Downey, Morad (175-186)

SUPERBRANCHING DEGREES[1]

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§1 *INTRODUCTION*

All sets, degrees, etc will be r.e. unless explicitly stated otherwise. A degree $\mathbf{a} \neq \mathbf{0}'$ is called *branching* if there exist **c**|**b** with $\mathbf{c} \cap \mathbf{b} = \mathbf{a}$. By constructing a minimal pair (i.e. **c**|**d** with $\mathbf{c} \cap \mathbf{d} = \mathbf{0}$), Lachlan [7] and Yates [13] showed that branching degrees exist, thereby giving a negative solution to Shoenfield's conjecture [10]. The technique introduced is now called the minimal pair method and relied on the conscious use of nested strategies. It can be viewed to provide the conceptual framework that lead to the tree method and ultimately to the much of our understanding of **R**, the usl of r.e. degrees.

Various other results concerning branching degrees have been proven. Notably Lachlan [7] showed that not all r.e. degrees are branching and later Fejer showed that the nonbranching degrees are dense in **R**([6]) and above every low r.e degree there exists a branching degree ([5]). This second result was improved by Slaman [11] who used a rather difficult $\mathbf{0}'''$ argument to show that the branching degrees are dense. We should point out that these were the first elementary classes of degrees shown to be dense in **R**. Obvious consequences (via Sacks splitting theorem) are that the branching and nonbranching degrees each generate the r.e. degrees under join. Another consequence of the Fejer density theorem (in [4] see Soare [12, chIX, 4.5]) is that the first Centor-Bendixon derivative of **R** has no isolated points.

Interest has begun to focus on fragments of **R**, and, in particular, upon intervals in **R**. A nice open question here is whether there exist $\mathbf{a} < \mathbf{b}$ with the first order theory of [**a**, **b**]

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decidable. a natural approach to such questions is to try to transport the machinery used to analyse **R** globally into various intervals. As **0** is branching and there are nonbranching degrees already we know that various intervals must be different.

We approached the above with the question of what sorts of constructions can be carried out relative to a given r.e. degree. A good and natural candidate was Lachlan's nonbounding construction [8] where Lachlan showed the existence of an r.e. degree $\mathbf{a} \neq \mathbf{0}$ such that for all **c**, **d** with $\mathbf{0} < \mathbf{c}, \mathbf{d} < \mathbf{a}$, $\mathbf{c} \cap \mathbf{d} \neq \mathbf{0}'$. In effect, permitting and the minimal pair method are incompatible. The more general question was whether if $\mathbf{a} < \mathbf{0}'$ there exists $\mathbf{b} > \mathbf{a}$ such that for all **c**, **d** if $\mathbf{a} < \mathbf{c}, \mathbf{d} < \mathbf{b}$ then $\mathbf{c} \cap \mathbf{d} \neq \mathbf{a}$.

In this paper we shall answer this question negatively by introducing a new class of r.e. degrees which we call the *superbranching degrees*. We define $\mathbf{a} \neq \mathbf{0}'$ to be superbranching if for all $\mathbf{b} > \mathbf{a}$ there exist **c** and **d** with $\mathbf{b} > \mathbf{c}, \mathbf{d} > \mathbf{a}$ and $\mathbf{c} \cap \mathbf{d} = \mathbf{a}$. The construction uses a concept we call *streaming* where nodes on a strategy tree process numbers into a well-behaved stream. We have found this viewpoint rather useful in other constructions (eg [1, 3] and believe it is a good way to view many tree arguments. Our argument blends with permitting. In fact, the first author has shown [2] that the superbranching degrees are dense in **R** and hence generate the r.e. degrees under join.

By Lachlan's results we know **0** is branching but not superbranching. We remark that it is possible (but remarkably tedious) to construct a nonzero branching-but-not-superbranching degree, using a variation on Lachlan's construction. It seems conceivable that the branching but not superbranching degrees are dense in **R**.

Notation is standard and follows Soare [12]. All uses, etc are bounded by s and stage s.

