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Splitting theorems in recursion theory *

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Abstract

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A splitting of an r.e. set A is a pair A_1, A_2 of disjoint r.e. sets such that $A_1 \cup A_2 = A$. Theorems about splittings have played an important role in recursion theory. One of the main reasons for this is that a splitting of A is a decomposition of A in both the lattice, $\mathcal{E}$, of recursively enumerable sets and in the uppersemilattice, $\mathbf{R}$, of recursively enumerable degrees (since $A_1 \leqslant_{\mathrm{T}} A$, $A_2 \leqslant_{\mathrm{T}} A$ and $A \leqslant_{\mathrm{T}} A_1 \oplus A_2$). Thus splitting theorems have been used to obtain results about the structure of $\mathcal{E}$, the structure of $\mathbf{R}$, and the relationship between the two structures. Furthermore it is fair to say that questions *about* splittings have often generated important new technical developments in recursion theory. In this article we survey most of the results and techniques associated with splitting properties of r.e. sets in ordinary recursion theory.

Contents

0. Introduction

A splitting of an r.e. set A is a pair A_1, A_2 of disjoint r.e. sets such that $A_1 \cup A_2 = A$. Theorems about splittings have played an important role in recursion theory. The main reason for this is that a splitting of A is a decomposition of A in both the lattice, $\mathcal{E}$, of recursively enumerable sets and in the uppersemilattice, **R**, of recursively enumerable degrees (since $A_1 \leqslant_T A$, $A_2 \leqslant_T A$ and $A \leqslant_T A_1 \oplus A_2$). Thus splitting theorems have been used to obtain results about the structure of $\mathcal{E}$, the structure of **R**, and the relationship between the two structures. Furthermore it is fair to say that questions *about* splittings have often generated important new technical developments in recursion theory. Examples include the development of the $0'''$-priority method, Lachlan's "diamond" theorem [64], and Shore's blocking technique in α-recursion theory.

In this article we survey many of the results and techniques associated with splitting properties of r.e. sets in ordinary recursion theory. We have attempted to include all important results and techniques related to ordinary recursion theory but have chosen to leave out other areas such as effective algebra and generalized recursion theory. For effective algebra, in which there are many important results concerning splittings, the surveys of Nerode and Remmel [81], Downey, Remmel and Welch [31], and Downey and Remmel [30] are adequate. With respect to generalized recursion theory, we would need to develop too much machinery and notation to fit in the current paper.

We have several reasons for writing this paper. The main reason is that many of the important results concerning splittings are scattered throughout the literature or are unpublished. Furthermore, the standard sources (Rogers [87], Soare [97], and Odifreddi [83]) do not contain many of the important results. It also seems timely to give such a survey since there have been many recent applications of splittings to major questions about **R** and $\mathcal{E}$ and many of these applications remain unpublished. Another reason for writing this paper is that many of the older results concerning the structure of **R** can be simplified and extended using splitting theorems.

This paper contains, as far as possible, all that we know about splitting theorems in ordinary recursion theory. We have also included enough proofs

to illustrate most major proof techniques used in this area. The only exceptions to this rule are in cases where the proofs are too long.

The paper consists of 10 sections of results. Section 1 is devoted to Friedberg splittings, their applications, and some classical generalizations. Section 2 consists of extensions of Friedberg splittings due to Downey and Stob. These extensions were introduced particularly to study the group, $\text{Aut}(\mathcal{E})$, of automorphisms of $\mathcal{E}$. In Section 3 we consider other splitting properties related to automorphisms of $\mathcal{E}$ including the splitting property of Maass, Shore and Stob [74] and Maass's outer splitting property. In Section 4 we consider the hemimaximal sets of Downey and Stob. Again, this splitting property was introduced to study automorphisms of $\mathcal{E}$. In Section 5 we examine the universal splitting property of Lerman and Remmel [69] and various extensions of it. Section 6 is devoted to other applications of splitting theorems to the study of the structure of **R**. Section 7 concerns sets without the universal splitting property and strengthenings of this. Section 8 concerns mitotic sets and degrees. Section 9 is devoted to generalizations of the notion of mitoticity. Also included in Section 9 is a study of array recursive splittings. Finally, Section 10 concerns splittings of d.r.e. sets culminating in Cooper's proof of the definability of the jump in the structure of the degrees.

There are quite a few new results, unpublished results, and new proofs of old results here. All results not otherwise credited are due to the authors. Some of the presentation in Section 9 (and a couple of other places) is reproduced with permission of North-Holland. Notation is standard and follows Soare [97]. In particular, use functions are monotone in stage and arguments and all computations and uses are bounded by s at any stage s. We use $\mathcal{L}(A)$ to denote the lattice of r.e. supersets of A and $\mathcal{L}^*(A)$ to denote $\mathcal{L}(A)$ modulo the ideal of finite sets. If A and B are r.e. sets, $A \searrow B$ denotes $\{x \mid (\exists s, t)[s < t \wedge x \in A_{\text{at } s} \wedge x \in B_{\text{at } t}]\}$. Also $A \backslash B = \{x \mid (\exists s)[x \in A_s - B_s]\}$. Note that $A \searrow B = (A \backslash B) \cap B$. We will refer to the *e*-state of x (at stage s) measured with respect to the standard enumeration of the r.e. sets as the standard *e*-state. That is, the standard *e*-state of x (at stage s) is $\{j \mid j \leqslant e \wedge x \in W_j\}$ ($\{j \mid j \leqslant e \wedge x \in W_{j,s}\}$). These *e*-states are denoted $\sigma(e, x)$ and $\sigma(e, x, s)$ respectively. We will also often have occasion to mention *length of agreement functions*. Given a functional Φ and r.e. sets A and B, the *length of agreement of* $\Phi(A) = B$ is the function l defined by $l(s) = \max\{x \mid (\forall y < x)[\Phi_s(A_s; y) = B_s(y)]\}$ and the *maximum length of agreement* is the function m defined by $m(s) = \max\{t < s \mid l(t)\}$.

1. Friedberg splittings

The earliest splitting theorem is due to Friedberg.

Theorem 1.1 (Friedberg's Splitting Theorem [43]). *If A is a nonrecursive r.e. set, then there exist disjoint nonrecursive sets A_1 and A_2 such that $A = A_1 \cup A_2$. Furthermore, these sets satisfy the property*

$$\text{for all r.e. sets } W, \text{ if } W - A \text{ is not r.e.,} \qquad (1.1)$$
$$\text{then } W - A_i \text{ is not r.e. for } i = 1, 2.$$

Proof. We have the following requirements for every $e \in \omega$ and $i = 1, 2$.

$$\mathbf{R}_{e,i}: \quad |W_e \setminus A| = \infty \Rightarrow W_e \cap A_i \neq \emptyset.$$

First, to see that the requirements suffice to prove (1.1), suppose that W is an r.e. set and i is such that $W - A_i$ is r.e. Let $U = W - A_i$. Supposing that $\mathbf{R}_{e,i}$ is met for $U = W_e$, we therefore have that $U \setminus A$ is finite. This implies that $U - A = W - A$ is r.e. (since $U - A =^* U \setminus A$ and the latter set is r.e.). Notice also that (1.1) implies that each set A_i is not recursive.

To meet the requirements $\mathbf{R}_{e,i}$, at stage s we search for the least $\langle e, i \rangle$, if any, such that $W_{e,s} \cap A_{i,s} = \emptyset$ and for which there is z such that $z \in W_{e,s} \cap (A_{s+1} - A_s)$. If such exists, we enumerate z in A_i at stage $s + 1$, enumerating all other elements of $A_{s+1} - A_s$ into A_1, say. To see that this strategy suffices to meet the requirements, notice that each requirement acts only finitely often but if $W_e \setminus A$ is infinite, requirement $\mathbf{R}_{e,i}$ has infinitely many opportunities to act. $\square$

We record the following useful definitions.

Definition 1.2. Suppose that A is a nonrecursive r.e. set.
(1) A *splitting* of A is a pair A_1, A_2 of disjoint r.e. sets such that $A_1 \cup A_2 = A$. We sometimes will write $A = A_1 \sqcup A_2$ if A_1, A_2 is a splitting of A.
(2) A *nontrivial splitting* of A is a splitting of A such that, in addition, the sets A_1, A_2 are nonrecursive.
(3) A *Friedberg splitting* of A is a nontrivial splitting of A such that, in addition, the sets A_1, A_2 satisfy (1.1).

Although Friedberg's splitting theorem is a very easy wait-and-see argument, it has a number of important consequences and extensions. For example, it plays a crucial role in Lachlan's decision procedure for the $\forall \exists$-theory of $\mathcal{E}^*$. As another example, we give an alternate proof of a result of Shore on nowhere simple sets.

Definition 1.3 (Shore [91]). An r.e. set A is *nowhere simple* if for every r.e. set W such that $W - A$ is infinite, there is an infinite r.e. set $C \subseteq W$ such that $C \cap A = \emptyset$. If an index for C can be found uniformly in an index for W, then A is *effectively nowhere simple*. (The terminology comes from the fact

that this property asserts that A is not simple in the lattice $\mathcal{E}(W \cup A)$ of r.e. sets restricted to $W \cup A$.)

Shore [91] showed that every r.e. set has a nontrivial splitting by a pair of nowhere simple sets. This fact follows from Theorem 1.5 which below. We first remark that there is an alternate characterization of effective nowhere simplicity which shows that the property is lattice-theoretic.

Theorem 1.4 (Miller and Remmel). *A is effectively nowhere simple if and only if there is an r.e. set B disjoint from A such that for all r.e. sets W, $W - A$ infinite implies that $W \cap B$ is infinite.*

Proof. ($\Rightarrow$) Suppose that f witnesses that A is effectively nowhere simple. That is, suppose that if $W_e - A$ is infinite then $W_{f(e)}$ is infinite and $W_{f(e)} \subseteq W_e - A$. Then $B = \bigcup_e W_{f(e)}$ is the desired set B.

($\Leftarrow$) Conversely, suppose that B satisfies the mentioned condition. Then f defined by $W_{f(e)} = B \cap W_e$ witnesses that A is effectively nowhere simple. $\square$

Theorem 1.5. *Suppose that A_1, A_2 is a Friedberg splitting of A.*
(1) *Then A_1 and A_2 are nowhere simple.*
(2) *If A is simple, then A_1 and A_2 are effectively nowhere simple.*
(3) *(with R. Shore) If B_1, B_2 is any other Friedberg splitting of A, then B_1 is effectively nowhere simple iff A_1 is effectively nowhere simple.*

Proof. (of 1) Suppose that W is an r.e. set such that $W - A_1$ is infinite. To produce the desired C, we consider two cases. In the case that $W \cap A_2$ is infinite, we have that $C = W \cap A_2$ has the necessary properties. On the other hand, suppose that $W \cap A_2$ is finite. Then $W - A_2$ is r.e., hence $W - A$ is r.e. (because A_1, A_2 is a Friedberg splitting of A). However $W - A =^* W - A_1$ in this case so that $C = W - A_1$ works.

(of 2) In the case that A is simple, the latter case above cannot happen so that we have $C = W \cap A_2$ always works. Thus it is clear that, if A is simple, the index for C can be produced effectively from that of W.

(of 3) Suppose that A_1 is effectively nowhere simple. Then by Theorem 1.4 there is an r.e. set C such that $C \cap A_1 = \emptyset$ and if $W - A_1$ is infinite, $C \cap W$ is infinite. Since C is disjoint from A_1 and A_1 is half of a Friedberg splitting of A, $C - A$ is r.e. Now let $D = (C - A) \cup B_2$. We claim that D witnesses the effective nowhere simplicity of B_1 (using the Miller–Remmel characterization of effective nowhere simplicity). For suppose that W is an r.e. set with $W - B_1$ infinite. We must show that $W \cap D$ is infinite. There are two cases. If $W \cap B_2$ is infinite, then $W \cap D$ is obviously infinite. Otherwise, if $W \cap B_2$ is finite, this implies that $W - A$ is an infinite r.e. set. Let $V = W - A$. Then, we must have that $V \cap C$ is infinite and this implies that $V \cap D$ is infinite. $\square$

Shore used his splitting theorem to establish the following interesting fact about automorphism bases of $\mathcal{E}^*$.

Theorem 1.6 (Shore [91]). *Suppose that $\varphi : \mathcal{E}^* \to \mathcal{E}^*$ is an elementary lattice injection and that φ is the identity on a nontrivial class C^* of r.e. sets closed under recursive permutations of ω. Then φ is the identity on $\mathcal{E}^*$.*

Proof. Suppose that φ is not the identity on $\mathcal{E}^*$. Then by Theorem 1.5 there is a nowhere simple set A such that $\varphi(A) \neq^* A$. We may assume that $|A - \varphi(A)| = \infty$. As φ is elementary, $\varphi(A)$ is nowhere simple. Hence, there exists a recursive set $R \subseteq A - \varphi(A)$. Now choose $C \in C^*$ with $\overline{C} \subseteq R \subseteq A - \varphi(A)$. (This is possible as C^* is closed and nontrivial.) Now $A \cup C = \omega$ but $\varphi(A) \subseteq C$. This is a contradiction since $\varphi(C) = C$ implies that $C = \varphi(A) \cup C = \varphi(A) \cup \varphi(C) = \varphi(A \cup C) =^* \omega$. $\square$

As we observed in Theorem 1.5, if A_1, A_2 is a Friedberg splitting of a simple set A, A_1 is effectively nowhere simple with A_2 playing the role of B in the Miller–Remmel characterization. This result suggested to us a converse of the second part of Theorem 1.5; perhaps it is the case that effective nowhere simplicity is equivalent to being half of a Friedberg splitting of a simple set. This suggestion fails.

Theorem 1.7. *There is an effectively nowhere simple nonrecursive set A such that A is not half of a Friedberg splitting of a simple set.*

Proof. We build disjoint sets A and B and auxiliary sets Q_e, $e \in \omega$, in stages. Let $a_{0,s} < a_{1,s} < \cdots$ list the elements of $\overline{A}_s$ in increasing order. We meet the following requirements.

$$\mathbf{P}_e: \quad \overline{A} \neq W_e,$$

$$\mathbf{R}_e: \quad |W_e - A| = \infty \Rightarrow W_e \cap B \neq \emptyset,$$

$$\mathbf{N}_{e,i}: \quad W_e \cap A \neq \emptyset \vee (|Q_e - W_e| \geq i \wedge Q_e \cap A = \emptyset),$$

$$\mathbf{N}'_e: \quad \lim_s a_{e,s} = a_e \text{ exists.}$$

The requirements $\mathbf{R}_e$ guarantee that B witnesses the effective nowhere simplicity of A. The requirements $\mathbf{N}_{e,i}$ for $i \in \omega$ ensure that W_e and A do not form a Friedberg splitting of a simple set. We also define for each $e \in \omega$ the sequence $q_{e,0,s} < q_{e,1,s} < \cdots$ to list in increasing order the elements of $\overline{A_s \cup B_s \cup Q_{0,s} \cup \cdots \cup Q_{e-1,s}}$. We will also use $e = -1$. The argument is a finite injury one so we will describe the strategies for each requirement independently and leave the details of the coherence to the reader.

Requirement $\mathbf{P}_e$ requires attention at a stage s if $W_{e,s} \cap A_s = \emptyset$ and there is $x \in W_{e,s}$ such that $x = q_{e,k,s}$ for some $k \geq e$. In this case we meet $\mathbf{P}_e$ by enumerating x into A. We shall also initialize and restart all requirements $\mathbf{N}_{e',j}$ for $e' > e$. We meet requirement $\mathbf{R}_e$ by waiting until we see some $x \in W_{e,s} - A_s$ while $W_{e,s} \cap B_s = \emptyset$ such that $x > q_{e,e,s}$ and then enumerating one such x into B. This meets $\mathbf{R}_e$ forever. Notice that this action does not directly interfere with $\mathbf{N}_{e',i}$ for $e' \leq e$ since it does not require enumeration of elements of $Q_{e'}$ into A. However it may indirectly injure such a requirement as we shall see below.

Requirement $\mathbf{N}_{e,i}$ can receive attention in two ways. First, if there is a stage s and $x \in W_{e,s}$ such that $x = q_{e,j,s}$ for some $j \geq e$, we can satisfy all requirements $\mathbf{N}_{e,i}$ for $i \in \omega$ forever (by satisfying the first disjunct of the requirement) by enumerating x into A. Otherwise while $W_{e,s} \cap A = \emptyset$, we must have that $x \in W_{e,s}$ implies that $x \in B \cup \bigcup_{j<e} Q_j$. In this case we will satisfy $\mathbf{N}_{e,i}$ by meeting the second disjunct. To do this we shall say that $\mathbf{N}_{e,i}$ requires attention also if $W_{e,s} \cap A_s = \emptyset$, $W_{e,s} \subseteq B \cup \bigcup_{j<e} Q_j$, and $|Q_{e,s} - W_{e,s}| < i$. We then chose a large fresh number y (say, $q_{s,s,s}$) currently in no set Q_j and enumerate y in $Q_{e,s+1}$. It is now not difficult to see that all the strategies combine properly by a standard application of the finite injury priority method. $\square$

There are several interesting open questions related to Friedberg splittings and nowhere simple sets. To state the first, we need the following definition.

Definition 1.8. A set X (not necessarily r.e.) is *semilow* if $\{e : W_e \cap X \text{ is finite}\} \leq_T 0'$ and *semilow*$_{1.5}$ if $\{e : W_e \cap X \text{ is finite}\} \leq_1 0''$.

It is easy to see that if A is effectively nowhere simple, then $\overline{A}$ is semilow$_{1.5}$. Maass used the automorphism machinery of Soare to show (generalizing Soare [96]) the following.

Theorem 1.9. *A is r.e. coinfinite with semilow$_{1.5}$ complement if and only if $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$.*

We discuss some aspects of the proof of Theorem 1.9 in Section 3. Here we note that it is an obvious consequence of Theorem 1.9 that for effectively nowhere simple A, $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$. This leads to the following open question, first posed by Shore.

Open Question 1.10. If A is coinfinite and nowhere simple, is $\mathcal{L}^*(A) \cong \mathcal{E}^*$?

Semilowness is related to the Blum–Marques machine-independent theory of computational complexity. Blum and Marques [12] define an r.e. set A to be speedable if A has no "fastest program modulo a recursive cost function". The

exact definition of this concept is not important here for Soare [95] showed that A is speedable if and only if $\bar{A}$ is not semilow. Blum and Marques [12] observed that any r.e. set A can be split into a pair of nonspeedable sets. Soare later observed that this result follows from a theorem of Sacks [88]. In fact, the Blum–Marques result follows from our proof of Friedberg's Splitting Theorem. To demonstrate this, we first notice that our proof produces a splitting with a stronger property than the defining property of a Friedberg splitting.

Definition 1.11. A_1, A_2 is a *true Friedberg splitting* of A if for every $e \in \omega$,

$$|W_e \smallsetminus A| = \infty \Rightarrow W_e \cap A_i \neq \emptyset, \quad i = 1, 2. \tag{1.2}$$

Our proof of the Friedberg splitting theorem actually showed that every nonrecursive r.e. set A has a true Friedberg splitting. We now obtain the Blum–Marques result on non-speedable splittings in the following way.

Theorem 1.12. *Suppose that A_1, A_2 is a true Friedberg splitting of A. Then both A_1 and A_2 are nonspeedable (i.e., $\bar{A}_i$ is semilow for $i = 1, 2$).*

Proof. Define a recursive function f by

$$f(e, s) = \begin{cases} 1 & \text{if } W_{e,s} \cap \bar{A}_{1,s} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\lim_s f(e, s) = f(e)$ exists and $f(e) = 1$ only if $W_e \cap \bar{A}_1 \neq \emptyset$. Hence $\bar{A}_1$ is semilow by the Shoenfield Limit Lemma. Suppose that there are infinitely many s such that $W_{e,s} \cap \bar{A}_{1,s} \neq \emptyset$. We must show that $W_e \cap A_1 \neq \emptyset$ and thus that there are cofinitely many such s. If otherwise, it must be the case that $W_e \smallsetminus A_1$ is infinite. However then it must be the case that $W_e \smallsetminus A$ is infinite and so that $W_e \cap A_2 \neq \emptyset$. But any element of $W_e \cap A_2$ is in $W_e \cap \bar{A}_1$ and so $W_e \cap \bar{A}_1 \neq \emptyset$ contrary to our assumption. $\square$

This result leads to the following dual question, first posed by Remmel.

Open Question 1.13. If A is speedable, does there exist a splitting A_1, A_2 of A such that A_1 and A_2 are each speedable?

The best result along these lines that we have is the next theorem. Recall that an r.e. set A is called hyperhypersimple (hhsimple) if for every weak array $\{V_j\}_{j \in \omega}$, there is a j such that $V_j \subseteq A$. (A weak array is an r.e. sequence of finite sets given by r.e. indices.) Blum and Marques [12] showed that all hhsimple sets are speedable. We have the following.

Theorem 1.14 (Downey, Jockusch, Lerman and Stob). *If A is hhsimple, there is a splitting A_1, A_2 of A such that each of A_1 and A_2 is speedable.*

Proof. Suppose that hyperhypersimple A is given. We show how to enumerate the splitting A_1, A_2. At each stage s, each number $x \in \overline{A}_s$ will be targeted for at most one of the sets A_1 or A_2. At stage $s + 1$, if $x \in A_{s+1} - A_s$, we will enumerate x in A_1 unless x is targeted for A_2, in which case we will enumerate x in A_2. This will ensure that A_1, A_2 is a splitting of A. If $x \in \overline{A}_s$ is targeted for a set at stage s then at a stage $t > s$ it might be retargeted for the other set or it may cease to be targeted for either set.

In order to state the requirements, the following notation will be useful. For a set X, define

$$\|X\| = \begin{cases} 1 & \text{if } X \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The requirements to make each set A_i speedable as follows.

$$\mathbf{R}_{e,i}: \ (\exists h)\,[\Phi_e(K;h) \neq \|W_h \cap \overline{A}_i\|\,].$$

The basic strategy for meeting one requirement $\mathbf{R}_{e,i}$ is the following. We enumerate a weak array $\{V_{e,i,j}\}_{j\in\omega}$ of r.e. sets. By the recursion theorem, we will assume that we know an index $h(e,i,j)$ for the set $V_{e,i,j}$. The intention is to guarantee that there is j such that if $h = h(e,i,j)$, $\mathbf{R}_{e,i}$ is satisfied with witness h. We show that if $\mathbf{R}_{e,i}$ is not satisfied, $\{V_{e,i,j}\}_{j\in\omega}$ is a weak array witnessing that A is not hhsimple.

The construction is on a tree.

Define the priority tree as $PT = (\omega \cup \{f\})^{<\omega}$. If $\alpha \in PT$ we refer to α as a *guess*. We order the guesses lexicographically which we denote by $\leqslant_L$. If $|\alpha| = \langle e, i\rangle$ we associate α with $R_{e,i}$. Instead of the sets $V_{e,i,j}$ described above, we will have sets $V_{\alpha,j,k}$. The extra index k represents the fact that we will have to sometimes "restart" $V_{\alpha,j}$ with a new version. For each α and j, at each stage s, there will be an *active* k denoted by $k(\alpha,j,s)$. That is, at stage s, we will be using the set $V_{\alpha,j,k(\alpha,j,s)}$. By the recursion theorem, we assume the existence of a recursive function h such that $W_{h(\alpha,j,k(\alpha,j,s))} = V_{\alpha,j,k(\alpha,j,s)}$. The life cycle of a set $V_{\alpha,j,k}$ is as follows. Initially it is empty. If it is never activated, then it remains empty forever. At some stage s, $V_{\alpha,j,k}$ may be activated and can potentially get elements to target for A_i where $|\alpha| = \langle e, i\rangle$. This activation will indicate that for each $k' < k$, $V_{\alpha,j,k'}$ has been deactivated. (Deactivation occurs when $\alpha^\frown j$ has been *initialized*. Initialization occurs in the construction, and if a node $\alpha^\frown j$ is initialized at stage s, we set $k(\alpha,j,s+1) = k(\alpha,j,s) + 1$ thereby activating the next set on the list.) It will be the case that $V_{\alpha,j,k}$ remains active until $\alpha^\frown j$ is again initialized. Though we are using the Recursion Theorem, we will act as if when we enumerate an element x into some $V_{\alpha,j,k(\alpha,j,s)}$ at

stage s, it immediately enters $W_{h(\alpha,j,k(\alpha,j,s))}$ at stage s. This convention makes the description of the construction easier and is easily justified (say by the Slowdown Lemma of Soare [97, p. 284]).

Define

$$l(\alpha,s) = \max\{x : (\forall y < x)[\Phi_{e,s}(K_s; h(\alpha,y,k(\alpha,y,s)))$$
$$= \|W_{h(\alpha,y,k(\alpha,y,s)),s} \cap \overline{A}_{i,s}\|]\}$$

and

$$m(\alpha,s) = \max_{t<s} l(\alpha,t).$$

Construction

Stage $s + 1$

Step 1. Let $z \in A_{s+1} - A_s$. If z is not targeted for either A_1 nor A_2 put z into A_2. Otherwise put z into its target set.

Step 2. In substages $t + 1$ $0 \leqslant t \leqslant s$, we define a string $\alpha(s+1, t+1)$ of length $t + 1$ and take action for $\mathbf{R}_{e,i}$ such that $\langle e, i \rangle = t$. We set $\alpha(s+1, 0) = \emptyset$. Suppose that we have defined $\alpha(s+1, t) = \alpha$. We describe substage $t + 1$. Let e, i be such that $\langle e, i \rangle = |\alpha|$.

Case 1. $l(\alpha,s) > m(\alpha,s)$. (In this case, we say that s is an α-expansionary stage.) First determine if there exists some j with $h(\alpha,j,k(\alpha,j,s)) < l(\alpha,s+1)$ and $W_{h(\alpha,j,k(\alpha,j,s)),s} \subseteq \overline{A}_{i,s}$. If such a j exists, choose the least such. Let x be the least element of $\overline{A}_s$ not already targeted by some $\beta <_{\mathrm{L}} \alpha^\frown\langle j+1\rangle$, and not in $\bigcup_{j,k} V_{\alpha,j,k,s}$. Target x for A_i, and put x into $V_{\alpha,j,k(\alpha,j,s)}$.

To define $\alpha(s+1, t+1)$, see if there exists $j < l(\alpha,s)$ such that the use of $\Phi_{e,s}(K_s, h(\alpha,j,k(\alpha,j,s)))$ has increased since the previous α-expansionary stage. (Here we assume that if a computation changes its value then its use increases.) In this case, let j be the least such, let $\alpha(s+1, t+1) = \alpha^\frown j$, and initialize all γ with $\alpha^\frown j \leqslant_{\mathrm{L}} \gamma$ and $\alpha^\frown j \not\subseteq \gamma$. If no such j exists, let $\alpha(s+1, t+1) = \alpha^\frown\mathbf{f}$ and initialize all γ with $\alpha^\frown\mathbf{f} \leqslant_{\mathrm{L}} \gamma$.

Case 2. s is not α-expansionary. Do nothing save to set $\alpha(s+1, t+1) = \alpha^\frown\mathbf{f}$.

Let TP denote the true path of the construction. That is, let TP be leftmost path of the priority tree which is visited infinitely often during the construction. Let $\alpha \subseteq TP$. Then $\alpha = \beta^\frown a$ for some $a \in \omega \cup \{\mathbf{f}\}$ and $\beta \subseteq TP$. Let $\langle e, i \rangle = |\beta|$. We prove by induction on $|\alpha|$ the following:

(1) The sets $U_j = \bigcup_{k,s} V_{\beta,j,k,s}$ form a weak array.
(2) α is initialized only finitely often.
(3) For every $j < a$, $\bigcup_{k,s} V_{\beta,j,k,s}$ is finite. (Here we say that $j < \mathbf{f}$ for every $j \in \omega$ because of the priority ordering.)
(4) If $\alpha = \beta^\frown j$ for some $j \in \omega$, then $k = \lim_s k(\beta,j,s)$ exists and the use of $\Phi_{e,s}(K_s, h(\beta,j,k))$ is unbounded in s.

(5) If $\alpha = \beta^\frown f$, then there are at most finitely many β-expansionary stages.

Notice that (4) and (5) together imply that requirement $\mathbf{R}_{e,i}$ is met. It is easy to see that (1) is true by construction. It is also clear that (2) is true by induction and by the fact that α is on the true path. Again by induction and the fact that α is on the true path, we see that if $j \leqslant a$, $\beta^\frown j$ is initialized only finitely often and so $\lim_s k(\beta, j, s)$ exists. Thus (4) holds since the fact that outcome $\beta^\frown j$ is on the true path implies that the use of $\Phi_{e,s}(K_s, h(\beta, j, k))$ changes infinitely often. It remains to see that (3) and (5) are true.

We next argue that (3) holds. Let $k = \lim_s k(\beta, j, s)$. We need to argue that $V_{\beta,j,k}$ is finite. If there are finitely many β-expansionary stages, $V_{\beta,j,k}$ is obviously finite. Suppose then that there are infinitely many such stages. Since $j < a$, it must be the case that $\lim_s \Phi_{e,s}(K_s; h(\beta, j, k))$ exists. If the value of this limit is 1, it is clear that (3) holds since we will only finitely often wish to enumerate x into $V_{\beta,j,k}$. Thus we may assume that the value of the limit is 0 and, by induction on j, that at each β-expansionary stage we wish to enumerate an element into $V_{\beta,j,k}$. By induction and the construction, almost all elements of $\overline{A}$ are available to be so enumerated. (There are only finitely many sets $V_{\gamma,j',k'}$ of higher priority to lay claim to such elements and at most one element of $\overline{A}$ can be prevented from entering $V_{\beta,j,k}$ by such a set. This argument relies on (5).) Thus, at some stage we will enumerate $x \in \overline{A}$ into $V_{\beta,j,k}$. But then $\|V_{\beta,j,k} \cap \overline{A}_i\| = 1$ contradicting the assumption that $\lim_s \Phi_{e,s}(K_s; h(\beta, j, k)) = 0$ and that there are infinitely many β-expansionary stages. Thus (3) holds.

To see that (5) holds, assume that there are infinitely many β-expansionary stages and that $\alpha = \beta^\frown f$. By the same reasoning as that for (3), we see that if $k(j) = \lim_s k(\beta, j, s)$, $\|V_{\beta,j,k(j)} \cap \overline{A}_i\| = 1$ for all j. Furthermore, by initialization, every element of $V_{\beta,j,k(j)} \cap \overline{A}_i$ is actually an element of $\overline{A}$. But this implies that the weak array U_j defined in (1) witnesses that A is not hyperhypersimple. This contradiction completes the proof of (5) and the theorem. □

The following is a corollary to this theorem and work of Downey and Stob reported on in Section 4.

