there exists a 1-topped r.e. degree* a $\neq$ **0** *with* a $\not\geqslant$ c.

We leave the proof of (2.9) to the reader. Another corollary is motivated by the following definition: recall from [LR] that an r.e. set A has USP (*universal splitting property*) if for all r.e. sets $B \leqslant_T A$ there is an r.e. splitting $A_1 \sqcup A_2 = A$ of A with $A_1 \equiv_T B$. Every nonzero r.e. degree contains an r.e. set without USP [Do1], and many r.e. degrees contain no r.e. set with USP (e.g. cf. [LR]). We obtain the following very strong USP existence theorem.

(2.10) COROLLARY. *There exists an r.e. set* $A \not\equiv_T \varnothing$, $\varnothing'$ *such that for any nonsimple coinfinite r.e. set* $B \leqslant_T A$ *there is an r.e. splitting* $A_1 \sqcup A_2 = A$ *with* $A_1 \equiv_1 B$. *Hence for any coinfinite r.e. set* $C \leqslant_T A$ *there is an r.e. splitting* $A_3 \sqcup A_4 = A$ *with* $A_3 \equiv_m C$. *Any 1-topped r.e. degree contains such an r.e. set.*

PROOF. Let A be the 1-top of a 1-topped r.e. degree. Let $B \leqslant_T A$ be r.e. and nonsimple. Then $B \leqslant_1 A$ and $B \oplus A \equiv_1 A$. Let γ be a recursive permutation of ω with $\gamma(B \oplus A) = A$. Let $A_1 = \gamma(B \oplus \varnothing)$ and $A_2 = \gamma(\varnothing \oplus B)$. Evidently $A_1 \sqcup A_2 = A$ and since B is not simple, $B \equiv_1 B \oplus \varnothing \equiv_1 A_1$. □

3. Classification of the 1-topped degrees. The concern of this section is to analyze the properties in **R** possessed by 1-topped degrees. The first one was found by Jockusch [Jo3] and we give it here for the reader's convenience.

(3.1) THEOREM (JOCKUSCH [Jo3, COROLLARY 8(i)]). *Let* $\mathbf{a} \neq \mathbf{0'}$ *be an r.e. 1-topped degree. Then* $\mathbf{a}$ *is low$_2$ (i.e.* $\mathbf{a}'' = \mathbf{0}''$*).*

PROOF. Let A be the r.e. set which is the 1-top of $\mathbf{a}$. Let $G (\leqslant_1 A) = \{e : W_e \leqslant_1 A\}$ and $G (\leqslant \mathbf{a}) = \{e : W_e \leqslant_T A\}$. Then clearly $G (\leqslant_1 A)$ is Σ_3^0. Also $G (\leqslant \mathbf{a})$ is Σ_3^0-complete by Yates [Ya]. Now $G (\leqslant_1 A) = G (\leqslant \mathbf{a})$ and as A is incomplete, this can only happen if A is low$_2$ by Yates [Ya]. □

A related result we would like to mention here is also due to Jockusch in [Jo3]. Let A and B be r.e. We write $A \leqslant_{pr} B$ if $A = f^{-1}(B)$ via some primitive recursive function f. In [Jo3] it is shown that if B is r.e., nonrecursive, and not creative then there is an r.e. set $B \equiv_T A$ such that $B \not\leqslant_{pr} A$.

Our next result shows that not all nonzero degrees bound nonzero 1-topped r.e. degrees. Indeed, no low r.e. degree $\neq \mathbf{0}$ is 1-topped.

(3.2) THEOREM. (i) *Suppose* A *is r.e. and semilow (that is,* $\{e : W_e \cap \overline{A} \neq \varnothing\} \leqslant_T$ $\varnothing'$*) and* E *is r.e. and nonrecursive. Then there is an r.e. set* $B \leqslant_{wtt} E$ *and* $B \not\leqslant_m A$.

(ii) *In particular, no low nonzero r.e. degree is 1-topped.*

PROOF. Let $A = \bigcup_s A_s$ be given by some recursive enumeration. We build $B = \bigcup_s B_s$ to satisfy the requirements

$$R_e : \neg(B \leqslant_m A \text{ via } \gamma_e).$$

Here γ_e denotes the eth partial recursive function. To meet R_e we use auxiliary sets $W_{f(e)}$ where f is recursive. Let $C = \{e : W_{f(e)} \cap \overline{A} \neq \varnothing\}$. Then $C \leqslant_T \varnothing'$ as A is semilow. Thus there is a recursive function $g(e, s)$ such that $\forall e(C(e) = \lim_s g(e, s))$ by the limit lemma. By the recursion theorem we can use g in the construction.

We now describe the strategy for a single R_e.

Look for x and s such that

(i) $x \notin B_s$,

(ii) $\gamma_{e,s}(x)\downarrow$, and

(iii) $E_{s+1}[x] \neq E_s[x]$. (We say E *permits* on x at s.)

Then we put $\gamma_{e,s}(x)$ into $W_{f(e),s+1}$ and search for the least $t \geqslant s$ such that $\gamma_e(x) \in A_t$ or $g(e, t) = 1$. This t exists, because if $\gamma_e(x) \notin A$ then $W_{f(e)} \cap \overline{A} \neq \varnothing$ and so $g(e, t) = 1$ for all sufficiently large t.

If $\gamma_e(x) \in A_t$, do nothing further for R_e since then $x \notin B$ and $\gamma_e(x) \in A$ and so R_e is satisfied.

If $g(e, t) = 1$ put x into $B_{s+1} - B_s$ and do nothing further to satisfy R_e unless $\gamma_e(x)$ is later enumerated into A. In this case, start over by searching for a new witness.

This strategy obviously succeeds if the "$\gamma_e(x) \in A_t$," case occurs, or if $\gamma_e(x)\uparrow$ for some witness, or if we put x into B and $\gamma_e(x) \notin A$. If none of these occurs then R_e receives attention infinitely often since E is nonrecursive and furthermore $W_{f(e)} \subset A$. However if we attack R_e at stage s and this attack is not successful, by assumption there exists $t \geqslant s$ with $g(e, t) = 1$, and yet $W_{f(e)} \subset A$. This contradicts the assumption that $C(e) = \lim_s g(e, s)$.

To combine strategies we simply require that a witness x for R_e be an element of $\omega^{(e)}$. The requirements then do not interfere with each other in any way. Notice that $B \leqslant_{wtt} E$ by permitting. The result follows. $\square$

A similar argument establishes

(3.3) THEOREM. (i) *Suppose A is r.e. and low, and E is r.e. and nonrecursive. Then there is an r.e. set $B \leqslant_{wtt} E$ such that $B \not\leqslant_{tt} A$.*

(ii) *Hence, no low nonzero r.e. degree is tt-topped.*

PROOF. In this argument, we now have that R_e is

$$R_e : \neg(B \leqslant_{tt} \text{ via } \gamma_e).$$

This time $W_{f(e)}$ is a set of canonical indices of finite sets. We let

$$C = \left\{ e : (\exists u \in W_{f(e)})\left[D_u \subset \overline{A}\right] \right\}.$$

Now, as A is low, it follows that $C \leqslant_T \emptyset'$ (cf. Soare [So1]) and may again be approximated by the limit lemma using $g(e, t)$.

Now attack R_e under essentially the same conditions as before, but put u into $W_{f(e),s+1}$ where D_u is the set of $y \in \overline{A}_s$ mentioned in the tt-condition $\alpha_{\gamma_e(x)}$. Search for $t \geqslant s$ such that either $D_u \cap A_t \neq \emptyset$ or $g(e, t) = 1$. If $D_u \cap A_t \neq \emptyset$ put u' into $W_{f(e),t+1}$ where $D_{u'}$ is the set of $y \in \overline{A}_t$ mentioned in $\alpha_{\gamma_e(x)}$. Repeat this until a stage $t' \geqslant t$ is found with $g(e, t') = 1$. Now put x into B if $A_{t'}$ does not satisfy $\alpha_{\gamma_e(x)}$. The remaining details are essentially the same. $\square$

It is interesting to compare this result and the existence of a 1-topped r.e. degree with the following result due to Jockusch [Jo3]. Recall from [Jo3] that if a is an r.e. degree then a class C of unary functions is called a-*subuniform* if there is a binary function f of degree $\leqslant$ a such that $C \subseteq \{ f_e : e \in \omega \}$. (Here f_e denotes $\lambda n(f(e, n))$.) Define an r.e. degree b to be 1-*bounding over an r.e. degree* a if there exists an r.e. set B of degree b such that if C is any r.e. set degree $\leqslant$ a, then $C \leqslant_1 B$. Finally define an r.e. degree b to be 1-*bounding* if it is 1-bounding over some lesser r.e. degree $a \neq 0$. (By (3.3) we see no low degree is 1-bounding.) We have

(3.4) THEOREM. (a) (JOCKUSCH [Jo3]). *If a and b are r.e. with b $\leqslant$ a and b < 0′ then the following three statements are equivalent:*

(i) *the r.e. sets of degree $\leqslant$ b are a-subuniform,*

(ii) *b″ = a′ = 0″,*

(iii) *there is an r.e. sequence of r.e. sets which is uniformly of degree $\leqslant$ a and consists exactly of the r.e. sets of degree $\leqslant$ b.*

(b) *In particular, every high r.e. degree a is 1-bounding (over any low_2 r.e. degree b $\leqslant$ a).*

PROOF. (b) Let **a** be high and **b** $\leqslant$ **a** be nonzero and low_2. By (a)(iii) we have an r.e. sequence of r.e. sets $\{A_i A\}_{i \in \omega}$ which consists exactly of the r.e. sets of degree $\leqslant$ **b** and this sequence is uniformly of degree $\leqslant$ **a**. Let A be an r.e. set of degree **a**. Then $A \oplus \bigoplus_{i \in \omega} A_i$ is an r.e. set of degree **a** which 1-bounds all r.e. sets of degree $\leqslant$ **b**.

It would seem an interesting project to classify the 1-bounding r.e. degrees. It is possible that the collection of 1-bounding r.e. degrees is exactly the nonlow r.e. degrees, but this seems unlikely.

Returning to the classification of the 1-topped r.e. degrees, combining (3.1) and (3.3) we see that the nontrivial 1-topped r.e. degrees form a subset of the $\text{low}_2 - \text{low}_1$ r.e. degrees. On the other hand, we have

(3.5) THEOREM. *Let* **c** *be a low r.e. degree. Then there exists an incomplete* 1-*topped r.e. degree* **a** $\geqslant$ **c**.

PROOF. Rather than give complete details, we explain how the proof of the existence of a nontrivial 1-topped degree (2.1) can be modified to give the result at hand. Thus we assume the reader is completely familiar with the proof of Theorem 2.1.

Let C be an r.e. set of the given low degree **c**. We construct an auxiliary r.e. set B to satisfy the same requirements as in (2.1) except that the requirements P_e are dropped and the requirements R_e are modified by replacing A by $A \oplus C$. Thus our requirements are the following:

$Q_e: \neg(B \leqslant_1 A \text{ via } \gamma_e)$.

$R_e: \Phi_e(A \oplus C) = W_e \text{ implies } W_e \leqslant_1 A$.

Note that these requirements imply that $C \leqslant_1 A$, by applying R_e to an e chosen so that $\Phi_e(A \oplus C) = W_e = C$. Thus the theorem holds for $\mathbf{a} = \deg(A) = \deg(A \oplus C)$ if the requirements are all satisfied.