§2. We shall construct a low superbranching degree. In fact we will supress the lowness requirements as the argument will be seen to be easily combined with (eg) permitting. We build $A = \bigcup_s A_s$, $C_e = \bigcup_s C_{e,s}$ and $B_e = \bigcup_s B_{e,s}$ to satisfy the requirements that C_e, $B_e \leq_T A \oplus W_e$ and

$$P_{e,i} : (W_e \leq_T A) \vee (\Phi_i(A) \neq C_e)$$

$$\hat{P}_{e,i} : (W_e \leq_T A) \vee (\Phi_i(A) \neq B_e)$$

$$N_{e,i} : \Phi_i(\hat{C}_e) = \Phi_i(\hat{B}_e) = f \text{ total} \Rightarrow f \leq_T A, \text{ where } \hat{C}_e = A \oplus C_e \text{ and } \hat{B}_e = A \oplus B_e.$$

To meet the $N_{e,i}$ we first attempt a minimal pair type strategy. That is, let

$$l(e,i,s) = \max\left\{x : (\forall y < x)\left[\Phi_{i,s}(\hat{C}_{e,s}; y) = \Phi_{i,s}(\hat{B}_{e,s}; y)\right]\right\}$$

$$ml(e,i,s) = \max\{l(e,i,t) : t < s\}.$$

We say a stage s is (e, i) - *expansionary* if $l(e,i,s) > ml(e,i,s)$. In a minimal pair type strategy we only allow one side of a computation to change between (e,i) - expansionary stages. Hence we know that one side always preserves computations we have seen. (We assume the reader familiar with the minimal pair technique.)

Should the minimal pair strategy fail, we will code the atomic fact that "both sides have changed" by enumeration into A along the lines of Lachlan [9]. As we will see this involves 'processing' the various follows of $P_{f,j}$, $\hat{P}_{f,j}$ by $N_{e,i}$ so that such a coding procedure coheres with the construction.

The basic idea used to meet the $P_{e,i}$ (and dually $\hat{P}_{e,i}$) is to define a *stream* of followers $x(e,i,0,s)$, $x(e,i,l,s)$, ... as follows. Pick $x(e,i,0,s)$. Wait till $L(e,i,s) > x(e,i,0,s)$ where $L(e,i,s) = \max\left\{x : (\forall y < x)\left[\Phi_{i,s}(A_s : y) = C_{e,s}(y)\right]\right\}$. We then appoint a follower $x(e,i,l,s)$ so that $x(e,i,l,s) > u(\Phi_{i,s}(A_s; x(e,i,0,s)))$. Continue this in the obvious way so that (for a single requirement $P_{e,i}$ alone) we have a stream $x(e,i,0,s)$, $x(e,i,l,s)$, ..., $x(e,i,n,s)$ with $x(e,i,j+1,s) > u(\Phi_{i,s}(A_s; x(e,i,k,s)))$ for $k \le j$.

We then wait till we see W_e permit some j and then act by enumerating $x(e,i,j,s)$ into C_e and restrain A on $u(\Phi_{i,s}(A_s; x(e,i,j,s)))$. Note that for a *single* $P_{e,i}$ by the above Friedberg action, W_e must be recursive: To compute $W_e[j]$ simply wait until $x(e,i,j,s)$ is defined.

In the context of more than one $P_{e,i}$ ($\hat{P}_{e,i}$), the fact that we cannot know if (eg) W_e is recursive causes some complications. The most naive approach would be to allow $P_{e,i}$ to initialise all lower priority $P_{f,j}$ whenever it acts. In that way the (e,i) - stream would be a permanent set-up and would be sure of winning $P_{e,i}$. However, if $W_e = \emptyset$ (say) such a procedure would cause us to initialise $P_{f,j}$ cofinally in the construction.

Our solution to this dilemma is to allow $P_{f,j}$ of lower priority to upset some of the $x(e,i,k,s)$ - set ups by enumerating these numbers into A, whilst we await W_e-permissions. We do this by asking that $P_{f,j}$ ($\hat{P}_{f,j}$) of lower priority than $P_{e,i}$ choose as their followers only numbers from $P_{e,i}$'s stream, so that they will only "refine" $P_{e,i}$'s stream. That is, if $P_{f,j}$ is a version guessing that $P_{e,i}$ has eventually infinitely many followers, $P_{f,j}$ will only select followers of $P_{e,i}$ as its followers.

The reader should note that there is no real conflict between $P_{e,i}$ and $P_{f,j}$ here since one targets such x for C_e and the other for C_f. The problems will occur due to the interaction of $P_{e,i}$, $P_{f,j}$ and $N_{f,k}$.