Corollary 1.15. *The properties of being a true Friedberg splitting and of being speedable are not lattice-theoretic (in the lattice $\mathcal{E}^*$).*

Proof. Let M be a maximal (and thus hhsimple) set. Let M_1, M_2 be a true Friedberg splitting of M. By Theorem 1.12, M_1 and M_2 are nonspeedable. However by Theorem 1.14, there is a splitting N_1, N_2 of M such that N_1 and N_2 are speedable and hence such that N_1, N_2 is not a true Friedberg splitting of M. However by a theorem of Downey and Stob (see Theorem 4.2), there is an automorphism Φ of $\mathcal{E}^*$ such that $\Phi(M_i) = N_i$ for $i = 1, 2$. □

Several natural extensions of the notion of Friedberg splittings have appeared in the literature. One important one, an extension to d.r.e. sets, is due to Owings.

Theorem 1.16 (Owings [84]). *Suppose that $A \subseteq B$ are r.e. sets such that $B - A$ is not co-r.e. Then there is a splitting B_1, B_2 of B such that $B_i - A$ is not co-r.e. for $i = 1, 2$. In fact, it is possible to guarantee that for every r.e. set W, if $A \cup (W - B)$ is not r.e., then $A \cup (W - B_i)$ is not r.e. for $i = 1, 2$. Such a splitting is called an Owing's splitting of B over A.*

The proof of Theorem 1.16 is quite similar to the proof of the Friedberg Splitting Theorem. A reference is Soare [97, Chapter X, 2.5]. Owing's splitting theorem plays an important role in Lachlan's decision procedure for the ∀∃-theory of $\mathcal{E}^*$ and in Lachlan's remarkable characterization of hhsimplicity given in the next theorem.

Theorem 1.17. *Let A be a coinfinite r.e. set. Then A is hhsimple if and only if $\mathcal{L}^*(A)$ is a Boolean algebra.*

Proof. ($\Rightarrow$) Suppose that $B \supseteq A$ is noncomplemented in $\mathcal{L}^*(A)$. By repeatedly applying Owings splitting theorem (the indices for an Owing's splitting can be found uniformly from those of A and B), we obtain an r.e. array $\{B_i\}_{i \in \omega}$ witnessing the non-hhsimplicity of A (it is well known that the array need not consist only of finite r.e. sets).

($\Leftarrow$) Suppose that A is not hhsimple. Let $\{V_n\}_{n \in \omega}$ witness this. Define $B = A \cup \bigcup_{n \in \omega} (W_n \cap V_n)$. We claim that B is not complemented in $\mathcal{L}^*(A)$. For if $B \cap \overline{A} = \overline{W_e} \cap \overline{A}$, let x be an element of $V_e - A$. Then by the definition of B, $x \in B$ if and only if $x \in W_e$. $\square$

Morley and Soare later extended Lachlan's Theorem to Δ_2^0 sets by showing that a Δ_2^0 set S is hhimmune if and only if $\mathcal{E}^*(S)$ is a Boolean algebra. Lachlan went on to characterize the Boolean algebras that may arise as $\mathcal{L}^*(A)$ for r.e. A; they are precisely those with a Σ_3 presentation. Lachan's result relativizes so that, using some work of Feiner and the automorphism machinery of Soare, Todd Hammond [44] showed that $\mathcal{E}^A$ is effectively isomorphic to $\mathcal{E}^B$ if and only if $A' = B'$. ($\mathcal{E}^A$ here denotes the lattice of sets r.e. in A.) This leaves the following question.

Open Question 1.18. Is there a set A such that $\mathcal{E}$ and $\mathcal{E}^A$ are not elementarily equivalent?

Harrington and Herrman independently proved that the elementary theory of the lattice of r.e. sets is undecidable using, in part, ideas along the lines of

Lachlan's result on Boolean algebras. They represented the theory of Boolean pairs, known to be undecidable, in the theory of $\mathcal{E}^*$.

2. Orbits

A major program in the study of $\mathcal{E}^*$ is the study of $\text{Aut}(\mathcal{E}^*)$, the group of automorphisms of $\mathcal{E}^*$. The most powerful technique here is the automorphism machinery of Soare [94,96], and its modifications by various authors including Maass [72,73], Maass and Stob [75], Stob [99], and Downey and Stob [39,40]. The main tool in this machinery is the Extension Lemma of Soare which allows dynamic constructions of automorphisms via partial matching of e-states. We give the statement of the Extension Lemma and some basic intuition here; the full proof and more intuition is in Soare [97].

Suppose that A and B are r.e. sets and we wish to build an automorphism of $\mathcal{E}$ taking A to B. The technique allowed by the Extension Lemma of Soare consists of a certain back-and-forth type argument. Soare constructs four recursive arrays, $\{U_e\}_{e\in\omega}$, $\{V_e\}_{e\in\omega}$, $\{\hat{U}_e\}_{e\in\omega}$, $\{\hat{V}_e\}_{e\in\omega}$, such that the map Φ defined by $\Phi(U_e) = \hat{U}_e$, and $\Phi^{-1}(V_e) = \hat{V}_e$ induces an automorphism of $\text{Aut}(\mathcal{E}^*)$. (It is enough to construct an automorphism of $\text{Aut}(\mathcal{E}^*)$ since Soare has also shown that if A and B are r.e. sets which are infinite and coinfinite, and there is $\Phi \in \text{Aut}(\mathcal{E}^*)$ such that Φ maps the equivalence class of A to B, then there is an automorphism of $\mathcal{E}$ which maps A to B.) To insure that Φ defined in this way is defined on all of $\text{Aut}(\mathcal{E}^*)$ and is onto $\text{Aut}(\mathcal{E}^*)$, Soare guarantees that

$$(\forall e)(\exists n)[W_e =^* U_n] \quad \text{and} \quad (\forall e)(\exists n)[W_e =^* V_n]. \tag{2.1}$$

To guarantee that Φ preserves inclusions, the only other requirement on Φ, Soare divides the problem into two subproblems, the so-called $\overline{A}$ to $\overline{B}$ part and the A to B part. To state exactly what each part requires, we need the following definition.

Definition 2.1. Let $\{X_e\}_{e\in\omega}$ and $\{Y_e\}_{e\in\omega}$ be recursive arrays of r.e. sets. The *full e-state, ν of x with respect to $\{X_e\}_{e\in\omega}$, $\{Y_e\}_{e\in\omega}$* is the triple $\langle e, \sigma, \tau \rangle$ where σ is the e-state of x with respect to $\{X_e\}_{e\in\omega}$ and τ is the e-state of x with respect to $\{Y_e\}_{e\in\omega}$. (Given x and s, $\nu_{e,s}(x)$ is the approximation to the full e-state of x at stage s in some fixed simultaneous enumeration of all the sets in the arrays $\{X_e\}_{e\in\omega}$, $\{Y_e\}_{e\in\omega}$.)

Now the $\overline{A}$ to $\overline{B}$ part of the requirement amounts to

for each full e-state ν,
infinitely many elements of $\overline{A}$ have e-state ν w.r.t. $\{U_e\}_{e\in\omega}, \{\hat{V}_e\}_{e\in\omega}$ (2.2)
iff
infinitely many elements of $\overline{B}$ have e-state ν w.r.t. $\{\hat{U}_e\}_{e\in\omega}, \{V_e\}_{e\in\omega}$.

Similarly, the A to B requirement is

for each full e-state ν,
infinitely many elements of A have e-state ν w.r.t. $\{U_e\}_{e\in\omega}, \{\hat{V}_e\}_{e\in\omega}$ (2.3)
iff
infinitely many elements of B have e-state ν w.r.t. $\{\hat{U}_e\}_{e\in\omega}, \{V_e\}_{e\in\omega}$.

It is clear that (2.1), (2.2), and (2.3) guarantee that Φ as defined above is an automorphism such that $\Phi(A) =^* B$. The most difficult of the three conditions (2.1), (2.2), and (2.3) is (2.3). The primary reason for this difficulty is the conflict between (2.2) and (2.3). To see why this is so, suppose that U_0 is given. (Suppose for instance, because of (2.1), that U_0 is enumerated to satisfy $U_0 = W_0$.) Then, as we observe elements in $U_0 \cap \overline{A}$, we must enumerate certain elements in $\hat{U}_0$ while they remain in $\overline{B}$. However, these elements may later enter B thereby threatening (2.3) with respect to U_0. For if $\hat{U}_0 \cap B$ is infinite, we must have that $U_0 \cap A$ is infinite but we have no control over U_0. Thus a necessary condition for meeting (2.3) seems to be that if infinitely many elements enter B while in $\hat{U}_0$, infinitely many elements of A must be in U_0. Soare extends this analysis to all e-states to get a sufficient condition on the enumeration on all the sets in the four arrays above for (2.3) to be met. Two preliminary definitions are needed.

Definition 2.2. Given full e-states $\nu = \langle e, \sigma, \tau \rangle$ and $\nu' = \langle e, \sigma', \tau' \rangle$, $\nu \leqslant \nu'$ if $\sigma \subseteq \sigma'$ and $\tau \supseteq \tau'$. (The relation $\leqslant$ is pronounced "is covered by".)

Definition 2.3. Suppose that a simultaneous enumeration of the r.e. sets A and $\{U_e\}_{e\in\omega}$ is given. For an e-state ν measured with respect to $\{U_e\}_{e\in\omega}$, we define the sets $\nu \searrow_{ex} A = \{x \mid (\exists s)[x \in A_{at\,s} \wedge \nu(e, x, s) = \nu]\}$. If $x \in \nu \searrow_{ex} A$, we say that ν is the *entry e-state of x at s*. The notation $\nu \searrow_{ex} A$ is defined similarly for full e-states.

Lemma 2.4 (Soare's Extension Lemma). *Assume that A and B are infinite r.e sets and $\{U_n\}_{n\in\omega}, \{\tilde{V}_n\}_{n\in\omega}, \{\tilde{U}_n\}_{n\in\omega}, \{V_n\}_{n\in\omega}$ are recursive arrays of r.e. sets. Suppose that there is a simultaneous enumeration of a recursive array including all the above such that $A \searrow \tilde{V}_n = \emptyset = B \searrow \tilde{U}_n$, for all n. Furthermore suppose that*

$$(\forall \nu)[\nu \searrow_{ex} B \text{ infinite} \Rightarrow (\exists \nu')[\nu \leqslant \nu' \wedge \nu' \searrow_{ex} A \text{ infinite}]]$$

and

$$(\forall \nu) [\nu \searrow_{\mathrm{ex}} A \text{ infinite} \Rightarrow (\exists \nu') [\nu' \leqslant \nu \wedge \nu' \searrow_{\mathrm{ex}} B \text{ infinite}]].$$

Then there are r.e. sets $\hat{U}_n$ extending $\tilde{U}_n$ and $\hat{V}_n$ extending $\tilde{V}_n$ such that (2.3) above is satisfied.

Downey and Stob, in [39], were interested in the question of whether an automorphism taking a set A to a set B could be extended to a splittings of A and B. It is clear that the condition corresponding to (2.3) in this case becomes

$$\begin{array}{c} \text{for each } i \text{ and for each full } e\text{-state } \nu, \\ \text{infinitely many elements of } A_i \text{ have } e\text{-state } \nu \text{ w.r.t. } \{U_e\}_{e\in\omega}, \{\hat{V}_e\}_{e\in\omega} \\ \text{iff} \\ \text{infinitely many elements of } B_i \text{ have } e\text{-state } \nu \text{ w.r.t. } \{\hat{U}_e\}_{e\in\omega}, \{V_e\}_{e\in\omega}. \end{array} \quad (2.4)$$

We have the following version of the Extension Lemma for this case.

Lemma 2.5. *Let A and B be infinite r.e. sets and A_1, A_2 and B_1, B_2 form splittings of A and B respectively. Suppose that $\{U_n\}_{n\in\omega}$, $\{V_n\}_{n\in\omega}$, $\{\tilde{U}_n\}_{n\in\omega}$, $\{V_n\}_{n\in\omega}$ are recursive arrays of r.e. sets and that there is a simultaneous enumeration of a recursive array including all the above such that $A_i \searrow \tilde{V}_n = \emptyset = B_i \searrow \tilde{U}_n$, for all n and i. Furthermore suppose that for each i, $i = 1, 2$,*

$$(\forall \nu) [\nu \searrow_{\mathrm{ex}} B_i \text{ infinite} \Rightarrow (\exists \nu') [\nu \leqslant \nu' \wedge \nu' \searrow_{\mathrm{ex}} A_i \text{ infinite}]] \quad (2.5)$$

and

$$(\forall \nu) [\nu \searrow_{\mathrm{ex}} A_i \text{ infinite} \Rightarrow (\exists \nu') [\nu' \leqslant \nu \wedge \nu' \searrow_{\mathrm{ex}} B_i \text{ infinite}]]. \quad (2.6)$$

Then there are r.e. sets $\hat{U}_n$ extending $\tilde{U}_n$ and $\hat{V}_n$ extending $\tilde{V}_n$ such that (2.4) above is satisfied.

Proof. Apply Soare's Extension Lemma 2.4 to the pair A_1, B_1, in place of A, B. The extension guaranteed there meets (2.4) with respect to A_1 and B_1. Further, the proof of Soare's Extension Lemma guarantees that $\hat{U}_n - \tilde{U}_n \subseteq B_1$ and $\hat{V}_n - \tilde{V}_n \subseteq A_1$ for all n. Now, renaming the sets $\hat{U}_n$ and $\hat{V}_n$ which result from this application of Lemma 2.4 to $\tilde{U}_n$ and $\tilde{V}_n$, we see that the hypotheses of the Lemma 2.4 are now satisfied with A_2 and B_2 in place of A and B. Thus, applying the Extension Lemma again, we get that (2.4) is satisfied with respect to A_2, B_2. $\quad \square$

The application for which Lemma 2.5 was introduced in [39] is to splittings of maximal sets; this result is discussed in Section 4. The conditions of Lemma 2.5 look very similar to those of the requirements that we satisfied in the

Friedberg Splitting Theorem. In particular, the conditions ask us to do for states what was done in that proof for single r.e. sets. This fact led Downey and Stob to ask the following question.

Open Question 2.6. Suppose that A_1, A_2 and B_1, B_2 are both Friedberg splittings of A. Under what further conditions is A_1 automorphic to B_1?

Suppose that Friedberg splittings, A_1, A_2 and B_1, B_2 of A are given and we are attempting to construct an automorphism Φ such that $\Phi(A_1) = B_1$. We might try to construct Φ so that in addition, $\Phi(A) = A$. To do this, it is natural to take $U_e = W_e$, $V_e = W_e$ for all e and to attempt to enumerate, say, $\hat{U}_e$ as follows. Whenever an element x appears in $U_{e,s} - A_s$, we enumerate x into $\hat{U}_e$. Playing this strategy guarantees that (2.2) above is met. However this strategy fails to meet (2.4) if the following happens. It could be the case that $U_e \searrow A$ is infinite but that almost every element of $U_e \searrow A$ enters A_2 and B_1. Then we would fail to meet (2.4) with respect to U_e. The minimal dynamic condition that insures that this strategy works is summarized in the following definition of e-Friedberg splittings.

Definition 2.7. A splitting A_1, A_2 of A is an *e-Friedberg splitting* if for every e-state ν and each $i = 1, 2$, if $\nu \searrow_{\text{ex}} A$ is infinite then $\nu \searrow_{\text{ex}} A_i$ is infinite. (Here we assume that e-states are measured with respect to the standard enumeration of all the r.e. sets and that some fixed enumeration of A, A_1, and A_2 is given. Further we assume that if $x \in A_{s+1} - A_s$, then either $x \in A_{1,s+1} - A_s$ or $x \in A_{2,s+1} - A_s$.)

Clearly, using the same argument as that in the proof of the Friedberg Splitting Theorem, we can show that every nonrecursive r.e. set has an e-Friedberg splitting. Also, by the above remarks, it is not hard to show the following.

Theorem 2.8. *Suppose that A_1, A_2 and B_1, B_2 are e-Friedberg splittings of A. Then there is an effective automorphism Φ of $\mathcal{E}$ such that $\Phi(A_i) = B_i$ for $i = 1, 2$.*

We can actually do slightly better than Theorem 2.8. Recall that a recursive array of r.e. sets $\{X_e\}_{e \in \omega}$ is called a *skeleton* for the r.e. sets is $(\forall e)(\exists n)[W_e =^* X_n]$. The obvious generalization to skeletons of e-Friedberg splitting can be made; we call such a splitting an e*-Friedberg splitting.

Theorem 2.9 (Downey and Stob [40]). *Suppose that A_1, A_2 and B_1, B_2 are e*-Friedberg splittings of A. Then there is an effective automorphism Φ of $\mathcal{E}$ such that $\Phi(A_i) = B_i$ for $i = 1, 2$.*

Downey and Stob had originally hoped to extend the result of Theorem 2.8 to Friedberg splittings or, at least, true Friedberg splittings. The problem with this extension is easy to describe. For a single r.e. set W, we know that for true Friedberg splittings, $W \setminus A$ infinite implies that $W \setminus A_i$ infinite for each i. The corresponding result for states is not true. It may be the case that $\nu \setminus A$ is infinite while $\nu \setminus A_i$ is empty. This problem turns out to be enough of an obstacle to produce an elementary difference between Friedberg splittings of a given r.e. set. We have the following definition which is a generalization of the notion of d-simplicity due to Lerman and Soare [67].

Definition 2.10 (Downey and Stob [40]). A splitting A_1, A_2 of A is a *d-Friedberg* splitting if for every r.e. set X there is an r.e. set $Y \subseteq X$ such that

(1) $X - A = Y - A$ and
(2) for all r.e. sets W, if $W - (X \cup A)$ is not r.e., then $(W - Y) \cap A_i \neq \emptyset$, $i = 1, 2$.

Note that a d-Friedberg splitting is a Friedberg splitting by setting $X = \emptyset$. We have the following theorem.

Theorem 2.11 (Downey and Stob [40]). *There is a simple r.e. set A with true Friedberg splittings A_1, A_2 and B_1, B_2 such that A_1, A_2 is a d-Friedberg splitting and B_1, B_2 is not. Consequently, Friedberg splittings (even true Friedberg splittings) of a single r.e. set can realize different elementary types.*

Proof. We will construct the splittings A_1, A_2, and B_1, B_2 together with auxiliary sets Q, Y_e and M_e ($e \in \omega$) to meet the following requirements for every $e, i \in \omega$ and $j = 1, 2$. Let $\{X_e\}_{e \in \omega}$ be an enumeration of all the r.e. sets.

$$\mathbf{D}_e: \quad W_e \text{ infinite} \Rightarrow W_e \cap A \neq \emptyset,$$

$$\mathbf{P}_{e,j}: \quad |W_e \setminus A| = \infty \Rightarrow W_e \cap B_j \neq \emptyset,$$

$$\mathbf{R}_{e,i}: \quad (Y_e \subseteq X_e) \wedge (X_e - A = Y_e - A),$$

$$\mathbf{R}_{e,i,j}: \quad W_i - (X_e \cup A) \text{ not r.e.} \Rightarrow (W_i - Y_e) \cap A_j \neq \emptyset,$$

$$\mathbf{P}_e: \quad |Q - A| \geqslant e,$$

$$\mathbf{N}_e: \quad (W_e \nsubseteq Q) \vee ((W_e - A) \neq (Q - A))$$

$$\vee \, [(M_e - W_e) \cap B_1 \text{ is finite} \wedge (\forall i)\mathbf{N}_{e,i}]$$

where

$$\mathbf{N}_{e,i}: \quad |M_e - (Q \cup A)| \geqslant i.$$

The requirements $\mathbf{P}'_{e,j}$ guarantee that the splitting B_1, B_2 is a true Friedberg splitting; $\mathbf{R}_{e,i}$ and $\mathbf{R}_{e,i,j}$ for $i, j \in \omega$ ensure that A_1, A_2 is a d-Friedberg splitting of A. Together, requirements $\mathbf{P}_e$ and $\mathbf{N}_e$ guarantee that B_1, B_2 is not a d-Friedberg splitting of A. Q here is supposedly the witness for a set X for which the witness Y in the definition of d-Friedberg splitting cannot be found.

We briefly describe the strategies for the basic requirements and the conflicts between them. The strategy for $\mathbf{R}_{e,i,j}$ is the following. While $(W_{i,s} - Y_{e,s}) \cap A_{j,s} = \emptyset$, we wait till we see $z \in W_{i,s} - (X_{e,s} \cup A_s)$. We can then put z in A_j and meet $\mathbf{R}_{e,i,j}$ forever. Note that if no such z exists, then $|W_e - (X_e \cup A_j)| < \infty$.

For the requirements $\mathbf{P}_{e,j}$, we do the following. If we see $z \in W_{e,s} - A_s$ such that $z \geqslant \langle e, j \rangle$ (this reflects the priority restraint we impose on z) and $W_{e,s} \cap B_{j,s} = \emptyset$, then we enumerate z into B_j.

For the requirement $\mathbf{P}_e$, we enumerate e numbers into Q and restrain them from enumeration into A with the priority of $\mathbf{P}_e$.

For the requirement $\mathbf{N}_e$, we actually attempt to meet each $\mathbf{N}_{e,i}$. We do this enumerating an element into M_e and restraining it from both Q and A. We will argue that if one of the requirements $\mathbf{N}_{e,i}$ fails, then $W_e \not\subseteq Q$ or $(W_e - A) \neq (Q - A)$ and so that requirement $\mathbf{N}_e$ is met.

The conflicts among the strategies are as follows. First, there are no conflicts between $\mathbf{P}_{e,j}$ and either $\mathbf{N}_e$ or any $\mathbf{R}_{e,i,j}$ since all these requirements only wish to enumerate numbers into A_j. Requirements $\mathbf{N}_e$ and $\mathbf{P}_{e,1}$ conflict however for $\mathbf{P}_{e,1}$ wishes to enumerate z into B_1 while some $\mathbf{N}_{e,i}$ may have enumerated z in M_e and therefore restrained z from A (and hence restrained z from B_1). Assume that $\mathbf{P}_{e,1}$ has higher priority than this $\mathbf{N}_{e,i}$ but lower "global" priority than $\mathbf{N}_e$ (this is the only ordering of priorities that gives us serious difficulty), we overcome this conflict by "squeezing" $\mathbf{N}_e$ in the following way. Note first that we can put z in B_1 provided that $z \in W_e$. Thus we try to force z into W_e. Now if $W_e - A \neq Q - A$, we get a global win on requirement $\mathbf{N}_e$. Thus, the idea is to put z into Q first and wait until z enters W_e before we put z into A. If z enters W_e, we are free to put z into B_1 and we win $\mathbf{P}_{e,1}$ without injuring $\mathbf{N}_e$. If z does not enter W_e we win $\mathbf{N}_e$ and then can win $\mathbf{P}'_{e,i,1}$ with a new witness without interference from $\mathbf{N}_e$.

All other conflicts are essentially finite injury and are adjudicated in the standard fashion. With the above-described conflict, this argument is a $\mathbf{0}''$-argument. $\square$

Downey and Stob go on to show in [40], that even being a d-Friedberg splitting is not enough to guarantee that splittings are automorphic. The property that they use to distinguish d-Friedberg splittings is called the inner splitting property. It is analogous to the splitting property of Maass, Shore, and Stob [74] (which distinguishes among d-simple sets).

Definition 2.12. A splitting A_1, A_2 of A is an *inner splitting* of A if for every

r.e. B, if $B - A$ is not r.e. then there are Friedberg splittings C_1, C_2 and D_1, D_2 of B such that $C_1 \subseteq A_1$ and $D_2 \subseteq A_2$.

It is shown in [40] that

Theorem 2.13. *Any inner splitting of A is a d-Friedberg splitting of A.*

and

Theorem 2.14. *There is an r.e. set A (of low promptly simple degree) and d-Friedberg splittings A_1, A_2 and B_1, B_2 of A such that A_1, A_2 is an inner splitting but B_1, B_2 is not.*

The above results suggest that it might be very hard to find a property of a set A which guarantees that all splittings with that property are automorphic. One of the reasons for studying automorphisms of splittings in [40] was that it gave a new approach to a number of interesting open questions about automorphisms. One important and interesting such question is the "fat orbit" problem. Downey and Stob hoped to show that there is an r.e. set A whose orbit contains sets of every r.e. Turing degree. The plan was to show that some class of splittings both formed and orbit and contained sets of every degree. The fat orbit problem was recently settled negatively by Harrington.

Theorem 2.15 (Harrington). *There is no r.e. set A such that the orbit of A contains sets of every r.e. Turing degree.*

Proof (sketch). We say set A has property S if

$$(\exists C)(\forall X \subseteq C)[(C \subseteq (A \cup X)) \Rightarrow (\exists B \subseteq X)[X \subseteq (A \cup B)$$

$$\Rightarrow (\forall Y)[\omega \subseteq A \cup Y \Rightarrow (X \cap Y) - B \neq^* \emptyset]]].$$

Harrington shows that

Lemma 2.16. *There is a r.e. Turing degree $\mathbf{a}$ such that all r.e. sets A of degree $\mathbf{a}$ have property S.*

and

Lemma 2.17. *There is a nonrecursive r.e. set B such that no set $A \leq_T B$ has property S.*

The theorem follows immediately from the two lemmas. $\square$

Though the Downey–Stob approach to the fat orbit problem did not solve it, it still yields some interesting and instructive results. For example we show in Section 4 that the hemimaximal sets form an orbit that realizes all possible jumps (see Corollary 4.9). The first approach to the problem in [40] was to construct an r.e. set A such that A has e-Friedberg splittings of every non-zero r.e. degree. The next theorem shows that this program fails.

Theorem 2.18. *Suppose that A is a nonrecursive r.e. set. Then there is an r.e. degree $\mathbf{b} \leqslant \deg(A)$ such that $\mathbf{b} \neq \mathbf{0}$ and for all e-Friedberg splittings A_1, A_2 of A, $\deg(A_1) \not\leqslant_T \mathbf{b}$.*

Proof (sketch). We construct B in stages to meet the following requirements for every $e \in \omega$. Let $\langle \Phi_e, U_e, V_e \rangle_{e \in \omega}$ be an enumeration of all triples consisting of a functional and a pair of disjoint r.e. sets.

$\mathbf{N}_e$: $(\Phi_e(B) \neq U_e) \vee (U_e \cup V_e \neq A) \vee (\exists \nu)[|\nu \searrow_{\text{ex}} A| = \infty \wedge \nu \searrow_{\text{ex}} U_e =^* \emptyset]$,

$\mathbf{P}_e$: $\overline{B} \neq W_e$.

To meet $\mathbf{N}_e$, we will also enumerate sets X_e, Y_e. We will assume that we know indices for X_e and Y_e by the Recursion Theorem; $X_e = W_{h(e)}$ and $Y_e = W_{k(e)}$ for every e. We will also assume that $h(e) > k(e)$. The state ν_e which will witness that $\mathbf{N}_e$ is satisfied will be a state such that $\nu_e(h(e)) = 1$ and $\nu_e(k(e)) = 0$.

Let

$$l(e,s) = \max\{x \mid (\forall y < x)[\Phi_{e,s}(B_s,y) = U_{e,s}(y) \\ \wedge U_{e,s} \cup V_{e,s}(y) = A_s(y)]\}$$

and define

$$m(e,s) = \max\{l(e,t) \mid t < s\}.$$

The basic strategy for $\mathbf{N}_e$ is as follows. We enumerate any x such that $x < l(e,s)$ into X_e. If we preserve B_s through the use of the computations establishing $l(e,s)$, we will have that if x later enters A, x must enter V_e rather than U_e. Thus infinitely many such x will witness the existence of the desired state ν_e.

The easiest way to meet $\mathbf{N}_e$ subject to the conflicts with the requirements $\mathbf{P}_e$ is to have infinitely many subrequirements $\mathbf{N}_{e,i}$ each of which attempts to insure that one element of the appropriate state is enumerated into A but not U_e. Then the only difficulty in meeting the requirements is the conflict of a positive requirement $\mathbf{P}_k$ of lower priority than $\mathbf{N}_e$ but of higher priority than $\mathbf{N}_{e,i}$.

In this case, we might have some follower of $\mathbf{P}_k$ which we desire to enumerate into B (because it is permitted by A). Obviously, this would cause us to lose

the control over $N_{e,i}$ that the B restraint affords. The solution is to then raise the state of the potentially injurious elements by enumerating them into Y_e thus raising their $h(e)$-state to one other than ν_e.

Combining these strategies is now finite injury. $\square$

For Friedberg splittings, the answer is different. We have the following.

Theorem 2.19 (Downey and Stob [40]). *There is an r.e. set A such that*
(1) *for all nonrecursive sets B, there is a true Friedberg splitting A_1, A_2 of A such that $A_1 \equiv_T B$ and*
(2) *for all promptly simple r.e. sets B, there is an e-Friedberg splitting A_1, A_2 of A such that $A_1 \equiv_T B$.*

Proof. To meet (1), we construct A and sets C_e, D_e for $e \in \omega$ to meet the following requirements.

$$\mathbf{R}_e: \quad W_e \equiv_T \emptyset \vee (C_e \cup D_e = A \wedge C_e \equiv_T W_e \wedge (\forall i)[N_{e,i} \wedge N'_{e,i}]),$$

$$\mathbf{N}_{e,i}: \quad |W_i \setminus A| = \infty \Rightarrow |W_i \setminus C_e| \neq \emptyset,$$

$$\mathbf{N}'_{e,i}: \quad |W_i \setminus A| = \infty \Rightarrow |W_i \setminus D_e| \neq \emptyset.$$

The strategy we use for coding W_e into C_e is this. If $x \in W_e$, we enumerate $\langle e+1, x, z \rangle$ into C_e (and hence A) for some $z \leq 2(e+x+1)$. Thus $W_e \leq_T C_e$. We will ensure that $C_e \leq_T W_e$ by permitting. That is, we only allow y to enter C_e at stage $s+1$ in case some element $x \in W_{e,s+1} - W_{e,s}$.