The basic strategies for satisfying Q_e and R_e are as in (2.1). These requirements conflict as in (2.1), but the added difficulty here is that in satisfying Q_e we may not use A alone to control W_i for the "appropriate" $i \leqslant e$, but rather we must use $A \oplus C$. Of course this is a new kind of obstacle because C is not under our control. This obstacle is overcome by using the "Robinson technique" (see [So4, Chapter XI, Theorem 3.2]) to avoid excessive reliance on C-false computations. The Robinson technique is combined with our guessing strategy so that, for each string σ of length $e + 1$, there is an auxiliary set $W_{h(\sigma)}$ used to meet Q_e, where h is a recursive function. (Actually we may use several "versions" of $W_{h(\sigma)}$.) Let

$$E = \left\{ \sigma : (\exists u \in W_{h(\sigma)}) \left[D_u \subset \overline{C} \right] \right\}.$$

Then $E \leqslant_T C' \leqslant_T \varnothing'$, so by the limit lemma there is a recursive function $g(\sigma, s)$ with $\lim_s g(\sigma, s) = E(\sigma)$ for all σ. By the recursion theorem we may assume that h and g are known during the construction. When Q_e requires attention via a follower x with guess σ at stage s, there will be at most one pair $\langle i, y \rangle$ such that $i \leqslant e$, $y \notin W_{i,s}$, and $f(y, \tau) = \gamma_{e,s}(x)$, where $\tau^\wedge\langle 0 \rangle \subset \sigma$ and τ has length i. As in (2.1), we would like to put x into B to meet R_e, but our attempt will be ruined if y later

enters W_i. We thus require (if such a pair exists) that $\Phi_{i,s}(A_s \oplus C_s; y) = 0$ and enumerate u into $W_{h(\sigma)}$, where D_u is the set of numbers whose nonmembership in C_s is used in this computation. We then find the least $t \geqslant s$ such that either $g(\sigma, t) = 1$ or $D_u \cap C_t \neq \varnothing$. If $g(\sigma, t) = 1$ and $\gamma_{e,s}(x) \notin A_t$, we put x into B, and otherwise we do nothing. If we later realize that $D_u \cap C \neq \varnothing$ (having put x into B as above), we cancel x and thus may choose a new follower for R_e with guess σ, but we continue to use the same version of $W_{h(\sigma)}$. However, at any stage t with $\sigma_t \leqslant_L \sigma$ and $\sigma_t \not\subseteq \sigma$, we start over with a new version of $W_{h(\sigma)}$.

We now give some further technical details of the argument, although these are quite straightforward, and the reader may prefer to work them out for himself.

The definitions prior to the construction are the same as in Theorem 2.1, except that all references to the P_e's are dropped, and A_s is replaced by $A_s \oplus C_s$ whenever A_s occurs as an oracle. We will also have auxiliary functions g and h, where $W_{h(\sigma, k)}$ should be thought of as the $(k + 1)$st version of $W_{h(\sigma)}$, and

$$\lim_s g(\sigma, k, s) = E(\sigma, k),$$

where

$$E = \left\{ \langle \sigma, k \rangle : (\exists u \in W_{h(\sigma, k)})[D_u \cap C = \varnothing] \right\}.$$

The construction is the same with the following exceptions. In Step 1 (cancellation), also cancel all followers x of any Q_e such that $x \in B_s$ and $\gamma_{e,s}(x) \in A_s$. Declare all such Q_e's to be *unsatisfied*. In Step 2 (trace assignment), require that $f(\tau {}^\wedge 0, x) \in \omega^{\langle \tau, x \rangle}$, so that the pair $\langle i, y \rangle$ in our previous discussion will be unique if it exists. Cases 1 and 2 of Substep 2 of Step 3 are omitted. The main change is in Case 4 of Substep 2 of Step 3. As before in this case, if $\gamma_e(x) \in A_s$, let $B_{s+1} = B_s$, and declare Q_e to be satisfied. Assume now that $\gamma_e(x) \notin A_s$, and ask if there exist y and τ such that $\tau {}^\wedge 0 \subset \sigma_s$, and $f(y, \tau) = \gamma_e(x)$. If no such y, τ exist, let $B_{s+1} = B_s \cup \{x\}$, and declare Q_e to be satisfied. Assume now that such y and τ exist. Then y and τ are uniquely determined, since $\gamma_e(x) \in \omega^{\langle y, \tau \rangle}$. By (2.7)(ii), $l(\tau {}^\wedge 0, s) > \gamma_e(x) = f(y, \tau) > y$. (We have $f(y, \tau) > y$ by the way we choose to define f.) Thus $\Phi_{i,s}(A_s \oplus C_s; y) = W_{i,s}(y)$ by a $\tau {}^\wedge 0$-correct computation. Let D_u be the set of numbers whose nonmembership in C_s is used in this computation, and let

$$k = |\{t < s : \sigma_t \leqslant_L \sigma_s \& \sigma_t \not\subseteq \sigma_s\}|.$$

Enumerate u into $W_{h(\sigma, k)}$, and let $t \geqslant s$ be minimal so that $g(\sigma, k, t) = 1$ or $D_u \cap C_t \neq \varnothing$. If $g(\sigma, k, t) = 1$, let $B_{s+1} = B_s \cup \{x\}$ and declare Q_e to be satisfied. Otherwise (in which case $D_u \cap C_t \neq \varnothing$), let $B_{s+1} = B_s$. In this final case, do not declare Q_e to be satisfied or even regard Q_e as having received attention, and pass to the least $e' > e$ (if any) such that $Q_{e'}$ requires attention at s, and treat it similarly. Repeat this process until either some $Q_{\hat{e}}$ receives attention at s, or no e' remains such that $Q_{e'}$ requires attention at s and has not been considered. This completes the description of the construction.

The verification that the R_e's are met is as before. To treat Q_e, let σ be the initial segment of the leftmost path of length $e + 1$. We must show that Q_e *receives* attention only finitely often at σ stages and is met. (However, it may happen that Q_e

requires attention infinitely often at σ stages.) Let $k = |\{t : \sigma_t \leqslant_L \sigma \;\&\; \sigma_t \not\subseteq \sigma\}|$. There are two cases.

Case 1. $\langle \sigma, k \rangle \in E$, so there exists u in $W_{h(\sigma,k)}$ with $D_u \subset \bar{C}$. Let s be the stage at which we added u to $W_{h(\sigma,k)}$. Then s is a σ-stage at which Q_e receives attention via some existing follower x. It is easy to see as in Theorem 2.1 that x witnesses the satisfaction of Q_e, and Q_e never again requires attention at a σ-stage.

Case 2. $\langle \sigma, k \rangle \notin E$. Pick s_5 so large that $\sigma \leqslant_L \sigma_s$ for all $s \geqslant s_5$, no Q_i for $i < e$ receives attention at any σ-stage after s_5, and $g(\sigma, k, t) = 0$ for all $t \geqslant s_5$. Then by construction, Q_e is never regarded as receiving attention at any σ-stage after s_5. It remains only to show that Q_e is met. Let x be a follower of Q_e which exists at some σ-stage $s > s_5$. If $\gamma_{e,s}(x)\uparrow$, then $\gamma_{e,s}(x)$ is obviously met. Suppose that $\gamma_{e,s}(x)\downarrow$.

Case 2A. $\gamma_{e,s}(x) \in A$. Then we are done unless $x \in B$. If $x \in B$, then x will be cancelled and replaced by a new follower appointed after s_5. Hence we may assume without loss of generality that x is appointed after stage s_5, so $x \notin B$. Thus (still assuming that $\gamma_{e,s}(x) \in A$), Q_e is met.

Case 2B. $\gamma_{e,s}(x) \notin A$. Assume for a contradiction that Q_e is not met, so in particular $x \notin A$. Since $\gamma_e(x) \notin A$ and x exists after stage s_5, x is never cancelled. Thus Q_e never has a follower other than x at any σ-stage after x is appointed. If Q_e is declared satisfied at any σ-stage after x is appointed, it is easily seen as in the proof of Theorem 2.1 that Q_e is met. Thus this never happens. Hence we may see as in the proof of Theorem 2.1 that Q_e requires attention at all sufficiently large σ-stages. Since this attention does not lead to Q_e's being declared satisfied at a large σ-stage, there must be a fixed pair $\langle i, y \rangle$ such that $\Phi_i(A \oplus C; y) = 0$ and yet, for all sufficiently large σ-stages s, $\Phi_{i,s}(A_s \oplus C_s; y) = 0$ via a computation which is C-incorrect. This contradiction shows that Q_e is met and concludes our sketch of the proof of Theorem 3.5. $\square$

Using this result we can establish several existence theorems for 1-topped r.e. degrees in several other degree classes. For example, recall that an r.e. degree $\mathbf{a}$ is *low cuppable* (cf. [AJSS]) if there is an r.e. degree $\mathbf{c}$ such that $\mathbf{c}$ is low and $\mathbf{c} \cup \mathbf{a} = \mathbf{0}'$, and an r.e. degree $\mathbf{a}$ is *contiguous* (cf. [LS, Do2]) if it consists of a single r.e. wtt-degree. We have

(3.6) COROLLARY. *Let* $\mathbf{c}$ *be any low r.e. degree. Then there exists an incomplete* 1-*topped low cuppable degree* $\mathbf{a} \geqslant \mathbf{c}$ *such that for all* $\mathbf{b} \geqslant \mathbf{a}$, $\mathbf{b}$ *is not contiguous.*

PROOF. Let K denote a creative set. Let C be an r.e. set of degree $\mathbf{c}$ with $\mathbf{c}$ low. Apply Robinson's [Ro] (cf. Soare [So4, Chapter XI, Theorem 3.2]) splitting theorem to find r.e. sets K_1, K_2 with $K_1 \sqcup K_2 = K$ an r.e. splitting of K and $K_1 \oplus C$, $K_2 \oplus C$ both low. Let $D = K_1 \oplus C$. Apply (3.4) to D to obtain an r.e. set A such that A is the 1-top of a degree $\mathbf{a} \geqslant T$-$\deg(D)$. Clearly $\mathbf{a}$ is low cuppable by K_2. Suppose that Q is an r.e. set of contiguous degree with $A \leqslant_T Q$. Then $K_1 \leqslant_{wtt} Q$ and hence $K_2 \oplus Q \equiv_{wtt} K$. Thus Q is wtt-cuppable. This contradicts the established result of [AJSS] that no contiguous r.e. set is wtt-cuppable. (See §4 for an extension of this result.) $\square$

To complete the picture suggested by (3.5) we point out that 1-topped r.e. degrees may be *cappable*. Recall that a is cappable if there exists an r.e. degree b $\neq$ 0 with a $\cap$ b = 0. In [AJSS] it is shown that every r.e. degree is either cappable or low cuppable (but not both). We have

(3.7) THEOREM. *There exists an r.e. degree* a $\neq$ 0 *such that* a *is* 1-*topped and cappable.*

PROOF. We build $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$ to satisfy

$$P_{2e} : \overline{A} \neq W_e, \qquad P_{2e+1} : \overline{B} \neq W_e,$$
$$N_e : \Phi_e(A) = \Phi_e(B) = f \text{ and } f \text{ total implies } f \text{ recursive,}$$
$$R_e : \Phi_e(A) = W_e \text{ implies } W_e \leqslant_1 A.$$

We again use a tree of strategies $2^{<\omega}$. N_e is associated with those $\sigma \in 2^{<\omega}$ with $\mathrm{lh}(\sigma) = 2e + 2$ and the R_e with those σ of length $2e + 1$. In this construction we define restraints $r(\tau, s)$ for those τ of even length $\geqslant 2$. At stage s, we automatically define $R(\sigma, s)$ via

$$R(\sigma, s) = \max\{r(\tau, s) : \tau \leqslant_L \sigma\}.$$

This is well defined since we initialize all $r(\tau, 0) = -1$, and only reset those $r(\sigma, s)$ for $\sigma \subset \sigma_s$, where as usual σ_s is the unique path of length s with s a σ_s-stage.

The reader should note that in this construction the important guessing is the minimal pair guessing. This forces us to only enumerate $f(\tau \wedge 0, s)$ into A for the sake of R_e (where $e = \mathrm{lh}(\tau)$) at τ-stages. This is the gist of the next series of definitions.

DEFINITION. We define the notions σ-*stage*, $l(\sigma, s)$, $r(\sigma, s)$, and σ-*correctness* simultaneously by induction on $\mathrm{lh}(\sigma)$.

(i) Every stage s is a $\varnothing$-stage and $r(\varnothing, s) = -1$.

(ii) If s is a τ-stage and $\mathrm{lh}(\tau) = 2e$, define

$$l(\tau \wedge 0, s) = \max\left\{x : \forall y < x\left(\Phi_{e,s}(A_s; y) = W_{e,s}(y)\right.\right.$$
$$\text{and this computation is } \tau \wedge 0\text{-correct})\}.$$

Here a computation $\Phi_{e,s}(A_s; y) = W_{e,s}(y)$ is $\tau \wedge 0$-correct if, for all z with $f(\gamma \wedge 0, z)$ defined for some $\gamma \wedge 0 \subset \tau \wedge 0$ we have that

$$R(\gamma, z) < f(\gamma \wedge 0, z) < u\left(\Phi_{e,s}(A_s; y)\right)$$
$$\Rightarrow z \in W_{i,s} \quad \text{iff } f(\gamma \wedge 0, z) \in A_s \text{ where } 2i + 2 = \mathrm{lh}(\gamma \wedge 0).$$

Now if

$$l(\tau \wedge 0, s) > \max\{l(\tau \wedge 0, t) : t \text{ is a } \tau\text{-stage and } t < s\}$$

we say that s is a $\tau \wedge 0$-stage. Otherwise s is a $\tau \wedge 1$-stage. Set $r(\tau \wedge i, s) = -1$.

