THE COMPLEXITY OF THE SUCCESSIVITY RELATION IN COMPUTABLE LINEAR ORDERINGS REVISITED

RODNEY G. DOWNEY, STEFFEN LEMPP, AND GUOHUA WU

Abstract. We indicate how to fix an error in the proof of the main theorem of our original paper, pointed out to us by Maxim Zubkov. In correcting this error, we are now able to give a uniform proof for coding into the adjacency relation for a computable η-like linear ordering with infinitely many adjacencies. The proof is quite unusual in the construction of the isomorphism on the priority tree and uses some other unusual combinatorics as well.

1. Introduction

The main result of our paper [2] read as follows:

Main Theorem. *Let $\mathcal{A}$ be an infinite computable linear ordering with infinitely many successivities. Suppose that C is any c.e. set with* $\mathrm{Succ}(\mathcal{A}) \leq_T C$. *Then there is a computable linear ordering $\mathcal{B}$ isomorphic to $\mathcal{A}$ whose successivity relation has Turing degree* $\deg_T(C)$.

Zubkov (personal communication) pointed out some errors in our original paper, which we attempted to fix in a corrigendum [3]. Unfortunately, as again pointed out to us by Zubkov, our corrigendum doesn't quite fix these errors. In the current paper, we not only address all the issues raised by Zubkov but also, in fact, develop a new strategy, which allows a uniform proof of our Main Theorem in the case that $\mathcal{A}$ is a computable linear ordering without infinite blocks but with infinitely many adjacencies. Orderings without infinite blocks are called *η-like*, and if there is a uniform bound on the size of blocks, then $\mathcal{A}$ is called *strongly η-like*.

First, recall that Chubb, Frolov and Harizanov [1] had already handled the case of successivities occurring arbitrarily far to the right

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in a linear order $\mathcal{A}$ without right endpoint; and their proof can easily be adapted to the case of any point in $\mathcal{A}$ (including the "virtual limit points" $+\infty$ and $-\infty$) being a limit point of successivities; more precisely, their proof can handle the case when, given a fixed $a \in A \cup \{+\infty\}$, for any $b <_{\mathcal{A}} a$, there are infinitely many successivities in the interval (b, a), and, symmetrically, the case when, given a fixed $a \in A \cup \{-\infty\}$, for any $b >_{\mathcal{A}} a$, there are infinitely many successivities in the interval (a, b).

Moreover, Frolov [4] had already handled the case of non-η-like and of strongly η-like linear orderings, so our paper [2] concentrated on the case of η-like linear orderings without infinite strongly η-like intervals.

However, as Zubkov pointed out to us, we only handled the case of an η-like linear order where there is an infinite interval without infinite dense subinterval; that part of our proof in [2] is correct. The remaining case, where each infinite interval contains an infinite dense subinterval, splits into two subcases: Either

(1) there is an infinite interval containing infinitely many successivities, but for each partition of this interval into two subintervals, one of the subintervals contains only finitely many successivities; or
(2) every infinite interval containing infinitely many successivities can be partitioned into two subintervals such that each contains infinitely many successivities.

It is now not hard to see that in the above two cases, our linear order must contain an interval of one of the following two forms, respectively (if we also exclude the case handled three paragraphs above, namely, that there is a limit point of successivities in $A \cup \{+\infty, -\infty\}$),

$$\sum_{i \in \omega} (\eta + m_i) + \sum_{i \in \omega^*} (n_i + \eta) \tag{1.1}$$

$$\sum_{i \in \omega} (\eta + m_i) + \sum_{j \in \eta} \sum_{i \in \zeta} (q_{j,i} + \eta) + \sum_{i \in \omega^*} (n_i + \eta) \tag{1.2}$$

where each m_i, n_i and $q_{j,i}$ is an integer ≥ 2.

In this paper we prove the result for the cases above. However, the proof does not explicitly use the hypothesis that the adjacencies of $\mathcal{A}$ are of types (1.1) or (1.2) above. In fact, we will be giving a uniform proof for a case properly encompassing our original paper [2], namely, that $\mathcal{A}$ is η-like with infinitely many adjacencies.

We will thus fix an infinite computable linear ordering $\mathcal{A} = (A, <_{\mathcal{A}})$ with no infinite blocks. We need to build a computable linear ordering $\mathcal{B} = (B, <_{\mathcal{B}})$ isomorphic to $\mathcal{A}$ and a non-computable (in fact a

Δ^0_3-)map $\iota : A \to B$, meeting, in increasing order of difficulty, the following

Overall Requirements:

$\mathcal{O}$: ι is order-preserving, i.e., for all $a, a' \in A$, $a <_{\mathcal{A}} a'$ implies $\iota(a) <_{\mathcal{B}} \iota(a')$ (and so in particular ι is injective);
$\mathcal{W}$: ι is well-defined (and in particular total);
$\mathcal{S}$: ι is surjective; and
$\mathcal{R}$: there is a Turing reduction Γ such that $\Gamma^{\mathrm{Succ}(\mathcal{B})} = C$, where $\mathrm{Succ}(\mathcal{B})$ is the successivity relation on $\mathcal{B}$ and C is a 1-complete c.e. set.

Note that as in [2], we may assume here without loss of generality that C is a given 1-complete c.e. set since the modifications for the general case can easily be handled as outlined in the last paragraph of [2].

So assume that we are given an effective enumeration of the linear order $\mathcal{A}$ as $\{\mathcal{A}_s\}_{s\in\omega}$. At each stage s, new elements enter A, and as usual, this will guide our beliefs about the cardinality of intervals in the ordering. Thus, we will believe at stage $s+1$ that an interval $[a, b]$ is infinite if *sufficiently many* (to be defined precisely in section 3.2) elements enter this interval.

The overall requirement $\mathcal{O}$ is clearly easy to satisfy by simply enforcing that $a <_{\mathcal{A}} a'$ implies $\iota(a) <_{\mathcal{B}} \iota(a')$ at any given stage. The remaining overall requirements can be split up into the following

Requirements:

$\mathcal{W}_a$: $\iota(a)$ is (well-)defined, for each $a \in A$;
$\mathcal{S}_b$: $\iota^{-1}(b)$ is defined, for each $b \in B$; and
$\mathcal{R}_m$: $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = C(m)$, for each $m \in \omega$.

(We will actually meet the $\mathcal{R}_m$-requirements only for cofinitely many m, which is good enough for our theorem and makes the description of the tree of strategies easier.)

2. The Intuition for the Construction

2.1. **Intervals: Guessing, Outcomes, and Anchors.** Our construction will take place on a tree T of strategies, where each path through T codes a guess as to which intervals of $\mathcal{A}$ are finite or infinite. Each node ("strategy") σ on T will work with a finite number of elements $a_0 <_{\mathcal{A}} a_1 <_{\mathcal{A}} \cdots <_{\mathcal{A}} a_n$ of $\mathcal{A}$ (where $a_0 = -\infty$ and $a_n = \infty$) handed down from all strategies $\tau \subset \sigma$. (If σ is the root of T, then σ works with the single interval $[-\infty, \infty]$.) The strategy σ will be given a guess as to whether each interval $[a_i, a_{i+1}]$ is infinite or finite; in the latter case, σ will be given a guess about the precise cardinality of the interval (a_i, a_{i+1}). The main task of any strategy σ will be to define the ι-image of the element a possibly newly introduced by σ if σ is a $\mathcal{W}$-strategy (and similarly for an $\mathcal{S}$-strategy), but only under its current outcome o; i.e., σ will define $\iota_{\sigma^\frown\langle o\rangle}(a)$. Thus the map $\iota_{\sigma^\frown\langle o\rangle}$ will consist of all the ι-images already defined at σ (namely, ι_σ) and the ι-image of possibly one new element a introduced by σ as defined under its outcome o. The true isomorphism ι is then defined by the strategies along the true path TP through T as the union of the finite maps ι_σ for $\sigma \subset TP$.