To meet $\mathbf{N}_{e,i}$, $(\mathbf{N}'_{e,i})$ we will choose infinitely many followers $y_j \in W_i$ targetted for C_e (D_e) in such a way that these requirements together leave some elements untargetted to be coding markers. In particular, let $B_{j,k} = \{y \mid y = \langle j+1, k, z \rangle, z \leq 2(j+k+1)\}$. Then $\mathbf{N}_{e,i}$ and $\mathbf{N}'_{e,i}$ may choose no followers from the sets $B_{j,k}$ such that $j, k < \langle e, i \rangle$ and at most one element from each of the other sets $B_{j,k}$. Any such follower can be enumerated into C_e (D_e) at any stage such that W_e permits. It is easy to see that this is a finite injury argument.

We omit the details of modifying the above construction to meet (2). $\square$

We end this section with some results on Friedberg splittings of creative sets. We still do not know if any two splittings of a creative set are automorphic; the following results might be taken as evidence for this.

Definition 2.20. An r.e. set A is *f-creative* if there is a creative set K and an r.e. set B such that A, B is a Friedberg splitting of K.

We first remark that no creative set is f-creative. In fact, we have the stronger theorem.

Theorem 2.21. *Suppose that A is creative. Then A is not half of a Friedberg splitting of any r.e. set.*

Proof. By Theorem 1.5, we know that any half of a Friedberg splitting is a nowhere simple set. But Shore has observed that no creative set is nowhere simple. □

We will show below that the Turing degrees of f-creative sets are constrained to be promptly simple degrees. This result follows from the next two theorems.

Theorem 2.22. *There is an r.e. set C such that if A, B is a Friedberg splitting of C then A has promptly simple degree.*

Proof. Let (A_e, B_e), $e \in \omega$, be an effective listing of pairs of disjoint r.e. sets. (We will need to know recursive functions f and g such that $A_e = W_{f(e)}$ and $B_e = W_{g(e)}$ but that is easy to arrange.) We construct C to meet the following requirements.

$$\mathbf{R}_e: \quad A_e, B_e \text{ is not a Friedberg splitting of } C,$$
$$\text{or } A_e \text{ has promptly simple degree.}$$

To meet the requirements $\mathbf{R}_e$, we will construct sets $V_{e,i}$ to meet the following requirements.

$$\mathbf{R}_{e,i,j}: \quad W_i \text{ infinite} \Rightarrow W_j \neq V_{e,i} - C.$$

Our construction will meet all the requirements $\mathbf{R}_{e,i,j}$. The construction will guarantee that if we meet all requirements $\mathbf{R}_{e,i,j}$ and if A_e, B_e is a splitting of C, then either for some i such that W_i is infinite, $V_{e,i}$ witnesses that A_e, B_e is not a Friedberg splitting of C or A_e has promptly simple degree. The construction is very simple. We let $V_{e,i} = \{\langle e, i, z \rangle \mid z \in \omega\}$.

Construction

Stage s

We say that requirement $\mathbf{R}_{e,i,j}$ needs attention if it has not previously received attention and there is an x such that $x \in W_{i, \text{at } s}$, $x \geq \langle e, i, j \rangle$, and $\langle e, i, j \rangle \in W_{j,s} - C_s$. If some requirement needs attention, find the highest priority such requirement $\mathbf{R}_{e,i,j}$ and enumerate $\langle e, i, j \rangle$ in C.

It is easy to see that each requirement $\mathbf{R}_{e,i,j}$ receives attention at most once and is satisfied. We now show that this implies that $\mathbf{R}_e$ is satisfied. Suppose that A_e, B_e is a splitting of C. If no requirement of form $\mathbf{R}_{e,i,j}$ for any i, j receives attention at stage s, define $p_e(s) = p_e(s-1)$. Otherwise, some number $y = \langle e, i, j \rangle$ is enumerated in C at stage s. Let t be the least stage such that

$y \in A_{e,t} \cup B_{e,t}$. Define $p_e(s) = t$. (We may assume that $t \geqslant s$ by enumerating A_e, B_e so that $A_e, B_e \subseteq C_e$.)

Now we show that either A_e is of promptly simple degree with witness p_e or that A_e, B_e is not a Friedberg splitting of C. The former means that

$$W_i \text{ infinite} \Rightarrow (\exists s)(\exists x)[x \in W_{i,\mathrm{at}\,s} \text{ and } A_e \text{ permits } x \text{ by stage } p_e(s)]. \quad (2.7)$$

Now the requirements $\mathbf{R}_{e,i,j}$ are all met. We thus consider two cases. First, suppose there is i such that W_i is infinite and $V_{e,i} \cap A_e = \emptyset$. Then $V_e - C$ is not r.e. by the requirements $\mathbf{R}_{e,i,j}$ but $V_{e,i} - A_e = V_{e,i}$ is r.e. and so $V_{e,i}$ witnesses that A_e and B_e is not a Friedberg splitting of C. Thus we may suppose that for every i such that W_i is infinite, $V_{e,i} \cap A_e \neq \emptyset$. Consider any $y = \langle e, i, j \rangle \in V_{e,i} \cap A_e$. At the stage s such that y was enumerated in C, we have that there is $x \geqslant y$ such that $x \in W_{i,\mathrm{at}\,s}$. Since y must appear in A_e by stage $p_e(s)$ (see the above definition of p_e), we have that Definition 2.7 is satisfied for every i. $\square$

The next theorem is very easy; we omit the proof.

Theorem 2.23. *Suppose that $D = A \oplus B$ and that D_1, D_2 is a Friedberg splitting of C. Then $D_1 = A_1 \oplus B_1$ and $D_2 = A_2 \oplus B_2$ such that A_1, A_2 form a Friedberg splitting of A.*

Theorem 2.24. *If A is f-creative, A has promptly simple degree.*

Proof. Since all creative sets are recursively isomorphic, we may choose the one we wish to work with. Let C be the set constructed in Theorem 2.22. Then $C \oplus K$ is creative. Let A, B be a Friedberg splitting of $C \oplus K$. Then by Theorem 2.23, $A = C_1 \oplus K_1$ such that C_1 is half of a Friedberg splitting of C. Therefore C_1 has promptly simple degree and so A has promptly simple degree. $\square$

We have sort of a converse to Theorem 2.24. We omit the proof.

Theorem 2.25. *Suppose that $\mathbf{a}$ is a promptly simple degree. Then there is a set A of degree $\mathbf{a}$ which is f-creative. (In fact, we can make A half of an e-Friedberg splitting of a creative set.)*

Corollary 2.26. *There is an orbit in $\mathcal{E}^*$ consisting of sets of precisely the promptly simple degrees.*

Proof. Let $\mathcal{O}$ be the orbit generated by the e-Friedberg splittings of creative sets. Then $\mathcal{O}$ contains sets of every promptly simple degree by the theorem.

However all sets in $\mathcal{O}$ are f-creative (since f-creativity is elementarily definable) and so promptly simple by Theorem 2.24.   □

We do not know (but we doubt) whether the property of being an e-Friedberg splitting of a creative set is elementarily definable. Related to the above result then is the following question.

Open Question 2.27. Does the orbit of every f-creative set A contain sets of all promptly simple degrees?

Related to this question of course is still our original one — do the f-creative sets form an orbit? We do know that we cannot distinguish among f-creative sets by using d-Friedberg splittings. The next three results show this. We omit the proofs.

Theorem 2.28. *There is an r.e. set C such that no splitting of C is a d-Friedberg splitting.*

Theorem 2.29. *Suppose that $D = A \oplus B$ and that D_1, D_2 is a d-Friedberg splitting of C. Then $D_1 = A_1 \oplus B_1$ and $D_2 = A_2 \oplus B_2$ such that A_1, A_2 form a d-Friedberg splitting of A.*

Theorem 2.30. *No creative set has a d-Friedberg splitting.*

3. Orbits, the splitting property, etc.

Already we have seen many examples of lattice-theoretic properties of r.e. sets which are closely related to the possible degrees of those sets. A natural notion to study is *invariant degree classes*, that is, degree classes generated by orbits in $\mathrm{Aut}(\mathcal{E}^*)$. The archetypical example is that the high r.e. degrees form an invariant class. This follows from Soare's result that the maximal sets form an orbit and Martin's result [77] that the high degrees are precisely the degrees of the maximal sets. In the next section, we look at another invariant class, the degrees of hemimaximal sets, but in this section we begin by summarizing the results of Maass, Shore and Stob [74] who showed that there are invariant classes that split all jump classes. The promptly simple sets played a crucial role in these results.

Definition 3.1. An r.e. set A is *promptly simple* if there is a recursive function p such that for every e, if W_e is infinite then there is x such that $x \in W_{e,\mathrm{at}\,s} \cap A_{p(s)}$.

Thus for a promptly set A, the witnesses to the simplicity of A enter A promptly. The degrees of promptly simple sets form an important degree class as witnessed by the following theorems.

Theorem 3.2 (Maass [72]). *If A and B are promptly simple and $\overline{A}$ and $\overline{B}$ are semilow, there is an automorphism Φ of $\mathcal{E}$ such that $\Phi(A) = B$.*

Theorem 3.3 (Maass, Shore and Stob [74]). *The promptly simple sets together with the cofinite sets form a filter in $\mathcal{E}^*$.*

Theorem 3.4 (Ambos-Spies, Jockusch, Shore and Soare [8]). *Let $\mathbf{P}$ denote the class of degrees of promptly simple sets and let $\mathbf{C}$ denote the class of cappable degrees (i.e., those degrees which are halves of minimal pairs). Then $\mathbf{P}$ and $\mathbf{C}$ form a decomposition of $\mathbf{R}$ into a strong filter and an ideal respectively.*

A result in the direction of establishing the existence of a large invariant class is the following.

Theorem 3.5 (Cholak, Downey and Stob [14]). *If A is a promptly simple r.e. set, then A is effectively automorphic to a complete set.*

Theorem 3.5 has recently been extended, quite considerably, by Harrington and Soare to the following.

Theorem 3.6 (Harrington and Soare [48]). *If A is an r.e. set of promptly simple r.e. degree, then A is effectively automorphic to a complete set.*

The property of being promptly simple is not invariant under automorphisms however. This follows from the fact that there maximal sets which are promptly simple and maximal sets which aren't. Maass, Shore and Stob were therefore led to the following property which is closely related to prompt simplicity but which is invariant under automorphisms.

Definition 3.7 (Maass, Shore and Stob [74]). An r.e. set A has the *splitting property* if for every nonrecursive r.e. set B there is a Friedberg splitting B_1, B_2 of B such that $B_1 \subseteq A$.

One relationship between prompt simplicity and the splitting property is the following.

Theorem 3.8 (Maass, Shore and Stob [74]). *Every promptly simple set has the splitting property.*

Proof. Suppose that A is promptly simple (with witness p). Given B we need to construct a Friedberg splitting B_1, B_2 of B such that in addition $B_1 \subseteq A$. As in the proof of Friedberg's Theorem, we must meet requirements such as

$$\mathbf{R}_{\langle e,i \rangle} : \quad |W_e \setminus B| = \infty \Rightarrow W_e \cap B_i \neq \emptyset.$$

Of course it is the requirements of form $\mathbf{R}_{\langle e,1 \rangle}$ that are difficult since if x occurs in $W_{e,s} \setminus B_s$ we may not freely be able to enumerate such an x into B_1. The solution is to use prompt simplicity in the following way. The set of all such x is an infinite r.e. set in the case that we are interested in. Using p, we know that some element of this infinite r.e. set must enter A promptly. That is our desired witness. □

Theorem 3.9 (Maass, Shore, and Stob [74]). *The collection of sets with the splitting property forms a filter in $\mathcal{E}^*$.*

Proof. It is immediate from the definition that this collection is upwards closed. Suppose that C and D have the splitting property. To verify that $A = C \cap D$ has the splitting property, suppose that B is a nonrecursive set. Let B_1, B_2 be a Friedberg splitting of B with the property that $B_1 \subseteq C$. Now let B_{11}, B_{12} be a Friedberg splitting of B_1 such that $B_{11} \subseteq D$. Then $B_{11} \subseteq A$ and it is easy to verify that B_{11}, $B_{12} \cup B_2$ is a Friedberg splitting of B. □

The splitting property is related to the property of d-simplicity introduced by Lerman and Soare [67].

Definition 3.10. An r.e. set A is *d-simple* if for every r.e. set X there is an r.e. set $Y \subseteq X$ such that
(1) $X \cap \overline{A} = Y \cap \overline{B}$,
(2) for all r.e. sets W, if $W - X$ is infinite then $(W - Y) \cap A$ is infinite.

The connection is the following.

Theorem 3.11 (Maass, Shore, and Stob [74]). *If A is a coinfinite r.e. set with the splitting property, then A is d-simple.*

Lerman and Soare introduced the d-simple sets as the first example of a definable class of r.e. sets the degrees of which split some jump class. In particular there are low, nonrecursive r.e. sets which have the property, and low, nonrecursive r.e. sets which don't. The property of d-simple sets arose from studying various blockages in extending the automorphism machinery to other cases. Since it is not the case that any two d-simple sets are automorphic (see [74]), the following question remains open.

Open Question 3.12. Find a property that characterizes the orbit of some low, simple r.e. set.

Theorem 3.13. *Every hyperhypersimple set has the splitting property.*

Proof. Suppose that A fails to have the splitting property and that B is a witness to that. Let B_1, B_2 be a Friedberg splitting of B. Then we have that $B_1 \cap \overline{A} \neq \emptyset$ and $B_2 \cap \overline{A} \neq \emptyset$. Now produce B_{11}, B_{12}, a Friedberg splitting of B_1. Again $B_{11} \cap \overline{A} \neq \emptyset$ and $B_{12} \cap \overline{A} \neq \emptyset$. We continue in this manner (splitting now B_{11}) to produce a weak array that witnesses that A is not hhsimple. $\square$

The key result concerning the relationship of the splitting property to prompt simplicity is the following.

Theorem 3.14 (Maass, Shore, and Stob [74]). *If A is an r.e. set which is cofinite, nonhyperhypersimple, and has the splitting property, then A has promptly simple degree.*

The proof of Theorem 3.14 uses the following lemma of Lachlan.

Lemma 3.15 (Lachlan [61]). *If A is not hhsimple and C is any r.e. set, there is an r.e. set B such that*
(1) $A \cap B \leqslant_T A$,
(2) $B \leqslant_T C$,
(3) *if B is recursive, then $C \leqslant_T A$.*

Proof. Let $\{U_e\}_{e \in \omega}$ be a weak array witnessing that A is not hhsimple and such that $U_e \cap \{0, 1, \ldots, e\} = \emptyset$. Suppose an enumeration of all the sets A, C, and U_e, $e \in \omega$ is given. Define B by

$$B = \{x \mid (\exists e)(\exists s)[x \in U_{e,s} - A_s \wedge e \in C_{s+1} - C_s]\}.$$

It is easy to see that B has the desired properties. $\square$

Proof of Theorem 3.14. Suppose that A is not hhsimple, and has the splitting property. We need to show that if C is a nonrecursive r.e. set, then A and C do not form a minimal pair. We may suppose that $C \not\leqslant_T A$ since otherwise the result is obvious. Let B be as in Lemma 3.15. Since B is nonrecursive, there is a Friedberg splitting B_1, B_2 of B such that $B_1 \subseteq A$. We claim that B_1 witnesses that A and C do not form a minimal pair. First, $B_1 \leqslant_T C$ since $B_1 \leqslant_T B$. Second, $B_1 \leqslant_T A$ since $B_1 \leqslant_T B \cap A$ and $B \cap A \leqslant_T A$ (by 3.15(3)). $\square$

Maass, Shore, and Stob observed that the proof of Theorem 3.14 only required a weaker property which they called the weak splitting property.

Definition 3.16. An r.e. set A has the *weak splitting property* if for every r.e. set B there is a splitting B_1, B_2 of B such that $B_1 \subseteq B$ and B nonrecursive implies that B_1 is nonrecursive.

Maass, Shore and Stob left open the question of whether the weak splitting property was actually weaker than the splitting property.

Theorem 3.17. *There is an r.e. set with the weak splitting property that does not have the splitting property.*

Proof (sketch). We construct such a set A and sets B, $\{V_e\}_{e \in \omega}$, $\{X_e\}_{e \in \omega}$, $\{Y_e\}_{e \in \omega}$, to meet the requirements $\mathbf{R}_e$ and $\mathbf{Q}_e$ below. Let $(C_e, D_e)_{e \in \omega}$ be an enumeration of all pairs of disjoint r.e. sets.

$\mathbf{R}_e$: W_e not recursive $\Rightarrow W_e = X_e \cup Y_e \wedge X_e \subseteq A \wedge (\forall i) \mathbf{R}_{e,i}$,

$\mathbf{R}_{e,i}$: $\overline{X}_e \neq W_i$,

$\mathbf{Q}_e$: $(B = C_e \cup D_e \wedge C_e \subseteq A) \Rightarrow (V_e - C_e$ is r.e. $\wedge \ (\forall i) \mathbf{Q}_{e,i})$ where

$\mathbf{Q}_{e,i}$: $V_e - B \neq W_i$.

Obviously the requirements $\mathbf{R}_e$ guarantee that A has the weak splitting property and the requirements $\mathbf{Q}_e$ guarantee that B witnesses that A does not have the splitting property (with V_e witnessing that C_e, D_e is not the appropriate splitting of B). The requirements above have actions which conflict so severely as to require the $\mathbf{0}'''$ priority method. To meet $\mathbf{R}_e$, we must insure that X_e, Y_e split W_e and also try to meet each requirement $\mathbf{R}_{e,i}$. For $\mathbf{R}_{e,i}$, let $l(e,i,s) = \max\{x \mid (\forall y < x)[\overline{X}_{e,s}(y) = W_{i,s}(y)]\}$. We wish to enumerate some z in W_i into X_e. The idea is to wait until some $z \le l(e,i,s)$ occurs in $W_{i,s}$ and enumerate this z into X_e. If no such z occurs, then W_e is recursive.

To meet the requirements $\mathbf{Q}_e$, we enumerate B and V_e. The most difficult thing to achieve is making $V_e - C_e$ r.e. The action is as follows. We wish to make $V_e - B \neq W_i$. To do this we first enumerate some x into $V_e - B$. If x later occurs in W_i, we could succeed by enumerating x into B. However, to ensure that $V_e - C_e$ is r.e., we keep x out of A until x enters C_e or D_e. In case that x enters C_e, we succeed by keeping x out of A (this is a global win for the requirement $\mathbf{Q}_e$). If x enters D_e, we can release x to allow x to enter A (if we wish) since x can no longer enter C_e.

The important conflict between these two basic strategies is as follows. There might be an x which we add to B for the sake of $\mathbf{Q}_f$ that we must keep out of A for the reasons described above. While we are waiting for x to enter D_f, x enters W_e. Now we need to enumerate x into either X_e or Y_e for the sake of requirement $\mathbf{R}_e$. If we enumerate x in X_e, we must also enumerate it into A.

It is at this point that we assume some familiarity with the $\mathbf{0}'''$ method. Suppose first the $\mathbf{Q}_f$ has higher global priority that $\mathbf{R}_e$ ($f < e$). Then we can delay the decision (X_e or Y_e) until x enters D_f. Of course in the ensuing time, we begin a new version of $\mathbf{R}_e$ predicated on the guess that $B \neq C_f \cup D_f$ (via x). This is finite injury (along the true path) and can be handled by standard techniques.

The other case, $e < f$, is more difficult. We focus on the subrequirements $\mathbf{Q}_{f,i}$ and $\mathbf{R}_{e,j}$. If the local priority of $\mathbf{Q}_{f,i}$ is higher than that of $\mathbf{R}_{e,j}$, we allow $\mathbf{Q}_{f,i}$ to injure $\mathbf{R}_{e,j}$ by enumerating x into Y_e. Otherwise, the idea is to use a different version of V_f to meet the requirements. If $\mathbf{R}_e$ acts infinitely often, we know then that Y_e is infinite. Therefore we allow $\mathbf{Q}_{f,k}$ to pick followers only from those integers already in Y_e ($-A$). In this way, $\mathbf{R}_e$ cannot injure this version of V_f.

Actually, although this argument uses a $\mathbf{0}'''$ strategy, it can be done as a $\mathbf{0}''$ argument as an analysis of the outcomes of the strategies will reveal. $\quad\square$

We close this section with a discussion of one more splitting property closed related to automorphisms of $\mathcal{E}^*$, the outer splitting property of Maass. In [73], Maass proved the following.

Theorem 3.18 (Maass [73]). *Let A be an r.e. set. Then $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$ if and only if $\overline{A}$ is semilow$_{1.5}$.*

This result improved Soare's earlier result that if $\overline{A}$ is semilow, $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$. The reverse direction used a modification of the Soare machinery via a splitting property as we shall see. That proof runs as follows. As in the automorphism machinery described in Section 2, we suppose that we are given arrays $\{U_e\}_{e\in\omega}$, $\{V_e\}_{e\in\omega}$ of all the r.e. sets and we wish to construct array $\{\hat{U}_e\}_{e\in\omega}$, $\{\hat{V}_e\}_{e\in\omega}$, so that the map Φ defined by $\Phi(U_e) = \hat{U}_e$, $\Phi^{-1}(V_e) = \hat{V}_e$ induces an isomorphism of $\mathcal{L}^*(A)$ onto $\mathcal{E}^*$. To insure that the isomorphism is effective, we take $U_e = V_e = W_e$ for every e. The full e-state requirement that we need to meet is the following.

$$
\begin{array}{c}
\text{for each full } e\text{-state } \nu, \\
\text{infinitely many elements of } \overline{A} \text{ have } e\text{-state } \nu \text{ w.r.t. } \{U_e\}_{e\in\omega}, \{\hat{V}_e\}_{e\in\omega} \\
\text{iff} \\
\text{infinitely many elements of } \omega \text{ have } e\text{-state } \nu \text{ w.r.t. } \{\hat{U}_e\}_{e\in\omega}, \{V_e\}_{e\in\omega}.
\end{array} \quad (3.1)
$$

As in the description of the Extension Lemma (Lemma 2.4) we need only achieve covering and the technique of the proof of the Extension Lemma will do the rest. So we concern ourselves with not "going out of covering".

What is the difficulty? Consider one set U_e. We need to make sure that if $U_e - A$ is infinite, then $\hat{U}_e$ is infinite. The problem is that we may see infinitely many x such that there is a stage s for which $x \in U_{e,s} - A_s$ but yet such that

$U_e - A = \emptyset$. We don't want to respond to such x by enumerating elements in $\hat{U}_e$. Similarly, if V_e is infinite, we need to find elements of $\overline{A}$ to enumerate into $\hat{V}_e$. And of course these conflicts get more severe when full e-states are taken into account. It is here that we use the fact that $\overline{A}$ is semilow$_{1.5}$ for this gives a procedure to test whether certain r.e. sets intersect $\overline{A}$ nontrivially. Specifically, there is a recursive function h such that for all j, $W_j \cap A$ is infinite if and only if $W_{h(j)}$ is infinite. Therefore we use h (and the Recursion Theorem) to test whether certain events happen infinitely often. This allows us to raise the states of elements safely and not get out of covering in this way. However there is another problem. Consider again the problem of one set V_e. Should it be the case that V_e is infinite, we will enumerate infinitely many elements of $\overline{A}$ into $\hat{V}_e$. However we must be careful so as to not enumerate all of the elements of $\overline{A}$ into $\hat{V}_e$ (unless $\hat{V}_e = \omega$). Thus we must be able to seek out "true" elements of $\overline{A}$ to protect them from enumeration into states ν. Crucial in Maass' solution to this problem was the following property.

Definition 3.19 (Maass [73]). An r.e. set A has the *outer splitting property* (o.s.p.) if there are recursive functions f and g such that
(1) $W_{f(e)}, W_{g(e)}$ is a splitting of W_e,
(2) $W_{f(e)} \cap \overline{A}$ is finite,
(3) If $W_e \cap \overline{A}$ is infinite then $W_{f(e)} \cap \overline{A} \neq \emptyset$.

Maass showed the following.

Theorem 3.20 (Maass [73]). *Every r.e. set which is semilow$_{1.5}$ has the outer splitting property.*

Maass used the outer splitting property as follows. For each condition for which we want to impose such restraint as described above, we enumerate an r.e. set, say W_e. Then we actually restrain only those elements in $W_{f(e)}$. If we attack such a requirement infinitely often we are assured of finding an element to restrain ((3) above) but of imposing only finitely much restraint to find that element ((2) above). We give now the (easy) proof to Theorem 3.20.

Proof of Theorem 3.20. Let A be semilow$_{1.5}$ with h witnessing that; $W_e \cap \overline{A}$ is infinite if and only if $W_{h(e)}$ is infinite. We will simultaneously enumerate the sets $W_{f(e)}$ and $W_{g(e)}$ using the recursion theorem. Since A is r.e. we can also fix a recursive function k such that $W_e \cap \overline{A} = \emptyset$ if and only if $W_{k(e)}$ is infinite. The construction is this. At stage $s + 1$ for any x and e such that $x \in W_{e,s+1}$, we enumerate x into $W_{f(e)}$ if $|W_{k(f(e)),s}| \geq |W_{h(f(e)),s}|$ and enumerate x into $W_{g(e)}$ otherwise. It is easy to see that $W_{f(e)}, W_{g(e)}$ is a splitting of W_e for any e; i.e., that the recursion theorem was used appropriately. Suppose that $W_{f(e)} \cap \overline{A}$ is infinite. Then $W_{h(f(e))}$ is infinite and $W_{k(f(e))}$ is finite. By the

construction, this implies $W_{f(e)} \cap \overline{A}$ is actually finite. Similarly, one can show that $W_{f(e)} \cap \overline{A} = \emptyset$ implies that $W_e \cap \overline{A}$ is finite. $\square$

Maass observed that there are sets with the outer splitting property that are not semilow$_{1.5}$ and Cholak [13] observed that such sets could be semilow$_2$. This is relevant to the following recent result of Cholak.

Theorem 3.21 (Cholak [13]). *if A is r.e. with the outer splitting property and $\overline{A}$ is semilow$_2$, then $\mathcal{L}^*(A)$ is isomorphic to $\mathcal{E}^*$.*

Of course the isomorphism that Cholak produces in Theorem 3.21 cannot be effective by Maass's Theorem. Cholak's result is proved by "putting the Extension Lemma on a tree". This method, first used by Harrington, was also used by Harrington, Lachlan, Maass, and Soare to prove the following extension of Maass's result.

Theorem 3.22 (Harrington, Lachlan, Maass and Soare [47]). *If A is low$_2$, then $\mathcal{L}^*(A)$ is isomorphic to $\mathcal{E}^*$.*

Theorem 3.22 is the best result possible in the sense of jump classes since Shoenfield showed [89] that every nonlow$_2$ degree contains a set with no maximal superset; this is a property of $\mathcal{L}^*(A)$ not enjoyed by $\mathcal{E}^*$.

4. Hemiproperties

In [39], Downey and Stob proposed a program for finding orbits of $\mathrm{Aut}(\mathcal{E}^*)$ through splittings. The usefulness of this program was suggested by Theorem 4.2 below.

Definition 4.1. Suppose that P is a property of r.e. sets. An r.e. set A is *hemi-P* if there are r.e. sets B and C such that A, B is a nontrivial splitting of C and C has property P.

Thus, in particular, a set A is hemimaximal if A is half of a nontrivial splitting of a maximal set. Soare's result, that the maximal sets form an orbit in $\mathcal{E}^*$, suggested the following result.

Theorem 4.2. *The collection of hemimaximal sets form an orbit under $\mathrm{Aut}(\mathcal{E}^*)$.*

The proof of Theorem 4.2 in [39] had two components. The first consisted of the version of the Extension Lemma for splittings stated as Theorem 2.5. The second component of the proof was to show that the hypotheses of

this Lemma could be satisfied. For this purpose, we extended Soare's "Order Preserving Enumeration Theorem" to this context. (See [39, Lemma 4]). Independently, Herrmann found the following proof of Theorem 4.2 which uses more information from Soare's original argument for maximal sets.

Proof of Theorem 4.2 (Herrmann). For every r.e. set A, let

$$\mathcal{C}(A) = \{W \mid \overline{A} \subseteq W \text{ or } W \subseteq^* A\}.$$

Note that if A is maximal, $\mathcal{C}(A)$ contains representatives of every equivalence class in $\mathcal{E}^*$. Thus to prove that the maximal sets form an orbit in $\mathcal{E}^*$, Soare actually showed the following. For every pair A, B of r.e. sets, there is an isomorphism Φ of the lattice $\mathcal{C}(A)$ onto the lattice $\mathcal{C}(B)$. Now suppose that A_1, A_2 and B_1, B_2 are nontrivial splittings of a maximal set M. Let Φ_1 and Φ_2 be isomorphisms of $\mathcal{C}(A_1)$ to $\mathcal{C}(B_1)$ and $\mathcal{C}(A_2)$ to $\mathcal{C}(B_2)$, respectively. Now we construct an automorphism Φ of $\mathcal{E}^*$ such that $\Phi(A_i) = B_i$ for $i = 1, 2$ by piecing together these two isomorphisms. We have two cases. If $W \subseteq M$, we define $\Phi(W) = \Phi_1(A_1 \cap W) \cup \Phi_2(A_2 \cap W)$. If $\overline{M} \subseteq^* W$, we define $\Phi(W) = \Phi_1(A_2 \cup W) \cap \Phi_2(A_1 \cup W)$. The maximality of M guarantees that this works. $\square$

The class of degrees of hemimaximal sets, **H**, is a very interesting class. By modifying Lachlan's version of Martin's result that every high r.e. degree contains a maximal set, we showed the following.

Theorem 4.3 (Downey and Stob [39]). *All high r.e. degrees contain hemimaximal sets.*

It is also true that **H** is downward dense.

Theorem 4.4 (Downey and Stob [39]). *If* $\mathbf{c} \neq \mathbf{0}$ *is an r.e. degree, there is a degree* $\mathbf{a} \leqslant \mathbf{c}$ *such that* $\mathbf{a} \in \mathbf{H}$.

Proof. The proof is a fairly easy modification of the maximal set construction. Given nonrecursive r.e. set C, we wish to construct disjoint nonrecursive r.e. sets A and B so that $M = A \cup B$ is maximal and $A \leqslant_T C$. We construct these sets to meet the following requirements.

$\mathbf{N}_e$: $|\overline{M}| \geqslant e$,

$\mathbf{P}_e$: $W_e \neq \overline{A}$,

$\mathbf{Q}_e$: $\overline{M}$ has almost constant e-state.