Consider $P_{f,j}$ and $\hat{P}_{f,g}$ cohereing with $N_{f,k}$ which is infinitely active. What $N_{f,k}$ does is to process numbers in the same way as $P_{e,i}$ above. It will be given a stream y_0, y_1 ..., and will process them into $z_{0,s}, z_{1,s}, \ldots$, so that at any stage s,

$$z_{i+1,s} > \max\left\{u(\Phi_{k,s}(\hat{C}_{f,s};\ z_{i,s})),\ u(\Phi_{k,s}(\hat{B}_{f,s};\ z_{i,s}))\right\}.$$

It will then hand these numbers to lower priority nodes to use. In particular let us suppose that $\hat{P}_{f,g}$ has lower priority then $P_{f,j}$ but is guessing $P_{f,j}$ is finitely active. Then this version of $\hat{P}_{f,g}$ will be initialised each time $P_{f,j}$ acts, but at any stage we may have refined the z_i - stream to look like

$$x(f,j,0,s),\ x(f,j,1,s),\ \ldots\ x(f,j,n,s),\ \hat{x}(f,g,0,s),\ \ldots,\ \hat{x}(f,g,m,s),$$

where x ($\hat{x}$) follows $P_{f,j}$ ($\hat{P}_{f,j}$). If W_f permits $i < n$ then $P_{f,j}$ will act and initialise all the $\hat{x}$. However if $\hat{P}_{f,g}$ acts first, we will enumerate some $\hat{x}(f,g,t,s)$ into B_f. Before the next (f,k) - expansionary stage it may be that W_f permits $i < n$ and so $P_{f,j}$ wishes to act to enumerate x(f,j,i,s) into C_f. This act will cause a conflict with $N_{f,k}$ which is resolved by enumerating $\hat{x} = \hat{x}(f,g,t,s)$ into A. The conflict is that both sides have changed at $\hat{x}$. By the order of events, since $N_{f,k}$ has processed the z_i, we know that this makes $N_{f,k}$ happy and also $\hat{x} > u(\Phi_{j,s}(C_{f,s}, x(f,j,i,s)))$ and so $P_{f,j}$ is also happy.

If additionally $N_{f,k}$ was guessing infinitary $P_{e,i}$ activity, we know that the z_i - stream is a refinement of the (e,i) - stream. The process above will injure the (e,i) - stream since for some q, $x(e,i,q,s) = \hat{x}(f,j,t,s)$. Thus we will need to pick a new x(e,i,q,s+1). The whole point is that if $x(e,i,q,s) \neq x(e,i,q,s+1)$, then $x(e,i,q,s) \in A$. Hence A can decide if x(e,i,q,s) is permanent. Care is taken to ensure that $\lim_s x(e,i,q,s) = x(e,i,q)$ for all q (should $P_{e,i}$ be infinitely active) but, once this is done we see that $W_e \leq_T A$.

<u>**Formal Details Now Follow.**</u>

First we will always *dump*. Thus if $x \in A_{s+1} - A_s$ then

$(\forall z)(x \leq z \leq s \rightarrow z \in A_{s+1} - A_s)$.

2.1 The Priority Tree Let T be the tree of all $\sigma \in \{f, 0, 1\}^{<\omega}$ with $f < 0 < 1$ such that if i is even then $\sigma(i) \in \{0, 1\}$ and if i is odd then $\sigma(i) \in \{f, 0, 1\}$. We refer to $\sigma \in T$ as *guesses*, with $lh(\sigma)$ denoting the length of σ. Let $\sigma \leq_L \tau$ denote lexicographical ordering. Assign priorities as follows.

<u>**lh (σ) = 1**</u> Let $e(\sigma) = i(\sigma) = 0$. Let

$$L_1(\sigma\hat{\ }j) = \begin{cases} \omega - \{\langle 0, 0\rangle\} \text{ if } j = 1 \text{ or } j = f, \\ \omega - \{\langle 0, k\rangle : k \in \omega\} \text{ if } j = 0. \end{cases}$$

$$L_3(\sigma\hat{\ }j) = L_2(\sigma\hat{\ }j) = \begin{cases} \omega \text{ if } j = 1 \text{ or } j = f, \\ \omega - \{\langle 0, k\rangle ; k \in \omega\} \text{ if } j = 0. \end{cases}$$

<u>**lh(σ) > 1 and lh(σ) odd**</u>

Case 1 $lh(\sigma) = 4n + 1$ for some n.