(iii) If s is a τ-stage and $\mathrm{lh}(\tau) = 2e + 1$ then let

$$l(\tau \wedge 0, s) = \max\{x : \forall y < x\left(\Phi_{e,s}(A_s; y) = \Phi_{e,s}(B_s; y)\right. \text{ and}$$
$$\text{these computations are } \tau \wedge 0\text{-correct})\}.$$

Here $\tau\,^\wedge 0$-correctness is defined as in (ii) *mutatis mutandis*. Then if

$$l(\tau\,^\wedge 0, s) > \max\{l(\tau\,^\wedge 0, t) : t \text{ is a } \tau\,^\wedge 0\text{-stage and } t < s\}$$

declare s to be a $\tau\,^\wedge 0$-stage, and set $r(\tau\,^\wedge 0, s) = -1$ and $r(\tau\,^\wedge 1, s) = s$. If s is not a $\tau\,^\wedge 0$-stage declare s as a $\tau\,^\wedge 1$-stage and define

$$r(\tau\,^\wedge 1, s) = \max\{t : t \text{ is a } \tau\,^\wedge 0\text{-stage and } t < s\}.$$

We say P_{2e} *requires attention* at stage $s + 1$ if $A_s \cap W_{e,s} = \varnothing$ and for all followers x of P_{2e} with guess $\subset \sigma_s$, $x \leqslant R(\sigma, s)$.

We say a number x *requires attention* if either

(a) (i) x is a follower of P_e for some e, and

(ii) x has guess $\tau \subset \sigma_s$ for some τ, and

(iii) $x > R(\tau, s)$, and

(iv) $x \in W_{e,s}$, or

(b) (i) $x = f(\tau\,^\wedge 0, y)$ for some $\tau \subset \sigma_s$, and

(ii) $y \in W_{i,s}$ and $f(\tau\,^\wedge 0, y) \notin A_s$.

CONSTRUCTION, STAGE $s + 1$.

Step 1. Cancel all $f(\tau, z)$ and $r(\tau, s)$ and followers with guess τ for $\tau \not\leqslant_L \sigma_s$.

Step 2. For each $\tau\,^\wedge 0 \subset \sigma_s$ and $x < l(\tau\,^\wedge 0, s)$ for lh(τ) even and $\geqslant 2$ if $f(\tau\,^\wedge 0, x)$ is undefined for all stages $t \leqslant s$, find a large fresh number $y(x)$ and set $f(\tau\,^\wedge 0, x) = y(x)$.

Step 3. Find the least i such that P_i requires attention and as above we may suppose $i = 2e$. Appoint a large fresh number as a follower with guess σ. Cancel all $f(\tau, z)$ and $r(\tau, s)$ currently defined for $\tau \not\leqslant_L \sigma$.

Step 4. Now find the least number x (if any) such that x requires attention. If x is a follower targeted for B, set $B_{s+1} = B_s \cup \{x\}\ \&\ A_{s+1} = A_s$. Otherwise x is a follower or a trace targeted for A. In this case set $B_{s+1} = B_s$ and $A_{s+1} = A_s \cup \{x\}$. Notice only one number is enumerated into $A \oplus B$ at this stage, so the $f(\tau\,^\wedge 0, z)$ get into A "slowly".

END OF CONSTRUCTION.

We now sketch the verification as most of it is a fairly standard minimal pair type one or similar to (2.1). Let β denote the leftmost path. By the same argument as (2.1), once we see that $\lim_s R(\tau\,^\wedge 0, s) = R(\tau\,^\wedge 0)$ exists for all $\tau\,^\wedge 0 \leqslant_L \sigma \subset \beta$ where lh(σ) $= 2e + 1$, we see that R_e is met since almost all $f(\sigma\,^\wedge 0, z)$ are free to enter in Step 4. Hence $\Phi_e(A) = W_e$ implies $W_e \leqslant_1 A$.

It is evident that for $\sigma \subset \beta$, $\lim_s R(\sigma, s) = R(\sigma)$ exists: Briefly, for $\gamma \leqslant_L \sigma$ the first possibility is that $\gamma = \tau\,^\wedge 1$ and lh(τ) $= 2j$ for some j, and so there are only finitely many stages where

$$l(\tau\,^\wedge 0, s) > \max\{l(\tau\,^\wedge 0, t) : t \text{ is a } \tau\,^\wedge 0\text{-stage and } t < s\}.$$

Hence $\lim_s r(\tau\,^\wedge 1, s) = r(\tau\,^\wedge 1)$ exists in this case. (Namely $r(\tau\,^\wedge 1) = \max\{t : t \text{ is a } \tau\,^\wedge 0\text{-stage}\}$.) The other possibility for $\gamma \leqslant_L \sigma$ is that $\gamma \neq \tau\,^\wedge 1$ for any τ with lh(τ) even and hence $r(\tau, s) = -1$ for all s. This means that $\lim_s R(\sigma, s) = R(\sigma)$ exists since there are only finitely many $\tau\,^\wedge 1$-stages with $\tau\,^\wedge 1 \leqslant_L \sigma$ but $\tau\,^\wedge 1 \not\subset \beta$. (The above is all fairly standard.)

In view of this it is clear that all the P_e are met since eventually P_e must get a follower x with guess $\sigma \subset \beta$ where $x > R(\sigma)$ and this follower will succeed.

It remains therefore to observe that all the N_e are met. Let $\sigma \subset \beta$ with $\text{lh}(\sigma) = 2e + 1$. Let s_0 be a stage after which $\sigma \leqslant_L \sigma_s$ and $r(\tau, s_0) = r(\tau)$ for all $\tau \leqslant_L \sigma$ and all the P_j activity for $j < e$ has ceased and all $f(\gamma^\wedge 0, z) \in A_{s_0}$ if $f(\gamma^\wedge 0, z) \in A$ for all $\gamma^\wedge 0 \leqslant_L \sigma$ with $\gamma^\wedge 0 \not\subset \sigma$. Let x be given. Find a σ-stage $s > s_0$ where $l(\sigma, s) > x$. Now at most one number not restrained at s may enter $A \oplus B$ between this and the next σ-stage, as in the construction of a high minimal pair. We refer to, e.g., [So2 or So4, Chapter XIV] for further details. $\square$

The above result is the simplest of a series of such results. For example we can easily modify the argument above to show

(3.8) COROLLARY. *There exists a minimal pair* $\mathbf{a}, \mathbf{b}$ *of* 1-*topped r.e. degrees with* $\mathbf{a} \cup \mathbf{b}$ *also* 1-*topped.*

PROOF. Left to reader. $\square$

(Other results would include embeddings (by 1-topped r.e. degrees) of boolean algebras, etc.)

To conclude this section we give some further limitations on the 1-topped degrees. One we should mention is

(3.9) THEOREM (LERMAN AND REMMEL [LR]). *The r.e. degrees without wtt-tops form a dense subcollection of the r.e. degrees.*

Our last result for this section is a partial answer arising from an attempt to connect the 1-topped degrees with another jump class. We do not know whether every high r.e. degree bounds a nonzero 1-topped r.e. degree. Our partial solution to this question is:

(3.10) THEOREM. *There exists an r.e. set* A *of high degree such that for all r.e. sets* B *with* $\varnothing <_T B \leqslant_{wtt} A$, wtt-deg($B$) *is not tt-topped.*

PROOF. Let $D = \bigcup_e D^{(e)}$ be a piecewise recursive set such that $D^{(e)} = \omega^{(e)}$ if card(W_e) $= \infty$ and D_e is a finite initial segment of $\omega^{(e)}$ otherwise. We recall that an r.e. set $Q \subset D$ with $Q^{(e)} =^* D^{(e)}$ for all e is called a *thick subset* of D. As in [So2] if Q is a thick subset of D then Q is high. Thus we construct a subset $A = \bigcup_s A_s$ of D and a collection $C_e = \bigcup_s C_{e,s}$ of auxiliary r.e. sets to satisfy

$$P_e : A^{(e)} =^* D^{(e)},$$

$R_{e,i}$: If $\hat{\Phi}_e(A) = W_e$ then W_e is recursive or,

$$C_e \leqslant_{wtt} W_e \text{ and } \neg(C_e \leqslant_{tt} W_e \text{ via } \gamma_i).$$

Here $(\hat{\Phi}_e, W_e)$ is a standard enumeration of pairs of wtt-functionals $\hat{\Phi}_e$ with use ϕ_e and r.e. sets W_e. As usual we consider $\Phi_e(A)$ as controlling W_e, and $\hat{\Phi}_{e,s}(A_s; x)\downarrow$ means $\Phi_{e,s}(A_s; x)\downarrow$ and has use $\leqslant \phi_{e,s}(x)$ where $\phi_{e,s}(x)\downarrow$. Associated with the $R_{e,i}$ will be a collection of restraints $r(\tau, s)$ for $\tau \in 2^{<\omega}$ with $\text{lh}(\tau) = \langle e, i \rangle + 1$.

(3.11) DEFINITION. Define σ-correct, $l(\sigma, s)$, and σ-stage by induction on $\mathrm{lh}(\sigma)$ as follows:

(i) Every stage s is a $\varnothing$-stage and all computations are $\varnothing$-correct.

(ii) If s is a τ-stage where $\mathrm{lh}(\tau) = e$ then if

$$\mathrm{card}\left(D_{s+1}^{(e)}\right) > \max\left\{\mathrm{card}\left(D_t^{(e)}\right) : t \text{ is a } \tau\text{-stage with } t < s\right\}$$

then s is a $\tau\,^\wedge 0$-stage. Otherwise s is a $\tau\,^\wedge 1$-stage. Now a computation $\hat{\Phi}_{e,s}(A_s; x)\downarrow$ is $\tau\,^\wedge i$-correct for $i = 0, 1$ if for all $\rho\,^\wedge 0 \subset \tau\,^\wedge i$ where $\mathrm{lh}(\rho) = j$ if $r(\rho\,^\wedge 0, s) < z \leqslant \gamma_{e,s}(x)$ and $z \in \omega^{(j)}$ then $z \in A_s$. Let $l(\tau\,^\wedge i, s)$ denote $\max\{x : \forall y < x(\hat{\Phi}_{e,s}(A_s; y) = W_{e,s}(y)$ and the computation is $\tau\,^\wedge i$-correct)\}$.

As usual σ_s denotes the unique string of length s with s a σ_s-stage. Recall α_a denotes the ath tt-condition.

(3.12) DEFINITION. We say $R_{e,i}$ requires attention at stage $s + 1$ if one of the following options hold:

(3.13) For some follower z of $R_{e,i}$ we have

(i) z is active and has guess $\tau \leqslant_L \sigma_s$, and

(ii) if we set

$$l(e, s) = \max\left\{x : \forall y < x(\hat{\Phi}_{e,s}(A_s; y) = W_{e,s}(y))\right\},$$

then

$$l(e, s) > \max\{l(\tau, t) : t \text{ is a } \tau\text{-stage and } t < s\},$$

and

(iii) $W_{e,s}[z] \neq W_{e,t}[z]$ where t is the last τ-stage less than s, that is

$$t = \max\{t' : t' < s \text{ and } t' \text{ is a } \tau\text{-stage}\}.$$

(3.14)(a) For all followers z of $R_{e,i}$ if z has guess $\tau \subset \sigma_s$ then z is active, and

(b) there does not exist a number z such that

(i) $\gamma_{i,s}(z)\downarrow$ and $z \in C_{e,s}$ iff $\neg W_{e,s} \vDash \alpha_{\gamma_i(z)}$.

(ii) For some $\rho \leqslant_L \tau$ with $\mathrm{lh}(\rho) = \mathrm{lh}(\tau)$ we have $r(\rho, s) > N$ and $l(e, s) > N$, where

(ii)(a) $N = \max\{\phi_e(y) : y \leqslant M\}$, and

(ii)(b) $M = \max\{p : p \in \mathrm{dom}\, \alpha_{\gamma_i(z)}\}$.

(3.15) R_e has an inactive follower z with guess $\tau \subset \sigma_s$, $\gamma_{i,s}(z)\downarrow$ and $l(\tau, s) > \max\{z, z'\}$ where $z' = \max\{p : p \in \mathrm{dom}\, \alpha_{\gamma_i(z)}\}$.

CONSTRUCTION, STAGE $s + 1$.