The key idea of the construction is that if σ believes an interval $[a_i, a_{i+1}]$ to be infinite, then it will, at stage s, keep the size of the image interval $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ in $\mathcal{B}$ "much" smaller. This will not cause problems if $[a_i, a_{i+1}]$ is truly infinite since the image interval $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ will grow to be infinite as well. On the other hand, this will be critical in satisfying the requirements since the real tension is between strategies $\sigma' <_L \sigma$ on the tree T where σ' may conceivably want to preserve an adjacency in $\mathcal{B}$, whereas σ may want that adjacency to be destroyed. We have to be concerned about σ' only as long as σ' has not been proved "wrong" yet (where σ' has been proved wrong if σ' guesses an interval of $\mathcal{A}$ to be finite of a size smaller than the current size of that interval).

The key property that we will need to maintain is the following: Each element $a \in A$, introduced by σ, say, is associated with an "anchor" $a^\dagger_{\sigma^\frown\langle o\rangle}$ (with respect to σ and its outcome o). The idea is that $\sigma^\frown\langle o\rangle$ believes that a and $a^\dagger_{\sigma^\frown\langle o\rangle}$ are only finitely far apart, and that $a^\dagger_{\sigma^\frown\langle o\rangle}$ is, among all elements introduced along $\sigma^\frown\langle o\rangle$, the first one introduced along $\sigma^\frown\langle o\rangle$ with this property; so the ι-image of $a^\dagger_{\sigma^\frown\langle o\rangle}$ is "stable" (as explained in more detail below). (The anchors of $-\infty$ and ∞ are $-\infty$ and ∞ themselves, respectively. Note that a, introduced by σ, is its own anchor (with respect to σ and its outcome o) iff $\sigma^\frown\langle o\rangle$ believes that a is infinitely far from any point in $\mathcal{A}$ handed down to σ.) So, for the version of ι defined along the true path, we have that for any anchor a along the true path, $\iota(a)$ is the first $b \in B$ chosen as the ι-image of a, and for any non-anchor a' along the true path, say, with anchor a, we have that $\operatorname{dist}(a, a') = \operatorname{dist}(\iota(a), \iota(a'))$ (where dist is the finite distance between two elements) and, of course, $a <_{\mathcal{A}} a'$ iff $\iota(a) <_{\mathcal{B}} \iota(a')$. So the point about an anchor is that the ι-image of an anchor a' is the same for all $\sigma^\frown\langle o\rangle$ which believe that a' is an anchor, and that this ι-image will never change (as long as $\sigma^\frown\langle o\rangle$ has not been proved wrong). Intuitively, when σ defines $\iota_{\sigma^\frown\langle o\rangle}(a)$ for the element a introduced by σ, there are two possibilities: Either a is an anchor with respect to σ and its outcome o, and then $\iota_{\sigma^\frown\langle o\rangle}(a)$ never changes and agrees with $\iota_{\sigma'^\frown\langle o'\rangle}(a)$ for all $\sigma'^\frown\langle o'\rangle$ with $\sigma'^\frown\langle o'\rangle \leq \sigma^\frown\langle o\rangle$ such that $\iota_{\sigma'^\frown\langle o'\rangle}(a)$ is defined and $\sigma'^\frown\langle o'\rangle$ has not yet been proved wrong (since a is an anchor with respect to σ' and its outcome o' as well, as we will show). Or else there is an anchor a' for a (with respect to σ and its outcome o), and so (eventually) $\iota_{\sigma^\frown\langle o\rangle}(a)$ is defined to be at the same finite distance from $\iota_\sigma(a')$ as a is from a'. Since our linear order $\mathcal{A}$ is η-like, each finite block in $\mathcal{A}$ will contain a unique anchor, which is the first element of that block enumerated into $\mathcal{A}$, and there are only finitely many elements for which that element is an anchor.

One important consequence of this setup, which will be crucial in the verification, is the following feature: Consider two incomparable strategies $\sigma <_L \sigma'$ of the same length such that σ has been eligible to act before σ', and neither has been proved wrong or initialized since. Then we will actually be able to achieve that both use the same partition of $\mathcal{A}$. Here is the sketch of an argument: Let τ be the longest common substring of σ and σ' and proceed by induction on the length of σ and σ'. If σ and σ' are both $\mathcal{W}_a$-strategies, then they clearly both introduce a; if they are both $\mathcal{R}$-strategies, then they don't introduce any new points; so consider the case when both are $\mathcal{S}_b$-strategies, and suppose that b was initially created as the image of an element $a \in A$ by a strategy ρ, which must necessarily be a $\mathcal{W}_{\hat{a}}$-strategy (although $\hat{a}$ may differ from a as we will see) or an $\mathcal{R}$-strategy. Now there are two cases: If both σ and σ' can map a to b, then they will use the same partition, of course. Otherwise, the reason why one or both of σ and σ' cannot map a to b is that they don't currently guess a to be an anchor, and that currently the distance between a and the anchor $a^\dagger$ of a is less than the distance between the ι-image of $a^\dagger$ and b, forcing us to shift the ι-image of a finitely far over from b. However, note that both σ and σ' can still "pretend" to use a as a new partitioning point since it is at most finitely far away from the ι-preimage of b, under the current guess.

Let us now make somewhat more precise how a strategy $\sigma \in T$ deals with intervals and guesses their size: First of all, each σ is handed a sequence $a_0 <_\mathcal{A} a_1 <_\mathcal{A} \cdots <_\mathcal{A} a_n$ of $\mathcal{A}$ (where $a_0 = -\infty$ and $a_n = \infty$) by σ^-, along with a guess for each interval $[a_i, a_{i+1}]$ whether it is infinite or finite (and in the latter case also its size). Now, by our hypothesis on $\mathcal{A}$ and by possibly speeding up the enumeration of $\mathcal{A}$, at least one of the intervals $[a_i, a_{i+1}]$ will be guessed to be infinite.

Now, when satisfying its requirement, σ may introduce a new point a in one of the intervals (a_i, a_{i+1}). The outcome of σ will now encode, for each new interval not yet measured by σ^- (i.e., for the intervals $[a_i, a]$ and $[a, a_{i+1}]$), whether it is infinite or finite (and in the latter case also the size of the open interval). For each such interval $[a', a'']$, say, σ will determine its guess as follows:

First, σ needs to guess whether $[a', a'']$ is infinite. For this, σ will use a computable threshold function $g_{\sigma,s}(a', a'')$ subject to the following rule: The function $g_{\sigma,s}(a', a'')$ is nondecreasing in s and can only increase at a stage when σ guesses $[a', a'']$ to be infinite. (The intuition is that $g_{\sigma,s}(a', a'')$ is set very large each time σ guesses $[a', a'']$ to be infinite. In addition, if $[a', a'']$ is a proper subinterval of $[\hat{a}', \hat{a}'']$ and σ^- has a guess about $[\hat{a}', \hat{a}'']$, then $g_{\sigma^-,s}(\hat{a}', \hat{a}'')$ will be far larger than $g_{\sigma,s}(a', a'')$. Thus, we may assume that whenever σ guesses $[a', a'']$ to be infinite, then σ^- also guesses $[\hat{a}', \hat{a}'']$ to be infinite.)

In the case that σ guesses $[a', a'']$ to be finite at a stage s, say (even if the size of $[a', a'']$ has increased since the last time σ was active but its size has not reached the current threshold $g_{\sigma,s}(a', a'')$), then σ will simply guess the current size of the open interval (a', a'') to be the true size of it; of course, if $[a', a'']$ is truly finite, then σ will cofinitely often guess this size correctly.