Let $\langle e(\sigma), i(\sigma)\rangle = \mu z(z \in L_1(\sigma))$. Define

$$L_1(\sigma\hat{\ }m) = \begin{cases} L_1(\sigma) - \{\langle e(\sigma), i(\sigma)\rangle\} \text{ if } m \neq 0, \\ L_1(\sigma) - \{\langle e(\sigma), k\rangle : k \in \omega\} \text{ if } m = 0. \end{cases}$$

For p = 2,3, define

$$L_p(\sigma\hat{\ }m) = \begin{cases} L_p(\sigma) \text{ if } m \neq 0, \\ L_p(\sigma) - \{\langle e(\sigma), k\rangle : k \in \omega\} \text{ if } m = 0. \end{cases}$$

Case 2 $lh(\sigma) = 4n+3$ for some n.

Let $\langle e(\sigma), i(\sigma)\rangle = \mu z(z \in L_2(\sigma))$. Define

$$L_2(\sigma\hat{\ }m) = \begin{cases} L_2(\sigma) - \{\langle e(\sigma), i(\sigma)\rangle\} \text{ if } m \neq 0 \\ L_2(\sigma) - \{\langle e(\sigma), k\rangle : k \in \omega\} \text{ if } m = 0. \end{cases}$$

For $p = 1, 3$, define

$$L_p(\sigma\hat{\ }m) = \begin{cases} L_p(\sigma) \text{ if } m \neq 0 \\ L_p(\sigma) - \{\langle e(\sigma), k\rangle : k \in \omega\} \text{ if } m = 0. \end{cases}$$

<u>lh(σ) even</u> If $lh(\sigma) = 0$ let $e(\sigma) = i(\sigma) = 0$. Otherwise let $\langle e(\sigma), i(\sigma)\rangle = \mu z\,[z \in L_3(\sigma)]$. Set $L_3(\sigma\hat{\ }j) = L_3(\sigma) - \{\langle e(\sigma), i(\sigma)\rangle\}$ and $L_p(\sigma\hat{\ }j) = L_p(\sigma)$ for $p \in \{1, 2\}$. This concludes the priority assignment.

2.2 Definition Let $\alpha \in T$.

i) We say a stage s is an *α-stage* if $s=0$ or $\alpha=\sigma(t,s)$ at some substage t of stage s. ($\sigma(t,s)$ is defined later)

ii) If $lh(\alpha) = 2e$, we say a stage q is *α-expansionary* if $q=0$ or q is an α-stage and (a) and (b) below hold

a) $l(e(\alpha), i(\alpha), q) > \max\{l(e(\alpha), i(\alpha), \hat{q})\} : \hat{q}$ is an α-stage $< q\}$.

b) If there exist followers with guesses $\rho \supseteq \alpha\hat{\ }0$ not yet α-confirmed and $x(\rho, k, q)$ is the least such, then $l(e(\alpha), i(\alpha), q) > x(\rho, k, q)$.

iii) If $lh(\alpha) \equiv 1 \pmod 4$ and $e = e(\alpha)$, $i = i(\alpha)$, we say that q is α–expansionary if $q=0$ or q is an α-stage where

a) $L(e,i,q) > \max\{L(e,i,t) : t \text{ is an } \alpha\text{-stage} < q\}$

b) $L(e,i,q) > \max\left\{\begin{array}{l} x(\alpha,k,\hat{q}) : x(\alpha,k,\hat{q}) \text{ is a follower of } \alpha \text{ at any} \\ \text{stage } \hat{q} < q \end{array}\right\}$

c) If there exists $\gamma\hat{\ }0 \subseteq \alpha$ with $lh(\gamma)$ odd, let ρ denote the longest such γ. Then $x(\rho,g,q)$ is defined for $g=r(\alpha,q)$ and there exists a follower $x(\rho,k,q)$ with $k > g$ such that.