Step 1. Cancel all followers and restraints with guesses $\tau \not\leqslant_L \sigma_s$.

Step 2. For each $\langle e, i \rangle < s$, adopt the first case below which holds.

Case 1 (SATISFYING $R_{e,i}$). (3.13) holds; set $r(\tau, s + 1) = s + 1$. Set $C_{e,s+1} = C_{e,s} \cup \{z\}$ if $\neg W_{e,s} \vDash \alpha_{\gamma_i(z)}$. Declare z to be no longer a follower. Cancel all followers with guess ρ for $\tau \leqslant_L \rho$ and $\mathrm{lh}(\tau) = \mathrm{lh}(\rho)$.

Case 2 (FOLLOWER ASSIGNMENT). (3.14) holds: appoint a large fresh number as an inactive follower of $R_{e,i}$ with guess σ where $\sigma \subset \sigma_s$ and $\mathrm{lh}(\sigma) = \langle e, i \rangle + 1$.

Case 3 (ACTIVATION). (3.15) holds; declare z as active.

Step 3. Finally for all e and $x \in D_{s+1}^{(e)}$ if $x \notin A_s$ enumerate $x \in A_{s+1}$ provided that $x \geqslant r(\tau, s + 1)$ for all $\tau \leqslant_L \sigma$ where $\sigma \subset \sigma_s$ and $\mathrm{lh}(\sigma) = e + 1$.

END OF CONSTRUCTION.

Let β denote the leftmost path. We verify by simultaneous induction that for $\sigma \leqslant_L \beta$ that $\lim_s r(\sigma, s) = r(\sigma)$ exists, the $R_{e,i}$ are met and all the P_e are met. Thus let σ be given, with $\mathrm{lh}(\sigma) = e + 1$ and let s_0 be a stage such that for all $s > s_0$

(i) $\sigma \leqslant_L \sigma_s$, and

(ii) $r(\tau, s) = r(\tau, s_0)$ for all $\tau \neq \sigma$ and $\tau \subset \sigma$.

We easily see that if $\rho^\wedge 1 \subset \sigma$ then $D_{s_0}^{(j)} = D^{(j)}$ where $j = \mathrm{lh}(\rho)$. Thus we may also suppose that

(iii) $A_{s_0}^{(j)} = A^{(j)}$ for all $\rho^\wedge 1 \subset \sigma$ with $j = \mathrm{lh}(\rho)$.

Finally we may also suppose that for all followers y and all j with guess $\rho \leqslant_L \sigma$ and $\rho \not\subset \sigma$ (so that ρ is to the left of the true path) we have that $y \in C_j$ iff $y \in C_{j,s_0}$. Since there are only finitely many such ρ-stages it follows that $\lim_s r(\rho, s) = r(\rho)$ exists. By assumption $\lim_s r(\tau, s) = r(\tau)$ exists for all $\tau \subset \sigma$ with $\tau \neq \sigma$. Thus it suffices to argue that $\lim_s r(\sigma, s) = r(\sigma)$ exists.

Now, by construction, $r(\sigma, s)$ is only reset when some follower with guess σ is enumerated into $C_{j,s+1} - C_{j,s}$ for the sake of $R_{j,i}$ where $\langle j, i \rangle = \mathrm{lh}(\sigma) - 1$. The only possible way for $\lim_s r(\sigma, s)$ not to exist is if $r(\sigma, s) \to \infty$ and $R_{j,i}$ receives attention infinitely often via (3.13). Thus take some follower z appointed to $R_{j,i}$ after stage s_0 such that (3.13) pertains to z.

Let t be the stage at which (3.13) pertains to z. We know that at some stage t_1 with $t_1 < t$ we have that (3.15) pertained to z and so z was activated. At stage t_1 we know that

$$ l(\sigma, s) > M, \quad \text{where } M = \max\{ p : p \in \mathrm{dom}\, \alpha_{\gamma_i(z)} \}. $$

Now since $\sigma \subset \beta$ and s_0 is chosen as above, we know that the $\hat{\Phi}_{j,t_1}(A_{t_1}; y)$ computations for $y \leqslant M$ are σ-correct at stage t_1.

The critical observation we need to make is that at any stage $s > t_1$, if $l(j, s) > M$, then the $\hat{\Phi}_{j,s}(A_s; y)$ computations for $y \leqslant M$ are also σ-correct. This is because the use function cannot change in a wtt-reduction. Therefore when (3.15) pertains to z at stage t we have that

(i) $l(j, t) > M$, and

(ii) the computations $\hat{\Phi}_{j,t}(A_t; y)$ for $y \leqslant M$ are σ-correct.

Now we apply Step 2, Case 1 and this ensures that

(a) $C_{j,t+1}(z) = 1$ iff $\neg W_{j,t} \vDash \alpha_{\gamma_i(z)}$, and

(b) $A_{t+1}[N] = A_t[N]$ where $N = \max\{\phi_j(y) : y \leqslant M\}$.

We ensure (b) because we reset $r(\sigma, t)$ to be $t + 1$ at stage $t + 1$. Now by choice of s_0 and $\sigma \subset \beta$ we know that (b) ensures that, in fact

(b)' $A[N] = A_t[N] = A_{t+1}[N]$.

This means that $\hat{\Phi}_{j,t}(A_t; y) = \hat{\Phi}_j(A; y)$ for all $y \leqslant M$. Therefore $W_{j,t}[M] = W_j[M]$. Thus $W_j \vDash \alpha_{\gamma_i(z)}$ iff $W_{j,t} \vDash \alpha_{\gamma_i(z)}$ iff $C_j(z) = 0$.

It is now clear that (3.14) is exactly the condition we need, since (3.14)(b) ensures that no new followers can be appointed to $R_{j,i}$ after stage t. Also all followers with guess σ are cancelled in Step 1. Therefore $R_{j,i}$ never again receives attention, and so $r(\sigma, t + 1) = t + 1 = r(\sigma)$. Thus we know that $\lim_s r(\sigma, s) = r(\sigma)$ exists. This easily implies that all P_e are met: If $\sigma = \tau ^\wedge 1$ there is nothing to prove. If $\sigma = \tau ^\wedge 0$, then if t is chosen such that $t > s_0$ and $r(\sigma, t) = r(\sigma)$ we see that at every σ-stage $t_1 \geq t$ we have

(i) $r(\rho, t_1) = r(\rho, t)$ for all $\rho \leq_L \sigma$ and

(ii) $r(\rho, t_1) = 0$ for all $\rho \not\leq_L \sigma_s$.

Thus in Step 3 any number x in $D_t^{(e)}$, with $x > \max\{r(\rho): \rho \leq_L \sigma\}$ is enumerated into A_{t_1+1} if $x \notin A_{t_1}$. Thus $D^{(e)} =^* A^{(e)}$.

Finally we need argue that all the $R_{j,i}$ are met. As above let $\sigma \subset \beta$ with $\mathrm{lh}(\sigma) = \langle j, i \rangle + 1$. Our earlier analysis reveals that if (3.13) ever pertains to $R_{j,i}$ after stage s_0, then $R_{j,i}$ will be met. Now if we assume that $R_{j,i}$ fails to be met, or receives attention infinitely often, it must be that $\hat{\Phi}_j(A) = W_j$ and $C_j \leq_{tt} W_j$ via γ_i. It is not difficult to see that W_j is recursive since (3.13) cannot pertain after stage s_0 to any follower with guess σ: To determine $W_j[z]$ find the least σ-stage $t = t(z)$ where $R_{j,i}$ has an active follower $q > z$. Since (3.13) cannot pertain to any number $\leq q$, we must have $W_j[q] = W_{j,t}[q]$. Finally if $\hat{\Phi}_j(A) = W_j$ then $C_j \leq_{wtt} W_j$. To compute if $z \in C_j$ find the least j-expansionary stage (i.e. where $l(j, s) > l(j, t)$ for all $t < s$) with $W_{j,s}[z] = W_j[z]$. Then $z \in C_j$ iff $z \in C_{j,s+1}$. $\square$

4. Local structure.

In this section we analyze the structure of 1- and other r-degrees within a single r.e. T-degree. Thus we focus on the more "local" behavior of (for example) 1-topped r.e. degrees. Various questions seem to suggest themselves. For example, we know that an r.e. degree $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ can have a 1-top. It would seem natural to ask if an r.e. degree $\mathbf{a} \neq \mathbf{0}$ can have a 1-bottom. This question has the following negative solution.

(4.1) THEOREM (KOBZEV [Ko1]). *Each nonzero r.e. degree contains an infinite antichain of minimal r.e. btt-degrees. Here "minimal" refers to both the r.e. btt-degrees and all Δ_2^0 btt-degrees.*

Since a contiguous r.e. T-degree contains a single r.e. wtt-degree, some nonzero r.e. T-degrees have wtt-bottoms. The best result along these lines was also established by Kobzev [Ko3]. He constructed a strongly tt-bottomed r.e. degree $\mathbf{a} \neq \mathbf{0}$. That is, he constructed an r.e. degree $\mathbf{a} \neq \mathbf{0}$ containing an r.e. set A of least tt-degree amongst all (i.e. not necessarily r.e.) sets of degree $\mathbf{a}$. In fact he obtained the following result.

(4.2) THEOREM (KOBZEV [Ko3]). *If A is an r.e. η-maximal semirecursive set, then for all $B \equiv_T A$, $A \leq_{tt} B$, and furthermore A has minimal tt-degree amongst all tt-degrees.*

The principal subgoal of this section is to analyze strongly tt-bottomed degrees, how they relate to 1-topped degrees and more generally, the structure of tt-degrees within an r.e. T-degree. For our purposes (4.2) is too indirect and limiting since it

necessarily constructs minimal tt-degrees. We first give a direct construction of a strongly tt-bottomed nonzero r.e. degree. We prove the following corollary to Kobzev's result.

(4.3) COROLLARY (KOBZEV). *If C is r.e. and nonrecursive, then there exist an r.e. set A with $\varnothing <_T A \leqslant_{wtt} C$ such that if B is any set with $B \equiv_T A$, then $A \leqslant_{tt} B$.*

PROOF. Our construction is along the lines of Downey's [Do2] construction of a nonzero r.e. strongly contiguous degree. (Recall that **a** is strongly contiguous if a consists of a single wtt-degree and $\mathbf{a} \neq \mathbf{0}$.) We build $A = \bigcup_s A_s$ in stages to satisfy

$$P_e: \overline{A} \neq W_e.$$
$$N_e: \text{If } \Gamma_e(\Phi_e(A)) = A \text{ and } \Phi_e(A) \text{ is total}$$
$$\text{then } A \leqslant_{tt} \Phi_e(A).$$

Here we remind the reader that $\Phi_e(A)$ is $\{0,1\}$-valued by convention. Before giving the formal details of the construction, we shall briefly discuss the technique we shall employ to meet the N_e.
Define

$$l(e,s) = \max\{x: \forall y < x(\Gamma_{e,s}(\Phi_{e,s}(A_s); y) = A_s(y))\}.$$

Note implicitly here we mean that for all $z < u$ $(\Gamma_{e,s}(\Phi_{e,s}(A_s); y))$, $\Phi_{e,s}(A_s; z)\downarrow$.

From N_e's point of view, the key points regarding the P_e will be that the P_e are finitary in nature, and will be satisfied by *followers* which are always appointed at stage s to be larger than s. In particular they exceed all computations, etc. by convention.

For a single N_e we keep in mind that our overall aim is to achieve $A \leqslant_{tt} \Phi_e(A)$. To do this, roughly speaking, for any follower x of P_j (say) we wait till the *first stage* where $l(e,s) > x$. At this stage we declare x as *e-confirmed* and attempt to ensure

$$x \in A - A_s \quad iff \quad \Phi_{e,s}(A_s)[Q] \neq \Phi_e(A)[Q]$$

where $Q = \max\{u(\Gamma_{e,s}(\Phi_{e,s}(A_s); y)): y \leqslant x\}$. (Note that this is a dtt-condition if $\Phi_e(A)$ is r.e.) To achieve this it is clearly in our interest to stop $\Phi_{e,s}(A_s)[Q]$ from changing once x is e-confirmed (unless x enters $A - A_s$). The first part of our strategy towards this aim is to *cancel* all lower priority followers than x when x is e-confirmed. (Such followers will be $> x$.) The net effect of this will be to ensure that there are no followers z with $x < z \leqslant s$, and thus if $A_s[x] = A[x]$ then $\Phi_e(A)[Q] = \Phi_{e,s}(A_s)[Q]$.