There is one fine point worth noting here: When introducing the new element a (in an interval (a_i, a_{i+1}), say), the strategy σ needs to measure the size of both $[a_i, a]$ and $[a, a_{i+1}]$ correctly, and so it needs to do so sequentially: σ will first measure the size of $[a_i, a]$, and only then the size of $[a, a_{i+1}]$. (This is to make sure that σ does not alternately measure one of these to be infinite but never realizes that they are both infinite.) This is easily arranged by first performing the measurements for $[a_i, a]$, and only then for $[a, a_{i+1}]$.

As for the ordering between the outcomes, the strategy σ will either introduce no new element in $\mathcal{A}$ (and then have only a single outcome, copying the guesses of σ^-), or it will introduce one new element a, resulting in two new intervals, one on either side of a, and the outcome of σ will code our new guesses about these two intervals. (The outcomes of the $\mathcal{R}$-strategies are finitary and can thus be collapsed into one outcome, with initialization, so the current discussion only concerns $\mathcal{W}$- and $\mathcal{S}$-strategies.) We will always arrange the outcomes of σ, roughly speaking, so that if $o_1 < o_2$ are two outcomes of σ and outcome o_1 is visited after outcome o_2 is visited, then there must be a new interval $[a', a'']$ introduced by σ such that

(1) outcome o_1 guesses $[a', a'']$ to be infinite, and outcome o_2 guesses it to be finite; or

(2) outcome o_1 guesses (a', a'') to be finite of larger size than outcome o_2.

This will ensure that if we move from outcome o_2 to outcome o_1 (i.e., from one outcome to an outcome to its left), then some adjacency guessed by σ to exist under outcome o_2 will have been destroyed when σ takes outcome o_1. More specifically, if the outcome o_2 guesses $[a', a'']$ to be finite, then outcome o_2 would suggest that every adjacency in $[a', a'']$ is stable, whereas outcome o_1 requires that at least one of these adjacencies is destroyed. — Any ordering of the outcomes respecting these constraints will do; for concreteness, let's say that for outcomes $o_1, o_2 \in (\{\infty\} \cup \omega^*)^2$ (coding the size of (a_i, a) and (a, a_{i+1}), respectively), we use the lex ordering with $\infty < \cdots < 2 < 1 < 0$ in each coordinate, with a default outcome *fin* rightmost.

One more important definition we need to make precise is that of a strategy being proved wrong: We say that σ has been *proved wrong at stage s* if σ guesses an interval (a_i, a_{i+1}) to be finite of a certain size k but by stage s, the interval (a_i, a_{i+1}) in $\mathcal{A}$ has already grown to a size $> k$. (Note that in that case, σ will never again be along the true path.) When this happens, no other strategy, including lower-priority strategies $\sigma' >_L \sigma$, has to respect any restraint that σ may have imposed in the past.

After the bird's eye view of the setup for the construction, let's take a closer look at how we will satisfy the individual requirements. Recall that we have to deal with three types of requirements.

2.2. **$\mathcal{W}_a$-strategy.** A $\mathcal{W}_a$-strategy σ will work to find an ι-image for a. Now it may be that $\iota_\sigma(a)$ is already defined, namely, if a is one of the a_i. Or a may be in an interval (a_i, a_{i+1}) guessed to be finite by σ; then $\iota_\sigma(a)$ is essentially determined already and we can rely on strategies $\sigma' \subset \sigma$ to have defined it. In either case, σ will end the substage immediately and have only the default outcome.

Otherwise, suppose that a is in one of the intervals (a_i, a_{i+1}) handed down to, and guessed to be infinite by, σ. Suppose first that σ guesses both of the intervals $[a_i, a]$ and $[a, a_{i+1}]$ to be infinite (call this outcome o); then we distinguish two cases: If there is a highest-priority τ with $\tau^- \leq \sigma$ such that $\iota_\tau(a)$ was defined before and τ has not been initialized or proved wrong since, then σ will define $\iota_{\sigma^\frown\langle o\rangle}(a) = \iota_\tau(a)$ for this τ. (Note that in this case, a is its own anchor (with respect to σ and its outcome o), and we will prove later that a is also its own anchor with respect to τ^- and its outcome $\tau(|\tau^-|)$. Note that we allow $\tau^- = \sigma$.) On the other hand, if there is no such τ, then let b be the first element that was enumerated into $(\iota_\sigma(a_i), \iota_\sigma(a_{i+1}))$, and define $\iota_{\sigma^\frown\langle o\rangle}(a) = b$. (If there is no such b, then enumerate a new element b into $(\iota_\sigma(a_i), \iota_\sigma(a_{i+1}))$.) We will prove later that if b already exists, then currently $|[a_i, a]| \geq |[\iota_\sigma(a_i), b]|$ and $|[a, a_{i+1}]| \geq |[b, \iota_\sigma(a_{i+1})]|$.

On the other hand, if σ currently guesses one of $[a_i, a]$ or $[a, a_{i+1}]$ to be finite (by symmetry, say, the former), then under its current outcome o, say, σ must define $\iota_{\sigma^\frown\langle o\rangle}(a)$ so as to ensure $|[a_i, a]| = |[\iota_\sigma(a_i), \iota_{\sigma^\frown\langle o\rangle}(a)]|$. Since $[a_i, a_{i+1}]$ has just achieved its threshold but $[a_i, a]$ has not, it must be that $[a, a_{i+1}]$ has also achieved its threshold, and so many new points must have entered $[a, a_{i+1}]$. Thus we can map the interval $[a_i, a]$ onto a proper initial segment of $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ unless there are currently not enough points in $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$. In that latter case, σ must determine where in the interval $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ to insert additional points. This step is rather involved and will be described precisely in the full construction, and in the verification in Lemma 4.5; the gist of it is that we try not to insert any new points into subintervals for which some τ with $\tau < \sigma^\frown\langle o\rangle$ or $\tau \supseteq \sigma^\frown\langle o\rangle$ guesses that it is the ι-image of a finite interval.

2.3. **$\mathcal{S}_b$-strategy.** An $\mathcal{S}_b$-strategy σ will work to find an ι-preimage for b. Now it may be that $\iota_\sigma^{-1}(b)$ is already defined, namely, if b is of the form $\iota_\sigma(a_i)$. Or b lies between $\iota_\sigma(a_i)$ and $\iota_\sigma(a_{i+1})$ for an interval $[a_i, a_{i+1}]$ guessed to be finite by σ, and we can rely on strategies $\sigma' \subset \sigma$ to have defined $\iota_\sigma^{-1}(b)$ already. In either case, σ will end the substage immediately and have only the default outcome. (We note that by the way we order the requirements, b will always have entered B by the time this strategy acts.)

Otherwise, fix the strategy τ (necessarily a $\mathcal{W}_a$-strategy for some $a \in A$ or an $\mathcal{R}$-strategy) which created the element b in $\mathcal{B}$. If τ is $\mathcal{W}_a$-strategy, it will have defined $\iota_{\tau^\frown\langle o\rangle}(a') = b$ for some $a' \in A$ (and note that in that case, possibly $a' \neq a$). Note first that we may assume that $|\tau| < |\sigma|$ since $|\tau| = |\sigma|$ is impossible by the way we assign requirements to strategies, and if $|\tau| > |\sigma|$ then $\tau \upharpoonright |\sigma|$ is an $\mathcal{S}_b$-strategy and so $\iota_\tau^{-1}(b)$ will already be defined. On the other hand, i.e., if τ is a $\mathcal{W}_a$-strategy, then τ already defines $\iota_\tau^{-1}(b)$ by the time we reach σ; and if $|\tau| < |\sigma|$ and τ is an $\mathcal{R}$-strategy, then τ and thus also $\sigma \upharpoonright |\sigma|$ guesses b to be contained in the ι-preimage of a finite interval as we will show in the verification, in Lemmas 4.7 and 4.3. So in either case, we are fine.