c i) $x(\rho,k,q)$ has never been assigned to α

cii) $x(\rho,k,q) > u(\Phi_{i,s}(A_s ; \hat{x}))$ where

$$\hat{x} = \max\left\{\begin{array}{l} x(\alpha,j,q),\ r(\alpha,q) : x(\alpha,j,\hat{q}) \text{ is the largest} \\ \text{follower of } \alpha \text{ at any } \hat{q} < q \end{array}\right\}$$

ciii) For all $\gamma\hat{\ }0$ with $lh(\gamma)$ even and $\rho \subseteq \gamma$ if ρ exists, and for all j with $x(\alpha,j,q)$ existing, if $m = m(\gamma, x(\alpha,j,q), q)$ is defined then $x(\rho,k,q) > m$. (r and m are defined in the construction)

Remark

The point of (b) and (c) of (iii) above is to guarantee that at α^0-stages we will get a new follower to add to the $P_{e,i}$-stream at guess α. The point of (ciii) is to ensure that $x(\rho,k,q)$ respects γ^0 in the sense that $x(\rho,k,q)$ can be used to trace $x(\alpha,j,q)$ for the sake of γ and yet not interfere with $x(\alpha,j,q)$'s γ^0-computations. As we will see this is achieved by our confirmation machinery. Note that (ii)(b) ensures that we don't believe a stage is α-expansionary until we can confirm some follower, should there exist some not yet confirmed follower.

iv) If $lh(\alpha) \equiv 3 \pmod 4$ we proceed as in (iii) above but with $\hat{L}$ in place of L and $\hat{P}$ in place of P (where $\hat{L}$ is used for the length of agreement between A and B.)

v) If $lh(\alpha)$ is odd, we say that α *requires attention at stage q via x(α, k+1,q)* if

a) q is an α-stage

b) If $\hat{q}$ is the stage where $x(\alpha, k+1, q) = x(\alpha, k+1, \hat{q})$ was appointed to α then

$$W_{e,q}[x(\alpha,k,q)] \neq W_{e,q}[x(\alpha,k,\hat{q})]$$

(The reader should note the use of k in place of k+1 here)

c) $L(e(\alpha), i(\alpha), q) > \max\{x, L(e(\alpha), i(\alpha), \hat{q}) : \hat{q} \text{ is an } \alpha\text{-stage} < q\}$.

Remark

There is a lot of room for variation here. For instance, we could simply wait for *any* stage where $x(\alpha,k,q)$ was still alive and we see $x(\alpha,k,q)$ permitted by W_e. Our version is more useful in the first author's proof of the density of superbranching degrees. The trade-off is that we need to work a little harder to get $C_e, B_e \leq_T W_e \oplus A$.

In the construction to follow we use the phrase "initialise α". As usual, this means reset $F(\alpha,s)$ (the current state of the α-module) to be 1 for $lh(\alpha)$ odd, cancel all followers of α, set $r(\alpha,s) = lh(\alpha) + s \geq \max\{r(\tau,s) : \tau \leq_L \alpha\}$ and cancel all $m(x,\alpha,s)$.

Construction

Stage 0 Set $r(\alpha,0)=0$ all $\alpha \in T$ and $\sigma_0 = \lambda$. (Here λ denotes the empty string).

Stage s+1 Refer to substage t of stage s+1 as stage (t, s+1). A parameter $Q \neq \sigma$ at stage (t, s+1) is denoted by Q_t.

Stage (0, s+1) Define $\sigma(0, s+1) = \lambda$.

Stage (t+1, s+1) We are given $\sigma(t, s+1)$ and for all α with $lh(\alpha)$ odd we are given $F_t(\alpha) = F_t(\alpha, s+1)$. Let $\alpha = \sigma(t, s+1)$. Adopt the first case below to pertain.

Case 1 $lh(\alpha)$ even.

Subcase 1 s+1 is not α-expansionary.

Action Let $\sigma(t+1, s+1) = \alpha\hat{}1$. If t=s set $\sigma_{s+1} = \alpha\hat{}1$ and initialise all τ with $\tau \nleq_L \sigma_{s+1}$. If $t \neq s$ go to stage (t+1, s+1).

Subcase 2 s+1 is α-expansioanry.