By definition of a use function, we know that if $A_s[x] \neq A[x]$ then $\Phi_e(A)[Q] \neq \Phi_{e,s}(A_s)[Q]$. Thus to complete our strategy (for a single N_e) we must ensure that if ever $A_s[x] \neq A_t[x]$ then $A_s(x) \neq A_t(x)$. This is fine if we add x to $A_t - A_{t-1}$ because x "receives attention," but is somewhat of a problem if some $y < x$ acts. Our solution is to use "dumping." That is, if $y \in A_{t+1} - A_t$ then we ensure that $\forall z(y \leqslant z \leqslant t \to z \in A_{t+1})$. (REMARK. In the terminology of [Jo1], this makes A semirecursive.) In particular, if y enters A then x enters too.

From a global point of view we must also incorporate various nested strategies according to whether or not $l(e, s) \to \infty$. This is again achieved by a tree of strategies argument. We now give the formal details of the argument.

A stage s is defined to be a σ-*stage* by induction on $\mathrm{lh}(\sigma)$.

(i) Every stage s is a $\varnothing$-stage.

(ii) If s is a τ-stage with $\mathrm{lh}(\tau) = e$ then if $l(e, s) > \max\{l(e, t) : t$ is a τ-stage and $t < s\}$, we say s is a $\tau{}^\wedge 0$-stage.

Otherwise, s is a $\tau{}^\wedge 1$-stage.

Let f be a 1-1 recursive function enumerating a given r.e. nonrecursive set C so that $f(\omega) = C$. We say that P_e *requires attention at stage* $s + 1$ if $W_{e,s} \cap A_s = \varnothing$ and one of the following options holds.

(4.4) P_e has no follower x with $x \notin W_{e,s}$.

(4.5) P_e has a follower x such that

(i) $x \in W_{e,s}$, and

(ii) $f(s) \leqslant x$.

CONSTRUCTION, STAGE $s + 1$.

Step 1. As usual, let σ_s denote the unique string with $\mathrm{lh}(\sigma_s) = s$ and s a σ_s-stage. Cancel all followers x with guess $\tau \not\leqslant_L \sigma_s$.

Step 2. Find the least follower x (if any) such that for some $\tau{}^\wedge 0 \subset \sigma_s$ we have

(i) x has guess γ and $\tau{}^\wedge 0 \subset \gamma$,

(ii) if $e = \mathrm{lh}(\tau)$ then $l(e, s) > x$, and

(iii) x is not yet $\tau{}^\wedge 0$-confirmed.

Declare x as $\tau{}^\wedge 0$-confirmed for each such τ and cancel all followers $y > x$.

REMARK. The reader should note that $\gamma \leqslant_L \sigma_s$ (as x is still alive) but it is not necessary for $\gamma \subset \sigma_s$ for Step 2 to apply.

Step 3. Find the least e such that P_e requires attention. If (4.4) holds, appoint $y = s + 1$ as a follower of P_e with guess $\sigma \subset \sigma_s$ where $\mathrm{lh}(\sigma) = e + 1$. Cancel all followers y with a guess γ for $\gamma \supset \sigma$ and $\gamma \neq \sigma$.

If (4.5) holds, set $A_{s+1} = A_s \cup \{z : x \leqslant z \leqslant s\}$. P_e is now met (forever).

END OF CONSTRUCTION.

(4.6) LEMMA. (i) *If* $z = \mu y$ $(y \in A_{s+1} - A_s)$ *then* z *is a follower*.

(ii) $A \leqslant_{\mathrm{wtt}} C$.

PROOF. (i) Numbers enter $A_{s+1} - A_s$ only in Step 3, and when (4.5) pertains to some follower x. We then set $A_{s+1} = A_s \cup \{z : x \leqslant z \leqslant s\}$.

(ii) Let z be given. To decide if $z \in A$, compute the least stage s such that $\forall t > s(f(t) > z)$. Then by (i) above and (ii) of (4.5) we see that $z \in A$ iff $z \in A_{s+1}$. $\square$

Now let β denote the leftmost path. Let $\sigma \subset \beta$ with $\mathrm{lh}(\sigma) = e + 1$.

(4.7) LEMMA. P_e *receives attention finitely often at* σ-*stages and* P_e *is met*.

PROOF. By induction, let s_0 be a σ-stage such that for all $s > s_0$

(i) For all $j < e$, P_j does not receive attention at stage s if s is a σ-stage.

(ii) $\sigma \leqslant_L \sigma_s$.

(iii) For all $j < e$, P_j does not receive attention via (4.5) at stage s.

(iv) No follower of P_j for $j < e$ with guess $\tau \leqslant_L \sigma$ receives any p-confirmation at stage s (for any ρ).

Define a follower x at P_e with guess σ to be *confirmed* if it is $\tau^\wedge 0$-confirmed for all $\tau^\wedge 0 \subset \sigma$. It is clear that choice of s_0 and induction ensures that if x is a confirmed follower of P_e with guess σ appointed after stage s_0, then x is uncancellable. Furthermore, (4.6)(i) ensures that $x \notin A$, unless P_e receives attention via some follower $y \leqslant x$.

Now if P_e fails to be met or receives attention infinitely often, it is quite easy to see that there exists an infinite recursive set of (uncancellable) followers $\{x_1 < x_2 < \cdots\}$ all appointed after stage s_0 such that for all i,

(i) x_i is confirmed,

(ii) x_i has guess σ, and

(iii) $x_i \in W_e$.

We claim that $C = f(\omega)$ is recursive. To compute $f(z)$ find the least stage $s > s_0$ such that P_e has a confirmed follower $x_i > z$ with guess σ and with $x_i \in W_{e,s}$. Then $z \in f(\omega) = C$ iff $z \in \{f(0), \ldots, f(s)\}$. Otherwise (4.5) would pertain to x_i at some stage meeting P_e.

To complete the proof, we argue that N_e is also met. Thus suppose ($\sigma = \tau^\wedge 0$ with $\sigma \subset \beta$ and $\mathrm{lh}(\sigma) = e + 1$. Let s_0 be a σ-stage as in the proof of (4.7). We need to show that $A \leqslant_{tt} \Phi_e(A)$ (assuming $\Phi_e(A)$ total). Let z be given. Find the least σ-stage s_1 with $l(e, s_1) > z$. Suppose there is no follower $x \leqslant z$ such that x has guess $\gamma \supset \sigma$. Then by (4.6)(i), choice of s_0, and cancellation at σ-stages it follows that $z \in A$ iff $z \in A_{s_1}$.

Thus we suppose there is some (σ-confirmed) follower $x \leqslant z$. Let x_1 be the largest such follower. Let

$$Q = \max\left\{u\left(\Gamma_{e,s_1}\left(\Phi_{e,s_1}\left(A_{s_1}\right); y\right)\right) : y \leqslant x_1\right\}.$$

We claim that if $z \notin A_{s_1}$, then

(4.8) $\qquad z \in A \quad \text{iff} \quad \Phi_e(A)[Q] \neq \Phi_{e,s_1}(A_{s_1})[Q].$

Note that (4.8) is a tt-condition. (Also if $\Phi_e(A)$ is r.e. then this is a dtt-condition. In any case it is a dtt-condition from $\Phi_e(A) \oplus \overline{\Phi_e(A)}$.)

Certainly as x_1 is the largest σ-confirmed follower $\leqslant z$ at stage s, the way we appoint followers ensures that

$$\text{if } A_{s_1}[x_1] = A[x_1] \text{ then } A_{s_1}[s_1] = A[s_1].$$

(By (4.6)(i) and minimality of s_1.) Therefore if $A_{s_1}[x_1] = A[x_1]$ then

(4.9) $\qquad \Phi_{e,t}(A_t)[Q] = \Phi_{e,s_1}(A_{s_1})[Q] \quad \text{for all } t \geqslant s_1,$

since $l(e, s_1) > z$ at stage s_1. By (4.9) and (4.6)(i) and the construction, we have

(4.10) $\qquad A_{s_1}(z) = A(z) \quad \text{implies} \quad \Phi_{e,s_1}(A_{s_1})[Q] = \Phi_e(A)[Q].$

Conversely, suppose $A_{s_1}(z) \neq A(z)$. Then by construction and (4.6)(i), $A_{s_1}[x_1] \neq A[x_1]$. Now since $\Gamma_{e,s_1}(\Phi_{e,s_1}(A_{s_1}); x_1) = 0 \neq A(x_1)$ it cannot be that $\Phi_e(A)[Q] = \Phi_{e,s_1}(A_{s_1})[Q]$ by definition of Q. Thus $\Phi_e(A)[Q] \neq \Phi_{e,s_1}(A_{s_1})[Q]$. Putting this together with (4.10) gives (4.8), as required, and so $A \leqslant_{tt} \Phi_e(A)$. $\square$

As we remarked at various points throughout the argument this construction gives (apparently) a little more than Kobzev's result:

(4.11) COROLLARY. *Let C be r.e. and nonrecursive. Then there exists an r.e. set A with $\varnothing <_T A \leqslant_{\text{wtt}} C$ such that*
(i) *$B \equiv_T A$ and B r.e. implies $A \leqslant_{\text{dtt}} B$, and*
(ii) *furthermore for any set $D \equiv_T A$, $A \leqslant_{\text{dtt}} D \oplus \overline{D}$.*

We remark that all known constructions of (strongly) tt-bottomed r.e. degrees seem to use semirecursive sets such as A above. We do not know if this is necessary. Concerning the T-degrees of tt-bottomed r.e. degrees, we do not know of any jump class classification although it seems probable that no high r.e. degree can be tt-bottomed. Later we show $\mathbf{0}'$ is not wtt-bottomed. In particular, the index set reasoning we used for 1-degrees in §3 fails since, as we show, $\{e: W_e \geqslant_{\text{wtt}} A\}$ is always different from $\{e: W_e \geqslant_T A\}$ for A r.e., nonrecursive, and incomplete.

We do have a partial classification in terms of wtt-cuppability. Recall that an r.e. set A is *wtt-cuppable* if there exists an r.e. set B such that $A \oplus B \equiv_{\text{wtt}} \varnothing'$ and $\varnothing' \not\leqslant_{\text{wtt}} B$.

E. L. Post showed [Po] that hypersimple sets cannot be tt-complete, and R. Friedberg and H. Rogers observed [FR, p. 124] that similar methods may be used to show that hypersimple sets cannot be wtt-complete. In the next theorem we extend this to show that no hypersimple set can be wtt-cuppable. This result will have as a corollary a result mentioned in [AJSS, p. 124]: no contiguous degree is wtt-cuppable. We thank Jeanleah Mohrherr for asking about the connection between hypersimplicity and wtt-cuppability.

(4.12) THEOREM. *No hypersimple set is wtt-cuppable.*

PROOF. Let H be hypersimple and A be r.e. Assume that $K \leqslant_{\text{wtt}} H \oplus A$, where K is creative. We must show that $K \leqslant_{\text{wtt}} A$. Using the technique devised by Lachlan for the nondiamond theorem [La1, Theorem 5], we enumerate an r.e. set E in such a way as to "force" many numbers into $H \oplus A$, and ultimately into A. Since $E \leqslant_{\text{wtt}} H \oplus A$ and our construction is uniform, we may assume that we know in advance a wtt-reduction procedure $\hat{\Gamma}$ such that $E = \hat{\Gamma}(H \oplus A)$, together with its (recursive) use function γ. Let

$$l(s) = \max\left\{ x : (\forall y < x)[\hat{\Gamma}_s(H_s \oplus A_s; y) = E_s(y)] \right\}.$$

Clearly if $u < l(s)$, $u < l(t)$, and $u \in E_t - E_s$, then

$$(4.13) \qquad (H_t \oplus A_t)[\gamma(u)] \neq (H_s \oplus A_s)[\gamma(u)].$$

We may assume without loss of generality that γ is nondecreasing and that $\langle n, j \rangle$ is nondecreasing as a function of j, so that $\gamma(\langle n, j \rangle)$ is a nondecreasing function of j.