2.4. **$\mathcal{R}_m$-strategy.** An $\mathcal{R}_m$-strategy σ will, jointly with all other $\mathcal{R}_m$-strategies on the tree T, try to define $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ and keep it correct. So σ has to accomplish two tasks: It must ensure that, if it is along the true path, it defines $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ with a correct use (unless some other

$\mathcal{R}_m$-strategy already does so). And, if m enters C while $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$, it must ensure that this computation is destroyed so that it can be corrected.

So, if $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is currently undefined and $m \in C$, then the $\mathcal{R}_m$-strategy σ defines $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 1$ with the empty use. If $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is currently undefined and $m \notin C$, then the $\mathcal{R}_m$-strategy σ defines $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$ based on all the adjacencies in all the intervals $[a_i, a_{i+1}]$ handed to, and guessed to be finite by, σ. (If there is no interval $[a_i, a_{i+1}]$ guessed to be finite by σ, then σ will not define $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$; this can happen for at most finitely many m along the true path. Note that if an $\mathcal{R}_m$-strategy is currently along the true path, then any definition of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ made earlier by an $\mathcal{R}_m$-strategy $\tau >_L \sigma$ will no longer apply.)

Now assume that an $\mathcal{R}_m$-strategy σ sees $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$ but m has entered C since the last time that $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ was defined by an $\mathcal{R}_m$-strategy $\tau \le \sigma$. Then τ needs to destroy an adjacency in $\mathcal{B}$ which is part of the use of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$. So τ selects an adjacency in the use of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ in the interval $[\iota_\tau(a), \iota_\tau(a')]$ handed to the longest $\tau' \subset \tau$ guessed infinite by τ. (Note that this may cause extra finite injury along the true path, but it will settle down once the next true adjacency is seen by all $\rho \supset \tau$.) Since the outcome of an $\mathcal{R}_m$-strategy is finitary, we will simply collapse the outcome on the tree into a single one.

3. The Full Construction

We now flesh out the remaining details from the above intuition into a full-blown construction. Whenever a point b is added to $\mathcal{B}$, we assume that the least number in ω not used before is chosen.

3.1. **Tree of Strategies.** The outcomes of the $\mathcal{W}$- and $\mathcal{S}$-strategies are somewhat involved: Besides the default outcome *fin* (denoting that the strategy does not add a new endpoint of an interval), each outcome codes a finite sequence of guesses about the cardinality of a finite sequence of intervals (a_i, a_{i+1}) (for some $n > 0$, where $a_0 = -\infty$ and $a_n = \infty$). We use a sequence of the form $\langle a_0, o_0, a_1, o_1, \dots, o_{n-1}, a_n \rangle$, where o_i denotes a *guess* about the cardinality of the interval (a_i, a_{i+1}), denoted $o_i \in \{\infty < \dots < 3 < 2 < 1 < 0\}$ (where ∞ denotes "infinite", and $n \in \omega$ denotes finite cardinality n of the open interval (a_i, a_{i+1})); the ordering of the outcomes for each new subinterval is arranged so that the leftmost outcome visited infinitely often is the true outcome for that subinterval, as explained before. If σ introduces no new elements and thus no new intervals, then σ will have only the default outcome *fin*, copying the previous guesses about the intervals; otherwise, σ introduces one new element and thus two new subintervals of an old interval, so σ's outcome will measure the size for each interval one after the other, in the end giving guesses about the two new intervals handed down to an immediate successor of σ, of the form $(o, o') \in (\{\infty < \dots < 3 < 2 < 1 < 0\})^2$. (Even though σ introduces guesses for two new intervals, we count the outcome of σ as a single outcome, adding only 1 to the length of its immediate successor nodes.)

Under this naming, we will ensure that the ordering of outcomes of σ observes the following general rule, which can be realized by an effective total ordering of all outcomes: For any outcomes (o_1, o_2) and (o'_1, o'_2) for the two new subintervals, we have that $(o_1, o_2) < (o'_1, o'_2)$ implies that some newly introduced interval $[a, a']$ (i.e., one not measured by σ^-) is guessed to be infinite by (o_1, o_2) and finite by (o'_1, o'_2); or guessed to be finite by both (o_1, o_2) and (o'_1, o'_2) but of a larger size by (o_1, o_2) than by (o'_1, o'_2).

More specifically, given that the outcomes of σ only differ on two subintervals of an interval (a_i, a_{i+1}), say (a_i, a) and (a, a_{i+1}), the outcomes of σ are fully determined by the status o and o' of the subintervals (a_i, a) and (a, a_{i+1}), respectively, and so we only have to fully specify the order type among such (o, o'); we stipulate that

$$(\infty, \infty) < \dots < (\infty, 2) < (\infty, 1) < (\infty, 0) < \\ \dots < (2, \infty) < (1, \infty) < (0, \infty) < \mathit{fin}.$$

List all $\mathcal{W}$-, $\mathcal{S}$- and $\mathcal{R}$-requirements in the order

$$\mathcal{W}_0 < \mathcal{S}_0 < \mathcal{R}_0 < \mathcal{W}_1 < \mathcal{S}_1 < \mathcal{R}_1 < \dots \tag{3.1}$$

Then assign the kth requirement in this list to all strategies $\sigma \in T$ of length k. Each $\sigma \in T$ is handed a guess about intervals from above. (The root node is handed just the guess $\langle \infty \rangle$, reflecting the fact that $\mathcal{A}$ (i.e., the interval $(-\infty, \infty)$, where we assume $-\infty$ and ∞ to be virtual endpoints of $\mathcal{A}$) is assumed to be infinite; we will also speed up the enumeration of $\mathcal{A}$ so that at each stage, the root node sees $\mathcal{A}$ to be of size exceeding its next threshold.) During its action, a strategy σ may refine the intervals handed to it by inserting a point in any of the intervals guessed to be infinite, as specified in detail for each kind of strategy below, and its outcome will then include a guess about the new intervals created. (Of course, for any interval $[a, a']$ into which no new point is inserted, σ will have the identical guess about cardinality and potential density as handed from above. For an interval $[a, a']$ guessed to be finite, this is clearly possible. If the interval $[a, a']$ is guessed to be infinite, then σ's threshold function for it will be smaller than that of σ^-, so σ and σ^- will agree on guessing it to be infinite. Thus the outcome of σ will automatically reflect the correct guesses about the other intervals and only needs to note the guesses about the two new intervals if there are such.)

3.2. **Construction.** Our construction proceeds in stages $s \in \omega$ such that at substage $t \leq s$ of stage s, some strategy of length t is eligible to act and decides its current outcome and thus also which strategy should be eligible to act at substage $t+1$, or whether to end stage s. The description of the action of each strategy σ is now determined by the requirement assigned to σ; at the same time, σ will also define the ι-image of the element in $\mathcal{A}$ which σ may introduce as a new endpoint of intervals being considered (this ι-image will depend on the current outcome of σ), and σ will define the threshold function $g_{\sigma,s}(a, a')$ for each interval $[a, a']$ newly measured by σ. Finally, at the end of stage s:

- We redefine, by *reverse* recursion on $|\sigma|$ (i.e., in *decreasing* order of length), the value of $g_{\sigma,s+1}(a, a')$ for all σ that were eligible to act at stage s and all intervals $[a, a']$ that σ guesses to be infinite and for which σ^- has no guess; in particular, $g_{\sigma,s+1}(a, a')$ exceeds 2^c where c exceeds the sum of all g-values used by any $\tau <_L \sigma$ and any $\tau \supset \sigma$; and
- we initialize all strategies to the right of the last strategy that was eligible to act at stage s.