Step 1 If there is a least follower $x = x(\rho,k,s)$ with $\rho \supset \alpha\hat{}0$ not yet $\alpha\hat{}0$-confirmed set $m(\alpha\hat{}0,x,s+1) = s+1$ and cancel all followers $x(\gamma,n,s)$ with $\rho \leq_L \gamma$ and $x(\rho,k,s) < x(\gamma,n,s)$. The reader should carefully note that some of these $x(\gamma,n,s)$ may also be $x(\eta,g,s)$ for some $\eta \leq_L \rho$. **The cancellation procedure will only cancel $x(\gamma,n,s)$ as a follower of γ and NOT of η.** On the other hand once $x = x(\rho,k,s)$ is confirmed by $\alpha\hat{}0$, if x is later assigned to some $\hat{\rho} \supseteq \rho$ we regard it as also confirmed as a $\hat{\rho}$-follower. (The reason being that x has done its duty to $\alpha\hat{}0$.) We will do this automatically.

If no such x exists, let q denote the last $\alpha\hat{}0$-stage < s. If no such q exists or α has been initialised at any stage u with $q \leq u \leq s$ go to step 2. Otherwise for each $x(\alpha,k,q) \in A_q - A_{q-1}$ define $m(\alpha, x(\alpha,k,q), s+1) = s+1$.

Step 2 Set $\sigma(t+1, s+1) = \alpha\hat{}0$ and go to stage (t+2, s+1) unless t=s. If t=s set $\sigma_{s+1} = \alpha\hat{}0$ and initialise all $\tau \nleq_L \sigma_{s+1}$.

Case 2 $lh(\alpha) \equiv 1 \pmod 4$. First reset $r_t(\alpha, s+1) = \max\{r(\alpha,s), lh(\alpha)\}$.

Subcase 1 $F_t(\alpha, s+1) = f$.

Action Define $\sigma(t+1, s+1) = \alpha\hat{}f$ and go to stage (t+2, s+1).

Subcase 2 $F_t(\alpha, s+1) = 1$ and α requires attention via $x = x(\alpha,k,s)$.

Action Define $F_{t+1}(\alpha, s+1) = F(\alpha, s+1) = \alpha\hat{}f = \sigma_{s+1}$, set $A_{s+1} = A_s \cup \{z : x \leq z \leq s\}$ (dumping) set $C_{e(\alpha), s+1} = C_{e(\alpha), s} \cup \{x(\alpha, k-1, s)\}$. Initialise all $\gamma \leq_L \alpha\hat{}f$. Cancel all (other) followers of α. (Here of course we mean cancel them *as followers* of α. Perhaps they follow $\delta \subsetneq \alpha$. We would not cancel them as followers of δ.) Set $r(\alpha, s+1) = r(\alpha, s) + s+1$.

Subcase 3 $F_t(\alpha, s+1) = 1$ and α does not require attention yet $s+1$ is α-expansionary. We know by (2.2)(c) that there is a follower x ready to be assigned to α if there exists $\rho\hat{}0 \subset \alpha$ with $lh(\rho)$ odd. If such a ρ exists and $x(\alpha,0,s)$ is not yet defined set $x(\alpha,0,s) = x$. Otherwise, if ρ exists, let $x(\alpha,j,s)$ denote the largest currently defined member of α's stream. Assign $x(\alpha, j+1, s+1) = x$.

If no such $\rho\hat{}0$ exists assign $x(\alpha, j+1, s+1) = r(\alpha, s) + s+1$ (or $x(\alpha, 0, s+1)$ as the case may be).

Set $\sigma(t+1, s+1) = \alpha\hat{}0$ and go to stage $(t+2, s+1)$.

Subcase 4 Otherwise.

Action Define $\sigma(t+1, s+1) = \alpha\hat{}1$. If t=s set $\sigma_{s+1} = \alpha\hat{}1$ and initialise all $\tau \nleq_L \sigma_{s+1}$. If $t \neq s$ go to stage $(t+2, s+1)$.

Case 3 $lh(\alpha) \equiv 3 \pmod 4$.

Action Proceed as in Case 2 with B_e in place of C_e.

End of construction

Verification

Let β denote the left most path.