If n enters K, we subsequently put the numbers $\langle n, 0 \rangle$, $\langle n, 1 \rangle$, $\langle n, 2 \rangle, \ldots$ into E in that order, but we do not put $\langle n, j + 1 \rangle$ into E until the change in $H \oplus A$ below $\gamma(\langle n, j \rangle)$—guaranteed by (4.13) with $u = \langle n, j \rangle$—has already occurred. More

precisely, we define $E = \bigcup_s E_s$, where $E_0 = \varnothing$ and

$$E_{s+1} = E_s \cup \{ \langle n, k \rangle : n \in K_s \,\&\, \langle n, k \rangle < l(s)$$
$$\&(\forall j < k)[\langle n, j \rangle \in E_s] \}.$$

It is trivial to show by induction on k that if $n \in K$, then $\langle n, k \rangle \in E$ for all k (so that $E = K \times \omega$). For fixed $n \in K$, let $t(k)$ be the unique s with $\langle n, k \rangle \in E_{s+1} - E_s$. By (4.13)

$$\left(H_{t(k)} \oplus A_{t(k)} \right)[\langle \gamma \langle n, k \rangle\rangle] \neq \left(H_{t(k+1)} \oplus A_{t(k+1)} \right)[\gamma(\langle n, k \rangle)].$$

Using the above, the assumption that $\gamma(\langle n, k \rangle)$ is nondecreasing in k, and the r.e.-ness of H and A, it follows that $(H_s \oplus A_s)[\gamma(\langle n, k \rangle)]$ has at least $k + 1$ distinct values for $s \in [t(0), t(k + 1)]$. Suppose for the moment that $\overline{H}_s$ has at most k elements $\leqslant \gamma(\langle n, k \rangle)$. Then

(4.14) $A_{t(0)}[\gamma(\langle n, k \rangle)] \neq A[\gamma(\langle n, k \rangle)]$

since not all of the changes in $H \oplus A$ can be due to changes in H.

The assumption that H is hypersimple is used to show that the hypothesis of (4.14) is satisfied sufficiently often. Let any number n be given. (We no longer assume that $n \in K$!) Then there exist k and s such that $\overline{H}_s$ has at most k elements $\leqslant \gamma(n, k)$. (Otherwise $\overline{H}$ has $> k$ elements $\leqslant \gamma(\langle n, k \rangle)$ for all k, in contradiction to the domination definition of hypersimplicity. We remind the reader that this definition reads that A is hypersimple if its complement is not majorized by any recursive function (see [Rg, p. 139]).) Let $\langle k_n, s_n \rangle$ be (say) the least pair $\langle k, s \rangle$ with this property. To compute whether $n \in K$ from A, find $\hat{s}$ so large that

$$A_{\hat{s}}[\gamma(\langle n, k_n \rangle)] = A[\gamma(\langle n, k_n \rangle)].$$

Then $n \in A$ iff $n \in A_{\hat{s}}$. To verify this, assume for a contradiction that $n \in A - A_{\hat{s}}$. Then $t(0) > \hat{s}$ where, as before, $t(k) = (\mu s)[\langle n, k \rangle \in E_{s+1}]$. By (4.14) with $k = k_n$, $A_{t(0)}[\gamma(\langle n, k_n \rangle)] \neq A[\gamma(\langle n, k_n \rangle)]$, so $A_{\hat{s}}[\gamma(\langle n, k_n \rangle)] \neq A[\gamma(\langle n, k_n \rangle)]$. This contradicts the choice of $\hat{s}$. Thus $K \leqslant_{wtt} A$ with use function $\gamma(\langle n, k_n \rangle)$. $\square$

(4.15) COROLLARY. *Every r.e. T-degree contains an r.e. set which is not wtt-cuppable. In particular, no contiguous r.e. degree* **a** *contains a wtt-cuppable set.*

REMARK. The latter result was mentioned without proof in [AJSS, p. 124], and a direct proof is given in [AS, Theorem 5.3].

PROOF. Let **a** be any r.e. degree. If **a** = **0**, then the result is trivial. If **a** $\neq$ **0**, then **a** contains a hypersimple set by [De]. The second sentence of the corollary follows immediately from the first since wtt-cuppability is invariant under $\equiv_{wtt}$. $\square$

Actually, the *proof* that contiguous degrees are not wtt-cuppable which is given in [AS] establishes somewhat more. This proof shows that *wtt-bottomed* r.e. degrees contain r.e. sets that are not wtt-cuppable. In particular, the *wtt-bottoms are not wtt-cuppable*. This result also follows from (4.12) in the manner of (4.15) since we know that *if A is any r.e. nonrecursive set then there exists a hypersimple r.e. set $B \equiv_T A$ with $B \leqslant_{wtt} A$*. The easiest way to see this is to consider the Dekker [De] deficiency set for B. (This actually achieves $B \leqslant_{tt} A$, see (e.g.) Soare [So4, V, Ex. 2.12].)

We now turn to the relationships between r-topped degrees and tt-bottomed ones. The fact that nonzero strongly contiguous r.e. degrees exist would seem to suggest that perhaps there exists an r.e. degree consisting of a single (r.e.) tt-degree. This suggestion fails by Cohen [Co]. In fact, building on earlier results of Lachlan [La2] and Cohen [Co], Kobzev [Ko2] has shown that each nonzero r.e. wtt-degree contains an infinite antichain of r.e. tt-degrees. By our results of §3 we have the following.

(4.16) THEOREM. *Below any nonzero r.e. degree there exists a nonzero r.e. strongly tt-bottomed T-degree which is not tt-topped.*

PROOF. Combine (4.3) with (3.3). □

We can do a little better by squeezing more information out of the construction of (4.3).

(4.17) THEOREM. *The r.e. set A constructed in (4.3) is also of strongly contiguous degree. Thus, if C is any r.e. nonrecursive set there exists an r.e. set A with $\varnothing <_T A \leqslant_{\text{wtt}} C$ such that for all sets B, if $B \equiv_T A$ then $A \leqslant_{tt} B$ and $B \leqslant_{\text{wtt}} A$. Hence $\deg(A)$ is strongly contiguous and tt-bottomed.*

PROOF. We verify that (4.3) gives the desired result. Thus suppose $\Gamma_e(\Phi_e(A)) = A$. We also claim that $\Phi_e(A) \leqslant_{\text{wtt}} A$.

Let σ, s_0 be as in (4.7) and (4.8). (Recall $\sigma \subset \beta$ with $\text{lh}(\sigma) = e + 1$ and s_0 is a stage "good for σ".) Let x be given. We show how to compute $\Phi_e(A)(x)$ from A. Find the least σ-stage that $l(e, s_1) > x$ and $s_1 > s_0$. Now compute the *least* σ-stage $s_2 > s_1$ with

(4.18) $$A_{s_2}[s_1] = A[s_1].$$

We claim that $\Phi_{e,s_2}(A_{s_2})(x) = \Phi_e(A)(x)$. We give the details although the argument is fairly standard by now. Suppose not. By (4.18), this means that there is some number y with

$$s_1 \leqslant y < u\left(\Phi_{e,s_2}(A_{s_2}; x)\right)$$

such that y enters A after stage s_2. By (4.6)(i) and (4.18) we may clearly suppose y is a follower at the stage t when $y \in A_t - A_{s_2}$. By cancellation at σ-stages, y must have guess $\gamma \supset \sigma$ since it is still alive at stage s_2. Now since y has guess $\supset \sigma$ it must have been appointed at a σ-stage s_3 with $s_1 \leqslant s_3 < s_2$. Since $y \notin A_{s_2}$ no number $\leqslant y$ has entered A *after* stage s_3 but *before* stage s_2. (The entry of such a number would cause y to also enter.) It follows that $A_{s_3}[s_1] = A_{s_2}[s_1]$ since $s_1 \leqslant y$. But then $A_{s_3}[s_1] = A[s_1]$ by (4.18). This specifically contradicts the minimality of s_2 since $s_3 < s_2$. Hence $A[u(\Phi_{e,s_2}(A_{s_2}; x))] = A_{s_2}[u(\Phi_{e,s_2}(A_{s_2}; x))]$ and so $\Phi_{e,s_2}(A_{s_2})(x) = \Phi_e(A)(x)$ as desired. Therefore $\Phi_e(A) \leqslant_{\text{wtt}} A$. □

(4.18) COROLLARY. *There exist r.e. T-degrees consisting of a single wtt-degree that are strongly tt-bottomed, but have no tt-top.*

On the other hand, in §3 we observed that an r.e. degree may be 1-topped but not tt-bottomed. For $0'_T$ in fact we have

(4.19) THEOREM. *$0'_T$ is 1-topped but if R is any r.e. set wtt below all r.e. sets of T-degree $0'$ then R is recursive.*

PROOF. Suppose $\varnothing <_T R$ and R satisfies the given hypotheses. We build a T-complete r.e. set A with $R \not\leq_{wtt} A$. Thus let $f(\omega)$ be a 1-1 enumeration of a creative r.e. set. We satisfy the requirements $Q_e : \hat{\Gamma}_e(A) \neq R$. Here $\hat{\Gamma}_e$ denotes the eth wtt-reduction and has monotone use γ_e. Now let

$$l(e,s) = \max\{x : \forall y < x(\hat{\Gamma}_{e,s}(A_s; y) = R_s(y))\},$$
$$ml(e,s) = \max\{l(e,s) : t < s\}, \text{ and}$$
$$ls(e,s) = \max\{0, t : t \text{ is a stage} < s \text{ with } l(e,t) > ml(e,t)\}.$$

We say that Q_e *requires attention* at stage $s + 1$ if e is least such that $l(e,s) > ml(e,s)$.

CONSTRUCTION.

STAGE 0. Set $A_0 = \varnothing$ and $a_{i,0} = i$ for all $i \in \omega$.

STAGE $s + 1$. Find the e (if any) such that Q_e requires attention. Let

$$n = \begin{cases} \min\{f(s), e\} & \text{if } e \text{ exists,} \\ f(s) & \text{otherwise.} \end{cases}$$

Set

$$A_{s+1} = A_s \cup \{a_{n,s}, \ldots, a_{n+s,s}\},$$

and set

$$a_{e,s+1} = \begin{cases} a_{i,s} & \text{for } i < n, \\ a_{i+n+s+1,s} & \text{otherwise.} \end{cases}$$

END OF CONSTRUCTION.

VERIFICATION. We argue that $\lim_s a_{i,s} = a_i$ exists and each Q_e receives attention at most finitely often (and is met). For an induction, let s_0 be a stage such that for all $s > s_0$

(i) $\forall j < e$ (Q_j does not receive attention at stage s),

(ii) $f(s) > e$, and

(iii) $a_{j,s} = a_j$ for $j < e$.

Suppose that Q_e receives attention infinitely often. Then $l(e,s) \to \infty$. We show that this implies that R is recursive.

To compute $R(x)$ find the least stage $s_1 > s_0$ with $l(e,s_1) > ml(e,s_1)$ and $l(e,s_1) > x$. Then Q_e receives attention at stage s_1 and

$$A_{s+1} = A_{s_1} \cup \{a_{e,s_1}, \ldots, a_{e+s_1,s_1}\}.$$

Now by convention $\gamma_e(x) < s_1$. By induction $a_{i,s_1} \geq i$. Thus we see

$$\forall z\Big(\big(z < \gamma_e(x) \& z \notin A_{s_1+1}\big) \to z < a_{e,s_1}\Big).$$

The assumptions on s_0 imply that this means

$$\forall z\Big(\big(z < \gamma_e(x) \& z \in A\big) \to z \in A_{s_1}\Big).$$

Consequently, if s_2 is the least stage with $s_2 > s_1$ and $l(e,s_2) > ml(e,s_2)$ we see that the computation $\Phi_{e,s_2}(A_{s_2}; x)$ is final. Hence $x \in R$ if $x \in R_{s_2}$. Hence R is recursive. Therefore Q_e receives attention finitely often. This in turn means $\lim_s a_{e,s} = a_e$ exists since after stage s_0, $a_{e,s}$ can only change when Q_e receives attention.

Finally, we need argue that $A \equiv_T f(\omega)$. To compute if $x \in f(\omega)$ A-recursively find the least stage s such that $a_{x,s} = a_{x,t}$ for all $t > s$. Then $x \in f(\omega)$ iff $x \in \{f(0), \ldots, f(s)\}$. $\square$

For our next results we establish that an r.e. degree can be strongly tt-bottomed but have no wtt-top. To complete the picture we would need to construct an r.e. degree $\mathbf{a} \ne \mathbf{0}$ with a 1-top and a (strong) tt-bottom. We do not know how to do this! In fact, it is an open question whether there exists an r.e. degree $\mathbf{a} \ne \mathbf{0}$ with an r.e. tt-top and an r.e. wtt-bottom. The infinitary nature of the positive requirements used to satisfy the 1-top requirements of (2.1) seem to interfere very strongly with the delicate contiguity-type machinery used to construct wtt-bottoms. On the other hand this machinery can be modified to admit certain infinitary positive requirements as witnessed by Ladner's [Ld] difficult construction of a $\text{low}_2 - \text{low}_1$ "completely mitotic" contiguous r.e. degree.