Note that this definition allows us to assume that if a strategy is handed an interval $[a_i, a_{i+1}]$ that it guesses to be infinite and its action is to split that interval into proper subintervals, then σ will necessarily measure at least one of these subintervals to be infinite. In particular, note that for any interval $[\iota_\sigma(a), \iota_\sigma(a')]$ for which σ^- introduces at least one endpoint and which σ guesses to be infinite, we will have at any stage s that

$$|[\iota_\sigma(a), \iota_\sigma(a')]_s| < g_{\sigma,s}(a, a'). \tag{3.2}$$

We now describe the action of each type of strategy at stage s in full detail. Recall that each strategy σ will be handed down from σ^- a finite sequence $a_0 <_{\mathcal{A}} a_1 <_{\mathcal{A}} \cdots <_{\mathcal{A}} a_n$ of $\mathcal{A}$ (where $a_0 = -\infty$ and $a_n = \infty$).

3.3. $\mathcal{W}_a$**-strategy.** A $\mathcal{W}_a$-strategy σ proceeds as follows: If $\iota_\sigma(a)$ is already defined, then σ will end the substage immediately and simply hand down the outcome of σ^- (augmented by the exact finite guesses for $|(a_i, a)|$ and $|(a, a_{i+1})|$ if a is not one of the a_i but in an interval (a_i, a_{i+1})).

Otherwise, let a be in one of the intervals (a_i, a_{i+1}) handed down to, and guessed to be infinite by, σ. If σ currently guesses one of $[a_i, a]$ or $[a, a_{i+1}]$ to be finite (by symmetry, say, the former), then under its current outcome o, say, σ will define $\iota_{\sigma\frown\langle o\rangle}(a)$ so as to ensure $|[a_i, a]| = |[\iota_\sigma(a_i), \iota_{\sigma\frown\langle o\rangle}(a)]|$. (Since $[a_i, a_{i+1}]$ has just achieved its threshold but $[a_i, a]$ has not, it must be that $[a, a_{i+1}]$ has also achieved its threshold, and so many new points must have entered $[a, a_{i+1}]$.)

In this case, σ may need to insert additional points into the interval $(\iota_\sigma(a_i), \iota_\sigma(a_{i+1}))$ so that $\iota_{\sigma\frown\langle o\rangle}(a)$ can be defined while ensuring $|[a_i, a]| = |[\iota_\sigma(a_i), \iota_{\sigma\frown\langle o\rangle}(a)]|$ without potentially injuring a higher-priority τ trying to keep a subinterval of $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ finite. For this, we determine if there are

(1) a strategy $\tau' < \sigma\frown\langle o\rangle$ such that
  - $\iota_{\tau'}(a')$ was defined for some $a' \in [a_i, a_{i+1}]$, and
  - τ' guessed $[a_i, a']$ to be finite at that stage and has not been initialized or proved wrong since then;

and

(2) a strategy $\tau'' < \sigma\frown\langle o\rangle$ such that
  - $\iota_{\tau''}(a'')$ was defined for some $a'' \in [a_i, a_{i+1}]$, and
  - τ'' guessed $[a'', a_{i+1}]$ to be finite at that stage and has not been initialized or proved wrong since then.

(The intervals $[a_i, a']$ and $[a'', a_{i+1}]$ above may be unions of intervals in the partitioning handed to τ' and τ''. Since neither τ' nor τ'' was proved wrong, we must have $a' <_{\mathcal{B}} a''$.)

We now distinguish five cases:

Case 1: There are no such τ' and τ'': Then insert sufficiently many points (if any) just left of $\iota_\sigma(a_{i+1}))$ to allow the above definition of $\iota_{\sigma\frown\langle o\rangle}(a)$.

Case 2: There is no such τ' but there is such τ'': Then choose τ'' with $\iota_{\tau''}(a'')$ leftmost in $\mathcal{B}$, and insert sufficiently many points (if any) just left of $\iota_{\tau''}(a'')$ to allow the above definition of $\iota_{\sigma\frown\langle o\rangle}(a)$ while ensuring that $\iota_{\sigma\frown\langle o\rangle}(a) <_{\mathcal{B}} \iota_{\tau''}(a'')$.

Case 3: There is such τ' but no such τ'': Then choose τ' with $\iota_{\tau'}(a')$ rightmost in $\mathcal{B}$, and insert sufficiently many points (if any) just right of $\iota_{\tau'}(a')$ to allow the above definition of $\iota_{\sigma\frown\langle o\rangle}(a)$.

Case 4: There are such τ' and τ'', and $\iota_{\tau'}(a') <_{\mathcal{B}} \iota_{\tau''}(a'')$ for all corresponding a' and a'': Then choose τ' with $\iota_{\tau'}(a')$ rightmost in $\mathcal{B}$ and insert sufficiently many points (if any) just right of $\iota_{\tau'}(a')$ to allow the above definition of $\iota_{\sigma\frown\langle o\rangle}(a)$ while ensuring that $\iota_{\sigma\frown\langle o\rangle}(a) <_{\mathcal{B}} \iota_{\tau''}(a'')$.

Case 5: In Lemma 4.5, we will show that Cases 1 to 4 exhaust all possibilities.

This allows us to define $\iota_{\sigma^\frown\langle o\rangle}(a)$, and indeed the $\iota_{\sigma^\frown\langle o\rangle}$-images for all the points in $[a_i, a]$, at this stage while ensuring that the interval $(\iota_\sigma(a_i), \iota_\sigma(a_{i+1}))$ is not covered by ι_τ-images of intervals guessed to be finite by strategies τ which have defined ι and not been initialized nor proved wrong since then. (Note that if the cardinality of $[a_i, a]$ has stopped changing, this map would work along the true path. If later on, this cardinality is proved wrong, then the map is discarded forever. As we will see below, if there is some other guess for both $[a_i, a]$ and $[a, a_{i+1}]$ being infinite, then this strategy allows us to return to the image of a under that other guess, assuming that the predecessor guess for the size of $[a_i, a_{i+1}]$ is on the true path.)

On the other hand, if σ guesses both of the intervals $[a_i, a]$ and $[a, a_{i+1}]$ to be infinite (under its current outcome o, thus assuming that a is an anchor), then σ can define $\iota_{\sigma^\frown\langle o\rangle}(a) = \iota_\tau(a)$ for the highest-priority τ with $\tau^- \leq \sigma$ such that $\iota_\tau(a)$ was defined before, τ guesses both $[a_i, a]$ and $[a, a_{i+1}]$ to be infinite, and τ has neither been initialized nor proved wrong since then (if such τ exists). If there is no such τ, then σ simply defines $\iota_{\sigma^\frown\langle o\rangle}(a)$ so as to ensure that

(i) there is no τ last guessing $[a_i, a]$ or $[a, a_{i+1}]$ to be finite unless τ has been initialized or proved wrong since then,
(ii) $|[\iota_\sigma(a_i), \iota_{\sigma^\frown\langle o\rangle}(a)]| \leq |[a_i, a]|$, and
(iii) $|[\iota_{\sigma^\frown\langle o\rangle}(a), \iota_\sigma(a_{i+1})]| \leq |[a, a_{i+1}]|$.

In addition, σ inserts a new element b into $(\iota_\sigma(a_i), \iota_\sigma(a_{i+1}))$ to define $\iota_{\sigma^\frown\langle o\rangle}(a) = b$ if for all current $b' \in (\iota_\sigma(a_i), \iota_\sigma(a_{i+1}))$, σ would not be able to ensure that both $[a_i, \iota^{-1}_{\sigma^\frown\langle o\rangle}(b')]$ and $[\iota^{-1}_{\sigma^\frown\langle o\rangle}(b'), a_{i+1}]$ are guessed to be infinite.

In either case, at the end of the substage, σ will have the outcome measuring the size of the intervals (a_i, a) and (a, a_{i+1}).