Lemma *Let* $\alpha \subseteq \beta$. *Then*

(i) *α receives attention only finitely often,*

(ii) *for all* $\gamma \leq_L \alpha$, $lim_s\, r(\gamma,s) = r(\gamma)$ *exists,*

(iii) $|\{s : \sigma_s \leq_L \alpha\}| < \infty$, *and*

(iv) *If* $lh(\gamma)$ *odd and* $\gamma\hat{}0 \subseteq \alpha$ *then* $lim_s\; x(\gamma,k,s) = x(\gamma,k)$ *exists.*

Proof

Assume (i), (ii) and (iii) for all $\hat{\alpha} \subsetneq \alpha$. Let $\alpha = \alpha^{+\hat{}j}$. Let s_0 be a stage where, for all $s \geq s_0$ and $\gamma \leq_L \alpha^+$

a) $r(\gamma,s) = r(\gamma)$

b) $\hat{\alpha} \leq_L \alpha \rightarrow \hat{\alpha}$ does not receive attention at s, and

c) $\sigma_s \not\leq_L \alpha^+$.

Note that (ii) $\Rightarrow$ (iii) since $\alpha \subset \beta$ and after stage s_0, $\sigma \subseteq \alpha$ only if α receives attention (via some $x=x(\alpha, k+1, s)$). If such a stage occurs we would enumerate $x(\alpha, k-1, s)$ into C_e or B_e (as the case may be) and initialise all $\gamma \not\leq_L \alpha$. Choice of s_0 ensure that all numbers to enter A after s must exceed s and so, in particular, exceed x. A number z can enter A only if some $x(\eta, m, s) \leq z$ simultaneously enters A and η receives attention. Such η can only have $\alpha \leq_L \eta$ after stage s. Thus to see that α never again receives attention it suffices to show that

$$(2.3)\ x(\alpha, k+1, s) > u(\Phi_{i(\alpha),s}(A_s ; x(\alpha, k, s))).$$

If (2.3) holds then $P_{e(\alpha), i(\alpha)}$ (or $\hat{P}_{e(\alpha), i(\alpha)}$ as the case may be) is met since by clause (c) of (2.2) we only attack α at $(e(\alpha), i(\alpha))$ - expansionary stages.

We claim that (2.3) holds as x is still alive at stage s. To see this let s_1 be the stage where $x = x(\alpha, k+1, s) = x(\alpha, k+1, s_1)$ was appointed to α. By initialisation, $s_1 \geq s_0$. We know s_1 is α-expansionary and $L(e(\alpha), i(\alpha), s_1) > x(\alpha, k, s)$. By the way we appoint, it follows that (2.3) holds at $s=s_1$. Because we dump since $x(\alpha, k+1, s) \notin A_s - A_{s1}$ no number $< x$ can have entered $A_s - A_{s1}$. Hence at s the relevant computations in (2.3) are the same since $A_s[x] = A_{s1}[x]$. Thus (2.3) holds. Thus (i), (ii) and (iii) hold for α.

Finally, to see that (iv) holds note that if $\gamma \not\leq_L \beta$ then $r(\gamma,s) \rightarrow \infty$ monotonically. We claim that for all $k \leq r(\alpha)$ and all $\gamma \subset \alpha$ with $\alpha\hat{}0 \subset \beta$ and $lh(\alpha)$ odd $\lim_s x(\gamma,k,s) = x(\gamma,k)$ exists. (This will suffice to give (iv).)

For an induction, additionally assume that this is true at stage s_0. That is $x(\gamma,k,s) = x(\gamma,k,s_0)$ for all $k \leq r(\hat{\gamma})$ and $s \geq s_0$ where $\hat{\gamma}$ is the longest γ with $\gamma\hat{}0 \subseteq \gamma$. Let $s_1 > s_0$ be the stage where α ceases receiving attention. Then $r(\alpha, s_1) = r(\alpha)$. If $\gamma\hat{}0 \subset \beta$ with $\gamma \subset \alpha$ then there exist infinitely many γ-expansionary stages. Choice of s_1 ensures that no follower $x < r(\gamma)$ can be cancelled except by confirmation. As any x is only confirmed finitely often we see that $x(\gamma,k,s)$ will eventually all exist in a confirmed state for all $k \leq r(\gamma)$. These are now never enumerated or cancelled, giving the result.

Lemma 2 $C_e \leq_T W_e \oplus A$, $B_e \leq_T W_e \oplus A$.