Of course the requirements $\mathbf{N}_e$ and $\mathbf{Q}_e$ are the standard maximal set construction requirements. Requirements $\mathbf{P}_e$ guarantee that A is nonrecursive. We shall also assume that C has low r.e. degree. This implies that A is of low r.e. degree and so B must be nonrecursive since $M = A \cup B$ is of high r.e. degree. At each stage s, we let $\overline{M}_s = m_{0,s} < m_{1,s} < \ldots$. The construction at stage $s + 1$ consists of two steps. First, to meet $\mathbf{P}_e$, we let e be least such that $W_{e,s} \cap A_s = \emptyset$ and such that for some $x \in W_{e,s}$, $x > m_{e,s}$ (priority), and C permits x at s. If such an x exists, we enumerate x into A. Second, to meet requirements $\mathbf{Q}_e$, find the least pair $i < j$ such that $\sigma_{i,s}(m_i^s) < \sigma_{i,s}(m_j^s)$ and such that $m_{j,s} < x$ for any x enumerated into A in the first step. If i, j exist, we enumerate $m_{i,s}$ into B.

We have that $A \leq_{\mathrm{T}} C$ by simple permitting (in the first step). Since each requirement P_e requires attention in a first step at most once, it is easy to argue as in the maximal set construction that all the requirements $\mathbf{N}_e$ and $\mathbf{Q}_e$ are met. Finally to see that $\mathbf{P}_e$ is satisfied, suppose to the contrary that $W_e = \overline{A}$. Let σ be the well-resided e-state guaranteed by the satisfaction of requirement $\mathbf{Q}_e$. Let i_0 be such that $j > i_0$ implies that $m_j = \lim_s m_j^s$ has e-state σ. let s_0 be such that for all $k \leq i_0$, $m_k = m_{k,s_0}$. Let x, $i > i_0$, and $s \geq s_0$ be such that $x = m_{i,s}$ and $\sigma_{e,s} = \sigma$. We claim that for all $s' > s$ there is $y \geq x$ and $j \geq i$ such that $\sigma_{e,s'} = \sigma$ and $y = m_{j,s'}$. This fact, together with the fact that $\overline{A} = W_e$ implies that C never permits x after stage s. Since infinitely many such x and s can be recursively generated, this implies that C is recursive contradicting our hypothesis. □

Not all r.e. degrees are degrees of hemimaximal sets however.

Theorem 4.5. *There is an r.e. set C such that if $A \equiv_{\mathrm{T}} C$, then A is not hemimaximal.*

Proof. Let $(\Phi_e, \Gamma_e, U_e, V_e)_{e \in \omega}$ be an effective listing of all quadruples where Φ_e, Γ_e are recursive functionals and U_e, V_e are disjoint r.e. sets. Then the requirements on C amount to the following

$$\mathbf{R}_e: \Phi_e(C) = U_e \wedge \Gamma_e(U_e) = C \Rightarrow U_e \cup V_e \text{ is not maximal.}$$

We will attempt to insure that $U_e \cup V_e$ is not maximal by enumerating an array $\{T_{e,i}\}_{i \in \omega}$ of disjoint, finite r.e. sets such that $T_{e,i} \not\subseteq U_e \cup V_e$. This guarantees that not only is $U_e \cup V_e$ not maximal, it is not even hyperhypersimple. Thus the requirements $\mathbf{R}_e$ will be divided into the following

$$\mathbf{R}_{e,i}: \Phi_e(C) = U_e \wedge \Gamma_e(U_e) = C \Rightarrow T_{e,i} \cap \overline{U_e \cup V_e} \neq \emptyset.$$

We will assume that the requirements are ordered in some ω-sequence, thereby inducing a priority ordering on them. We first give the strategy for

meeting a single requirement; it will be convenient in describing it to drop all subscripts. The requirement thus becomes

$$\mathbf{R}: \Phi(C) = U \wedge \Gamma(U) = C \Rightarrow T \cap \overline{U \cup V} \neq \emptyset.$$

Let $\phi(x,s)$ and $\gamma(x,s)$ be the use functions associated with the computations $\Phi_s(C_s;x)$ and $\Gamma_s(U_s;x)$ respectively and let $l_\Phi(s)$ and $l_\Gamma(s)$ be the corresponding lengths of agreements of these functions. To attack $\mathbf{R}$ we proceed as follows.

Step 1. Wait for a stage s such that there is an x satisfying

$$l_\Gamma(s) > x, \tag{4.1}$$

$$(\exists y \leqslant \gamma(x,s))[y \notin U_s \cup V_s \cup T_s], \tag{4.2}$$

$$l_\Phi(s) > \gamma(x,s). \tag{4.3}$$

(Such x and s must exist if the hypotheses of $\mathbf{R}$ are satisfied and $U \cup V$ is not cofinite.) Given x and s, our action is to enumerate into T_{s+1} all $y \leqslant \gamma(x,s)$ such that $y \notin U_s \cup V_s$ and to restrain from C all $z \leqslant \phi(\gamma(x,s),s)$.

Notice that after step 1, $\mathbf{R}$ is satisfied temporarily since $T_{s+1} \cap \overline{U_s \cup V_s} \neq \emptyset$. $\mathbf{R}$ will be satisfied forever (with the finite restraint imposed by step 1) unless there is a stage $t > s$ such that $(U_t \cup V_t) \supseteq T_t = T_{s+1}$. Now if any element, say z, of $T_{s+1} - (U_s \cup V_s)$ is enumerated into $U_t - U_s$, we have, by the restraints imposed on C at step 1, that

$$\Phi_t(C_t;z) = \Phi_s(C_s;z) = U_s(z) \neq U_t(z)$$

and this disagreement is preserved forever with finite restraint. Thus we may assume that each element enumerated in T at stage $s + 1$ is later enumerated into V by stage t. Now at stage $t + 1$ we perform

Step 2. Remove the restraint on C imposed by step 1. Enumerate x (the x of step 1) into C_{t+1}.

Step 2 wins requirement $\mathbf{R}$ forever since we have that

$$\Phi(U;x) = \Phi_s(U_s;x) = C_s(x) \neq C_{t+1}(x).$$

The first equality is the crucial one and is true since $U_s[\gamma(x,s)] = U[\gamma(x,s)]$ because U and V are disjoint sets, $U_t \cup V_t \supseteq \{x \mid x \leqslant \gamma(x,s)\}$, and $U_t = U_s$.

To see that the strategies for the various $\mathbf{R}_{e,i}$ cohere, notice that each $\mathbf{R}_{e,i}$ imposes only finitely much restraint on C and thus $\mathbf{R}_{e,i}$ may be restarted for the sake of $\mathbf{R}_{e',i'}$ of higher priority as in standard arguments of Friedberg-Muchnik type. The only restraints on the sets $T_{e,i}$ are to make $T_{e,i}$ disjoint from $T_{e,j}$ if $i \neq j$ and it is clear that this can be done. We will omit the details of combining the strategies for meeting the requirements $\mathbf{R}_{e,i}$ since this is a straightforward application of the finite injury priority method. $\square$

Theorem 4.5 refutes many conjectures that had previously been made concerning the structure of degree classes of sets forming an orbit. For instance, it refutes conjectures that (1) an orbit containing sets of all high degrees contains only sets of high degree, and (2) the degrees of sets in an orbit are closed upwards in **R**. We remark also that Theorem 4.5 also shows that the lattice-theoretic properties of an r.e. set may have consequences for the degrees of its subsets. Previous investigations had focused only on the degrees of supersets of a set. Another example of this phenomenon is Downey's result [21] that no hypersimple set can have the universal splitting property; we examine that result in Section 5. This result is that if A is hypersimple, there is a degree **b** such that **b** < deg(A) such that if A_1, A_2 is a splitting of A, then A_1 does not have degree **b**. This result is an analogue for splittings of Stob's [98] result that an r.e. set is simple if and only if it does not have supersets of every r.e. degree. Downey and Stob extended Theorem 4.5 to show that **H** is nowhere dense in the low degrees. We will not prove that result here as the essential new ingredient is the "Robinson trick" for exploiting lowness which we discuss in Section 6. Actually, using techniques of Shore and Slaman [93] for working below a low$_2$ degree, it may be possible to extend this result to answer positively the following question.

Open Question 4.6. Is **H** nowhere dense in the low$_2$ degrees?

There are nonhemimaximal degrees which are low$_2$ but not low.

Theorem 4.7 (Downey and Stob [38]). *There is a degree* **a** *which is low$_2$ but not low such that* **a** $\notin$ **H**.

We omit the proof of Theorem 4.7 as it is quite tricky. We do know the range of the jump operator on **H**; it is all possible degrees.

Theorem 4.8 (Downey and Stob [38]). *If S is a set r.e. in $0'$ and above $0'$, then there is a hemimaximal set A such that $A' = S$.*

In particular, we have the following corollary which should be contrasted with the result of Harrington that there is no "fat orbit" (Theorem 2.15).

Corollary 4.9. *There is an elementarily definable (in $\mathcal{E}$) orbit realizing all possible jumps.*

The proof of Theorem 4.8 involves a combination of the Sacks jump inversion theorem and the maximal set construction in a Π_2 argument. We do not give this here but refer the reader to [38, Theorem 1.1].

A complete characterization of the degree class **H** is still not known. Recently, Downey, Lempp, and Shore [29] have shown that there is a high$_2$ degree h not in **H**.

One of the reasons the authors were interested in the class **H** was its possible connection to Post's Program. In his investigation of degrees of unsolvable problems, Post [85] noticed that all of the r.e. problems that he considered were either recursive (decidable) or were of the same degree as the halting problem. This lead him to pose his famous question of whether all r.e. problems were of these two types or whether there were r.e. degrees intermediate between **0** and **0'**. As is well known, the solution of Post's problem twelve years later by Friedberg and Muchnik independently lead to the invention of the priority method. This powerful tool has been one of the main weapons of modern recursion theory ever since and has found direct applications in other areas such as complexity theory and descriptive set theory as well as reflections in such areas as combinatorics and game theory.

The solution of Friedberg and Muchnik was not along the lines suggested by Post in what has come to be called Post's Program. Post suggested finding a "thinness" property of the complement of an r.e. set which would guarantee its incompleteness. The strongest notion along the lines suggested by Post is certainly that of maximality. If Post's Program in its original form was to succeed, it would imply that maximal sets are incomplete. In this form, Post's Program was refuted by Yates [100] who demonstrated the existence of a complete maximal set and then completely destroyed by Soare who showed that all maximal sets are automorphic so that there is not even a $\mathbf{L}_{\omega_1,\omega}$ property that can distinguish among them. Thus no "extra" definable property together with maximality guarantees completeness.

Post's Program does have a partial realization in the work of Marchenkov [76] who showed that in the lattice of r.e. sets modulo a certain r.e. equivalence relation, there is a maximal element which is not complete. The r.e. equivalence relation used by Marchenkov is however not elementarily definable in the lattice $\mathcal{E}$. An interesting but completely unexplored line of investigation is the study of the automorphism groups of $\mathcal{E}/R$ for r.e. equivalence relations R.

A more general version of the question implied by Post's Program is whether there is a definable property of r.e. sets which guarantees incompleteness. A very strong negative answer has been given by Harrington and Soare who showed the following.

Theorem 4.10 (Harrington and Soare [50]). *There is an elementary property P such that if A is an r.e. set with property P, then A is nonrecursive and incomplete.*

Proof. We will only state the property P here and omit the proof. We need one definition.

Definition 4.11. An r.e. set $A \subseteq C$ is a *major subset* of the r.e. set C (written $A \subseteq_m C$) if for every r.e. set W, if $\overline{C} \subseteq W$ then $\overline{A} \subseteq^* W$.

Major subsets were first defined by Lachlan and play a crucial role in his decision procedure for the $\forall\exists$ theory of $\mathcal{E}^*$. We will write $A \sqsubset B$ for A is half of a splitting of B. The property $P(A)$ can now be defined as follows:

$$(\exists C)_{A \subseteq_m C} (\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S \sqsubset C}$$

$$[B \cap (S - A) = D \cap (S - A)$$

$$\Rightarrow (\exists T)_{\overline{C} \subset T}[A \cap (S \cap T) = B \cap (S \cap T)]].\qquad\Box$$

It is very interesting therefore to determine exactly which sets are automorphic with complete sets. As we have already seen in Section 3, all sets of promptly simple degree are automorphic to complete sets. Downey and Stob exploited their results about hemimaximal sets to find more classes of r.e. sets all of which are automorphic to complete sets.

Definition 4.12. Suppose that Q is a property of r.e. sets. We say A is half-Q if there is a splitting A_1, A_2 of A such that A_1 has property Q.

Suppose that r.e. sets A_1, A_2 form a splitting of an r.e. set A and that Φ is an automorphism such that $\Phi(A_1)$ is complete. Then it is easy to see that $\Phi(A)$ is complete. Thus every half-hemimaximal set is automorphic to a complete set. Thus we have the following theorem.

Theorem 4.13 (Downey and Stob [39]). *The following classes of r.e. sets are* (Σ_3-) *automorphic with complete sets:*
(1) *low$_2$ simple sets,*
(2) *simple sets with semilow$_{1.5}$ complements,*
(3) *d-simple sets with maximal supersets.*

Each of the parts of the preceding theorem follow from the fact that the corresponding sets are half-hemimaximal. We note that (1) of the theorem implies that Marchenkov's incomplete set is nevertheless automorphic to a complete set. This follows since Miller [79] showed that all sets which are maximal modulo some r.e. equivalence relation are low$_2$ and Marchenkov's incomplete set is simple (Odifreddi [83]).

We do not have a classification of the halfhemimaximal sets. We know that there is no nontrivial degree-theoretic classification. For instance, by Theorem 4.13 all nonzero r.e. degrees contain halfhemimaximal sets. It is easy to see [39, Theorem 10] that if A is halfhemimaximal then A has a maximal superset and

therefore that all nonlow$_2$ degrees contain nonhalfhemimaximal sets. However we have shown the following.

Theorem 4.14 (Downey and Stob [39]). *If* $\mathbf{a} \neq \mathbf{0}$, *then* A *contains a non-halfhemimaximal set.*

Proof. Suppose that B is nonrecursive. We construct A nonhalfhemimaximal so that $A \equiv_T B$. Let g be a 1-1 recursive function such that $g(\omega) = B$.

Let $\{F_e\}_{e \in \omega}$ be a recursive sequence of disjoint finite sets such that $\bigcup_n F_n = \omega$ and $|F_n| = n + 2$ for every n. We will ensure that $A \equiv_T B$ in the following way. At stage s we will enumerate exactly one element into A chosen from the set $F_{g(s)}$. Thus $|A \cap F_n| \leq 1$ and $|A \cap F_n| = 1$ iff $n \in B$. It is easy to see from this that $A \equiv_T B$. Let $(U_e, V_e)_{e \in \omega}$ be a recursive listing of all pairs of disjoint r.e. sets. The requirements to make A nonhalfhemimaximal are the following.

$\mathbf{N}_e$: $(U_e \subseteq A \wedge U_e \cup V_e \supseteq A) \Rightarrow (U_e \cup V_e$ is not maximal $\vee U_e$ is recursive).

We may assume in the light of the first hypothesis of $\mathbf{N}_e$ that no element of U_e is enumerated in U_e before it is enumerated in A.

Construction
Stage s
Let $n = g(s)$. We must enumerate one element of F_n into A. Choose the least element z of F_n such that for all $e \leq n$, $|F_n \cap V_{e,s}| \geq n + 1$ implies that $z \in V_{e,s}$. This is possible since $|F_n| = n + 2$ and the condition on z requires it only to be in the intersection of at most $n + 1$ subsets of F_n each of cardinality at least $n + 1$. This intersection is nonempty.

To complete the verification, we need only show that $\mathbf{N}_e$ is satisfied for each e. So suppose that the hypotheses of $\mathbf{N}_e$ are satisfied and that $U_e \cup V_e$ is maximal. We will show that U_e is recursive. Since $U_e \cup V_e$ is maximal, there is an integer n_0 such that for all $n \geq n_0$, $|F_n \cap (U_e \cup V_e)| \geq n + 1$ (see [86, Chapter 12, Theorem XIII]). Let z be fixed such that $z \in F_n$, $n \geq n_0$. we show how to decide if $z \in U_e$. Let s be a stage such that $|F_n \cap V_{e,s}| \geq n + 1$ or $|F_n \cap A_s| = 1$. One or the other must happen since if $F_n \cap A = \emptyset$, then $|F_n \cap V_e| \geq n + 1$. In the former case, $z \notin U_e$ since if z is later enumerated in A, $z \in V_e$. In the latter case, if $z \in A$ we can enumerate U_e and V_e until z appears in one or the other (this must happen since $A \subseteq U_e \cup V_e$) and if $z \notin A$ then $z \notin U_e$. $\square$

The notion and analysis of hemimaximal set can obviously be extended to other properties. As an example, we have the following.

Theorem 4.15 (Downey and Stob [39]). *For every* k, *the class of Friedberg splittings of* k-*quasimaximal sets form an orbit.*

It is also possible to classify the automorphism types of hemi-k-quasimaximal sets. For instance, if $k = 2$, the four types of hemi-k-quasimaximal sets are the following: sets which are Friedberg splittings of a 2-quasimaximal set, sets which are maximal in some infinite-coinfinite recursive set, hemimaximal sets, and sets which are not hemimaximal but which are halfs of nontrivial splittings of a 2-quasimaximal set for which the other half of the splitting is hemimaximal. It is also possible to extend Maass' proof that any two hh-simple sets with lattices of supersets that are isomorphic by a Σ_3-isomorphism are automorphic to show

Theorem 4.16 (Downey and Stob [39]). *If C is any class of hh-simple sets for which the lattices of supersets are pairwise Σ_3 isomorphic, then any two hemi-C sets are automorphic.*

All of the above still leaves open the question of Section 3:

Open Question 4.17. Under what conditions are all of the Friedberg splittings of a set A automorphic?

Here we are looking for an elementary lattice-theoretic property of such a set A. Perhaps it is the case that only k-quasimaximal sets have this property.

Independent of the automorphism question, we also think that there might be some interest in studying the classes of hemi-P and half-P sets for various P. Along these lines we have the following.

Theorem 4.18 (Downey and Stob [40]).
(1) *There is a complete r.e. set A that is not halfhemisimple.*
(2) *If fact every high degree contains a set that is not halfhemisimple.*
(3) *However there is a completely halfhemisimple degree.*

Proof. We discuss only the proof to (1). We first sketch the proof of the existence of a nonrecursive r.e. set A which is not halfhemisimple (without the requirement that A be complete). Let $\langle X_e, Y_e, Z_e \rangle$, $e \in \omega$ list triples X, Y, Z of r.e. sets such that $X \cap Y = \emptyset$ and $X \cap Z = \emptyset$. We have the following requirements for every $e \in \omega$.

$\mathbf{P}_e$: $\overline{A} \neq W_e$,

$\mathbf{R}_e$: $X_e \cup Y_e = A \Rightarrow (X_e \cup Z_e \text{ is not simple } \vee X_e \text{ is recursive})$.

The basic strategy for $\mathbf{P}_e$ (which we will need to modify to make A complete) is the Friedberg strategy. Namely we have a follower x such that if x occurs in W_e, we enumerate x in A.

The strategy for $\mathbf{R}_e$ is as follows. Consider $\mathbf{R}_0$. We will enumerate a certain auxiliary set Q_0 which is intended to witness that $X_e \cup Z_e$ is not simple. To begin our attack on $\mathbf{R}_0$, we will enumerate $Q_0 = \omega^1$ (the first column of ω). We will begin choosing all of our witnesses for the positive requirements from ω^0 so that $A \cap Q_0 = \emptyset$. Now we wait for a stage s_0 such that some number z_0 occurs in $Z_0 \cap Q_0$. We then restrain A so that if $y \leqslant z_0$ and $y \notin A_{s_0}$, then $y \notin A$. We now have that for all $y \leqslant z_0$, if the hypothesis that $X_0 \cup Y_0 = A$ is correct, we can recursively compute a stage $f(s_0)$ such that no element $y \leqslant z_0$ may enter X_0 after stage $f(s_0)$. Supposing now that we have defined z_i, s_i, and $f(s_i)$ to have the property that after stage $f(s_i)$, no $y \leqslant z_i$ may enter X_0 after stage $f(s_i)$. We then define $z_{i+1} > z_i$, s_{i+1} and $f(s_{i+1})$ as we did for z_0. Now either the above module acts infinitely often (and so X_0 is recursive by the definition of f) or it acts finitely often (so that $Q_0 \cap (Z_0 \cup A) =^* \emptyset$ and so $X_0 \cup Z_0$ is not simple). Note that in the former case, the set $T_0 = \{z_0 < z_1 < \cdots\}$ is an infinite recursive set such that $T \subseteq \overline{A}$. Thus other requirements of lower priority than $\mathbf{R}_0$ must work inside of T_0. This makes the full construction and e-state construction. To be specific consider the interactions of $\mathbf{R}_1$ and $\mathbf{P}_1$ with $\mathbf{R}_0$. There are two versions of each. A version of $\mathbf{P}_1$ guessing that $\mathbf{R}_0$ acts finitely often chooses its followers from ω^0. However one guessing that $\mathbf{R}_0$ acts infinitely often chooses its followers from T_0. A version of $\mathbf{R}_1$ guessing that $\mathbf{R}_0$ acts finitely often uses $Q_1 = \omega^2$ in place of $Q_0 = \omega^1$ as above. Of course $\mathbf{R}_1$ cannot restrain followers of $\mathbf{P}_0$ but this is only finite restraint. A version of $\mathbf{R}_1$ guessing that $\mathbf{R}_0$ acts infinitely often uses T_0 as its universe. Namely, it uses for Q_1 the set $g(\omega^2)$ where g is a recursive bijection of $g : \omega \to T_0$. An e-state construction clearly suffices to put the requirements together.

Turning now to the proof of (1), we replace the positive requirements by the requirement that we must code an arbitrary r.e. set C into A. To do this, we have coding markers A_e. At stage $s + 1$, if $x \in C_{s+1} - C_s$, we enumerate the current position $A_{x,s}$ of the coding marker A_x into A. The markers will move only finitely often and their final positions will be computable from A. To begin with, the coding markers are in ω^0. We need to move the coding markers to elements of $\overline{A}$ which are in higher states (more of the sets T_i, essentially). We describe how this strategy interacts with our strategy above for $\mathbf{R}_0$. The idea is to move a coding marker A_x to T_0 if there is an available element of T_0 for it. Now to move A_x to T_0 at stage $s + 1$ requires us to enumerate $A_{x,s}$ into A. We will do that at a stage s_i such that $\mathbf{R}_0$ receives attention and then we wait in defining $f(s_i)$ until that marker position is enumerated in X_0 or Y_0. □

Perhaps such notions as halfhemisimple may be related to invariant degree classes. Recently, Kummer [58] has pointed out a new direction in such studies. He defines two interesting classes of r.e. sets as follows.

Definition 4.19. An r.e. set A is *semihyperhypersimple* if there is no recursive function f such that $\{W_{f(n)}\}_{n \in \omega}$ is a weak array of disjoint sets such that $W_{f(n)} - A$ is not r.e.

An r.e. set A is *semimaximal* if for any pair of disjoint r.e. sets C, D, either $\overline{A} \cap C$ or $\overline{A} \cap D$ is r.e.

Some interesting facts about these sets and connections with our notions are summarized in the next theorem.

Theorem 4.20 (Kummer [58]).
(1) *A simple set B is shhs iff it is hhsimple and it is sm iff it is maximal.*
(2) *Every half of a splitting of a shhs (sm) set is shhs (sm).*
(3) *All hemi-hhs (hemimaximal) sets are shhs (sm).*
(4) *The class of shhs sets is closed under intersection.*
(5) *The index sets $S_{\mathrm{sm}} = \{n \mid W_n$ is sm$\}$ and $S_{\mathrm{shhs}} = \{n \mid W_n$ is shhs$\}$ are Π_4 complete.*

Kummer introduced these notions in connection with his study of non-standard numberings of the partial recursive functions. The class of partial recursive functions is said to be *numbered* by a recursive function n if for every e there is i such that $\varphi_e = \lambda x n(i, x)$. Kummer extended notions about the standard numbering of the recursive functions to such nonstandard numberings. For example, define $K_n = \{e \mid n(e, e){\downarrow}\}$. Kummer showed that an r.e. set A is K_n for some n if and only if A is not semihyperhypersimple.

In a surprising and interesting result, Herrmann and Kummer [51] showed that semihyperhypersimplicity is lattice-theoretic. They show this by considering certain quotient lattices in $\mathcal{E}^*$.

Definition 4.21. For A r.e., let $\mathcal{L}(A) = \{B \mid A \subseteq B\}$, $\mathcal{D}(A) = \{B \mid B \in \mathcal{L}(A) \land B - A$ r.e.$\}$, and let $C(A) = \mathcal{L}(A)/\mathcal{D}(A)$. Then A is called $\mathcal{D}$-*hyperhypersimple* if $C(A)$ is a Boolean algebra.

Theorem 4.22 (Herrmann and Kummer [51]). *An r.e. set A is semihyperhypersimple iff A is recursive or $\mathcal{D}$-hyperhypersimple.*

Kummer generalized our theorem that $\mathbf{H}$ is nowhere dense in the low degrees by showing the following.

Theorem 4.23 (Kummer [58]). *The degrees of shhs sets are nowhere dense in the low degrees.*

It is not clear whether this is a true generalization of our theorem since Kummer leaves open the following question.

Open Question 4.24 (Kummer). Is it the case that **a** is the degree of a shhs set if and only if it is the degree of a hemimaximal set?

Kummer has shown however that there are nonrecursive shhs sets that are not hemisimple.

5. Degrees of splittings

We can naturally associate to an r.e. set A, the class of degrees of splittings of A:

$$\mathcal{S}(A) = \{\mathbf{c} : (\exists A_1)[A_1 \text{ is half of a splitting of } A \text{ and } \deg(A_1) = \mathbf{c}]\}.$$

We also use $\mathcal{N}(A)$ to denote $\mathbf{R} - \mathcal{S}(A)$. We are interested in determining the structure of $\mathcal{S}(A)$ for various A. The first result along these lines was the splitting theorem of Sacks.

Theorem 5.1 (Sacks Splitting Theorem [88]). *Suppose that A is a nonrecursive r.e. set and let $\mathbf{c}$ be any nonrecursive degree (not necessarily r.e.). Then there exists a splitting A_1, A_2 of A such that $\deg(A_i) \not\geqslant \mathbf{c}$ for $i = 1, 2$.*

Proof (sketch). Let C be an r.e. set of degree $\mathbf{c}$. We will construct a splitting A_1, A_2 of A to meet the following requirements for every $e \in \omega$ and $i = 1, 2$.

$$\mathbf{R}_{e,i}\colon \Phi_e^{A_i} \neq C.$$

Fix enumerations of A and C such that $|A_{s+1} - A_s| \leqslant 1$. Let $l(\langle e, i \rangle, s)$ be the length of agreement function for $\Phi_e^{A_i} = C$ and let $r(\langle e, i \rangle, s)$ be the restraint imposed on A_i to preserve $l(\langle e, i \rangle, s)$.

Construction
Stage $s + 1$
Let x be the unique element of $A_{s+1} - A_s$ if any. Let $\langle e, i \rangle$ be least such that $x \leqslant r(\langle e, i \rangle, s)$. If $\langle e, i \rangle$ exists, enumerate x in A_{2-i}. Otherwise enumerate x in A_1.

Obviously A_1, A_2 is a splitting of A. To prove that $\mathbf{R}_{e,i}$ is satisfied for every i and e, one shows by induction on $\langle e, i \rangle$ that

$$\Phi_e^{A_i} \neq C, \tag{5.1}$$

$$\lim_s r(\langle e, i \rangle, s) \text{ is finite.} \qquad \square \tag{5.2}$$

The above theorem was the first result for which Sacks' preservation strategy was used, and this strategy is a key ingredient of most of the early infinite

injury priority arguments. Indeed Mytilinaios and Slaman [80] have given an analysis of priority arguments which shows that the proof of Sacks' Splitting Theorem is an infinite injury argument in a certain precise sense although it is not usually viewed as such.

A consequence of Theorem 5.1 is that for nonrecursive A, $S(A)$ always contains infinitely many incomparable r.e. degrees. During his investigations into effective algebra, Remmel asked if $S(A) = \{\mathbf{b} \mid \mathbf{b} \leqslant \deg(A)\}$. Later, Lerman and Remmel addressed this question with the following definition and results.

Definition 5.2. An r.e. set A has *universal splitting property* (USP) if $S(A) = \{\mathbf{b} \mid \mathbf{b} \leqslant \deg(A)\}$. A is *non-USP* otherwise. Furthermore, if $\mathbf{b} \leqslant \deg(A)$ and $\mathbf{b} \notin S(A)$, then $\mathbf{b}$ is a *nonsplitting witness* for A.

It is easy to see that K has the USP. Lerman and Remmel showed the following.

Theorem 5.3 (Lerman and Remmel [70]). *There is an r.e. set A without the USP. In fact, the degrees of the nonUSP sets are dense in the r.e. degrees and include $\mathbf{0}'$.*

Theorem 5.4 (Lerman and Remmel [70]). *If A is r.e. and nonrecursive, there is a nonrecursive $B \leqslant_{\mathrm{T}} A$ such that B has the USP.*

Actually Lerman and Remmel proved a stronger result than Theorem 5.3 in [70]. Recall that an r.e. set B is weak-truth-table reducible to A ($B \leqslant_{\mathrm{wtt}} A$) if $B \leqslant_{\mathrm{T}} A$ via a Turing reduction Φ for which there is a recursive function φ such that $u(\Phi(A;x)) \leqslant \varphi(x)$. Lerman and Remmel studied this stronger notion of reducibility with respect to universality of splittings.

Definition 5.5. An r.e. set A has the *universal weak-truth-table reduction property* (*UWP*) if for every $B \leqslant_{\mathrm{T}} A$ there is $C \equiv_{\mathrm{T}} B$ such that $C \leqslant_{\mathrm{wtt}} A$.

That a relationship exists between UWP and USP follows from the fact that if A_1, A_2 is a splitting of A then $A_1 \leqslant_{\mathrm{wtt}} A$. Thus if a set has the USP then it has the UWP. Lerman and Remmel proved Theorem 5.3 with UWP in place of USP. We sketch their proof of the existence of a nonUWP set.