3.4. **$\mathcal{S}_b$-strategy.** An $\mathcal{S}_b$-strategy σ will work to find an ι-preimage for b as follows. (Note that by (3.1), b will already be in $\mathcal{B}$ since at least $b + 1$ many points will have been introduced by higher-priority $\mathcal{W}$-strategies.) If $\iota^{-1}_\sigma(b)$ is already defined, say, $\iota^{-1}_\sigma(b) = a$, then σ will end the substage immediately and simply hand down the outcome of σ^- (augmented by the exact finite guesses for $|(a_i, a)|$ and $|(a, a_{i+1})|$ if a is not one of the a_i but in an interval (a_i, a_{i+1})).

Otherwise, fix the interval $[a_i, a_{i+1}]$ handed down to σ such that b lies in the interval $(\iota_\sigma(a_i), \iota_\sigma(a_{i+1}))$, and so σ guesses the interval $[a_i, a_{i+1}]$ to be infinite. (As we will show in Lemma 4.7, if there is some τ with $\tau^- \leq \sigma$ such that $\iota_\tau(a) = b$ and τ has neither been initialized nor proved wrong since $\iota_\tau(a)$ was last defined by τ^-, then $a \in (a_i, a_{i+1})$ and so σ can define $\iota_{\sigma\hat{}\langle o\rangle}(a) = b$.)

Otherwise, σ defines $\iota_{\sigma\hat{}\langle o\rangle}(a)$ by choosing the element $a \in (a_i, a_{i+1})$ to be the ι-preimage of b satisfying

(i) $|[\iota_\sigma(a_i), b]| = |[a_i, a]|$ if σ guesses $[a_i, a]$ to be finite, and
(ii) $|[b, \iota_\sigma(a_{i+1})]| = |[a, a_{i+1}]|$ otherwise.

(Since $[a_i, a_{i+1}]$ is guessed to be infinite, it will certainly be non-empty, and one of $[a_i, a]$ or $[a, a_{i+1}]$ must be guessed infinite by σ.)

In any case, σ will measure the size of the intervals (a_i, a) and (a, a_{i+1}) to determine its current outcome o. If σ guesses under this outcome that $[a_i, a]$ or $[a, a_{i+1}]$ is finite, then it will also define $\iota_{\sigma\hat{}\langle o\rangle}(a')$ for all a' in (a_i, a) or (a, a_{i+1}), respectively.

3.5. **$\mathcal{R}_m$-strategy.** An $\mathcal{R}_m$-strategy σ will, jointly with all the other $\mathcal{R}_m$-strategies on the tree T, try to define the single computation $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ and keep it correct. (Since σ does not introduce a new element of $\mathcal{A}$ and has only a single finitary outcome *fin*, we have $\iota_{\sigma\hat{}\langle fin\rangle} = \iota_\sigma$.)

If $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = C(m)$ then σ ends the substage. If $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is currently undefined and $m \in C$, then σ defines $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 1$ with use 0.

Otherwise, suppose first that $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is currently undefined and $m \notin C$. Then σ will define $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$ using all the adjacencies currently in $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ for any interval $[a_i, a_{i+1}]$ guessed to be finite by σ. (If σ guesses no interval $[a_i, a_{i+1}]$ to be finite, then σ does not define $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$.)

Finally, suppose that $m \in C$ and $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is currently defined to be 0. By induction, as discussed before and proved in Lemma 4.11 later, we may assume that the current definition was made either by σ itself or by an $\mathcal{R}_m$-strategy $\tau <_L \sigma$. So let $\tau \leq \sigma$ be the $\mathcal{R}_m$-strategy that defined the current computation $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$. We now let τ destroy this computation by inserting an element into an adjacency in the use of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ as follows:

Consider the interval $[a_i, a_{i+1}]$ newly guessed along τ to be finite by the longest $\tau' \subseteq \tau$, say. In that interval, let $[a, a']$ be the adjacency with the largest Gödel number. Then τ inserts a new element into $[\iota_\sigma(a), \iota_\sigma(a')]$, thus making $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ undefined. Then τ defines $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 1$ with use 0, initializes all $\rho \geq \tau'$ and ends the stage. (Note that this feature causes extra injury along the true path.)

In any case in which σ does not explicitly end the stage, it simply ends the substage and takes outcome *fin*.

4. The Verification

In this section, we make formal the intuitive verification provided in the earlier sections. The first lemma is essentially forced by the construction.

Lemma 4.1. *Let $\tau <_L \sigma$ be two strategies of the same length on T such that at a stage s, neither has been initialized nor proved wrong between the last stage $< s$ at which it acted and stage s. Then the endpoints of the intervals of $\mathcal{A}$ handed to τ and σ coincide.*

Proof. By induction on the length of τ and σ: If both are a $\mathcal{W}_a$-strategy, then they introduce the same new endpoint a (if any); if both $\mathcal{R}$-strategies, then neither introduces a new endpoint; and if both are $\mathcal{S}_b$-strategies, then σ introduces the same endpoint as τ (since σ must act after τ, and neither has been initialized nor proved wrong). □

The next lemma is almost routine for priority arguments, with a small wrinkle in that there is additional finite injury along the true path (namely, when an $\mathcal{R}$-strategy corrects Γ, it may injure higher-priority strategies).

Lemma 4.2. *There is an infinite* true path $TP \in [T]$ *such that*

- *each $\sigma \subset TP$ is eligible to act infinitely often;*
- *for each $\sigma \subset TP$, there is a stage s_σ after which no $\tau <_L \sigma$ is along the current true path; and*
- *each $\sigma \subset TP$ is initialized only finitely often.*

Proof. We argue by induction on the length of $\sigma = TP \restriction k$. The lemma clearly holds for $\sigma = \langle\rangle$, i.e., for $k = 0$. (The third clause holds since $\langle\rangle$ guesses no interval to be finite and thus cannot be initialized by the Γ-correction of an $\mathcal{R}$-strategy.)

Suppose it holds for $\sigma = TP \restriction k$. We will first show that σ does not end the stage at infinitely many stages at which it is eligible to act. For this, note that σ will end the stage after stage k early only if σ is an $\mathcal{R}_m$-strategy and either σ itself, or some $\mathcal{R}_m$-strategy $\tau < \sigma$, destroys a computation $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$; this can happen at most once.

Now observe that σ will measure the size of at most two intervals not already guessed by σ^-, and by our definition of outcomes, there will be a leftmost outcome taken by σ infinitely often.

This establishes that σ has a leftmost successor $\sigma^\frown\langle o\rangle = \sigma^+$, say, which is along the current true path infinitely often. By induction, after stage s_σ, no $\tau <_L \sigma$ can be along the current true path. So there can only be finitely many stages after stage s_σ at which an $\mathcal{R}$-strategy $\tau <_L \sigma^+$ can act to correct Γ since eventually, after a stage

$s_{\sigma^+} \geq s_\sigma$, say, no such τ can be along the current true path, and after that stage s_{σ^+}, only $\mathcal{R}_m$-strategies correcting $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$ can cause initialization, which can happen at most finitely often since any such strategy must have defined $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ by stage s_{σ^+}. Finally, it is possible that σ^+ itself is initialized due to the Γ-correction by some $\mathcal{R}$-strategy $\tau \leq \sigma^+$. However, since $\mathcal{A}$ contains infinitely many adjacencies and thus not all have been found by strategy σ^+, there will eventually be a stable adjacency not found by σ^+ but seen by all $\tau > \sigma^+$, and so Γ-correction will eventually not use the ι-image of an adjacency found by σ^+, so σ^+ will eventually also not be initialized.