(4.20) THEOREM. *There exists an r.e. degree* $\mathbf{a}$ *such that* $\mathbf{a}$ *is strongly tt-bottomed but has no wtt-top. In fact for all* (*not necessarily r.e.*) *sets B of degree* $\mathbf{a}$ *there is an r.e. set C of degree* $\mathbf{a}$ *with* $C \not\leq_{\text{wtt}} B$.

PROOF. We build $A = \bigcup_s A_s$ together with auxiliary r.e. sets $C_e = \bigcup_s C_{e,s}$ to satisfy

$$R_{e,i} : \Gamma_e(\Phi_e(A)) = A \text{ implies } C_e \leq_T A \text{ and } \hat{\Phi}_i(\Phi_e(A)) \ne C_e.$$
$$N_e : \Gamma_e(\Phi_e(A)) = A \text{ implies } A \leq_{\text{tt}} \Phi_e(A).$$

Again, let

$$l(e, s) = \max\{x : \forall y < x (\Gamma_{e,s}(\Phi_{e,s}(A_s); y) = A_s(y))\}.$$

Now as in (4.3) define "σ-stage" by induction on $\text{lh}(\sigma)$ using $l(e, s)$: every stage is a $\varnothing$-stage and if s is a τ-stage with $\text{lh}(\tau) = e$ define s to be a $\tau^\wedge i$ stage where $i = 0$ if $l(e, s) > \max\{l(e, t) : t < s\}$ and $i = 1$ otherwise.

Define σ_s as usual. Also define $ml(e, s) = \max\{l(e, t) : t < s\}$. We say a stage s is *e-expansionary* if $l(e, s) > ml(e, s)$. Now, define

$$L(e, i, s) = \max\{x : \forall y < x (\hat{\Phi}_{i,s}(\Phi_{e,s}(A_s); y) = C_{e,s}(y)$$
$$\& \ l(e, s) > \phi_{i,s}(\Phi_{e,s}(A_s); y))\}.$$

Here the reader should recall that ϕ_i is the use of $\hat{\phi}_i$ by convention (and is nondecreasing). We say a stage s is (e, i)-*expansionary* if $L(e, i, s) > mL(e, i, s)$ where $mL(e, i, s) = \max\{L(e, i, t) : t < s\}$.

We briefly describe the method we employ to satisfy the $R_{e,i}$ in conjunction with the N_e. First for the N_e we proceed *almost* exactly as we did in (4.3). Of course we cannot proceed exactly as we did since we know that this gives a strongly contiguous degree. The crucial difference will be that we will allow numbers to be appointed to some $R_{e,i}$ at some guess σ with $\text{lh}(\sigma) = \langle e, i \rangle$ *at other than σ-stages*. (These numbers will be traces and $R_{e,i}$ will be "waiting".)

We satisfy the $R_{e,i}$ by a Friedberg-Muchnik type procedure. Basically we wait till $L(e, i, s) > x$ for some follower x of $R_{e,i}$ targeted for C_e. At this stage we enumerate x into A and declare $R_{e,i}$ as *waiting*. "Waiting" indicates that whenever possible we

wish to give x a trace (to ensure $C_e \leqslant_T A$). The problem is that we cannot immediately give x a trace $T(x)$ targeted for A. Our idea will be to be able to add x to $\dot{C}_e$ and $T(x)$ to A at the same time to create a disagreement. Thus, we must know that $T(x)$'s entry will be good from the point of view of not injuring "$\dot{\Phi}_{e,s}(\Phi_{e,s}(A_s); x) = 0$" computations. Now when x entered A_t the "$\Phi_{e,t}(A_t; x)$" computations may have been injured and perhaps the use changed. Therefore we must wait until the "$\Phi_{e,t}(A_t; x)$" computations recover before we know how big the use is, and so how large $T(x)$ needs to be. Thus, we choose to wait until the next e-expansion stage $s > t$ and set $T(x) = s + 1$. We note that this is fine because $C_e \leqslant_T A$ is predicated upon $\Gamma_e(\Phi_e(A)) = A$.

Now at stage s when $T(x)$ is set, perhaps s is not a σ-stage where x has guess σ. Nevertheless, $T(x)$ will inherit x's guess (so $T(x)$ has guess σ) to cooperate with the N_j requirement.

The remainder of the argument is straightforward. We declare x as active and wait till $L(e, i, s) > x$ again. Then we create a disagreement by setting $C_{e,s+1} = C_{e,s} \cup \{x\}$ and enumerating $T(x)$ into A.

Formal details now follow, although we suspect that the reader may wish to supply them himself.

We say that $R_{e,i}$ *requires attention* at stage $s + 1$ if $R_{e,i}$ is not currently declared satisfied. One of the following options holds.

(4.21) $R_{e,i}$ is active via x and s is (e, i)-expansionary.

(4.22) $R_{e,i}$ is waiting via x and s is e-expansionary.

(4.23) $R_{e,i}$ is inactive but has a follower x with $L(e, i, s) > x$ and s is (e, i)-expansionary.

(4.24) $R_{e,i}$ is inactive, and has no follower.

CONSTRUCTION, STAGE $s + 1$.

Step 1. Let σ_s denote the unique string with $\mathrm{lh}(\sigma_s) = s$ and s a σ_s-stage. Cancel all followers or traces x with guesses $\tau \not\leqslant_L \sigma_s$. Also for each e, i if $R_{e,i}$ is active or waiting via such x, declare $R_{e,i}$ as inactive.

Step 2. Find the least follower or trace z not already $\tau^\wedge 0$-confirmed for some $\tau^\wedge 0 \subset \sigma_s$ such that

(i) z has guess $\gamma \supset \tau^\wedge 0$, and

(ii) $l(e, s) > z$, where $e = \mathrm{lh}(\tau)$.

Declare z as $\tau^\wedge 0$-confirmed for each such $\tau^\wedge 0$ and cancel all followers or traces $z' > z$.

Step 3. Find the least $\langle e, i \rangle$ (if any) such that $R_{e,i}$ requires attention. If none exists go to stage $s + 2$. Let $\sigma \subset \sigma_s$ with $\mathrm{lh}(\sigma) = \langle e, i \rangle + 1$. Cancel all followers or traces with guesses $\gamma \supset \sigma$ and $\gamma \neq \sigma$. Adopt the appropriate case below.

Case 1. (4.21) holds. Declare $R_{e,i}$ as currently satisfied via x. Set $A_{s+1} = A_s \cup \{z : T(x) \leqslant z \leqslant s\}$. Set $C_{e,s+1} = C_{e,s} \cup \{x\}$. (This step temporarily satisfies $R_{e,i}$.) Cancel all followers or traces of R_g for $g > e$ and declare these as inactive.

Case 2. (4.22) holds. Declare $R_{e,i}$ as active via x and set $T(x) = s + 1$. Declare $T(x)$ to have guess γ where γ is the guess of x. (Note $T(x)$ is not yet ρ-confirmed for any ρ.) Cancel all followers or traces with guess η for $\eta \not\leqslant_L \gamma$. Declare the appropriate R_g as inactive.

Case 3. (4.23) holds. Declare $R_{e,i}$ as waiting. Set $A_{s+1} = A_s \cup \{z : x \leqslant z \leqslant s\}$. Cancel all followers or traces of R_g for $g > e$. Declare these R_g as inactive.

Case 4. (4.24) holds. Appoint $x = s + 1$ as a follower of $R_{e,i}$ with guess σ. Cancel all followers and traces with guess γ for $\gamma \supset \sigma$ and $\gamma \neq \sigma$. For any R_g corresponding to these followers or traces, declare R_g as inactive.

END OF CONSTRUCTION.

VERIFICATION. Let β denote the leftmost path. Let $\sigma \subset \beta$ with $\mathrm{lh}(\sigma) = \langle e, i \rangle$. For an induction, suppose that s_0 is a σ-stage such that for all $s > s_0$ we have:

(i) For all $k < \langle e, i \rangle$, R_k does not receive attention at stage s, nor do any numbers associated with R_k receive any confirmation at stage s.

(ii) $\sigma \leqslant_L \sigma_s$.

(iii) All followers or traces with guess $\tau \not\geqslant_L \sigma$ cease acting.

We now verify that $R_{e,i}$ receives attention at most finitely often and is met. After stage s_0, once $R_{e,i}$ gets a follower x with guess $\gamma \leqslant_L \sigma$ this follower x is evidently uncancellable. We claim that this follower succeeds in meeting $R_{e,i}$. Notice that such x will be the last follower $R_{e,i}$ ever receives. This follows since either we get stuck in some state such as waiting, or via x, $R_{e,i}$ eventually is *declared* satisfied. This declaration can only be cancelled by higher priority activity, which cannot happen by choice of s_0. Thus for the $R_{e,i}$ it remains to verify that the strategy actually works.

Thus suppose $\Gamma_e(\Phi_e(A)) = A$. This clearly means that (4.23) pertains to x at some stage s_1, say. At this stage, we know $L(e, i, s_1) > x$ and in particular

(4.25) $\hat{\Phi}_{i,s_1}\big(\Phi_{e,s_1}(A_{s_1}); x\big) = C_{e,s_1}(x) = 0.$

Thus at stage $s_1 + 1$, we enumerate x into $A_{s_1+1} - A_{s_1}$, and declare $R_{e,i}$ as waiting. Now at the first e-expansionary stage s_2 exceeding s_1 (4.22) pertains and we set $T(x) = s_2 + 1$. By convention this means

(4.26) $T(x) > u$ for $u = \max\big\{u\big(\Phi_{e,s_2}(A_{s_2}; z)\big) : z \leqslant \phi_e(x)\big\}.$

Furthermore, the cancellation procedure and choice of s_0 will ensure that

(4.27) $\Phi_{e,s_2}\big(A_{s_2}\big)[u] = \Phi_e(A)[u]$ (with u as in (4.26)).

Now, either (4.21) never pertains to x and so $l(e, i, s) \nrightarrow \infty$, or (4.21) pertains and (4.25), (4.26) and (4.27) together ensure that then

$$\hat{\Phi}_i\big(\Phi_e(A); x\big) = 0 \neq 1 = C_e(x).$$

Finally, $C_e \leqslant_T A$ by traces. To decide if $x \in C_e$, see if x is a follower of some $R_{e,i}$ by stage $x + 1$. If $x \notin A_x$ and x is not a follower by stage $x + 1$ then $x \notin C_e$. If x is a follower, then $x \in C_e$ only if $x \in A$. If $x \in A$ find the stage s when this occurs. Now at the least e-expansionary stage $s_1 > s$ either x has been cancelled or x has a trace $T(x)$ at stage $s_1 + 1$. Finally if x is still alive, then $x \in C_e$ only if $T(x) \in A$. If $T(x) \in A$ go to stage $s_2 > s_1$ where $T(x) \in A_{s_2+1} - A_{s_2}$. Then $x \in C_e$ iff $x \in C_{e,s_2+1}$.

The verification for the N_e is almost the same as in (4.3). For example (4.6)(i) now reads "if $z = \mu y$ ($y \in A_{s+1} - A_s$) then z is either a follower or a trace". Thus we ask the reader to check for himself that these details go through unchanged save for minor changes like the above. $\square$

We point out that again this argument blends with permitting to construct such an r.e. degree below any given nonzero r.e. degree. We remark that to answer our question "does there exist a tt-topped and tt-bottomed r.e. degree $\neq$ 0?" does not necessitate the construction of a *contiguous degree* with these properties. It is possible for an r.e. degree $a \neq 0$ to be noncontiguous and yet wtt-topped and bottomed. This is witnessed by the next result which also has several nice corollaries.