Theorem 5.6 (Lerman and Remmel [69]). *There is an r.e. set A such that A does not have the UWP.*

Proof. We construct two sets A and B to meet the requirements that $B \leqslant_{\mathrm{T}} A$ and

$$\mathbf{R}_{e,i}: \Phi_e(B) = U_e \wedge \Gamma_e(U_e) = B \Rightarrow \Delta_i(A) \neq U_e.$$

Here, (Φ_e, Γ_e, U_e) lists triples consisting of two functionals and an r.e. set and Δ_i denotes the ith wtt-functional with partial recursive use function δ_i. (That is, Δ_i denotes a Turing reduction such that $\Delta_{i,s}(A_s; x)$ is considered to converge only if its use is bounded by $\delta_{i,s}(x)$.) The argument is a finite injury one. Define

$$l(e,s) = \max\{x \mid (\forall y \leq x)[\Gamma_{e,s}(U_{e,s}; y) = B_s(y)$$
$$\wedge(\forall z \leq u(\Gamma_{e,s}(U_{e,s}; y)))[\Phi_{e,s}(B_s; z) = U_{e,s}(z)]]\}$$

and

$$l(e,i,s) = \max\{x \mid (\forall y \leq x)[\Delta_{i,s}(A_s; y) = U_{e,s}(y)]\}.$$

Note that if $l(e,s) \geq z$, we can restrain $U_{e,s}[u]$ such that $u = u(\Gamma_{e,s}(U_{e,s}; z))$ by restraining B.

The action is as follows.

(1) Choose follower z targeted for B.
(2) Wait until $l(e,s) \geq z$. Then restrain $U_{e,s}[u]$ such that $u = u(\Gamma_{e,s}(U_{e,s}; z))$ by restraining B.
(3) Wait for $t > s$ such that $l(e,i,t) > u$. Then enumerate z into A and define a trace for z, $T(z)$, so that $T(z) > \delta_i(u)$.
(4) If a stage $r > t$ occurs for which $l(e,i,r) > u$, then enumerate $T(z)$ into A and z into B and restrain A so that $\Delta_i(A; z)$ does not change after r for $z \leq u$.

Note that by the restraints in (2), $U_{e,s}[u] = U_{e,r}[u]$. By the subsequent enumeration into B in (4), $U_{e,r}[u] \neq U_e[u]$. Since $T(z) > \delta_i(u)$, and the restraint imposed on A at (4), $\Delta_i(A_r; y) = U_{e,r}(y)$ for all $y \leq u$. This causes the desired disagreement; $\Delta_i(A)[u] \neq U_e[u]$. Also, $B \leq_T A$ by the traces. The construction puts together the basic strategies described above in the standard finite injury manner. $\square$

The proof of Theorem 5.3 involves the general technique of delayed permitting and coding (which can be extracted from the proof of Sacks Splitting Theorem). This general technique can be found in many results. Lerman and Remmel also extended the proof of Theorem 5.6 to show the following.

Theorem 5.7 (Lerman and Remmel [69]). *There is a degree **a** which is completely nonUWP. That is, if A is of degree **a**, A is nonUWP.*

Ambos-Spies and Fejer extended this result using the Robinson Trick.

Theorem 5.8 (Ambos-Spies and Fejer [7]). *The low degrees containing sets with the UWP are nowhere dense in **R**.*

It is not clear if there is an analogue of Theorem 5.3 for the completely USP degrees.

Open Question 5.9. Are the completely nonUSP (or nonUWP) degrees dense?

This question is nontrivial since not all incomplete r.e. degrees contain nonUWP sets. This follows from a very important result of Ladner and Sasso.

Definition 5.10. An r.e. Turing degree **a** is *contiguous* if for every pair A, B of r.e. sets in **a**, $A \equiv_{\text{wtt}} B$. An r.e. Turing degree **a** is *strongly contiguous* if for every pair C, D of sets (not necessarily r.e.) of degree **a**, $C \equiv_{\text{wtt}} D$.

Theorem 5.11 (Ladner and Sasso [67]). *For every nonzero r.e. degree* **b**, *there is a nonzero degree* **a** $\leqslant$ **b** *such that* **a** *is contiguous.*

Downey has strengthened the theorem of Ladner and Sasso to show that **a** can be made strongly contiguous. It is not know however whether there are contiguous degrees which are not strongly contiguous.

Theorem 5.12 (Downey [20]). *There is a strongly contiguous r.e. Turing degree.*

Proof. Our proof is a modification of Ambos-Spies [2]. We construct A to satisfy the following requirements. Let (Φ_e, Γ_e) enumerate all pairs (Φ, Γ) of Turing reductions.

$$\mathbf{P}_e: \overline{A} \neq W_e,$$

$$\mathbf{N}_e: \Phi_e(A) \text{ total } \wedge \Gamma_e(\Phi_e(A)) = A \Rightarrow A \leqslant_{\text{wtt}} \Phi_e(A),$$

$$\mathbf{Q}_e: \Phi_e(A) \text{ total } \wedge \Gamma_e(\Phi_e(A)) = A \Rightarrow \Phi_e(A) \leqslant_{\text{wtt}} A.$$

We shall describe the construction to meet a single requirement $\mathbf{N}_e$. The strategy is very similar to that of Ladner and Sasso. The only difficulty is to see that it works for $\Phi_e(A)$ not necessarily r.e.

Each requirement $\mathbf{P}_e$ will have followers. Let l be the length of agreement function for $\Gamma_e(\Phi_e(A)) = A$; i.e.,

$$l(e,s) = \max\{x \mid (\forall y < x)[\Gamma_{e,s}(\Phi_{e,s}(A_s; y); y) = A_s(y)]\}.$$

Let m be the maximum length of agreement function for l. Each follower of $\mathbf{P}_j$ for $j > e$ is equipped with a guess as to whether $l(e,s) \to \infty$. If a follower is guessing that $l(e,s) \not\to \infty$, then we shall cancel x at stage s is $l(e,s) > m(e,s)$. The other key follower rules are the following.
(1) If x is appointed at stage s then $x = s$. If $l(e,s) > m(e,s)$ we assign the guess $l(e,s) \to \infty$ to x, otherwise we assign x the guess that $l(e,s) \not\to \infty$.
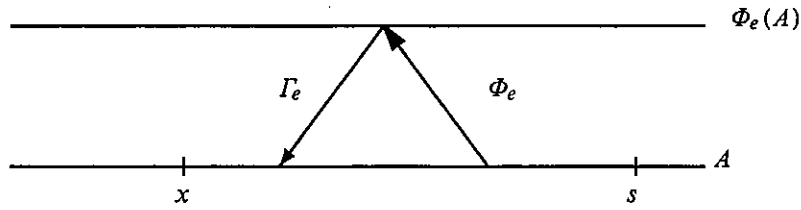
Fig. 1. At the stage that x is e-confirmed.

(2) If $x < y$ are followers and x enters A at stage s then y is cancelled at stage s.

(3) If x and y are followers and $y > x$ and x is uncancelled at stage y (the stage y is appointed as a follower by (1)), then y has lower priority than x.

The basic idea for N_e is this. For each follower x following some requirement P_j, $j > e$, and guessing $l(e,s) \to \infty$, we wait for the first stage such that $l(e,s) > x$. At this stage, we declare x e-confirmed and cancel all followers $y > x$. This gives us the situation of Fig. 1.

The crucial point is that if this situation occurs, x is guessing that $l(e,s) \to \infty$ and there are no followers left uncancelled in the interval $(x, s]$. we claim that this insures that $A \leqslant_{\mathrm{wtt}} \Phi_e(A)$. For let $u = \max\{u(\Phi_{e,s}(A_s; y)) \mid y \leqslant x\}$. To determine whether $x \in A$, compute the least stage $t > s$ such that $l(e,t) > m(e,t)$ and $\Phi_{e,t}(A_t)[u] = \Phi_e(A)[u]$. (Note that we do not necessarily have that $(\forall t' > t)[\Phi_{e,t'}(A_{t'})[u] = \Phi_{e,t}(A_t)[u]]$ as in the r.e. case.) We claim that $x \in A$ if and only if $x \in A_t$. There are two cases.

Case 1. $\Phi_{e,s}(A_s)[u] = \Phi_{e,t}(A_t)[u]$. In this case, the situation of Fig. 1 is unchanged at t and so, because u measures a use function, it must be the case that $A[x] = A_s[x] = A_t[x]$.

Case 2. Otherwise. Since there were no uncancelled numbers z such that $x < z \leqslant s$ after stage s, the only way this case could occur is if some follower $y < x$ enters A after stage s. This follower either cancels x or is x. In either case $x \in A$ if and only if $x \in A_t$.

The cancellation/confirmation procedure also serves to meet the requirements Q_e. To show this, we must show that the cancellation of numbers between x and s in Fig. 1 also allows A to compute $\Phi_e(A)$ via a w-reduction. Let z be given. To compute whether $z \in \Phi_e(A)$ using A, let s be least such that $l(e,s) > m(e,s)$ and $l(e,s) > z$. Let $t > s$ be least such that $l(e,t) > m(e,t)$ and $A_t[s] = A[s]$. We claim that $\Phi_{e,t}(A_t)[z] = \Phi_e(A)[z]$. To show this, we will argue that if follower $x < t$ is uncancelled at stage t, then $x \leqslant s$ (and hence never enters A). Thus $A_t[t] = A[t]$ and the claim follows.

Suppose then for a contradiction that x is a follower which is uncancelled at stage t and $s < x < t$. Then by definition of t, x must have guess $l(e,s) \to \infty$. Now x must be appointed at stage x such that $l(e,x) > m(e,x)$. Since $x < t$, it must be the case that $A_x[s] \neq A[s]$ so that there must be $y \leqslant s$ such

that y enters A after stage x but before stage t. However, at the stage that y enters A, x is cancelled contrary to the assumption on x. This is the desired contradiction.

The strategies described above can be organized on a Π_2 tree. $\square$

As we shall see in subsequent sections, Theorem 5.12 and properties of the weak truth table degrees are useful in analyzing splittings of r.e. sets and the structure of **R**. One example of this is the proof of Ambos-Spies and Fejer of an extension of Theorem 5.3.

Definition 5.13. An r.e. set A has the *strong universal splitting property* (SUSP) if for every pair of r.e. degrees $\mathbf{c}, \mathbf{d}$ such that $\mathbf{c} \cup \mathbf{d} = \deg(A)$, there is a splitting C, D of A such that $\deg(C) = \mathbf{c}$ and $\deg(D) = \mathbf{d}$.

An r.e. set A has the *w-strong universal splitting property* if for every pair $C, D \leqslant_{\text{wtt}} A$, with $C \oplus D \equiv_{\text{wtt}} A$, there is a splitting A_1, A_2 of A such that $A_1 \equiv_{\text{wtt}} C$ and $A_2 \equiv_{\text{wtt}} D$.

Theorem 5.14 (Ambos-Spies and Fejer [7]). *Suppose that A is an r.e. cylinder (i.e., $A = C \times N$ for some r.e. set C). Then A has the w-SUSP.*

The proof of Theorem 5.14 uses some intermediate results. The first is a very important lemma due to Lachlan about the structure of the wtt-degrees.

Lemma 5.15 (Lachlan's Lemma). *Suppose that $A \leqslant_{\text{wtt}} B_1 \oplus B_2$. Then there exists a splitting A_1, A_2 of A such that $A_i \leqslant_{\text{wtt}} B_i$ for $i = 1, 2$.*

Proof. Let $A = \Gamma(B_1 \oplus B_2)$ and γ be the recursive use function of Γ. Without loss of generality, A is infinite. Let $l(s) = \max\{x \mid (\forall y < x)[\Gamma_s(B_{1,s} \oplus B_{2,s}; y) = A_s(y)]\}$. We suppose that the reduction and sets are enumerated sufficiently quickly so that

$$l(s + 1) > l(s) \quad \text{and} \quad (\exists y < l(s))[y \in A_{s+1} - A_s].$$

Let $z_s = (\mu z)[z \in (B_{1,s+1} - B_{1,s}) \cup (B_{2,s+1} - B_{2,s})]$ Note that z_s exists by the hypothesis on the enumeration above. If $z_s \in (B_{1,s+1} - B_{1,s})$, let $A_{1,s+1} = A_{1,s} \cup (A_{s+1} - A_s)$ and let $A_{2,s+1} = A_{2,s}$. If $z_s \in (B_{2,s+1} - B_{2,s})$, then let $A_{2,s+1} = A_{2,s} \cup (A_{s+1} - A_s)$ and let $A_{1,s+1} = A_{1,s}$. Obviously A_1, A_2 form a splitting of A. It is also not difficult to see that $A_i \leqslant_{\text{wtt}} B_i$ for $i = 1, 2$. $\square$

The import of Lemma 5.15 is that **W** is a distributive upper semilattice.

Lemma 5.16 (Ambos-Spies and Fejer [7]). *If A is an r.e. cylinder, then A has the w-USP if and only if A has the w-SUSP.*

Proof. If A is an r.e. cylinder, $A \equiv_1 A \oplus A$ so it suffices to prove that if $A \oplus A$ has the w-USP then $A \oplus A$ has the w-SUSP. Suppose that $A \equiv_{wtt} B_1 \oplus B_2$. Since A has the w-USP, we have splittings $A_{1,1}$, $A_{1,2}$ and $A_{2,1}$, $A_{2,2}$ of A such that $A_{1,1} \equiv_{wtt} B_1$ and $A_{2,1} \equiv_{wtt} B_2$. By Lachlan's Lemma (5.15), since $A_{1,2} \leqslant_{wtt} B_1 \oplus B_2$ $(A_{2,2} \leqslant_{wtt} B_1 \oplus B_2)$ there is a splitting $C_{1,1}$, $C_{1,2}$ of $A_{1,2}$ $(C_{2,1}, C_{2,2}$ of $A_{2,2})$ such that $C_{i,j} \leqslant_{wtt} B_j$ for $j = 1,2$. Now we have that $A \oplus A = A_1 \cup A_2$ where $A_1 = (A_{1,1} \cup C_{1,1}) \oplus C_{2,1}$ and $A_2 = C_{1,2} \oplus (A_{2,1} \cup C_{2,2})$. Note that $A_1 \equiv_{wtt} B_1$ and $A_2 \equiv_{wtt} B_2$. $\square$

Proof of Theorem 5.14. By Lemma 5.16, it suffices to show that if A is a cylinder and $B \leqslant_{wtt} A$, then A has a splitting A_1, A_2 such that $A_1 \equiv_{wtt} B$. Let $A = \omega \times C$. Let f be an enumeration of A such that

$$(\forall x, y, z, s)[f(s) = \langle x, z \rangle \wedge x < y \Rightarrow (\exists t > s)[f(t) = \langle y, z \rangle]]. \quad (5.3)$$

Suppose that $\Gamma(A) = B$ with recursive use function γ; we suppose also that γ is strictly increasing. We enumerate A_1 and A_2 in stages as follows.

Stage $s + 1$

Step 1. Do nothing unless there is x such that $B_s(x) = \Gamma_s(A_s; x) = 0$ but it is not the case that $B_{s+1}(x) = \Gamma_s(A_{s+1}; x) = 0$. Find $t > s$ minimal with $B_t(x) = \Gamma_t(A_t; x)$. If $B_t(x) = 0$ do nothing, otherwise let $\langle w, y \rangle$ be the least element of $A_t - A_s$ and enumerate $\langle \gamma(x), y \rangle$ into A_1. (Note that $w \leqslant \gamma(x)$.)

Step 2. If $f(s) \notin A_{1,s+1}$, enumerate $f(s)$ into $A_{2,s+1}$.

Condition 5.3 and step 2 guarantee that A_1, A_2 is a splitting of A since $A_{2,s} \subseteq A_s = \{x \mid (\exists t \leqslant s)[f(t) = x]\}$. To see that $B \leqslant_{wtt} A_1$, let

$$s(x) = (\mu s > x)[B_s(x) = \Gamma_s(A_s; x)$$
$$\wedge A_{1,s}[\langle \gamma(x), \gamma(x) \rangle] = A_1[\langle \gamma(x), \gamma(x) \rangle]].$$

We claim that $B(x) = B_{s(x)}(x)$. If not, let $t > s(x)$ be least with $B_t(x) = 1$. It is not difficult to see that no number of the form $\langle \gamma(x), z \rangle$ for $z \leqslant \gamma(x)$ has entered A_1 (as γ is strictly increasing). The construction ensures that $A_{1,t+1}[\langle \gamma(x), \gamma(x) \rangle] \neq A_{1,s}[\langle \gamma(x), \gamma(x) \rangle]$, a contradiction. Therefore $B \leqslant_{wtt} A_1$. Finally, as $x \leqslant \langle \gamma(x), z \rangle$ for all z, if follows that $A_1 \leqslant_{wtt} B$ by simple permitting. $\square$

Corollary 5.17 (Ambos-Spies and Fejer [7]). *If* **a** *is contiguous, then* **a** *contains a set with the SUSP.*

Proof. Let C be any r.e. set of degree **a**. Let $A = C \times \omega$. Note that $C \equiv_{wtt} A$. Since **a** is contiguous, $B \leqslant_T A$ implies that $B \leqslant_{wtt} A$. The result follows from this fact and Lemma 5.14. $\square$
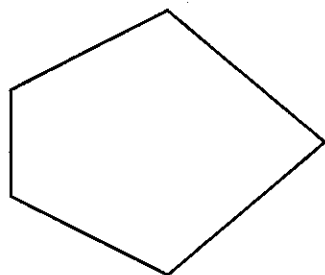
Fig. 2. N_5

Not all sets with the USP have the SUSP. That is a consequence of the next theorem.

Theorem 5.18 (Ambos-Spies and Fejer [7]). *No set of degree $0'$ has the SUSP.*

Theorem 5.18 follows from the next two lemmas.

Lemma 5.19. *Suppose that A has the SUSP. Then $\mathbf{a} = \deg(A)$ is locally distributive. That is,*

$$\forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}\,[\,(\mathbf{a}_0 \cup \mathbf{a}_2 = \mathbf{a} \wedge \mathbf{b} \leqslant \mathbf{a})$$
$$\Rightarrow (\exists \mathbf{b}_1, \mathbf{b}_2)\,[\mathbf{b} = \mathbf{b}_1 \cup \mathbf{b}_2 \wedge \mathbf{b}_i \leqslant \mathbf{a}_i, i = 1, 2]\,].$$

Proof. Suppose that A has the SUSP. Fix degrees $\mathbf{a}_1$ and $\mathbf{a}_2$ such that $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} = \deg(A)$. Suppose that $\mathbf{b} \leqslant \mathbf{a}$. Since A has the SUSP, there are sets A_1 and A_2 of degrees $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively and B of degree $\mathbf{b}$ such that $B \leqslant_{\mathrm{wtt}} A_1 \oplus A_2$. By Lemma 5.15, there are sets B_1 and B_2 such that $B_i \leqslant_{\mathrm{wtt}} A_i$ and such that B_1 and B_2 is a splitting of B. The degrees of these sets are the desired degrees $\mathbf{b}_0$ and $\mathbf{b}_1$. $\square$

Lemma 5.20 (Ambos-Spies [1]). $0'$ *is not locally distributive.*

Ambos-Spies proved Lemma 5.20 by showing that the lattice N_5 depicted in Fig. 2 embeds into $\mathbf{R}$ with top $0'$. This implies that $0'$ is not locally distributive. The proof is a natural extension of techniques due to Shoenfield and Soare [90] but is too long to include here. Recently, Ambos-Spies, Lempp and Lerman have shown that the other five element non-distributive lattice, M_5, can be embedded into $\mathbf{R}$ with top $0'$.

Index set arguments due to Jockusch [55] show that if A has the UWP or the SUSP, then $\deg(A)$ is low$_2$ or complete. Since no USP cylinder is complete, this implies that any cylinder with the USP is low$_2$. The only other constructions of sets with the USP are due to Downey [21], Downey and Jockusch [27], and Lerman and Remmel [70]. Downey's construction makes a low set with the USP, the Downey and Jockusch set is low$_2$, and it seems

that the original Lerman and Remmel set is also low$_2$. Thus a natural question is the following.

Open Question 5.21. Are all incomplete sets with the USP low$_2$?

There are low$_2$ sets with the USP that are not low (Ladner [65] and Ambos-Spies and Fejer [7]). The main question left by the results above is the possibility of the existence of a completely USP degree. This was solved negatively by Downey.

Theorem 5.22 (Downey [19]). *If* $\mathbf{a} \neq \mathbf{0}$ *then* $\mathbf{a}$ *contains a non-USP set.*

Later, Downey improved Theorem 5.22 to the following.

Theorem 5.23 (Downey [21]). (1) *No hypersimple set has the USP.*
(2) *Indeed, if* A *is hypersimple and* B *is a nonrecursive r.e. set, then there is an r.e. set* $Q \leqslant_T B$ *such that for any splitting* A_1, A_2 *of* A, $A_1 \not\equiv_T Q$.
(3) *If* A_1, A_2 *is a splitting of* A *and* A *is hypersimple, then* A_1 *does not have the USP.*

Proof. We sketch the proof of (2). Let A hypersimple and B nonrecursive be given. We construct the r.e. set Q. Let $(U_e, V_e, \Gamma_e, \Phi_e)$ list tuples consisting of a pair of disjoint r.e. sets U_e, V_e and a pair of functionals. The requirements are as follows.

$$\mathbf{R}_e: \quad U_e \cup V_e \neq A \vee \Phi_e(U_e) \neq Q \vee \Gamma_e(Q) \neq U_e.$$

The construction to meet $\mathbf{R}_e$ will be finite injury so we will assume in our description of the strategy that all higher priority requirements have ceased acting. To meet $\mathbf{R}_e$, we will have followers x. Our intention is to enumerate x into Q to arrange that $\Phi_e(U_e) \neq Q$. Associated with $\mathbf{R}_e$ will be a marker Λ_e. The position of Λ_e at stage s, $\Lambda_{e,s}$ will denote the upper bound of the segment of Q devoted to meeting $\mathbf{R}_e$. The strategy is as follows. We wait for a number x to occur so that

$$x > \Lambda_{e,s}, \tag{5.4}$$

$$(\forall y \leqslant x)\,[\Phi_{e,s}(U_{e,s}; y) = Q_s(y) \wedge (\forall z \leqslant u(\Phi_{e,s}(U_{e,s}; y))$$

$$[\Gamma_{e,s}(Q_s; z) = U_{e,s}(z) \wedge U_{e,s} \cup V_{e,s}(z) = A_s(z)], \tag{5.5}$$

$$(\forall z \leqslant \Lambda_{e,s})\,[x > u(\Gamma_{e,s}(Q_s; z))]. \tag{5.6}$$

At this stage we appoint x as a follower of $\mathbf{R}_e$, we reset $\Lambda_{k,s+1} = s + 1$ for $k \geqslant e$, and we define T_x to be the finite set consisting of the interval $(\Lambda_{e,s}, \Lambda_{e,s+1}]$. We also cancel all lower priority followers than x. We would

now like to enumerate x into Q and cause $\Phi_e(U_e; x) \neq Q(x)$. To do this, we need a way of preserving the computation $\Phi_{e,s}(U_{e,s}; x)$. Now if we never enumerate any $y < x$ into Q, we have (by (5.6)) that $U_{e,s}[A_{e,s}] = U_e[A_{e,s}]$. Thus we need only restrain the interval $(A_{e,s}, s]$ of $U_{e,s}$. Note that this interval is what we called T_x. Of course since x is in this interval, we cannot restrain it by holding T_x out of Q. Rather, we wait for a stage $t > s$ $T_x \subseteq A_t$ and $(U_{e,t} \cup V_{e,t}) \cap T_x = A_t \cap T_x$. After stage t, U_e may no longer change on the interval $(A_{e,s}, s]$ or else the hypothesis that U_e, V_e is a splitting of A is false.

If such a stage t occurs, then we declare that x is confirmed. Once x is confirmed, we may later enumerate x into Q (if permitted by B) and create the necessary disagreement. We use the hypersimplicity of A to guarantee that we can get a infinite sequence of confirmed followers (the sets T_x as x ranges over all followers form a strong array) and then it is easy to see that one of the confirmed followers will be permitted by B, else B is recursive. □

The last theorem shows another close connection between the lattice $\mathcal{E}$ of r.e. sets and the structure **R**. It says that if A has a "thin" complement, then it cannot have splittings of all r.e. degrees. One might conjecture that Theorem 5.23 could be extended to simple sets. However we have the following.

Theorem 5.24 (Downey [21]). *There is a promptly simple set of low r.e. degree with the SUSP. Consequently, SUSP is not invariant under automorphisms of $\mathcal{E}$.*

Proof. We omit the proof of the existence of a set, A, with the SUSP which is promptly simple. The noninvariance of the SUSP can be seen as follows. Let B be a promptly simple, hypersimple set of low r.e. degree. (By [8], the deficiency set of A is such a set B.) By Theorem 5.23, B does not have the SUSP. But by Maass [72], there is an automorphism Φ of $\mathcal{E}$ such that $\Phi(A) = B$. □

In the next theorem, we give results on an even stronger notion than SUSP.

Definition 5.25. If r and s are reducibilities such that r is stronger than s, an s degree **a** is called r-topped if there is a set A of s-degree **a** such that for all r.e. sets B such that $B \equiv_s A$, $B \leqslant_r A$. (Similarly for r-bottomed.)

Note that if A witnesses that **a** is r-topped, then for all $B \leqslant_s A$, $B \leqslant_r A$. Note also that contiguous r.e. Turing degrees are both w-topped and w-bottomed. Note that by an index set argument, it is easy to see that if **a** is 1-topped,

then it is low$_2$. Recently, Downey and Shore have used this fact to obtain the following definability result.

Theorem 5.26 (Downey and Shore [33]). *An r.e. set A is low$_2$ if and only if it is bounded in degree by an incomplete 1-topped degree. Hence, the property low$_2$ is definable in the structure of sets with the two orderings $\leqslant_T$ and $\leqslant_m$. Also, an r.e. set is low$_2$ if and only if it has a minimal cover in the r.e. tt-degrees. Hence the property low$_2$ is definable in the r.e. tt-degrees.*

Downey and Jockusch [27] have shown that there is a 1-topped Turing degree **a** such that $0 < \mathbf{a} < \mathbf{0}'$. A consequence of this is the following.

Theorem 5.27 (Downey and Jockusch [27]). *There is a nonrecursive, incomplete r.e. set A such that for any coinfinite, nonsimple r.e. set $B \leqslant_T A$, there is a splitting A_1, A_2 of A such that $A_1 \equiv_1 B$. Hence, for any coinfinite r.e. set $C \leqslant_T A$, there is a splitting A_3, A_4 of A such that $A_3 \equiv_m C$.*

Proof. Let **a** be a 1-topped Turing degree which is incomplete and nonrecursive and let $A \in \mathbf{a}$ be the witness to this. Let $B \leqslant_T A$ be nonsimple. Then $B \leqslant_1 A$ and $B \oplus A \equiv_1 A$. Let γ be a recursive permutation of ω such that $\gamma(B \oplus A) = A$. Let $A_1 = \gamma(B \oplus \emptyset)$ and $A_2 = \gamma(\emptyset \oplus A)$. Obviously, A_1, A_2 is a splitting of A and, as B is nonsimple, $B \equiv_1 B \oplus \emptyset \equiv A_1$. □

Theorem 5.27 can be used together with the next theorem to give an easy proof of the existence of low$_2$ sets with the USP which are not low.

Theorem 5.28 (Downey and Jockusch [27]). *Suppose that A is r.e. and semilow and that B is nonrecursive. Then there is an r.e. set $C \leqslant_{wtt} B$ such that $C \not\leqslant_m A$.*

Corollary 5.29. *No low nonzero r.e. degree is 1-topped.*

The conclusion of Theorem 5.28 can be improved to $C \not\leqslant_{tt} A$ so that no low nonzero r.e. degree is even tt-topped.

Theorem 5.30 (Downey [19]). *There are r.e. degrees $0 < \mathbf{b} < \mathbf{a}$ such that if $A \in \mathbf{a}$, there is a splitting A_1, A_2 of A such that $\deg(A_1) = \mathbf{b}$.*

We will prove Theorem 5.30 in the next section.

We know that all nonzero r.e. degrees contain sets without the USP. We could ask the dual question about the degrees of the splitting witnesses. Along these lines we have the following.

Theorem 5.31 (Downey [24]). *There is an incomplete r.e. set A such that for any r.e. set B with $A \leqslant_{\mathrm{T}} B$, there is a splitting B_1, B_2 of B such that $B_1 \equiv_{\mathrm{T}} A$.*

Proof (sketch). We construct sets A, C_e, and D_e, to meet the following requirements.

$$\mathbf{P}_e: \overline{A} \neq W_e,$$

$$\mathbf{N}_e: \Phi_e(A) \neq K,$$

$$\mathbf{Q}_e: \Gamma_e(V_e) = A \Rightarrow (C_e \cup D_e = V_e \wedge C_e \equiv_{\mathrm{T}} A).$$

Here (Γ_e, V_e) enumerates pairs (Γ, V) consisting of a functional Γ and a set V. We meet the requirements $\mathbf{P}_e$ and $\mathbf{N}_e$ by the standard Friedberg procedures. There are two basic strategies for meeting the requirements $\mathbf{Q}_e$. In [24], Downey uses a construction with an $\omega + 2$ branching tree. Another strategy is the more standard $0'''$ method with the requirements $\mathbf{Q}_e$ spread out over the tree. We describe this latter strategy. To meet requirement $\mathbf{Q}_e$, we must define Turing reductions Δ_e and Λ_e such that $\Delta_e(C) = A$ and $\Lambda_e(A) = C$. Let δ_e and λ_e be the associated use functions which we must also define. We will divide requirement $\mathbf{Q}_e$ into infinitely many subrequirements, $\mathbf{Q}_{e,i}$, each of which has an associated follower $x(e, i, s)$ at stage s. $\mathbf{Q}_{e,i}$ is devoted to insuring that $x = \lim_s x(e, i, s)$ exists and that $\lim_s \lambda_e(\delta_e(x, s))$ exists. $\mathbf{Q}_{e,i}$ can have a Π_2 outcome or a Σ_2 outcome; the Π_2 outcome is that $\lim_s \lambda_e(\delta_e(x, s))$ fails to exist and this outcome will ensure that $\mathbf{Q}_e$ is won absolutely via $\Gamma_e(V_e) \neq A$. The basic strategy for $\mathbf{Q}_{e,i}$ is this; let $x = x(e, i, s)$. Let l_e be the length of agreement function for $\Gamma_e(V_e) = A$. We wait for the first stage such that $l_e(s) > x$. We then define $\delta_e(x, s) = \gamma_e(x, s)$ (where γ_e is the use function of Γ_e) and define $\lambda_e(\delta_e(x, s)) = \langle e + 1, x, s\rangle$. Whenever stage t occurs such that $t > s$ is e-expansionary and $V_{e,s}[\gamma_e(x, s)] \neq V_{e,t}[\gamma_e(x, s)]$, we enumerate this change into $C_{e,t+1} - C_{e,t}$ and $\gamma_e(\delta_e(x, s))$ into $A_{t+1} - A_t$. At stage $t + 1$, we also redefine $\delta_e(x, t + 1) = \gamma_e(x, t + 1)$ and $\gamma_e(\delta_e(x, t + 1)) = \langle e + 1, x, t\rangle$. Note that if $\Gamma_e(V_e) = A$, then $\lim_s \delta_e(x, s)$ exists and hence so does $\lim_s \gamma_e(\delta_e(x, s))$. Unfortunately this procedure, though it makes $A \equiv_{\mathrm{T}} C_e$, quite possibly makes both of these sets complete. The basic difficulty results from the requirements $\mathbf{N}_e$ because these requirements place restraint on A. In particular, we might not be able to enumerate $\gamma_e(\delta_e(x, s))$ into A when we wish to for the sake of $\mathbf{Q}_e$. Since V_e is not under our control, V_e may change when $\gamma_e(\delta_e(x, s))$ is restrained by a higher priority $\mathbf{N}_i$. In this case, our only option is to enumerate the relevant change into D_e rather than C_e. However now $\mathbf{Q}_{e,i}$ may not enumerate an element into A since C_e can no longer comprehend this fact. (Since $V_e[\delta_e(x, s)]$ has changed and hence $\delta_e(x, t)$ may have changed.) We now describe in more detail the interaction of the various requirements and how this basic difficulty in meeting $\mathbf{Q}_e$ is met.