This establishes the lemma. □

Lemma 4.3. *If a strategy σ guesses an interval $[a_i, a_{i+1}]$ to be infinite at a stage s, then no τ can have guessed it to be finite before unless τ was initialized or proved wrong since then.*

Proof. Immediate by the fact that, by initialization, we have $\tau <_L \sigma$; now σ guesses $[a_i, a_{i+1}]$ to be infinite since its size has increased since it was last initialized, which happened after τ last guessed, so τ would have been proved wrong if it had guessed $[a_i, a_{i+1}]$ to be finite then. □

Lemma 4.4. *Suppose a strategy σ is handed an interval $[a_i, a_{i+1}]$ and guesses it to be infinite at a stage s. Then for any τ, τ' and τ'' which satisfy $\tau <_L \sigma$ or $\tau \supset \sigma$, and similarly for τ' and τ'':*

(1) Let a be rightmost in the interval $[a_i, a_{i+1}]$ such that some such τ guessed $[a_i, a]$ to be finite when it last acted (up to the end of stage s) and has not been initialized or proved wrong since then until the end of stage s. (Assume here that if stage s ends before substage s, then the guessing of interval sizes continues to a strategy of length s for accounting purposes. Also, here and below, we allow $[a_i, a]$ to be the union of several intervals measured by τ but all to be guessed finite.) Then any such τ'' cannot guess any interval to be finite which contains a unless a is a right endpoint of that interval.

(2) Symmetrically, let a' be leftmost in the interval $[a_i, a_{i+1}]$ such that some such τ' guessed $[a', a_{i+1}]$ to be finite when it last acted (up to the end of stage s) and has not been initialized or proved wrong since then until the end of stage s. Then any such τ'' cannot guess any interval to be finite which contains a' unless a' is a left endpoint of that interval.

(3) For τ, τ', a and a' as above, we have $a <_{\mathcal{A}} a'$.

Proof. (1) Clearly, we may assume that $a_i <_{\mathcal{A}} a$. So the only way that (1) can fail would mean the existence of some τ'' with $\tau'' <_L \sigma$ or

$\tau'' \supset \sigma$ which last guessed $[\hat{a}, a'']$ to be finite for $a_i <_{\mathcal{A}} \hat{a} \leq_{\mathcal{A}} a <_{\mathcal{A}} a''$. By Lemma 4.1, we know that $|\tau''| > |\sigma|$, and so by Lemma 4.3, τ'' must also guess whether $[a_i, \hat{a}]$ is infinite; in fact, by the maximality of a, τ'' must guess it to be infinite. Similarly, by Lemma 4.1, we know that $|\tau'| > |\sigma|$, and so again by Lemma 4.3, τ'' must also guess whether $[a, a'']$ is infinite; in fact, again by the maximality of a, τ'' must guess it to be infinite. But then at least of τ or τ'' must have been proved wrong by stage s, namely, whichever last acted before the other, a contradiction.

(2) The proof is symmetric to the previous case.

(3) This follows immediately from (1) and (2). □

The following lemma is critical:

Lemma 4.5. *Let $[a_i, a_{i+1}]$ be an interval guessed to be infinite by a strategy σ at a stage s. Then $[\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ is not contained in the union of intervals of the form $[\iota_\tau(a), \iota_\tau(a')]$ for various τ with $\tau < \sigma$ or $\tau \supset \sigma$ such that τ last guessed $[a, a']$ to be finite and has not been initialized or proved wrong since then.*

Proof. Suppose the lemma first fails at a stage s; so at that stage, some σ determines an interval $[\iota_\sigma(a_i), \iota_{\sigma\hat{}\langle o\rangle}(a)]$ to be the image of a finite interval $[a_i, a]$ (in the notation of intervals for σ; and assuming without loss of generality that σ determines the left half of the interval $[a_i, a_{i+1}]$ to be finite).

Then σ is either a $\mathcal{W}_a$- or an $\mathcal{S}_b$-strategy. Now in the case of a $\mathcal{W}_a$-strategy, we explicitly ensured in the construction that $\iota_{\sigma\hat{}\langle o\rangle}(a)$ is not defined to be in an interval guessed to be the ι-image of a finite interval by some τ with $\tau < \sigma$ or $\tau \supset \sigma$ that has not been initialized or proved wrong.

So suppose σ is an $\mathcal{S}_b$-strategy, and so σ must define $\iota^{-1}_{\sigma\hat{}\langle o\rangle}(b)$. There were two cases how σ defines $\iota^{-1}_{\sigma\hat{}\langle o\rangle}(b)$: In the first case, it defined it as $\iota^{-1}_\tau(b)$, and so the claim holds by induction. In the second case, since b is currently not an ι_τ-image, there can be no τ covering the ι-image of an interval containing b except as a right endpoint. □

Here is the most critical property of the construction that we need to enforce at every stage:

Lemma 4.6. *At any stage s at which $\iota_\tau(a)$ and $\iota_\tau(a')$ are defined for $a <_{\mathcal{A}} a'$ and such that τ (actually, τ^- under its outcome $\tau(|\tau| - 1)$) guesses $[a, a']$ to be finite and such that at that stage, τ has neither been initialized nor proved wrong since these definitions were made, we have that $|[a, a']| \geq |[\iota_\tau(a), \iota_\tau(a')]|$.*

Proof. Assume that $\iota_\tau(a)$ was last defined by $\tau' \subset \tau$ at a stage s', and that $\iota_\tau(a')$ was last defined by $\tau'' \subset \tau$ at a stage s''; by symmetry assume that $\tau' \subseteq \tau''$. Then $s' \leq s''$, and τ'' will ensure $|[a, a']| \geq |[\iota_\tau(a), \iota_\tau(a')]|$ at stage s''. So we need to show that this cannot fail after stage s'', due to some strategy σ inserting a new element into the interval $[\iota_\tau(a), \iota_\tau(a')]$ of $\mathcal{B}$.

Now no strategy $\sigma <_L \tau$ can act between stages s'' and s (or else τ would be initialized); and no $\mathcal{S}$-strategy inserts any number into $\mathcal{B}$. Any $\mathcal{R}$-strategy $\sigma \subset \tau$ inserting a number into $\mathcal{B}$ would also initialize τ. Any $\mathcal{R}$-strategy $\sigma > \tau$ can insert a new number only into an interval of $\mathcal{B}$ which σ, and thus also τ, guesses to be the ι-image of an infinite interval of $\mathcal{A}$ unless σ is initialized at the same time. Any $\mathcal{W}$-strategy $\sigma \subset \tau$ inserts a number into the interval $(\iota_\tau(a_i), \iota_\tau(a_{i+1}))$ (under the endpoint notation according to σ) only when σ takes its current outcome the first time. Any $\mathcal{W}$-strategy $\sigma > \tau$ inserts a number into a subinterval of the interval $(\iota_\tau(a_i), \iota_\tau(a_{i+1}))$ of $\mathcal{B}$ (under the endpoint notation according to σ) only if there is no ρ with $\rho^- \leq \sigma$ that last guessed an interval containing a in its interior as finite and has not been initialized or proved wrong since then. $\square$

Lemma 4.7. *If a strategy τ^- defines $\iota_\tau(a) = b$ and later a strategy σ finds $b \in [\iota_\sigma(a_i), \iota_\sigma(a_{i+1})]$ for an interval $[a_i, a_{i+1}]$ handed to σ while τ has not yet been initialized nor proved wrong, then $a \in [a_i, a_{i+1}]$.*

Proof. Suppose the lemma fails; by symmetry assume $a <_\mathcal{A} a_i$. But then $\iota_\sigma(a_i) \leq_\mathcal{B} \iota_\tau(a) = b <_\mathcal{A} \iota_\sigma(a_i)$ and so in particular, $\iota_\sigma(a_i) <_\mathcal{B} \iota_\tau(a_i)$. But then some $\rho \subset \sigma$ must have defined $\iota_\sigma(a_i)$ in the belief that an interval containing $[a_i, a_{i+1}]$ is finite, contradicting our assumption on σ. $\square$

Lemma 4.8. *Suppose at a stage s, a strategy σ defines $\iota_{\sigma\widehat{\ }\langle o\rangle}(a)$ for some a in an interval (a_i, a_{i+1}) handed to σ for some outcome o. Then:*

(1) If σ guesses both $[a_i, a]$ and $[a, a_{i+1}]$ to be infinite at stage s (i.e., guesses a to be an anchor), then for any τ for which $\iota_\tau(a)$ is defined, we have $\iota_\tau(a) = \iota_{\sigma\widehat{\ }\langle o\rangle}(a)$ unless τ has been initialized or proved wrong since it defined $\iota_\tau(a)$. (In this case, a is an anchor with respect to σ and its outcome o.)