Proof

Let x be given. To decide if $x \in C_e$ or not compute a stage s where

$W_{e,s}[x] = W_e[x]$. If x does not yet follow some $P_{e,i}$ then $x \in C_e$ iff $x \in C_{e,s}$. If x follows some $P_{e,i}$ then $x = x(\alpha,k,s)$ for some k. If $x(\alpha,k+1,s)$ is currently undefined or $x(\alpha,k+1,s) \notin A$ then $x \notin C_e$. If $x(\alpha,k+1,s) \in A$ compute u where $x(\alpha,k+1,s) \in A_u$. Then $x \in C_e$ iff $x \in C_{e,u}$. Hence $C_e \leq_T W_e \oplus A$ and $B_e \leq W_e \oplus A$ is the same.

<u>Lemma 3</u> $(\forall e)(P_e)$ *and* $(\forall e)(\hat{P}_e)$.

<u>Proof</u>

(For P_e) Suppose that for some i, $P_{e,i}$ fails to be met by a disagreement. Then $\alpha\hat{}0 \subset \beta$ with $e = e(\alpha)$ and $i = i(\alpha)$ for some α. By lemma 1, there are eventually infinitely many stable followers $x(\alpha,k,s) = x(\alpha,k)$ following α with $x(\alpha,k) \notin A$. Once $x(\alpha,k,s)$ is confirmed, $x(\alpha,k,s+1) \neq x(\alpha,k,s)$ iff $x(\alpha,k,s) \in A$. Thus, as we outlined in the introduction, to compute $W_e[k]$ simply find a stage $s(> s_0$ of lemma 1) where $x = x(\alpha,k+1,s)$ is defined, confirmed and $x \notin A$. Then $W_{e,s}[k] = W_e[k]$ and so $W_e \leq_T A$.

<u>Lemma 4</u> $(\forall e,i)(N_{e,i})$

<u>Proof</u>

Let $\alpha\hat{}0 \subset \beta$ with $lh(\alpha)$ even and $e=e(\alpha)$, $i=i(\alpha)$. Let $\Phi_i(\hat{B}_e) = \Phi_i(\hat{C}_e) = f$. Let z be given. Let s_0 be a stage good for α in the sense of Lemma 1.

To compute f(z) find an $\alpha\hat{}0$-stage $s > s_0$ where $l(e,i,s) > z$. Now compute an a $\alpha\hat{}0$-stage $\hat{s} > s$ with $l(e,i,\hat{s}) > s$ and $A_{\hat{s}}[s] = A[s]$. We claim that $f_{\hat{s}}(z) = f(z)$. Indeed, we claim that for all stages $u > \hat{s}$.

$$(2.4) \quad \begin{cases} \text{one of } \Phi_{i,u}(\hat{B}_{e,u};\ z) = \Phi_{i,\hat{s}}(\hat{B}_{e,\hat{s}};\ z) \text{ or} \\ \Phi_{i,u}(\hat{C}_{e,u};\ z) = \Phi_{i,\hat{s}}\ \hat{C}_{e,\hat{s}};\ z) \text{ holds.} \end{cases}$$

If (2.4) is to fail, there must exist $\alpha\hat{}0$-stages $s_1 > s_2 \geq \hat{s}$ with s_2 the preceding α – expansionary stage before s_1 such that two numbers y_1 and y_2 entered respectively the $\hat{C}_e$ and $\hat{B}_e$ sides below the z-uses at s_2. We argue that this cannot happen.

We know that the construction ensures that we can take y_1 and y_2 as followers. By the way we appoint (and cancel) at $\alpha\hat{}0$-stages y_1 and y_2 must have both been present at stage $\hat{s}$. This means that if y_1 enters first (wlog) then y_2 must have higher priority than y_1 (let it be cancelled). By (c)(iii) of (2.2) we know that y_1 must respect y_2's confirmations. That is, when y_1 was assigned

$$s > y_1 > m(\alpha\char`^0, y_2, s).$$

In particular, $y_1 > u(\Phi_{i,p}(\hat{B}_{e,p}; y_2)), u(\Phi_{i,p}(\hat{B}_{e,p}; y_2))$ where p is the stage that y_1 was assigned to its node. By the dump in the construction it follows that the $(\Phi_{i,p}; y_2)$-computations are unchanged at stage s and indeed at stage s_2. But now we are done. When y_2 enters y_1 is automatically put into A causing $A_{\hat{s}}[s] \neq A[s]$ as $s_2 > s$. ⊗

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