Suppose then that we are concerned with a version of N_e, say N_τ, and a version of Q_k with primary node Q_η. The first case we consider is that $e < k$ and $\tau \subset \eta$. In this case, N_τ will have absolute control over Q_η. Thus, using the standard Sacks strategy we preserve a τ correct length of agreement between $\Phi_\tau(A)$ and K. The versions of Q_η will work in τ stages and also believe that the effect of τ on them is finite; thus each time a τ correct length of agreement arises, these requirements are initialized. Requirements P_y of lower priority than N_τ can also be met in this way.

Now suppose that the situation is the other way; that $k < e$ and $\eta \subseteq \tau$. We consider the cases of the priority of the subrequirements Q_y of the global requirement Q_η. Suppose first that Q_y is of higher local priority than the requirement N_τ. That is, we have $\eta \subset y \subset \tau$. Requirement Q_y is attempting to get $x(\eta, i, s)$ defined. In this case, Q_y functions exactly as in the basic module. Namely, once Q_y defines $x(\eta, i, s)$ it is committed to the appropriate values of λ_η and δ_η. So each time V_k changes we reset λ_η and enumerate the appropriate change in C_η. There are two cases according as to whether N_τ is guessing that Q_y has the Σ_2 or the Π_2 outcome. In the former case of course, it simply gets initialized each time Q_y acts. In the case of the Π_2 outcome, as in the thickness lemma, it waits for a stage such that $\lambda_\eta(\delta_\eta(x(\eta, i, s)))$ exceeds the use of the computation to declare a τ-correct computation.

The hard case is the case where the global priority of N_τ exceeds the local priority of Q_y. That is we have that $\eta \subset \tau \subset y$. (This is the case that usually makes $0'''$ arguments difficult.) The obstacles are as follows. At some τ stage, Q_y gets to assign a number $x_1 = x(\eta, i, s)$. Now we will not let τ act until the next y-expansionary stage when we define axioms for x_1. (We can do this using the technique of links.) It may happen that τ later cancels x_1, but we only need to argue that $A[\lambda_\eta(\delta_\eta(x_1))]$ can figure out $C_\eta[\delta_\eta(x_1)]$ and $C_\eta[\delta_\eta(x_1)]$ can figure out if x_1 enters A. Now if τ does exert control over x_1 and so cancels it, then each time we get a change in $V_{k,t}[\delta(x_1, t)]$, we must put the change into D_η and not C_η at the next η stage. We will consider the effect of this. First, at the next η stage u we can get to redefine $x(\eta, i, u)$ to exceed all previously seen numbers. Note that a fixed N_τ can cancel $x(\eta, i, s)$ only a finite number of times. Either the true outcome of N_τ is finitary or this version of N_τ is incorrect and some infinitary node causes us to move left of N_τ. But in the latter case we will get to define some $x(\eta, i, u)$ at some node ψ such that $\psi <_L \tau$ which N_τ must respect.

The only problem with all of the above is the following. Consider a later incarnation of $x = x(\eta, i, s)$ at y. We will not have complete control to enumerate numbers $\leqslant \lambda_\eta(\delta_\eta(x(\eta, i, s)))$ into A nor numbers $\leqslant \delta_\eta(x(\eta, i, s))$ into C_η. This is because of earlier injury. As a representative illustration, at stage s_0 we have promises for numbers $< z, \delta_\eta(z), \gamma_\eta(\delta_\eta(z))$. Now at stage s_1 we define $x_1 = x(\eta, i, s_1)$ and $\lambda_\eta(\delta_\eta(x_1))$. At stage $s_2 > s_1$, nothing has changed but N_τ asserts control. At $s_3 > s_2$, V_k changes and the change is

enumerated into D_η not C_η. Now at $s_4 \geqslant s_3$, we define a new $x_2 = x(\eta, i, s_4)$. The diagram shows the situation with dotted lines for reductions that have vanished. (We have omitted some subscripts and stages on use functions.)
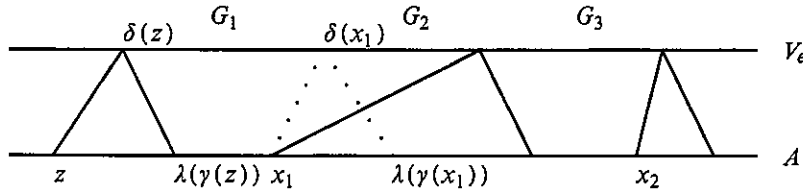
Now in $G_2 \cup G_3$, we promise that if we get a change then we put it into C_η and $\lambda(\gamma(x_1))$ into A. Now remember that the G_1 change has gone into D_η. Suppose now that we get a V_k change below $\delta(x_1)$ yet N_τ still has control over G_1. So again we cannot put this change into C_η so that it must go into D_η. This in general we cannot enumerate $\gamma(\delta(x_2))$ into A. The key to remember is that this process can happen only finitely often as G_1 is finite and so can reset x_2's use only finitely often. This means that we can still ensure that $C_\eta \leqslant_T A$. That $V_k \leqslant_T A$ requires more work. Basically, when we reset, we need a new $x(\eta, i, \cdot) = x_3$, say that exceeds all the rest. Suppose $V[G_3]$ has reached its final state. Now we will not put either of x_1, x_2 into A so V_k cannot be wrong about them. We only need to check that $\delta(x_3)$ can be moved for any change above G_1 but this is clearly possible since $\lambda(\delta(x_2))$ can legally be added to A. Thus it follows that we get a version of $x(\eta, i, \cdot)$ that becomes stable and for which C_η can comprehend its entry. Note that C_η can comprehend $\lambda(\gamma(x_j))$'s entry as we need a V_k (and thus C_η) change below $\delta(x_j)$. Thus $A \leqslant_T V_k$ too. The remaining details go together in the usual $\mathbf{0}'''$ way. $\square$

6. Embeddings into R and the structure of R

In this section we examine the ways that splitting properties can be used to obtain results about the structure of **R** and that of **W**. The earliest example of such a result is Sacks Splitting Theorem, Theorem 5.1; this theorem implies that if **a** is a nonzero r.e. degree, there are incomparable r.e. degrees $\mathbf{a}_1$, $\mathbf{a}_2$ such that $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a}$. In particular there are no minimal r.e. degrees. Sacks Splitting Theorem was extended by Robinson who showed the following.

Theorem 6.1. *Suppose that* $\mathbf{a} > \mathbf{b}$ *and that* $\mathbf{b}$ *is low. There there are degrees* $\mathbf{a}_1$, $\mathbf{a}_2$ *such that* $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a}$, $\mathbf{b} < \mathbf{a}_1, \mathbf{a}_2$ *and* $\mathbf{a}_1, \mathbf{a}_2 < \mathbf{a}$.

In fact, Theorem 6.1 was proven by a splitting theorem. The corresponding splitting theorem, in which Robinson introduced what is now referred to as the "Robinson Trick", is the following.

Theorem 6.2. *Suppose that A and B are r.e. sets with $B <_T A$ and B low. Then there is a splitting A_1, A_2 of A such that $A_i \oplus B <_T A$ for $i = 1, 2$.*

Proof. It is easy to see that the following requirements suffice.

$$\mathbf{R}_{e,i}: \quad \Phi_e(B \oplus A_i) \neq A_{2-i}.$$

As in the Sacks Splitting Theorem, we let $l(\langle e, i\rangle, s)$ be the length of agreement function for $\Phi_e(B \oplus A_i) = A_{2-i}$ and let $r(\langle e, i\rangle, s)$ be the restraint imposed on A_i to preserve $l(\langle e, i\rangle, s)$. Of course the difficulty here is that we do not have complete control over $r(\langle e, i\rangle, s)$ since we do not control the enumeration of B. Instrumental in the proof is the following lemma.

Lemma 6.3. *Suppose that B is a low r.e. set and $\{D_n\}_{n\in\omega}$ is the canonical indexing of finite sets. Then there is a recursive function f such that for all j*

$$W_j \cap \{n : D_n \subseteq \overline{B}\} = W_{f(j)} \cap \{n : D_n \subseteq \overline{B}\} \quad and \tag{6.1}$$

$$W_j \cap \{n : D_n \subseteq \overline{B}\} = \emptyset \Rightarrow W_{f(j)} \text{ is finite.} \tag{6.2}$$

The lemma is used to modify the basic construction of the Sacks Theorem in the following way. Suppose that we are attempting to define $r(\langle e, i\rangle, s)$. We will pay attention to a computation used in establishing $l(\langle e, i\rangle, s)$ only if we can "B-certify" the computation in the following way. When we wish to certify a computation, say $\Phi_{e,s}(B_s \oplus A_{i,s})(x)$, we let u be the use in establishing this computation. We then let n be such that $D_n = \overline{B}_s[u]$ and enumerate n into a set $V_{\langle e,i,x\rangle}$ which we construct. By the recursion theorem, we assume that we know an index, j, for $V_{\langle e,i,x\rangle}$. We then simultaneously enumerate $W_{f(j)}$ and B until either n occurs in $W_{f(j)}$ or some number $z \leq u$ appears in B. Such must happen by Lemma 6.3. In the former case, we say that the computation is certified and we can use it in establishing the length of agreement. In the latter case, the computation is not correct so we ignore it. Equation (6.1) says that we will certify any B-correct computation; equation (6.2) says that we will not certify infinitely many incorrect computations at a single argument x. Thus we will not have the $\lim_s r(\langle e, i\rangle, s) = \infty$ if $\Phi_e(B \oplus A_i) \neq A_{2-i}$. Of course when a B-certified computation becomes invalid because of injury (i.e., our enumeration into A_i rather than enumeration into B), we must restart the certification process with a new version of $V_{\langle e,i,x\rangle}$. $\square$

A natural question arising from Theorem 6.1 is whether the hypothesis that the degree **b** is low be removed. The next result, the celebrated "monster" result of Lachlan, answered this question negatively.

Theorem 6.4 (Lachlan [62]). *There exist degrees* $\mathbf{b} < \mathbf{a}$ *such that if* $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a}$ *and* $\mathbf{b} < \mathbf{a}_1, \mathbf{a}_2$ *then the degrees* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *are comparable.*

The proof of Theorem 6.4 is very difficult and we omit it. This proof was particularly important since Lachlan introduced the $\mathbf{0}'''$ priority method with it. This was the key technique that led to the eventual proof that $\mathrm{Th}(\mathbf{R})$ is undecidable by Harrington and Shelah [48] (see also [49]).

Harrington [46] showed that one may take $\mathbf{a} = \mathbf{0}'$ in Theorem 6.4 and Jockusch and Shore [56] showed how these results are related using pseudo-jumps. Finally Slaman (unpublished) has claimed that one may make $\mathbf{b}$ low$_2$ and $\mathbf{a}$ low$_3$.

For the structure of the weak-truth-table degrees, $\mathbf{W}$, the situation is different. Ladner and Sasso showed that splitting and density could be combined.

Theorem 6.5 (Ladner and Sasso [67]). *Suppose that A and B are r.e. sets with $B <_{\mathrm{wtt}} A$. Then there is a splitting A_1, A_2 of A such that $B \oplus A_i <_{\mathrm{wtt}} A$ for $i = 1, 2$.*

Proof (sketch). The requirements are

$$\mathbf{R}_{e,i}: \quad \Phi_e(B \oplus A_i) \neq A_{2-i}.$$

Here Φ_e is the eth wtt-reduction. As usual, we let $l(e, i, \cdot)$ be the length of agreement function for $\Phi_e(B \oplus A_i) = A_{2-i}$ and let $r(e, i, s)$ be the restraint necessary to preserve the computations through $l(e, i, s)$ (as in Sacks Splitting Theorem). One now performs exactly the Sacks construction. The reason that $\lim_s r(e, i, s) < \infty$ is that we are using wtt-reductions so that if $\Phi_e(B \oplus A_i) \neq A_{2-i}$, then $\lim_s l(e, i, s) < \infty$ and the use function of Φ_e is bounded by a recursive function. $\square$

Theorem 6.5 and the distributivity of $\mathbf{W}$ implies more regularity in the structure of $\mathbf{W}$ than is present in $\mathbf{R}$. As a consequence, the Harrington–Shelah techniques alone do not suffice to prove that $\mathrm{Th}(\mathbf{W})$ is undecidable. The undecidability of the theory of $\mathbf{W}$ was recently established using the distributivity of $\mathbf{W}$ in an essential way by Ambos-Spies, Nies, and Shore [11]. The degree of this theory is still unknown.

Another interesting nonsplitting result for $\mathbf{R}$ is also due to Lachlan.

Theorem 6.6 (Non-Diamond Theorem, Lachlan [59]). *There are no r.e. degrees $\mathbf{a}_1$, $\mathbf{a}_2$ such that $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{0}'$ and $\mathbf{a}_1 \cap \mathbf{a}_2 = \mathbf{0}$.*

The proof of Theorem 6.6 is well-known and we omit it. A pair $\mathbf{a}_1$, $\mathbf{a}_2$ of nonzero r.e. degrees is called a *minimal pair* if $\mathbf{a}_1 \cap \mathbf{a}_2 = \mathbf{0}$. In contrast to

Theorem 6.6, minimal pairs of r.e. degrees exist. Lachlan [59] and Yates [100] were the first to construct such pairs. A slight variation of their construction can be used to show:

Theorem 6.7 (Downey and Welch [41], Ambos-Spies [5]). *There is a nonrecursive r.e. set A such that if A_1, A_2 is a splitting of A then $\deg(A_1) \cap \deg(A_2) = \mathbf{0}$.*

Proof (sketch). Let (U_e, V_e, Φ_e) enumerate triples consisting of a disjoint pair U_e, V_e of r.e. sets and functional. It suffices to meet the following requirements.

$\mathbf{P}_e$: $\overline{A} \neq W_e$,

$\mathbf{N}_e$: $U_e \cup V_e = A \wedge \Phi_e(U_e) = \Phi_e(V_e) = f \wedge f$ total $\Rightarrow f$ recursive.

We employ the following variation of the standard minimal pair construction. First, we establish length of agreement in the computations $\Phi_{e,s}(U_{e,s}), \Phi_{e,s}(V_{e,s})$ by only admitting computations $\Phi_{e,s}(U_{e,s}; x), \Phi_{e,s}(V_{e,s}; x)$ such that if $z \leqslant u(\Phi_{e,s}(U_e, s; x))$ and $z \leqslant u(\Phi_{e,s}(U_e, s; x))$ then $U_{e,s} \cup V_{e,s}[x] = A_s[x]$. This implies that we can preserve such computations by restraining A. Then we use the usual minimal pair strategy. Namely, for the sake of $\mathbf{N}_e$, at an e-expansionary stage we allow the restraint to drop and then allow at most one number to enter A for the sake of a lower priority P_i and then reimpose the restraint. $\square$

Sets with the property of Theorem 6.7 are called strongly atomic sets.

Definition 6.8. A nonrecursive set A is *strongly atomic* if for every splitting A_1, A_2 of A, $\deg(A_1) \cap \deg(A_2) = \mathbf{0}$. (Such sets were called *antimitotic* by Ambos-Spies.)

Corollary 6.9 (Lachlan [59] and Yates [100]). *There are minimal pairs.*

Proof. Let A be a strongly atomic set and A_1, A_2 a Sacks splitting of A. Then $\deg(A_1)$, $\deg(A_2)$ form a minimal pair. $\square$

The existence of strongly atomic sets gives much information about the structure of **R**. We first prove the following lemma.

Lemma 6.10 (Downy and Welch [41]). *Suppose that A is strongly atomic and that A_1, A_2 and B_1, B_2 are two different splittings of A. Then*

(1) $A_i \cap B_j \leqslant_T A_i, B_j$, $i, j = 1, 2$. (*This doesn't depend on the strong atomicity of A.*)

(2) *If $A_1 \leqslant_T B_1$ then $A_1 \cap B_2$ is recursive.*

(3) *If* $A_1 \leqslant_T B_1$ *then* $B_2 \leqslant_T A_2$.
(4) *If the splittings consist of nonempty sets and* $A_1 \equiv_T B_1$, *then* $A_1 \equiv_m B_1$.
(5) A_1 *is strongly atomic.*

Proof. (1) To show that $A_1 \cap B_1 \leqslant_T B_1$, for example, suppose x is given. If $x \notin B_1$ then $x \notin A_1 \cap B_1$. If $x \in B_1$, enumerate A_1 and A_2 until x enters either A_1 or A_2 and answer accordingly.

(2) Suppose that $A_1 \leqslant_T B_1$. By (1), $A_1 \cap B_2 \leqslant_T A_1, B_2$. Thus $A_1 \cap B_2 \leqslant_T B_1, B_2$ and hence is recursive since A is strongly atomic and B_1, B_2 is a splitting of A.

(3) If $A_1 \leqslant_T B_1$ then $A_1 \cap B_2$ is recursive by (2). Hence $B_2 = (A_1 \cap B_2) \cup (A_2 \cap B_2) \equiv_T (A_2 \cap B_2) \leqslant_T A_2$.

(4) Suppose that $A_1 \equiv_T B_1$. Then by (2), $A_1 \cap B_2$ and $A_2 \cap B_1$ are recursive. Let b_1, b_2 be fixed elements of B_1 and B_2 respectively. Define a function f by

$$
f(n) = \begin{cases} b_1 & \text{if } n \in A_1 \cap B_2, \\ b_2 & \text{if } n \in A_2 \cap B_1, \\ n & \text{otherwise.} \end{cases}
$$

It is easy to see that f witnesses that $A_1 \leqslant_m B_1$. Similarly, one can show that $B_1 \leqslant_m A_1$.

(5) Suppose that C_1, C_2 is a splitting of A_1. Suppose that $D \leqslant_T C_1, C_2$; we must show that D is recursive. Note that C_1, $C_2 \cup A_2$ is a splitting of A. Also, $D \leqslant_T C_2 \cup A_2$ since C_2 and A_2 are disjoint. Thus D is recursive since A is strongly atomic. $\square$

Downey and Welch extended Lemma 6.10 to the following.

Theorem 6.11 (Downey and Welch [41]). *Let A be strongly atomic. Let $\mathcal{H}(A) = \{A_1 \mid A_1$ is half of a splitting of $A\}$. Let $f : \mathcal{H}(A) \to \mathcal{S}(A)$ be the map defined by $f(A) = \deg(A)$. Then f is a surjective homomorphism of Boolean algebras.*

Proof. That f is surjective follows trivially from the definition of $\mathcal{S}(A)$ (recall that $\mathcal{S}(A)$ consists of the degrees of halves of splittings of A). That f preserves inclusions follows directly from Lemma 6.10(1). We need only show that f preserves the join and meet operators of the Boolean algebra.

Note first of all that $\mathcal{H}(A)$ is a Boolean Algebra. That is, if A_1 and B_1 are halves of splittings of A, then so are $A_1 \cup B_1$ and $A_1 \cap B_1$. Now to show that f preserves supremums, it suffices to show that $\deg(A_1 \cup B_1) = \deg(A_1 \oplus B_1)$ for this shows that $\mathcal{S}(A)$ inherits the join operator from **R**. Obviously, $A_1 \cup B_1 \leqslant_T A_1 \oplus B_1$. The other direction, that $A_1, B_1 \leqslant_T A_1 \cup B_1$ follows by an argument similar to that for Lemma 6.10(1). To show that f preserves infimums, we need to show that if C_1, C_2 is another splitting of A and $C_1 \leqslant_T A_1, B_1$, then

$C_1 \leqslant_T A_1 \cap B_1$. Since $C_1 \leqslant_T A_1, B_1$, we have that $A_2, B_2 \leqslant_T C_2$ by Lemma 6.10(3). Thus $A_2 \cup B_2 \leqslant_T C_2$. But $A_1 \cap B_1$, $A_2 \cup B_2$ is a splitting of A and thus, again by Lemma 6.10(3), we have that $C_1 \leqslant_T A_1 \cap B_1$. □

For strongly atomic sets of contiguous degree, we get an even stronger embedding theorem.

Theorem 6.12 (Downey and Welch [41] and Ambos-Spies [5]). *Suppose that A is a strongly atomic set and that $\deg(A)$ is contiguous. Then the function f as defined in Theorem 6.11 above defines an embedding of the countable atomless Boolean algebra into $\mathbf{R}$ preserving supremums and infimums.*

The proof relies on the following lemma.

Lemma 6.13 (Ambos-Spies [5]). *Suppose that $\mathbf{a}$ is contiguous, $\mathbf{b} \cap \mathbf{c} = 0$ and $\mathbf{b} \cup \mathbf{c} = \mathbf{a}$. Then $\mathbf{b}$ is contiguous.*

Proof. Let B and C be r.e. sets of degrees $\mathbf{b}$ and $\mathbf{c}$ respectively and let D be another set of degree $\mathbf{b}$. It suffices to show that $D \leqslant_{wtt} B$. We have that $D \leqslant_{wtt} B \oplus C \equiv_T A$ so that D has a splitting D_1, D_2 such that $D_1 \leqslant_{wtt} B$ and $D_2 \leqslant_{wtt} C$. However, $D_2 \leqslant_T B, C$ implies that D_2 is recursive and hence that $D \equiv_{wtt} D_1 \leqslant_{wtt} B$. □

Proof of Theorem 6.12. In view of Lemma 6.10, it suffices to show that if A_1 and B_1 are halves of splittings of A, then $\deg(A_1 \cap B_1) = \deg(A_1) \cap \deg(B_1)$.

Obviously, $\deg(A_1 \cap B_1) \leqslant \deg(A_1) \cap \deg(B_1)$. So suppose that $C \leqslant_T A_1, B_1$; we must show that $C \leqslant_T A_1 \cap B_1$. By Lemma 6.13, $C \leqslant_{wtt} A_1, B_1$. Now $A_1 \cap B_1$ and $A_1 \cap B_2$ is a splitting of A_1. Thus $C \leqslant_{wtt} (A_1 \cap B_1) \oplus (A_1 \cap B_2)$. Thus, C has a splitting C_1, C_2, such that $C_1 \leqslant_{wtt} A_1 \cap B_1$, $C_2 \leqslant_{wtt} A_1 \cap B_2 \leqslant_{wtt} B_2$. But then C_2 must be recursive since $C_2 \leqslant_{wtt} B_1, B_2$. Thus $C \equiv_{wtt} C_1 \leqslant_{wtt} A_1 \cap B_1$. □

To make Theorem 6.12 useful, we need to construct a contiguous strongly atomic set. A direct construction is possible (see [41]) but there is a simpler argument due to Ambos-Spies based on the following lemma.

Lemma 6.14 (Ambos-Spies [5]). *Suppose that $B \leqslant_{wtt} A$. Then there is an r.e. set $C \equiv_{wtt} B$ such that for every splitting C_1, C_2 of C, there is a splitting A_1, A_2 of A such that $C_i \leqslant_{wtt} A_i$, $i = 1, 2$.*

Proof. Let $\Gamma(A) = B$ be the reduction which witnesses that $B \leqslant_{wtt} A$ and let γ be the recursive use function. Let l be the associated length of agreement function. We will suppose that the sets and Γ are enumerated so that $l(s+1) > l(s)$ for every s. Define C by $C_{s+1} = C_s \cup \{(\mu z)[z \in B_{s+1} - B_s]\}$.

Suppose then that C_1, C_2 is a splitting of C; suppose also that these sets are enumerated so that $C_{1,s} \cup C_{2,s} = C_s$ for every s. We define A_1 and A_2 as follows. If $C_{s+1} - C_s \subseteq C_{1,s+1}$, then define $A_{1,s+1} = A_{1,s} \cup (A_{s+1} - A_s)$ and $A_{2,s+1} = A_{2,s}$. Otherwise, define $A_{1,s+1} = A_{1,s}$ and $A_{2,s+1} = A_{2,s} \cup (A_{s+1} - A_s)$. It is easy to see that if t is such that $A_{1,t}[\gamma(x)] = A_1[\gamma(x)]$, then $C_{1,t}[x] = C_1[x]$. $\square$

Corollary 6.15. *Suppose that A is strongly atomic and that $B \leqslant_{\mathrm{wtt}} A$. Then there is $C \equiv_{\mathrm{wtt}} B$ such that C is strongly atomic.*

We now have

Theorem 6.16 (Ambos-Spies [5] and Downey and Welch [41]). *There is a contiguous strongly atomic degree. Indeed, every strongly atomic degree bounds a contiguous strongly atomic degree.*

Strongly atomic sets have several other applications. For example, Lachlan [63] showed that there is a nonzero degree **a** such that every nonzero **b** $\leqslant$ **a** bounds a minimal pair. This result can be extended to the following.

Theorem 6.17 (Ambos-Spies [5] and Downey and Welch [41]). *There is a nonzero r.e. degree **a** such that for all **b** with $0 <$ **b** $\leqslant$ **a**, **b** is the supremum of a minimal pair.*

Proof. Let A be strongly atomic and of contiguous degree. The theorem follows immediately from Corollary 6.15. $\square$

Another easy application of strongly atomic sets of contiguous degree is the following. We say a degree **a** bounds a 1–3–1 lattice with least element **b** $<$ **a** if there are incomparable degrees a_1, a_2, $a_3 <$ **a** such that **b** is the infimum of any pair of them and each pair of the three has the same supremum. Lachlan has shown that $0'$ bounds a 1–3–1 lattice with least element **0**. He has also show that there are degrees which bound no minimal pairs.

Theorem 6.18 (Downey [19]). *There is a degree **a** $\neq$ **0** such that every nonzero degree **b** $\leqslant$ **a** is the supremum of a minimal pair but such that **a** bounds no 1–3–1 lattice with least element **0**.*

Proof. Again, let A be strongly atomic and of contiguous degree. Suppose that B_1, B_2, B_3 are sets of the degrees witnessing that $\deg(A)$ bounds a 1–3–1 lattice with least element **0**. By Lachlan's Lemma, $A = A_1 \cup A_2 \cup A_3$ where $A_i \leqslant_{\mathrm{wtt}} B_i$ for $i = 1, 2, 3$. We claim that, in fact, $A_i \equiv_{\mathrm{wtt}} B_i$. This would be a contradiction for it would imply that $A_1 \equiv_{\mathrm{wtt}} B_i \leqslant_{\mathrm{T}} B_2 \oplus B_3 \equiv_{\mathrm{wtt}} A_2 \oplus A_3$. This cannot happen since A_1, $A_2 \cup A_3$ is a splitting of a strongly atomic set.

To see that $B_1 \leqslant_{\text{wtt}} A_1$, say, note first that $B_1 \leqslant_{\text{wtt}} A_1 \oplus A_2 \oplus A_3$. Thus B_1 has a splitting D_1, D_2, D_3, with $D_i \leqslant_{\text{wtt}} A_i \leqslant_{\text{wtt}} B_i$. This implies that D_2 and D_3 are recursive since $D_2 \leqslant_{\text{wtt}} B_1, B_2$ and $D_3 \leqslant_{\text{wtt}} B_1, B_3$. Thus $B_1 \equiv_{\text{wtt}} D_1 \leqslant_{\text{wtt}} A_1$. $\square$

The preceding theorem can be extended using a $0'''$ argument to the following.

Theorem 6.19 (Downey [25]). *If* $\mathbf{a} \neq \mathbf{0}$ *then there is a degree* $\mathbf{b}$ *such that* $\mathbf{0} < \mathbf{b} \leqslant \mathbf{a}$ *and such for every* $\mathbf{c} \leqslant \mathbf{b}$, $\mathbf{b}$ *bounds no* 1–3–1 *lattice with least element* $\mathbf{c}$.

However, it is not known whether Theorem 6.19 holds for every strongly atomic contiguous degree. Downey and Shore [34] have shown that if $\mathbf{a}$ is low$_2$, then $\mathbf{a}$ bounds a 1–3–1 lattice (not necessarily preserving $\mathbf{0}$).

Open Question 6.20. Are there degrees $\mathbf{a}$ and $\mathbf{c}$ such that $\mathbf{a}$ is strongly atomic and contiguous and such that $\mathbf{a}$ bounds a 1–3–1 lattice with least element $\mathbf{c}$?

As a final result using contiguous strongly atomic degrees, we supply the promised proof of Theorem 5.30.

Theorem 5.30 (Downey [19]). *There are degrees* $\mathbf{0} < \mathbf{b} < \mathbf{a}$ *such that if A is any set of degree* $\mathbf{a}$, *then* $\mathbf{b} \in \mathcal{S}(A)$.

Proof. Let C be a strongly atomic set of contiguous degree. Let $\mathbf{a} = \deg(C)$. Let $\mathbf{b}$ be any degree in $\mathcal{S}(C)$. Suppose that A is of degree $\mathbf{a}$. Then $A \equiv_{\text{wtt}} C$ by contiguity of $\mathbf{a}$. Let C_1, C_2 be any splitting of C such that $C_1 \in \mathbf{b}$. Then A has a splitting A_1, A_2 such that $A_1 \leqslant_{\text{wtt}} C_1$ and $A_2 \leqslant_{\text{wtt}} C_2$. We show that $A_1 \equiv_{\text{wtt}} C_1$ so that A_1 witnesses that $\mathbf{b} \in \mathcal{S}(A)$.