(2) If σ guesses one of $[a_i, a]$ or $[a, a_{i+1}]$ to be finite at stage s (by symmetry, say, the former), then there is $j \leq i$ such that σ guesses $[a_j, a_i]$ to be finite, and for any τ for which $\iota_\tau(a_j)$ is defined, we have $\iota_\tau(a_j) = \iota_\sigma(a_j)$ unless τ has been initialized or proved wrong since it defined $\iota_\tau(a_j)$. (In this case, a_j is an anchor of a with respect to σ and its outcome o.)

Proof. (1) Here, σ is either a $\mathcal{W}_a$- or an $\mathcal{S}_b$-strategy. In either case, the claim is obvious from the definition of $\iota_{\sigma^\frown\langle o\rangle}(a)$.

(2) The proof is the same as for (1). □

We now define the true map $\iota = \iota_{TP}$ as the union of all ι_σ for $\sigma \subset TP$. We can then verify that each strategy $\sigma \subset TP$ satisfies its requirement. For convenience, we define $\sigma^+ = TP \upharpoonright (|\sigma|+1)$, the immediate successor of σ along the true path.

Lemma 4.9. *Each $\mathcal{W}_a$-requirement is satisfied by the $\mathcal{W}_a$-strategy $\sigma \subset TP$.*

Proof. This is trivial if $\iota_\sigma(a)$ is already defined. Otherwise, fix an interval $[a_i, a_{i+1}]$ handed down to σ containing a in its interior. If σ guesses at least one of $[a_i, a]$ and $[a, a_{i+1}]$ to be finite, say, the former, then σ ensures that $|[a_i, a]| = |[\iota_\sigma(a_i), \iota(a)]|$ once the former size stabilizes.

Otherwise, σ guesses both $[a_i, a]$ and $[a, a_{i+1}]$ to be infinite, and so, by construction, σ can find some $b \in (\iota_\sigma(a_i), \iota(a_{i+1}))$ and define $\iota_{\sigma^+}(a) = b$. □

Lemma 4.10. *Each $\mathcal{S}_b$-requirement is satisfied by the $\mathcal{S}_b$-strategy $\sigma \subset TP$.*

Proof. This is trivial if $\iota_\sigma^{-1}(b)$ is already defined. Otherwise, fix an interval $[a_i, a_{i+1}]$ handed down to σ such that b is in $(\iota_\sigma(a_i), \iota(a_{i+1}))$. If σ guesses at least one of $[a_i, a]$ or $[a_{i+1}]$ to be finite, say, the former, then σ ensures that $|[a_i, a]| = |[\iota_\sigma(a_i), \iota(a)]|$, which easily defines $\iota_{\sigma^+}^{-1}(b)$.

Otherwise, σ guesses both $[a_i, a]$ and $[a, a_{i+1}]$ to be infinite, and so, by construction, σ can find some $a \in (a_i, a_{i+1})$ and define $\iota_{\sigma^+}^{-1}(b) = a$. □

Lemmas 4.9 and 4.10 combined now show that ι is a bijection from $\mathcal{A}$ onto $\mathcal{B}$, defined by

$$\iota = \bigcup_{\sigma \subset TP} \iota_\sigma$$

Clearly, since ι preserves the ordering, it is in fact an isomorphism, and it is also computable from $\mathbf{0}''$ (but not necessarily from TP alone since there is some finite injury along the true path due to $\Gamma^{\mathrm{Succ}(\mathcal{B})}$-correction).

Lemma 4.11. *Each $\mathcal{R}_m$-requirement is satisfied by the $\mathcal{R}_m$-strategy $\sigma \subset TP$ (at least for cofinitely many arguments m, which suffices for our proof).*

Proof. We need to ensure two things: Firstly, that $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is eventually defined with a permanent use; and secondly, that if $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is defined when m enters C, then $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ is corrected.

First of all, recall that we already indicated above that no definition of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ made by any $\mathcal{R}_m$-strategy $\tau >_L \sigma$ can be permanent, so here is the formal proof: Let $\rho \subset \sigma, \tau$ be the longest common substrategy. Then there is an interval $[a, a']$ created by ρ such that σ and τ disagree either about its cardinality: If τ guesses that $[a, a']$ is finite then σ guesses that it is either finite of larger size or infinite; in either case, when σ is next eligible to act at a stage s after τ has defined $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$, then τ used every adjacency in $[a, a']$ in the use of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$, and one of these must have been destroyed by stage s.

Now assume that no $\mathcal{R}_m$-strategy $\tau <_L \sigma$ defines $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ permanently. Then we claim that σ must eventually do so since σ has the correct guesses about all the intervals handed to it, as long as at least one interval handed to σ is guessed to be finite by σ: The intervals that σ guesses to be finite are truly finite of the right size; so σ eventually enumerate a Γ-axiom with correct use in $\mathrm{Succ}(\mathcal{B})$.

Finally, suppose that some $\mathcal{R}_m$-strategy τ has defined $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$ and then m enters C. (By the argument two paragraphs above, we may assume that $\tau \leq \sigma$ since otherwise the definition of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m) = 0$ will have been destroyed by the next time σ is eligible to act.) Then τ will make $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ undefined by inserting an element into an adjacency in the use of $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ in an interval which τ guesses to be finite, thus destroying the computation $\Gamma^{\mathrm{Succ}(\mathcal{B})}(m)$ as desired. □

Note that in the proof above, and in the construction, we could have ensured that $\Gamma^{\mathrm{Succ}(\mathcal{B})}$ is indeed total by rearranging the $\mathcal{R}_m$-strategies so that none is below an outcome guessing all intervals handed to the strategy being infinite.

This completes the proof of our Main Theorem.

References

[1] Chubb, Jennifer C.; Frolov, Andrey N.; and Harizanov, Valentina S., *Degree spectra of the successor relation of computable linear orderings*, Arch. Math. Logic **48** (2009), 7–13.

[2] Downey, Rodney G.; Lempp, Steffen; and Wu, Guohua, *On the complexity of the successivity relation in computable linear orderings*, J. Math. Logic **10** (2011), 1–17.

[3] Downey, Rodney G.; Lempp, Steffen; and Wu, Guohua, *Corrigendum: On the complexity of the successivity relation in computable linear orderings*, J. Math. Logic **17** (2017), paper 1792002, 4 pages.

[4] Frolov, Andrey N., *Presentations of the successor relation of computable linear orderings*, Izv. Vyssh. Uchebn. Zaved. Mat. **54**, no. 7 (2010), 73–85; translated in: Russian Math. (Iz. VUZ) **54**, no. 7 (2010), 64–74.

School of Mathematics, Statistics and Operations Research, Victoria University of Wellington, Wellington, New Zealand
Email address: Rod.Downey@vuw.ac.nz
URL: http://www.mcs.vuw.ac.nz/Main/RodDowney

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706-1325, USA
Email address: lempp@math.wisc.edu
URL: http://www.math.wisc.edu/~lempp

School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Republic of Singapore
Email address: guohua@ntu.edu.sg
URL: http://www3.ntu.edu.sg/home/guohua/