(4.28) THEOREM. (a) *Let* $C \not\equiv_T \varnothing$ *be r.e. Then there exist r.e. sets A and B with* $A \equiv_T B$ *and* $A \leqslant_{wtt} C$ *such that*
   (i) *for all r.e. sets D if* $D \leqslant_T B$ *then* $D \leqslant_{wtt} A$,
   (ii) *for all sets E if* $B \equiv_T E$ *then* $B \leqslant_{tt} E$, *and*
   (iii) $A \not\equiv_{wtt} B$.
   (b) *Hence* $\deg(A)$ *is wtt-topped and strongly tt-bottomed but not contiguous.*

PROOF. For simplicity, we drop the $A \leqslant_{wtt} C$ requirements which are achieved by an easy permitting argument along the lines considered earlier. Also, since the argument is essentially an amalgam of earlier ones ((4.3) and (4.17)) and a wait-and-see argument, we feel fairly free to merely sketch some details, and also not to discuss the various strategies. We build $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$ to satisfy

$$N : A \equiv_T B.$$
$$R_e : \Phi_e(A) = W_e \text{ implies } W_e \leqslant_{wtt} A.$$
$$N_e : \Gamma_e(\Phi_e(B)) = B \text{ implies } \Phi_e(B) \geqslant_{tt} B.$$
$$P_e : \hat{\Phi}_e(B) \neq A.$$

We ensure $A \equiv_T B$ by traces. Numbers may be targeted for A or B or both. Let

$$l(e, s) = \max\{x : \forall y < x(\Phi_{e,s}(A_s; y) = W_{e,s}(y))\},$$

and let

$$L(e, s) = \max\{x : \forall y < x(\Gamma_{e,s}(\Phi_{e,s}(B_s); y) = B_s(y))\}.$$

We associate R_e with those $\sigma \in 2^{<\omega}$ with $\mathrm{lh}(\sigma) = 2e + 1$. We associate N_e with those $\sigma \in 2^{<\omega}$ with $\mathrm{lh}(\sigma) = 2e + 2$. A stage s is called a σ-stage by induction on $\mathrm{lh}(\sigma)$.
   (i) Every stage s is a $\varnothing$-stage.
   (ii) If s is a τ-stage with $\mathrm{lh}(\tau) = 2e$ for some e, then if

$$l(e, s) > \max\{l(e, t) : t < s \text{ and } t \text{ is a } \tau\text{-stage}\},$$

we say s is a $\tau^\wedge 0$-stage; otherwise s is a $\tau^\wedge 1$-stage.
   (iii) If s is a τ-stage with $\mathrm{lh}(\tau) = 2e + 1$ for some e, then if

$$L(e, s) > \max\{L(e, t) : t \text{ is a } \tau\text{-stage and } t < s\}$$

then s is a $\tau^\wedge 0$-stage; otherwise s is a $\tau^\wedge 1$-stage.

Followers of P_e and their traces are only given guesses σ with $\mathrm{lh}(\sigma)$ even and $\geqslant 2$. We say that P_e requires attention if P_e is not currently (declared) satisfied and one of the following options holds:

(4.29) $q(e, s) > \max\{q(e, t) : t < s\}$

where $q(e, s) = \max\{x : \forall y < x(\hat{\Phi}_{e,s}(B_s; y) = A_s(y))\},$

and either

    (a) P_e is active via x, or

    (b) P_e is inactive but for some follower x, $q(e, s) > x$, or

(4.30) P_e is inactive and has no follower.

CONSTRUCTION, STAGE $s + 1$.

Step 1. Define σ_s as usual. Cancel and inactivate as usual for guesses $\tau \not\leqslant_L \sigma_s$.

Step 2. Find the least follower or trace x (if any) targeted for A for which there exist γ and τ such that

    (i) x has guess γ,

    (ii) $\mathrm{lh}(\tau)$ is even, say, $\mathrm{lh}(\tau) = 2e$,

    (iii) $\tau {}^{\wedge}0 \subset \gamma$ and $\tau {}^{\wedge}0 \subset \sigma_s$,

    (iv) x is not yet $\tau {}^{\wedge}0$-confirmed, and

    (v) $l(e, s) > x$.

Declare x as $\tau {}^{\wedge}0$-confirmed and cancel all traces or followers targeted for A or B greater than x. Declare all P_j affected by this as inactive.

Step 3. Find the least (if any) follower or trace x targeted for B for which there exist γ and τ such that

    (i) x has guess γ,

    (ii) $\mathrm{lh}(\tau)$ is odd, say, $\mathrm{lh}(\tau) = 2e + 1$,

    (iii) $\tau {}^{\wedge}0 \subset \gamma$ and $\tau {}^{\wedge}0 \subset \sigma_s$,

    (iv) x is not yet $\tau {}^{\wedge}0$-confirmed, and

    (v) $L(e, s) > x$.

Cancel and inactivate as in Step 2, above. Declare x as $\tau {}^{\wedge}0$-confirmed.

Step 4. Now find the least e such that P_e requires attention. Cancel all followers and traces with guess τ for $\tau \supset \sigma$ and $\tau \neq \sigma$ where $\sigma \subset \sigma_s$ and $\mathrm{lh}(\sigma) = 2e + 2$. Inactivate appropriate P_j. Adopt the appropriate case below.

Case 1. (4.30) holds. Appoint $y = 2s + 1$ as a follower of P_e with guess σ. P_e remains inactive. Declare x and $x + 1$ as both targeted for A, declare $x + 1$ as a trace, and declare $x + 1$ as targeted for B.

Case 2. (4.29) holds.

Subcase (i). (b) holds. Declare P_e as active. Set $A_{s+1} = A_s \cup \{x + 1\}$. Set $B_{s+1} = B_s \cup \{z : x + 1 \leqslant z \leqslant 2s\}$. Set $y(x) = 2s + 1$ as x's new trace. Give $y(x)$ guess γ where γ is the guess of x. Declare $y(x)$ to be targeted for both A and B. Cancel all followers and traces with guess η for $\gamma \leqslant_L \eta$ and $\gamma \neq \eta$. Inactivate appropriate P_j.

Subcase (ii). (a) holds. Set $A_{s+1} = A_s \cup \{x, y(x)\}$ and $B_{s+1} = B_s \cup \{z : y(x) \leqslant z \leqslant 2s\}$. Declare P_e as satisfied. Cancel, etc., as in subcase (i).

END OF CONSTRUCTION.

VERIFICATION (SKETCH). Due to the evident similarities with (4.3) and (4.17), we sketch some of the details.

Let β denote the leftmost path. We first verify the R_e. Let s_0 be a stage such that all the P_j for $j < e$ cease acting and $\sigma \leqslant \sigma_s$ for all $s > s_0$ where $\sigma \subset \beta$ and $\mathrm{lh}(\sigma) = 2e + 1$.

Let z be given. We wish to compute if $z \in W_e$ or not (from A). Find the least σ-stage $s_1 > s_0$ with $l(e, s_1) > z$. We claim that, as in (4.17), if s_2 is the least σ-stage with $s_2 > s_1$ and $A_{s_2}[s_1] = A[s_1]$, then

(4.31) $z \in W_e$ iff $z \in W_{e,s_2}$.

This follows by virtually the same argument as in (4.17). The crucial observation is that $y(x)$ may only enter A at the same time as x. Briefly, to see (4.31) it suffices to observe that the only numbers p left alive at stage s_2 with

(4.32) $s_1 < p < u$ where $u = u\big(\Phi_{e,s_2}(A_{s_j}; z)\big)$

must be traces $y(x)$ and must be traces of some follower x already present at stage s_1. Thus they can only enter A at the same time as x. The rest of the argument mimics (4.17). Since we have transported the N_e machinery virtually unchanged, the argument given in (4.3) will suffice for the N_e. Again we leave this to the reader.

We thus turn to the verification of the P_e. Let s_0 be an appropriate σ-stage as above (with $\mathrm{lh}(\sigma) = 2e + 2$ and $\sigma \subset \beta$). Now if R_e fails to be met then $q(e, s) \to \infty$. Thus find a σ-stage $s_1 > s_0$ where (4.30) holds. Then P_e is given an uncancellable follower x. Find the least stage $s_2 > s_1$ with $q(e, s_2) > x$. At such a stage we set $A_{s_2+1} = A_{s_2} \cup \{x + 1\}$ and $B_{s_2+1} = B_{s_2} \cup \{x + 1\}$. We also cancel all lower priority followers and traces.

The crucial observation is that the way we appoint followers and choice of s_0 ensures that

$$B_{s_2+1}[\phi_e(x)] = B[\phi_e(x)]$$

since $\phi_e(x) = u(\hat{\Phi}_{e,s_1}(B_{s_1}; x)) = u(\hat{\Phi}_e(B; x))$. (Any number targeted for B appointed after stage s_2 must exceed s_2 and there is no number left alive targeted for B and $< s_2$ (except those that never get into B covered by s_0).)

Thus, at the least stage $s_3 > s_2$, with $q(e, s_3) > x$, (4.29)(a) pertains to x and we create a (permanent) disagreement

$$\hat{\Phi}_e(B; x) = 0 \neq 1 = A_{s_3+1}(x).$$

Finally, we verify that $A \equiv_T B$. To compute if $x \in A$ or not (from B) see if x or $x - 1$ is a follower or trace by stage x. If not and $x \notin A_x$ then $x \notin A$. If x is a trace then $x \in A$ only if $x \in B$. If $x \in B$ find the stage s where $x \in B_s - B_{s-1}$. Then $x \in A$ iff $x \in A_s$. If x is a follower, $x \in A$ only if $x + 1 \in B$ first. If $x + 1 \in B$ find the stage s where $x + 1 \in B_s - B_{s-1}$. If x is still alive at stage s, x is given a trace $y(x)$. To then see if $x \in A$, find the stages (if any) where $y(x) \in B$. If none exists then $x \notin A$. If $y(x) \in B_t - B_{t-1}$ then $x \in A$ if $x \in A_t$. Hence $A \leqslant_T B$. On the other hand $B \leqslant_T A$ by simple permitting. $\square$

We now give one surprising consequence of (4.28).

(4.33) COROLLARY. *There exists an r.e. degree* **a** $\neq$ **0** *such that* **a** *contains infinitely many r.e. wtt-degrees and the structure of the r.e. wtt-degrees of degree* **a** *is a lattice with least and greatest elements.*

PROOF. In [LS], Ladner and Sasso showed that if an r.e. degree is noncontiguous, then it contains infinitely many r.e. wtt-degrees. In [Fi] Paul Fischer constructed an r.e. nonzero wtt-degree f such that the r.e. wtt-degrees below f form a lattice. Now apply (4.28). □

We remark that we do not know if (4.33) may be improved to say that the collection of *all* wtt-degrees of degree **a** form a lattice. (Note: By Exercise 3.5 of Chapter IX of Soare [So4], if a pair of r.e. wtt-degrees have an infimum in the wtt-degrees, the infimum is r.e.) We cannot use the technique above since for all r.e. wtt-degrees $f \neq 0$ the wtt-degrees below f never form a lattice (cf. e.g. [Do3]).

We ask whether or not it is possible for the (r.e.) tt-degrees of an r.e. (wtt-) degree to also form a lattice. We also point out that degrees satisfying the properties of **a** of (4.32) would seem rare. We refer to [DS, Fi and Do3] for results on pairs without infimum in the r.e. wtt-degrees. We do not know what countable lattices can be embedded in this way, although Downey [Do3] has shown that they are not Boolean algebras.

As our final result we use the flexibility of the construction of (4.3) to answer a question of Odifreddi (personal communication) by showing that (strongly) tt-bottomed degrees do not necessarily contain minimal (r.e.) tt-degrees.

(4.34) THEOREM. *There is a nonzero strongly contiguous strongly tt-bottomed degree* **a** *containing no sets of minimal* $(r.e.)$ *tt-degree.*

PROOF (SKETCH). Build $A = \bigcup_s A_s$ and an auxiliary r.e. set $B = \bigcup_s B_s$ to satisfy

$N : B \leqslant_{tt} A.$

$P_e : \overline{B} \neq W_e.$

$N_e : \Gamma_e(\Phi_e(A)) = A$ implies $A \leqslant_{tt} \Phi_e(A)$ and $\Phi_e(A) \leqslant_{wtt} A.$

$R_e : \Phi_e(B) \neq A.$

We meet the N_e exactly as we did in (4.3) and (4.17). We actually ensure that $B \leqslant_m A$ by asking that a follower x targeted for B enters B iff $x \in A$. By meeting the R_e we ensure that $A \nleqslant_T B$. To meet these requirements we use a standard Friedberg-Muchnik procedure. Thus we first pick a follower y targeted for A, cancelling all lower priority followers when we do so. We next wait till $\Phi_{e,s}(B_s; y) = A_s(y)$. At this stage we enumerate y into A and cancel all lower priority followers targeted for B. This clearly combines with our (4.3) strategy and creates a disagreement $\Phi_e(B_0; y) = 0 \neq 1 = A(y)$.

We meet the P_e by appointing followers x and waiting till $x \in W_{e,s}$. Then add x to both A and B. The only interactions we must be careful with are those where if x is a follower of P_e that is later cancelled, we must still add x to B if ever x enters A, because of the tt-condition we need to achieve. Nevertheless, it is really quite easy to blend the above with (4.3) and we leave this to the reader. □

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