To see this, note that since $C_1 \leqslant_{\text{wtt}} A \equiv_{\text{wtt}} A_1 \oplus A_2$, C_1 has a splitting D_1, D_2 such that $D_1 \leqslant_{\text{wtt}} A_1$, $D_2 \leqslant_{\text{wtt}} A_2$. Since $D_2 \leqslant_{\text{wtt}} C_1$ and $D_2 \leqslant_{\text{wtt}} A_2 \leqslant_{\text{wtt}} C_2$ and C_1, C_2 form a minimal pair, D_2 is recursive. Thus $C_1 \equiv_{\text{wtt}} D_1 \leqslant_{\text{wtt}} A_1$ and thus $C_1 \equiv_{\text{wtt}} A_1$. $\square$

One of the limitations in using contiguous degrees to gain information on embeddings into **R** is that all contiguous degrees are low$_2$. Thus these degrees are only useful in analyzing certain initial segments of **R**. A possible line of inquiry then is the extension of the definition of contiguity to one useful for larger portions of **R**. One way to do this is to replace $\leqslant_{\text{wtt}}$ by another reducibility notion. We suggest two such possibilities here.

Definition 6.21. Define $A \leqslant_{\text{T}}^{f} B$ if there is a recursive function g such that $A \leqslant_{\text{T}} B$ by a reduction Γ such that the use function, u, of Γ satisfies

$$|\{s \mid u(\Gamma_s(B_s; x)) \neq u(\Gamma_{s+1}(B_{s+1}; x))\}| \leqslant g(x).$$

It is easy to see that $\leqslant_T^f$ is a reduction and many of the properties of wtt-degrees carry over to this setting. The reason for this is that disagreements give finite restraints rather than just restraints with finite lim inf.

Open Question 6.22. What is the structure of the degrees under $\leqslant_T^f$ and what is the relationship of this structure to that of **R**?

Another such notion is this.

Definition 6.23. Let C be an r.e. set. We say that $A \leqslant_{C\text{-wtt}} B$ if $A \leqslant_T B$ via a reduction Γ such that the use function of Γ is bounded by a C-recursive function.

Given a Turing reduction φ, a C-recursive function φ, and an enumeration of C, we denote by Φ_C the wtt-C-reduction with use function φ defined as follows. Given x, let $u = \varphi_s^{C_s}(x)$. Then define $\Phi_{C,s+1}(B_{s+1}; x)$ to be equal to $\Phi_{C,s}(B_s; x)$ unless $B_{s+1}[u] \neq B_s[u]$ In this way, we can obviously generate a simultaneous enumeration of all such reductions $\{\Phi_{e,C} \mid e \in \omega\}$. The key fact that we need in carrying results for wtt-reducibility over to this new reducibility is the following easy analogue of Lachlan's Lemma.

Theorem 6.24. *Let B, C, A_1, and A_2 be r.e. sets with $B \leqslant_{C\text{-wtt}} A_1 \oplus A_2$. Then there is a splitting B_1, B_2 of B such that $B_1 \leqslant_{C\text{-wtt}} A_1 \oplus C$ and $B_2 \leqslant_{C\text{-wtt}} A_2 \oplus C$.*

Proof. The proof is very similar to that of Lachlan's Lemma. Suppose that $\Phi_C(A_1 \oplus A_2) = B$ with use function bounded by φ^C. Let l denote the associated length of agreement function and $u(x, s)$ the use of the approximation to this functional at s. We will assume that u is monotone in both variables and that the enumerations of the sets involved guarantee that

$$(\exists x)[(x \in A_{1,s+1} \oplus A_{2,s+1}) \wedge (x \notin A_{1,s} \oplus A_{2,s})] \, l(s + 1) > l(s).$$

Construction
Stage $s + 1$
Let x be the least element of $(A_{1,s+1} \oplus A_{2,s+1}) - (A_{1,s} \oplus A_{2,s})$. If x is even (corresponding to enumeration in $A_{1,s+1}$) let $B_{1,s+1} = B_{1,s} \cup (B_{s+1} - B_s)$ and $B_{2,s+1} = B_{2,s}$. Otherwise let $B_{2,s+1} = B_{2,s} \cup (B_{s+1} - B_s)$ and $B_{1,s+1} = B_{1,s}$.

Obviously, B_1, B_2 is a splitting of B. To see that $B_1 \leqslant_{C\text{-wtt}} A_1 \oplus C$, let x be given. To compute $B_1(x)$, use C to find a stage t such that $u = \varphi^C(x)$ is correctly computed. Now find $s \geqslant t$ such that $A_{1,s}[t] = A[t]$ and $l(s) > x$. We claim that $x \in B_1$ if and only if $x \in B_s$. For if x enters B at a stage

later than s, it must cause a change in $A_1 \oplus A_2$ below $u(x,s) \leqslant u$ and this change must be in A_2 rather than in A_1 by choice of s. Consequently, x is not enumerated in B_1. □

We can now extend the notion of contiguity to this case.

Definition 6.25. An r.e. set A is *C-contiguous* if for every r.e. set B such that $B \equiv_T A$, $B \equiv_{C\text{-wtt}} A$.

Given an r.e. degree c, A is *c-contiguous* if A is C-contiguous for some (every) r.e. set C of degree c.

A is *strongly C-contiguous* if for every set X (not necessarily r.e.) such that $X \equiv_T A$, $X \equiv_{C\text{-wtt}} A$.

We now have the following analogue of distributivity.

Definition 6.26. A degree $\mathbf{a}$ is *locally distributive over* $\mathbf{b}$ if $\mathbf{a}$ is distributive relative to the interval $[\mathbf{b}, \mathbf{a}]$. That is, $\mathbf{b} < \mathbf{a}$ and

$$(\forall \mathbf{a_1}, \mathbf{a_2}, \mathbf{c}) [(\mathbf{a_1} \cup \mathbf{a_2} = \mathbf{a} \wedge \mathbf{b} \leqslant \mathbf{c} \leqslant \mathbf{a})$$
$$\Rightarrow (\exists \mathbf{c_1}, \mathbf{c_2}) [\mathbf{c_1} \cup \mathbf{c_2} = \mathbf{c} \wedge \mathbf{c_1} \leqslant \mathbf{b} \cup \mathbf{a_1} \wedge \mathbf{c_2} \leqslant \mathbf{b} \cup \mathbf{a_2}]].$$

$\mathbf{a}$ is *weakly distributive* if $\mathbf{a}$ is distributive over $\mathbf{b}$ for some degree $\mathbf{b} < \mathbf{a}$.

The next lemma follows easily from Theorem 6.24.

Lemma 6.27. *Suppose that $\mathbf{a}$ is C-contiguous and that $\deg(C) < \mathbf{a}$. Then $\mathbf{a}$ is locally distributive over $\deg(C)$.*

The following lemma is also easy.

Lemma 6.28. *$\mathbf{a}$ is contiguous if and only if $\mathbf{a}$ is c-contiguous for all $\mathbf{c} \leqslant \mathbf{a}$.*

If $\mathbf{d} < \mathbf{a}$ then $\mathbf{a}$ is d-contiguous if and only if $\mathbf{a}$ is c-contiguous for all $\mathbf{c}$ such that $\mathbf{d} \leqslant \mathbf{c} < \mathbf{a}$.

It is also fairly easy to construct degrees $\mathbf{a}$ which are c contiguous for some $\mathbf{c} < \mathbf{a}$ but which are not contiguous. It is well known that $\mathbf{0}'$ is not contiguous (see [27]). But we have

Theorem 6.29. *There is an r.e. degree $\mathbf{c} < \mathbf{0}'$ such that $\mathbf{0}'$ is strongly c contiguous.*

Proof. Downey's construction of a strongly contiguous r.e. degree (Theorem 5.12) gives the following relativized version.

$$(\exists e)(\forall X)[X <_T W_e^X \wedge (\forall C)[C \equiv_T W_e^X \Rightarrow C \equiv_{X\text{-wtt}} W_e]].$$

Now by the Jockusch–Shore pseudo-jump theorem (see [56]), there is an r.e. set X such that $W_e^X \equiv_T \emptyset'$. It is clear that $0'$ is then X-contiguous. □

There is some restriction on the degree of X. We have the following theorem by straightforward relativization of the corresponding result for contiguous degrees.

Theorem 6.30. *Suppose that* $\mathbf{a}$ *is c-contiguous. Then* $\mathbf{a}'' = \mathbf{c}''$.

All of the previous results suggest that the notion of wtt-C reducibility might be quite useful just as wtt-reducibility proved useful. However we have not investigated much further than this. Using a rather difficult delayed permitting argument, the first author has shown the following.

Theorem 6.31 (Downey, unpublished). *Suppose that* $B <_T A$. *Then there is an r.e. set* C *such that* $B \oplus C \leqslant_T A$ *and such that* $B \oplus C$ *is strongly B-contiguous.*

We think that the most interesting question here is

Open Question 6.32. Suppose that $\mathbf{a} \neq \mathbf{0}$. Is $\mathbf{a}$ weakly distributive? Indeed, is $\mathbf{a}$ c-contiguous for some degree $\mathbf{c} < \mathbf{a}$?

We turn now to a splitting theorem of Lachlan which has consequences for embedding lattices into $\mathbf{R}$ preserving the greatest element (rather than the least).

Theorem 6.33 (Lachlan [64]). *Suppose that* A *is nonrecursive. Then there is a splitting* A_1, A_2 *of* A *and an r.e. set* C *such that* $\deg(C) = \deg(A_1 \oplus C) \cap \deg(A_2 \oplus C)$ *and* $A_1 \oplus C, A_2 \oplus C <_T A$.

Corollary 6.34 (Lachlan [64] and Shoenfield and Soare [90]). *Every nonzero degree* $\mathbf{a}$ *is the top of a diamond in* $\mathbf{R}$.

Proof (sketch). We have the Sacks requirements

$$\mathbf{N}_{e,i}: \Phi_e(A_i \oplus C) \neq A_{2-i}$$

and the infimum requirements

$$\mathbf{R}_e: \Phi_e(A_1 \oplus C) = \Phi_e(A_2 \oplus C) = f \wedge f \text{ total} \Rightarrow f \leqslant_T C.$$

The main new idea of Lachlan's proof is to meet the requirements $\mathbf{R}_e$ by enumeration rather than by restraint. That is, if both A_1 and A_2 change on a computation between e-expansionary stages, then we allow C to recognize this

fact by enumerating a number into C. This strategy conflicts with the requirements $N_{e,i}$ since these requirements restrain $A_i \oplus C$ to preserve computations according to the Sacks strategy. The conflicts are resolved by a Π_2 tree. Rather than prove this result, we give the reader an idea of the proof by proving a weaker result for wtt-reducibility. That is, we will suppose that the reductions mentioned in the requirement are wtt-reductions. We use $l(e,s)$ to denote the length of agreement function for the requirement R_e and $L(e,i,s)$ to denote the length of agreement function for requirement $N_{e,i}$. We will use $r(\langle e,i\rangle, s)$ to denote the restraint function necessary to preserve $l(e,i,s)$, except that we will also constrain $r(f,s)$ to be nondecreasing in s and f.

Construction
Stage s

Rule 1. For each x, at the first stage s such that $l(e,s) > x$, define $\{\langle e,x,s\rangle, \langle e,x,s+1\rangle, \ldots, \langle e,x,s+s\rangle\}$ to be the set of traces for x and let $\langle e,x,s\rangle$ be the *active* one.

Rule 2. If at stage s, $l(e,s) > m(e,s) > x$ (here m is the maximum length of agreement function for l), and there is no active trace for x, declare the least trace $\langle e,x,i\rangle \notin C_s$ to be active.

Rule 3. Suppose $t < s$ is the greatest stage such that $l(e,t) > x$. Suppose that the trace active for x is $\langle e,x,u\rangle$. If $(A_{1,t} \oplus C_t)[\varphi_e(x)] \neq (A_{1,s} \oplus C_s)[\varphi_e(x)]$ and $(A_{2,t} \oplus C_t)[\varphi_e(x)] \neq (A_{2,s} \oplus C_s)[\varphi_e(x)]$ and $u > r(e,s)$, then enumerate $\langle e,x,u\rangle$ into C_{s+1}.

Rule 4. Enumerate elements of $A_s - A_{s-1}$ into A_1 or A_2 so as to preserve the Sacks requirements of highest priority.

Lemma 6.35. *For all e, $\lim_s r(e,s)$ exists and is finite and so the requirements $N_{e,i}$ are satisfied.*

Proof. Fix $e = \langle f,i\rangle$. First note that for any x, all of the traces u appointed for x satisfy $u > \varphi_e(x)$. Furthermore, there are more than $\varphi_e(x)$ such traces. To see this note that at the stage s that these traces are appointed, $\varphi_e(x) < s < \langle e,x,s\rangle, \langle e,x,s+1\rangle, \ldots, \langle e,x,s+s\rangle$. This implies that not all traces for x are enumerated into C since both sides must change before a trace can be so enumerated (Rule 3). This also implies that such traces can injure a computation $\Phi_{f,s}(A_{i,s} \oplus C_s; y)$ if $\varphi_e(x) < \varphi_f(y)$. $\square$

Now by induction let s_0 be a stage such that for all $i < e$, $\lim_s r(i,s) = r(i,s_0)$ and such that $x \in A - A_{s_0}$ implies that $x > r(i,s_0)$. Now let $s_1 \geq s_0$ be a stage such that if $j \leq e$ and $\lim_s l(f,s) = \infty$, then $l(j,s_1) > \varphi_j(x)$.

Lemma 6.36. *The requirements R_e are satisfied.*

Proof. Fix e. By the previous Lemma and Rule 3, almost all traces for $\mathbf{R}_e$ which wish to enter C do. Suppose then we are given x such that all the traces for x exceed the restraint imposed on $\mathbf{R}_e$. We show how to compute $\Phi_e(A_1 \oplus C; x) = \Phi_e(A_2 \oplus C; x)$. Let s be the stage such that the traces for x are appointed at stage s. These traces are $\langle e, x, s \rangle, \langle e, x, s+1 \rangle, \ldots, \langle e, x, s+s \rangle$. Let t be the least e-expansionary stage such that $C_t[\langle e, x, s+s \rangle] = C[\langle e, x, s+s \rangle]$. Then it is not difficult to see (using the argument that after stage t, at most one side of the computation changes between e-expansionary stages) that $\Phi_e(A_{1,t} \oplus C_t; x) = \Phi_e(A_1 \oplus C; x)$. $\quad \square$

Shore and Slaman have asked whether Theorem 6.33 can be extended to show that the diamond in question can be embedded above any fixed low degree $\mathbf{b} < \mathbf{a}$. This question was recently answered positively by Downey and Shore.

Theorem 6.37 (Downey and Shore [32]). *Suppose that $\mathbf{a} \mid \mathbf{b}$. Then there is a degree $\mathbf{c}$ such that $\mathbf{a} \cup \mathbf{c}, \mathbf{b} \cup \mathbf{c} < \mathbf{a} \cup \mathbf{b}$ and $(\mathbf{a} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c}) = \mathbf{c}$.*

The proof of Theorem 6.37 is a very difficult $0'''$ argument and is omitted. Theorem 6.37 has the following consequences.

Corollary 6.38 (Slaman Density Theorem). *If $\mathbf{b} < \mathbf{a}$, then there are degrees $\mathbf{c}$, $\mathbf{d}$ such that $\mathbf{b} < \mathbf{c} < \mathbf{a}$, $\mathbf{b} < \mathbf{d} < \mathbf{a}$ and such that $\mathbf{c} \cap \mathbf{d}$ exists.*

Proof. First, let $\mathbf{e}, \mathbf{f}$ be incomparable degrees such that $\mathbf{b} < \mathbf{e} < \mathbf{a}$ and $\mathbf{b} < \mathbf{f} < \mathbf{a}$. Such exist by a routine variation of the Sacks Density Theorem. Now let $\mathbf{e}$ and $\mathbf{f}$ play the roles of $\mathbf{a}$ and $\mathbf{b}$ of Theorem 6.37. $\quad \square$

Corollary 6.39. *If $\mathbf{a}$ splits over $\mathbf{b}$ then $\mathbf{a}$ splits over $\mathbf{b}$ by a pair with an infimum.*

Theorem 6.37 was proved for wtt-degrees in place of T-degrees much earlier by Downey [23]. Corollary 6.39 cannot be extended to splittings of sets as the corollary to the next theorem shows.

Theorem 6.40 (Ambos-Spies [5]). *There is a complete r.e. set A such that for every splitting A_1, A_2 of A, either A_1 or A_2 is low.*

Proof. Suppose that f is a 1-1 recursive enumeration of K. We enumerate A in stages; let $a_{0,s} < a_{1,s} < \cdots$ enumerate the complement of A_s in increasing order. We shall define certain markers $\Lambda_{e,s}$ to rest on some (but not all) members of $\overline{A}_s$. Let (U_e, V_e) list all pairs of disjoint r.e. sets. Given $n = \langle e, i, j \rangle$, define the following:

$$u(n,s) = \min\{u(\Phi_{i,s}(U_{e,s}; i)), u(\Phi_{j,s}(V_{e,s}; j))\}, \tag{6.3}$$

$$m(n,s) = \max\{u(\Phi_{i,s}(U_{e,s};i)), u(\Phi_{j,s}(V_{e,s};j))\}, \tag{6.4}$$

$$q(n,s) = \begin{cases} 1 & \text{if } \Phi_{i,s}(U_{e,s};i)\downarrow, \ \Phi_{j,s}(V_{e,s};j)\downarrow \\ & \text{and } (U_{e,s} \cup V_{e,s})[m(n,s)] = A_s[m(n,s)], \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

The requirements on A are

$$\mathbf{R}_n\colon \ (\exists^\infty s)[q(n,s) = 1] \Rightarrow (\Phi_i(U_e;i)\downarrow \vee \Phi_j(V_e;j)\downarrow).$$

To see that these requirements suffice, suppose that U_e, V_e is a splitting of A. If U_e is not low, then there is a fixed i such that $\Phi_i(U_e;i)\uparrow$ but for which there are infinitely many s such that $\Phi_{i,s}(U_{e,s};i)\downarrow$. But then, since we meet $\mathbf{R}_{n(j)}$ for all j where $n(j) = \langle e,i,j\rangle$, we have that for all j, $(\exists^\infty s)[\Phi_{j,s}(V_{e,s};j)\downarrow] \Rightarrow (\Phi_j(V_e;j)\downarrow)$. This in turn implies that V_e is low.

Construction

Stage 0

Define $A(i,0) = a_{i,0} = i$ for all i.

Stage $s + 1$

Find the least n if any, such that $q(n,s) = 1$ and $A_{n,s} < u(n,s)$. Let $m = \min\{n, f(s)\}$. Define $A_{s+1} = A_s \cup \{A_{m,s}\}$ and define

$$A_{y,s+1} = \begin{cases} A_{y,s} & \text{if } y < m, \\ A_{y+s+1,s} & \text{otherwise.} \end{cases}$$

Note that if each requirement $\mathbf{R}_n$ receives attention only finitely often, then $\lim_s A_{y,s} = A_y$ exists and thus A is coinfinite and by the construction $K \leqslant_T A$. Thus it suffices to show that each requirement $\mathbf{R}_n$ receives attention only finitely often and is satisfied. Suppose that this is true for all requirements $\mathbf{R}_p$, $p < n$. By the construction, $A(n,s+1) \neq A(n,s)$ only if requirement $\mathbf{R}_p$, $p \leqslant n$ receives attention or a number $x \leqslant n$ is enumerated into K. Thus, by induction, let s_0 be a stage such that no requirement $\mathbf{R}_p$, $p < n$ receives attention and such that $K_{s_0}[n] = K[n]$. Suppose that $s > s_0$ is a stage at which $\mathbf{R}_n$ receives attention. Then $q(n,s) = 1$, $A(n,s) < u(n,s) \leqslant m(n,s)$. Now $A(n,s)$ is enumerated into A_{s+1} and we have $A_{n,s+1} > s + 1 > m(n,s)$. $A_{n,s}$ can enter only one of U_e or V_e, thus the other of the two computations mentioned in the definition of $q(n,s)$ is preserved forever. This implies that $\mathbf{R}_n$ is met and never again receives attention. $\quad\square$

Corollary 6.41. *There is a complete r.e. set A such that if A_1, A_2 is any splitting of A with $A_1 \mid_T A_2$, then $\deg(A_1) \cap \deg(A_2)$ does not exist.*

Proof. Choose A as in the Theorem. Lachlan [59] (or Ambos-Spies [3]) have shown that there are no incomparable low degrees, $\mathbf{a}_1$, $\mathbf{a}_2$ with $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{0}'$ and such that $\mathbf{a}_1 \cap \mathbf{a}_2$ exists. $\quad\square$

We also remark that Ambos-Spies [4] has shown in a dual result that for every nonrecursive $\mathbf{a}$, there are degrees $\mathbf{a}_1$, $\mathbf{a}_2$ such that $\mathbf{a}_1 \cap \mathbf{a}_2$ does not exist. This fails for splittings of sets since if A is strongly atomic, any set splitting of A has an infimum in the degrees. Theorem 6.40 suggests the following question of Remmel: Is there a set A such that for every nontrivial splitting A_1, A_2 of A, A_1', $A_2' <_T A'$? This was recently resolved by Ingrassia and Lempp.

Theorem 6.42 (Ingrassia and Lempp [54]). *There is an r.e. set A such that for all nontrivial splittings A_1, A_2 of A, $A_1', A_2' <_T A'$.*

We omit the proof of Theorem 6.42. Theorem 6.42 leads to a general question about the jumps of splittings.

Open Question 6.43. If A is a nonrecursive r.e. set, what are the possibilities for the degrees of jumps of splittings of A?

As concrete special cases we propose

Open Question 6.44. Is there an r.e. set of high degree which has no nontrivial splitting A_1, A_2 such that A_1 is high?

and

Open Question 6.45. Is there a nonlow set A such that for every nontrivial splitting A_1, A_2 of A, both of A_1 and A_2 are low?

Other reasonable questions and conjectures could be made by extending the Ingrassia–Lempp idea or these questions above to nth jumps.

7. Antisplitting and other strong nonsplitting properties

In this section, we examine further the structure of $\mathcal{S}(A)$. The first result shows that if $\mathcal{S}(A)$ is not all of $[\mathbf{0}, \mathbf{a}]$, then in fact $\mathcal{S}(A)$ misses a whole interval $[\mathbf{b}, \mathbf{c}]$ of degrees in the interval $[\mathbf{0}, \mathbf{a}]$.

Theorem 7.1 (Downey and Welch [41]).
(1) *Suppose that A does not have the USP. Then there exist sets B, C, such that $\emptyset <_T B <_T C <_T A$ and such that if $B \leqslant_T D \leqslant_T C$, then $\deg(D) \notin \mathcal{S}(A)$.*

(2) *Suppose that A does not have the UWP. Then there exist sets B, C, such that $\emptyset <_T B <_T C <_T A$ and such that if $B \leqslant_T D \leqslant_T C$, then $D \nleqslant_{wtt} A$.*

Proof. We prove (2) as (1) is similar. Let $C <_T A$ be such that if $D \equiv_T C$, then $D \nleqslant_{wtt} A$. Let C_1, C_2 be a Sacks' Splitting of C. Then we claim that either C_1 or C_2 is the desired set B. For suppose that there are sets D_1 and D_2 such that $C_i \leqslant_T D_i <_T C$ and $D_i \leqslant_{wtt} A$ for $i = 1, 2$. Then we have that $D_1 \oplus D_2 \leqslant_{wtt} A$ but $D_1 \oplus D_2 \equiv_T C$ contradicting the choice of C. □

The best extension of Theorem 7.1 from one point of view is the following.

Theorem 7.2 (Downey and Welch [41]). *Suppose that A is strongly atomic and of contiguous degree $\mathbf{a}$. Then $\mathcal{N}(A)$ is dense in $[\mathbf{0}, \mathbf{a}]$ and hence in $\mathbf{R}$.*

Theorem 7.2 is a corollary of Theorem 6.11 and the following result of Fejer.

Theorem 7.3 (Fejer [42]). *The nonbranching degrees are dense in $\mathbf{R}$.*

Proof of Theorem 7.2. Let A be strongly atomic and of contiguous degree $\mathbf{a}$. Suppose that $[\mathbf{b}, \mathbf{c}]$ is an interval of $[\mathbf{0}, \mathbf{a}]$. We may suppose by Theorem 7.3 that $\mathbf{b}$ is not branching. Let $\mathbf{d}$ be such that $\mathbf{b} < \mathbf{d} < \mathbf{c}$. If all of $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are in $\mathcal{S}(A)$, then there is a degree $\mathbf{e} \in \mathcal{S}(A)$ which is a complement in the Boolean algebra $\mathcal{S}(A)$ for $\mathbf{d}$ on the interval $[\mathbf{b}, \mathbf{c}]$. But then $\mathbf{d} \cap \mathbf{e} = \mathbf{b}$ which contradicts that $\mathbf{b}$ is nonbranching. □

Theorem 7.4 (Downey and Welch [41]). *There is a nonrecursive r.e. set A such that $\mathcal{S}(A)$ is nowhere dense in $\mathbf{R}$.*

Proof. Let A be strongly atomic and of contiguous degree. We may assume that A is low. Let $[\mathbf{b}, \mathbf{c}]$ be an interval of $[\mathbf{0}, \mathbf{a}]$. By Theorem 7.1, let $\mathbf{d} \in \mathcal{N}(A)$ be such that $\mathbf{d} \in [\mathbf{b}, \mathbf{c}]$. Then $\mathbf{d}$ can be split over $\mathbf{b}$ since $\mathbf{b}$ is low; let $\mathbf{d}_1$, $\mathbf{d}_2$ be such a splitting. Then using similar reasoning as in Theorem 7.1, either $[\mathbf{d}_1, \mathbf{d}]$ or $[\mathbf{d}_2, \mathbf{d}]$ is an interval entirely contained in $\mathcal{N}(A)$. □

By Theorem 6.42 and the Jump Interpolation Theorem, there are r.e. sets $B <_T A$ such that for any splitting A_1, A_2 of A, if $B \leqslant_T A_1$ then $A_1 \equiv_T A$. In fact we get a somewhat stronger result as follows.

Theorem 7.5. *Suppose A is strongly atomic. Then there is a degree $\mathbf{b}$ such that $\mathbf{0} < \mathbf{b} < \deg(A)$ and such that for any splitting A_1, A_2 of A, if $\mathbf{b} \leqslant \deg(A_1)$ then $A_1 \equiv_T A$.*

Proof. Downey and Stob have shown [36] that

$$(\forall a \neq 0)\,(\exists b < a)\,(\forall c \leqslant a)\,[c \cap b = 0 \Rightarrow c = 0]. \tag{7.1}$$

Now let A be strongly atomic; $a = \deg(A)$. Let b be given by the existential quantifier in (7.1). Let A_1, A_2 be any splitting of A with $b \leqslant \deg(A_1)$. We must have $b \cap \deg(A_2) = 0$ since A_1 and A_2 form a minimal pair. Thus, by (7.1), we have that $\deg(A_2) = 0$. $\square$

We turn now to consider how splitting combines with permitting.

Definition 7.6. A set A has *the antisplitting property* if there is a degree $b < \deg(A)$ such that for any splitting A_1, A_2 of A such that $\deg(A_1) \leqslant b$, A_1 is recursive. That is, there is an interval $(0, b]$ in $[0, a]$ which does not intersect $\mathcal{S}(A)$.

Theorem 7.7 (Downey and Welch [41]). *Suppose that A is strongly atomic. Then*
(1) *if A has high degree, then A has the antisplitting property,*
(2) *if A is of contiguous degree then A has the antisplitting property.*

Proof. If a is high or a is contiguous, then there is b $(0 < b < a)$ such that $c \cup b = a$ implies that $c = a$. For high degrees this is a result of Harrington ([45], see Miller [78]) and for contiguous degrees a result of Ladner and Sasso [67]. If A is strongly atomic of degree a, then for this degree b, it must be the case that $(0, b]$ is an interval missing $\mathcal{S}(A)$. $\square$

Not all sets with the antisplitting property are atomic.

Theorem 7.8 (Downey and Stob [37]). *There is a complete set A such that A has the antisplitting property.*

Proof. Let f be a 1–1 enumeration of K. Let $a_{0,s} < a_{1,s} < a_{2,s} < \cdots$ enumerate the complement of A_s in increasing order. We guarantee that A is complete by enumerating $a_{f(s),s}$ into A at stage $s + 1$. If A is also constructed to be coinfinite, this guarantees that A is complete. We also enumerate the antisplitting witness, B, for A. The requirements are

$$\mathbf{P}_e\colon \overline{B} \neq W_e,$$

$$\mathbf{N}_e\colon (U_e \cup V_e) = A \wedge \Phi_e(B) = U_e \Rightarrow U_e \text{ is recursive.}$$

Here, as usual, (U_e, V_e) lists pairs of disjoint r.e. sets. In light of requirements $\mathbf{N}_e$, we may also assume that U_e and V_e are enumerated so that $U_{e,s}, V_{e,s} \subseteq A_s$. The construction of B is arranged on a pinball machine as in Fig. 3.
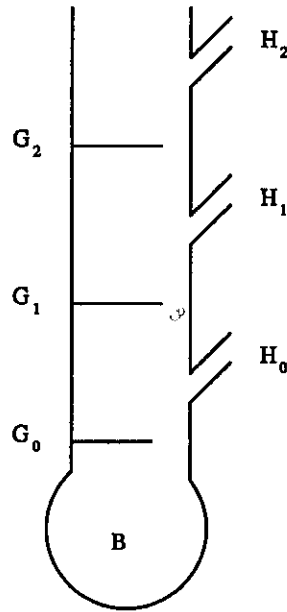
Fig. 3. The standard pinball machine.

Balls leaving hole H_e are followers of $\mathbf{P}_e$. Gate G_e corresponds to the negative requirements $\mathbf{N}_e$. The negative requirements together will at any stage apply a restraint $r(j, s)$ to requirement $\mathbf{P}_j$ as follows.

Define a length of agreement function, l, by

$$l(e, s) = \max\{x : (U_{e,s} \cup V_{e,s})[x] = A_s[x] \wedge \Phi_{e,s}(B_s)[x] = U_{e,s}[x]\}.$$

Define $m(e, s)$ to be the maximum length of agreement for $l(e, s)$. Now define

$$r(e, j, s) = \begin{cases} \max\{u(\Phi_{e,s}(B_s); y)) \mid y \leqslant a_{j,s}\} & \text{if } l(e, s) \geqslant a_{j,s}, \\ r(e, j, s - 1) & \text{otherwise} \end{cases}$$

and

$$r(j, s) = \max\{r(e, j, s) \mid e \leqslant j\}.$$

Construction

Stage $s + 1$

Step 1. If x is a ball on the surface of the machine associated with requirement $\mathbf{P}_e$, then cancel x if either $f(s) \leqslant e$ or $x \leqslant r(j, s)$ for some $j \leqslant e$.

Step 2. Find the highest priority requirement $\mathbf{P}_e$ such that $W_{e,s} \cap B_s = \emptyset$ and which requires attention according to one of the cases below, choose the first case which pertains to that requirement, and perform the indicated action.

Case 1. There is a follower of x of $\mathbf{P}_e$ at gate G_j and $l(j,s) > m(j,s)$. Cancel any lower priority followers than x (either a follower of P_i for $i > e$ or of $\mathbf{P}_e$ and appointed later than x.)

Allow x to drop from gate G_j to the next unoccupied gate. If no such gate exists, enumerate $x \in B$ and $a_{f(s),s}$ in A. If such a gate exists, let $B_{s+1} = B_s$ and enumerate $a_{f(s),s}$ into A and also enumerate $a_{e+1,s}, a_{e+2,s}, \ldots, a_{s,s}$ into A. (Note that as x is not cancelled in step 1, we have that $f(s) > e$.)

Case 2. There is a follower x of $\mathbf{P}_e$ above hole H_e and $x \in W_{e,s}$. Perform the same action as in Case 1.

Case 3. There is no follower of $\mathbf{P}_e$ above hole H_e. Cancel any follower of requirement $\mathbf{P}_i$, $i > e$. Appoint s as a follower of $\mathbf{P}_e$ and place it above hole H_e. Let $B_{s+1} = B_s$ and enumerate $a_{f(s),s}$ into A.

Lemma 7.9. *For every* e, $\lim_s a_{e,s} = a_e$ *exists* (*implying that* A *is complete*), $\lim_s r(e,s)$ *exists, and* $\mathbf{P}_e$ *receives attention only finitely often.*

Proof. The proof is by simultaneous induction on e. Suppose that the lemma is true for $i < e$ and let s_0 be a stage such that for all $s \geqslant s_0$ and $i < e$, $r(i,s) = r(i,s_0)$, $a_{i,s} = a_{i,s_0}$, $\mathbf{P}_i$ doesn't require attention at s, and $f(s) > e$. Now $a_{e,s}$ can be enumerated in A only for coding (if $f(s) = e$) or by requirement $\mathbf{P}_i$ for some $i < e$ receiving attention. Therefore we have that for all stages $s \geqslant s_0$, $a_{e,s} = a_{e,s_0}$. Also after s_0 no follower can injure the computations mentioned in establishing $r(j,s)$ for $j \leqslant e$, so we can see that $\lim_s r(e,s)$ exists. Once this limit is attained, followers for $\mathbf{P}_e$ can no longer be cancelled. Thus after at most $e + 1$ followers for $\mathbf{P}_e$ are appointed, one follower must succeed in being enumerated in B so that $\mathbf{P}_e$ no longer receives attention. $\square$

Lemma 7.10. *For every* e, *requirement* $\mathbf{N}_e$ *is met.*

Proof. Suppose that $U_e \cup V_e = A$ and $\Phi_e(B) = U_e$. We must argue that U_e is recursive. Suppose that x is given. To determine if $x \in U_e$, let s_0 be a stage such that $l(e,s_0) \geqslant x$; for all gates G_j if $j \leqslant e$ and G_j has a resident at stage s_0, then that resident is a permanent one; no requirement $\mathbf{P}_i$ receives attention after s_0 for $i \leqslant e$; and $f(s) > e$ for all $s \geqslant s_0$. We claim that $x \in U_e$ if and only if $x \in U_{e,s_0}$. By the choice of s_0, the only possible counterexample x is an element of form a_{j,s_0} for some $j > e$. Suppose that $x = a_{j,s_0}$ is such a counterexample. It must be the case that there is a follower $y \leqslant u(\Phi_{e,s_0}(B_{s_0};x))$ that enters B at some stage later than s_0 (since $l(e,s_0) \geqslant x$ and $\lim_s l(e,s_0) = \infty$). Let y be the first such follower. Then there is a stage $s > s_0$ such at stage s, y stopped at gate G_e or passed gate G_e because it was occupied.

Suppose first that gate G_e was occupied at stage s. Then some element z occupies gate G_e at stage s. At the stage that z arrived at gate G_e, say t,

$s_0 < t < s$, it must be the case that follower y is not yet in existence since otherwise z would cancel it. Thus $y > t$ and so could not possibly injure the computation established at s_0.

On the other hand suppose that y stopped at gate G_e at stage s and later left gate G_e at stage $t > s$. Then $l(e,t) > l(e,s_0)$. Furthermore, the computation $\Phi_{e,t_0}(B_t;x)$ and the computation $\Phi_{e,s_0}(B_{s_0};x)$ are the same since y has not yet entered B. But then by the restraints and Step 1, if must be the case that y exceeds the use of this computation contrary to assumption.   □

Theorem 7.8 can be extended to show that every high r.e. degree contains an r.e. set with the antisplitting property. To do this, we combine the pinball machine technique above with the standard method of high permitting in the same way that Cooper showed that every high degree bounds a minimal pair.

Not all r.e. degrees can be antisplitting witnesses. This follows easily from Theorem 5.28 and was proven earlier by Downey using a much easier argument. From this argument, we get a stronger antisplitting result.

Definition 7.11 (Downey). An r.e. degree **a** is *persistent* if for every r.e. set B such that **a** $<_T \deg(B)$, there is a splitting B_1, B_2 of B such that $0 < \deg(B_1) \leq$ **a**.

Theorem 7.12 (Downey [19]). *There is a low persistent r.e. degree.*

Proof. Let K be a creative set. Let K_1, K_2 be a Sacks splitting of K with K_1 low. Let **a** $= \deg(K_1)$. Then **a** is persistent. For let C be any r.e. set such that $K_1 <_T C$. Then $C \leq_{wtt} K$ so that C has a splitting C_1, C_2 such that $C_1 \leq_{wtt} K_1$ by Lachlan's Lemma. Since $K_1 <_T C$, C_1 must be nonrecursive, else $K_1 \leq_T C \equiv_T C_2 \leq_T K_2$ which contradicts that K_1, K_2 is a Sacks Splitting of K.   □

We leave the following interesting question open.

Open Question 7.13. Are all cappable degrees antisplitting witnesses?

The following theorem shows that there is a degree without a set with the antisplitting property.

Theorem 7.14. *There is an r.e. degree* **a** $\neq$ **0** *such that for every set A of degree* **a**, *A does not have the antisplitting property.*

Proof. We construct a set A so that **a** $= \deg(A)$ has the desired property. The requirements to make A nonrecursive are, of course,
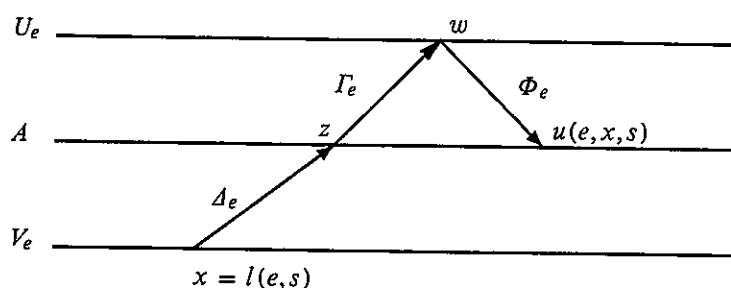
Fig. 4. Length of agreement.

$$\mathbf{P}_e: \overline{A} \neq W_e.$$

To insure that all sets of degree **a** do not have the antisplitting property we will meet the requirements $\mathbf{R}_e$ below. here $(\Phi_e, \Gamma_e, \Delta_e, U_e, V_e)$ is an enumeration of all 5-tuples of three reductions and two r.e. sets. The sets B_e and C_e are sets that we enumerate.

$\mathbf{R}_e: (\Phi_e(A) = U_e \wedge \Gamma_e(U_e) = A \wedge \Delta(A) = V_e) \Rightarrow (V_e$ is recursive

$\vee \ (B_e, C_e$ is a splitting of $U_e \wedge B_e \leqslant_T V_e \wedge B_e$ is not recursive)).

To meet the last clause of $\mathbf{R}_e$ if necessary we have the following requirements

$$\mathbf{R}_{e,i}: \overline{B}_e \neq W_i.$$

We describe the construction for one requirement $\mathbf{R}_e$. It is a tree of strategies construction. To meet $\mathbf{R}_e$, we will have two types of nodes. Some nodes will be devoted to meeting the requirements $\mathbf{R}_{e,i}$. Such a node seeks to enumerate an element x of U_e into B_e if $x \in W_i$ and V_e permits (we must have $B_e \leqslant_T V_e$). We will also have nodes devoted to the requirement $\mathbf{R}_e$ itself. Such a node is devoted to insuring that B_e, C_e is a splitting of U_e if the hypotheses of $\mathbf{R}_e$ are satisfied.

Let $\varphi_e(x, s)$ (resp. $\gamma_e(x, s)$, $\delta_e(x, s)$) denote the use of the computation $\Phi_{e,s}(A_s; x)$ (resp. $\Gamma_{e,s}(U_{e,s}; x)$, $\Delta_{e,s}(A_s; x)$). To measure a length of agreement in the hypothesis of $\mathbf{R}_e$ we use

$$l(e, s) = \max\{x \mid (\forall y \leqslant x)[\Delta_{e,s}(A_s; y) = V_{e,s} \wedge (\forall z \leqslant \delta_e(y, s))$$

$$[\Gamma_{e,s}(U_{e,s}; z) = A_s(z) \wedge (\forall w \leqslant \gamma_e(z, s))[\Phi_{e,s}(A_s; w) = W_{e,s}]]]. \quad (7.2)$$

Define also $u(e, x, s)$ as the use of A_s in establishing $l(e, s)$. Fig. 4 might be helpful in understanding the length of agreement $l(e, s) = x$.

For the sake of $\mathbf{R}_e$, we will construct a sequence of followers, a "stream", $x_0, x_1, \ldots$, as follows. We start with x_0. We then wait for a stage s_0 such that $x_0 < l(e, s_0)$ and appoint follower $x_1 > u(e, x_0, s_0)$. At s_0 we cancel all (lower priority) numbers in the interval (x_0, x_1). Each succeeding x_{i+1} is appointed at s_i such that $l(e, s_i)$ and is appointed so that $x_{i+1} > u(e, x_i, s_i)$. The only

numbers that $\mathbf{R}_e$ will use are tied to these x_i. In particular, the requirements $\mathbf{R}_{e,i}$ choose followers to enter B_e numbers associated with the x_i as followers. At the stage s_i that x_{i+1} is appointed, $\gamma_e(\delta_e(x_i, s_i), s_i)$ is defined and we declare that numbers $y \leqslant \gamma_e(\delta_e(x_i, s_i), s_i)$ cannot enter B_e at $s + 1 > s_i$ unless $V_{e,s+1}[x_i] \neq V_{e,s}[x_i]$. (To see what this means, refer to Fig. 4. Read x_i for x. Then B_e is not allowed to change through w unless V_e changes through x. For later ease of notation, define $w_i = \gamma_e(\delta_e(x_i, s_i), s_i)$.) This implies that $B \leqslant_T V_e$. Note also that, because of the cancellation involved in the appointment of the x_i, we have that $V_e[x_i]$ cannot change unless $A[x_i]$ changes. Note that a node for $\mathbf{R}_{e,i}$ may have infinitely many followers assigned to it as perhaps V_e is recursive. Then we need have no other nodes associated with $\mathbf{R}_e$ later on the tree, but of course other nodes must live with this outcome.

We discuss the strategies for meeting the requirements subject to the conflicts among them. The simplest configuration of nodes is $\tau \subset \sigma \subset \gamma$ with γ working on $\mathbf{P}_f$, σ working on $\mathbf{R}_{e,i}$, and τ assigned to $\mathbf{R}_e$ such that $\tau = \tau(\sigma)$. For σ at any stage, we will have one largest follower x_j which is waiting for realization, that is waiting for $W_{i,s}[w] = \overline{B}_{e,s}[w]$ (here again w is as in Fig. 4 where x_j is x). When this happens, $\mathbf{R}_{e,i}$ requests that the next member x_k of the stream that τ is building be defined. (This is done using the method of "links"; see Soare [97].) Then σ cancels all members of this stream between x_j and x_k. That is σ refines the stream of $\mathbf{R}_e$. In this way we also have that for each x_j in the stream for $\mathbf{R}_e$, we also have $W_{i,s}[w_j] = \overline{B}_{e,s}[w_j]$. Now what we would like to do is to put some x_j into A causing U_e to change on w_j. We would then enumerate any change between w_{j-1} and w_j into B_e causing the desired disagreement, $W_i[w_j] \neq \overline{B}_e[w_j]$. The difficulty with this strategy is that we also need a change in V_e. The idea is to allow $\mathbf{P}_f$ (at γ) to enumerate x_j into A thereby hoping to cause a V_e change. Roughly speaking, γ chooses its followers of $\mathbf{P}_f$ from the stream generated at σ. When we see x_j devoted to $\mathbf{P}_f$ at σ occur in W_f we would like to put x_j into A and at the next τ stage argue that if we see a V_e change we could win $\mathbf{R}_{e,i}$. This won't quite work. For though we get a V_e change, it might be a change in V_e on some large number (say x_n, $n \gg j$) but the corresponding U_e change is only on w_j. Then we can't use this V_e change to comprehend the relevant U_e change. For this, we need V_e to change on x_j. The key idea is to force U_e to change big and then later enumerate x_j into A so that no matter where the V_e change occurs, we. can use such a change. The basic module then is as follows.

(1) We see x_k following $\mathbf{P}_f$ at γ such that x_k occurs in $W_{f,s}$. We wish to put x_k into A. This happens at a γ stage so that the only live numbers at this stage have guess $\supset \sigma$.

(2) At stage s, the stream at σ looks like $x_0, \ldots, x_k, \ldots, x_p, x_{p+1}$ with x_{p+1} unrealized. Put x_p into A. Note that $W_{i,s}[w_p] = \overline{B}_{e,s}[w_p]$.

(3) Wait until the least τ stage $t > s$. Note that $U_{e,s}[w_p] \neq U_{e,t}[w_p]$. This is the big change that we desire. Now we do not yet decide whether

to enumerate $U_{e,t}[w_p] - U_{e,s}[w_p]$ into B_e or C_e but we immediately enumerate the interval $[x_k, t]$ into A. (This meets $\mathbf{P}_f$.)

(4) Wait until the least τ stage $u > t$. Then either
  (a) (Success) $V_{e,u}[x_p] \neq V_{e,s}[x_p]$. In this case put $U_{e,t}[w_p] - U_{e,s}[w_p]$ into B_e (meeting $\mathbf{R}_{e,i}$ at σ) and $U_{e,u}[w_p] - U_{e,t}[w_p]$ into C_e.
  (b) (Failure) otherwise. Enumerate all of $U_{e,u}[w_p] - U_{e,s}[w_p]$ into C_e.

(5) Don't allow $x_0, x_1, \ldots, x_{k-1}$ to enter A until the stream reaches its former length. (The point of this is not to allow $V_e[w_n]$ to change until we are in a position to take advantage of it.)

The above argument works for many $\mathbf{P}_f$ and one $\mathbf{R}_e$. Now we must argue that the various $\mathbf{R}_e$ cohere. Suppose now that we have two nodes τ_1, τ_2 devoted to $\mathbf{R}_{e_1}$ and $\mathbf{R}_{e_2}$ respectively with $\tau_1 \subset \tau_2$. The first thing to notice is that only part of the τ_1 stream is in the τ_2 stream. This is because τ_2 must "process" the τ_1 numbers. Now the problem with the basic module is that we get to step 3 for τ_1 but not for τ_2. But now τ_1 demands that we fulfill its commitment to build $B_{e_1} \cup C_{e_1} = U_{e_1}$. The basic module delays this decision one expansionary stage. Nevertheless, τ_2 is expecting us to wait for a τ_2 stage so that it can see a large V_{e_2} change. There are fundamentally two cases to deal with.

Case 1. $\tau_1 \subset \sigma\hat{\ }g \subset \tau_2 \subset \eta\hat{\ }g \subset \gamma$ where $\tau(\sigma) = \tau_1$ and $\tau(\eta) = \tau_2$

This is handled as follows. The η stream will appear as $y_1, \ldots, y_n, x_1, \ldots, x_q$ with each y_i processed by both η and σ. and each x_j processed by σ but not η. We see $y_k \in W_{i(\gamma),s}$. We wait to put y_k into A. We do the following.

(1) We drop links from γ to τ_2 and τ_1. We enumerate x_q into A.
(2) We wait until the next τ_1 stage. We have now seen a U_{e_1} change at w_q.
(3) At the next stage we immediately enumerate y_n into A. (Using the basic module inductively, in essence.)
(4) At the next τ_1 stage, we attend to the pending τ_1 commitments. (If there is no helpful V_{e_1} change, we put all of the changes in C_{e_1}, etc.)
(5) Now we really should wait for a τ_2 stage. We cannot know one happens, but if it does, it will be a $\sigma\hat{\ }g$ stage. Thus we won't access τ_2 until σ has produced a new realized $\hat{x}_1$ beyond all the things that we've seen that can cause yet another U_e change if we need it.
(6) Should another τ_2 stage occur, we delay decisions on C_{e_2} but immediately enumerate $\hat{x}_1$ into A. But now we have seen a U_{e_2} change.
(7) At the next τ_1 stage, we can immediately enumerate y_k into A as we have pending U_{e_1} and U_{e_2} changes and so can use any V_{e_1} or V_{e_2} changes.

Case 2. $\tau_1 \subset \tau_2 \subset \eta\hat{\ }g \subset \sigma\hat{\ }g \subset \gamma$ with the nodes as in Case 1.

Now the situation is different and easier. The stream at γ looks like $x_1, \ldots, x_n, y_1, \ldots, y_m$ with the followers x_i both $\sigma\hat{\ }g$ and $\eta\hat{\ }g$ realized but the followers y_i only $\eta\hat{\ }g$ realized. Now we begin by putting y_m into A. We don't really now need to act unless there is another τ_2 stage (not just a τ_1 stage). At the next τ_2 stage we can safely put x_n into A and then finish at the next τ_1 stage. $\quad\square$

There are also antisplitting theorems for the wtt-degrees. An easy, but strong, result along these lines is the following.

Theorem 7.15 (Downey and Stob [37]). *There are sets A and B such that $A \equiv_T B$ but such that the wtt-degrees of A and B form a minimal pair in* **W**.

Proof. It is easy to embed the 1–4–1 lattice into **R** with infimum **0**. Let representatives of the four degrees be A_1, A_2, A_3, A_4. Then $A_1 \oplus A_2$ and $A_3 \oplus A_4$ are Turing equivalent sets but form a minimal pair in the wtt-degrees. The former fact is obvious; for the latter suppose that $B \leqslant_{\text{wtt}} A_1 \oplus A_2$ and $B \leqslant_{\text{wtt}} A_3 \oplus A_4$. Then there is a splitting B_1, B_2 of B such that $B_i \leqslant_{\text{wtt}} A_i$ for $i = 1, 2$. Now $B_1 \leqslant_{\text{wtt}} A_3 \oplus A_4$ so that B_1 has a splitting $B_{1,3}$ $B_{1,4}$ such that $B_{1,i} \leqslant_{\text{wtt}} A_i$ for $i = 3, 4$. But then $B_{1,3} \leqslant_T A_1, A_3$ and so is recursive. Similarly $B_{1,4}$ is recursive. Hence B_1 is recursive. Finally, the same argument shows B_2 is recursive. This implies that B is recursive. $\square$

In [37], Downey and Stob proved Theorem 7.15 by a direct construction. Downey noticed the simple proof later. The result hints that the semilattice structure of degrees below **a** in **R** has implications for the structure of wtt-degrees in **a**. It might be an interesting program to carry this observation further. For instance,

Open Question 7.16. Suppose that A and B are nonrecursive sets such that $A \equiv_T B$ and A and B form a minimal pair in the wtt-degrees. Then is $\deg(A)$ the top of a 1–4–1 lattice with least element **0** in **R**?

A related question is whether contiguity can be defined by a failure of antisplitting. Namely

Open Question 7.17. Is **a** contiguous if and only if

$$(\forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b})\,[\,(\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} \wedge \mathbf{b} \leqslant \mathbf{a}) \Rightarrow (\exists \mathbf{b}_1, \mathbf{b}_2)\,[\mathbf{b}_i \leqslant \mathbf{a}_i \wedge \mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{b}]\,]?$$

Related questions suggested by Theorem 7.15 include the following two.

Open Question 7.18. Is there a degree $\mathbf{a} \neq \mathbf{0}$ such that for all A of degree $\mathbf{a}$ there is a set B such that $A \equiv_T B$ and A and B form a minimal pair in the wtt-degrees?

Open Question 7.19. Are there r.e. sets $B <_T A$ such that for all C, if $B \leqslant_T C <_T A$, then C and A form a minimal pair in the wtt-degrees?

We have a partial answer to Question 7.18.

Theorem 7.20 (Downey and Stob [37]). *There is an r.e. degree* $\mathbf{a} \neq \mathbf{0}$ *such that for all r.e.sets* $A \in \mathbf{a}$, *there is* $B \leqslant_T A$ *such that* B *and* A *form a minimal pair in the wtt-degrees.*

Proof. The proof is a pinball machine argument similar to those used to prove that certain nondistributive lattices can be embedded in **R**. This technique was invented by Lerman [68] and appears with essentially all that is known about which lattices can be so embedded in [9]. The pinball machine of Fig. 3 is the model. Let (Φ_e, U_e) enumerate pairs of Turing reductions and sets. We enumerate sets A and B_e, $e \in \omega$ such that if $\Phi_e(A) = U_e$, then $B_e \leqslant_T A$ and satisfies the requirements

$\mathbf{P}_{e,i}$: $\overline{B}_e \neq W_i$,

$\mathbf{N}_{e,i}$: $(\Gamma_i(U_e) = \Delta_i(B_e) = g \wedge g$ is total $) \Rightarrow g$ is recursive.

Here (Γ_e, Δ_e) lists pairs of wtt-reductions. We use $l(\langle e, i \rangle, s)$ to measure the length of agreement of the reductions in requirement $\mathbf{N}_{e,i}$. We also use $L(e, s)$ to measure the length of agreement of $\Phi_e(A) = U_e$. We will assume that U_e is restrained by A in the sense that if $\Phi_{e,s}(A_s; x) = U_{e,s}(x)$ then we do not allow x to be enumerated into $U_{e,s+1}$ unless some element is enumerated into A at $s + 1$ below the use of the computation $\Phi_{e,s}(A_s; x)$. Requirement $\mathbf{P}_{e,i}$ is associated with hole $H_{\langle e,i \rangle}$ and requirement $\mathbf{N}_{e,i}$ is associated with gate $G_{\langle e,i \rangle}$. To insure that $B_e \leqslant_T A$ in case that $\Phi_e(A) = U_e$, we not only have follower balls associated with the positive requirements that are attempting to reach the bottom of the machine to be enumerated into B_e, we also have trace balls that are to be enumerated into A. We will identify a ball with the number it is attempting to enumerate. A follower ball may move on the surface of the machine only in company of a corresponding trace ball. However, a trace ball may be separated from its associated follower ball and move down the surface of the machine alone. Follower balls can be either *active, frozen,* or *waiting.* Trace balls are always *active.* If y is a trace ball, we write $f(y)$ for the follower ball of which y is a trace, Follower ball x has higher priority than follower ball z if the positive requirement associated with x has higher priority than that associated with z or if the positive requirements have the same priority but $x < z$. Trace balls have the same priority as their associated follower ball.

Construction
Stage $s + 1$
Requirement $\mathbf{P}_{e,i}$ requires attention at stage $s + 1$ if one of the following cases obtains.

Case 1. There is a gate G_n an active ball z associated with requirement $\mathbf{P}_{e,i}$ at gate G_n, and $l(n,s) > m(n,s)$.

Case 2. There is an active follower z of requirement $\mathbf{P}_{e,i}$ such that z is above hole $H_{\langle e,i\rangle}$ such that $z \in W_{i,s}$.

Case 3. There is a follower z of requirement $\mathbf{P}_{e,i}$ which is waiting and $L(e,s) > M(e,s)$.

Case 4. There is no follower above hole $H_{\langle e,i\rangle}$, $W_{i,s} \cap B_{e,s} = \emptyset$, and $L(e,s) > M(e,s)$.

Let $\mathbf{P}_{e,i}$ be the requirement of highest priority which requires attention according to one of the Cases above. Cancel (remove from the machine) all balls associated with requirements of lower priority. If Case 1, 2, or 3 obtains, let z be the highest priority such ball. Cancel all balls associated with $\mathbf{P}_{e,i}$ of priority lower than that of z. Perform the appropriate action below.

Case 1 action. Let $p < n$ be maximal such that there is no active ball at gate G_p.

Let z (and if z is a follower, the trace ball currently associated with z) fall down to gate G_p. If z is a follower, z becomes frozen. (In effect, this separates z from its trace ball y at this stage since y may later move but frozen balls may not.) If p does not exist, enumerate z (and the trace, if it exists) in the appropriate set and, if z is a trace, change $f(z)$ from frozen to waiting.

Case 2 action. Let $p < \langle e,i\rangle$ be maximal such that there is no active ball at gate G_p. Now perform the same action as in the second paragraph of Case 1 action.

Case 3 action. Appoint $s + 1$ as a trace ball for z and change z from waiting to active.

Case 4 action. Appoint $s + 1$ as a follower ball, declare it active, and appoint $s + 1$ as a trace ball for $\mathbf{P}_{e,i}$ and place then above hole $H_{e,i}$.

We omit the proofs of the first two lemmas as they are easy.

Lemma 7.21. *If $\Phi_e(A) = U_e$, then $B_e \leqslant_T A$.*

Lemma 7.22. *For every e, i, $\mathbf{P}_{e,i}$ receives attention only finitely often. Furthermore, if $\Phi_e(A) = U_e$, then $\mathbf{P}_{e,i}$ is satisfied.*

Lemma 7.23. *For every e and i, if $\Phi_e(U_e) = A$, then $\mathbf{N}_{e,i}$ is satisfied.*

Proof. Let e and i be given; let $n = \langle e,i\rangle$. Suppose that $\Gamma_i(U_e) = \Delta_i(B_e) = g$ and g is total. We show how to compute g. Fix p; we describe how to compute $g(p)$. For convenience, let γ, δ be recursive functions bounding the use of Γ_i and Δ_i. (Recall that these are wtt-reductions.) Also let $\varphi(x,s)$ denote the use of $\Phi_{e,s}(A_s; x)$.

Let s_0 be a stage such that

$$l(n,s) > p, \tag{7.3}$$

for all $\langle f, j \rangle < n$ $\mathbf{P}_{f,j}$ does not receive attention after s_0, (7.4)

$$L(e,s) > \varphi(\delta(p),s), \tag{7.5}$$

for all $m < n$, any active ball at G_m is permanently at G_m. (7.6)

That such a stage exists can be seen as follows. Clauses (7.3), (7.4) and (7.5) occur at cofinitely many stages. To see that clause (7.6) occurs infinitely often, let s be any stage. Let x be the ball of highest priority which receives attention after stage s and let t be the last stage at which it receives attention. Then it is easy to see that t satisfies (7.6).

Now let $q = \Gamma_{i,s_0}(U_{e,s_0};p) = \Delta_{i,s_0}(B_{e,s_0};p)$ We claim that $g(p) = q$. We prove this by showing that for all $s \geqslant s_0$,

$$\Delta_{i,s}(B_{e,s};p) = q \vee (\Gamma_{i,s}(U_{e,s};p) = q$$
$$\wedge \Phi_{e,s}(A_s)[\gamma(p)] = U_{e,s}[\gamma(p)]). \tag{7.7}$$

To see this, suppose otherwise and let $s_1 + 1$ be the least counterexample. Then it must be the case that at stage $s_1 + 1$, either a follower $x \leqslant \delta(p)$ is enumerated into B_e or a trace $y \leqslant \varphi(\gamma(p),s_1)$ is enumerated into A.

Suppose first that a follower $x \leqslant \delta(p)$ is enumerated into B_e at stage $s_1 + 1$. Since $x < \delta(p) \leqslant s_0$, it must be the case that x was on the surface of the machine at stage s_0 (necessarily above gate G_n). Since x is not cancelled by stage $s_1 + 1$ no higher priority ball receives attention between stages s_0 and $s_1 + 1$. Let s_2 be the stage that x arrives at gate G_n and stops there. (x does not pass G_n without stopping since no follower of higher priority can already be there). At s_2, x becomes frozen. Let $s_3 + 1$ be the stage at which the trace of x, say y leaves gate G_n. Then we have that (7.7) holds for $s = s_3$. Let $s_4 + 1$ be the stage at which y enters A so that x changes from frozen to waiting at stage $s_4 + 1$. We claim also that (7.7) also holds for $s = s_4$ since no ball of higher priority or the same priority as y can move between stages s_3 and s_4 or else y is cancelled. Balls of lower priority than y are numbered with numbers greater than s_3 and so can not interfere with the computations mentioned by (7.7) at s_3. Let $s_5 + 1$ be the stage such that x changes from waiting to active at $s_5 + 1$. Then $z = s_5 + 1$ is appointed as a trace for x. Note that $\varphi_e(\gamma(p),s_5)$ converges by the action at stage $s_5 + 1$ so that $s_5 + 1 > \varphi_e(\gamma(p),s_5)$. Now let $s_6 + 1$ be the stage such that x and z together leave gate G_n. Again, no higher priority ball than x (or z) moves between stages s_5 and s_6 so that (7.7) holds for stage s_6 and $\varphi_e(\gamma(p),s_6) = \varphi_e(\gamma(p),s_5)$. Now between stages $s_6 + 1$ and $s_1 + 1$ (the stage at which x enters B_e) the only balls which can injure the computations on (7.7) for $s = s_6$ are balls of higher priority than x or of the

same priority. But no ball of higher priority than x may move during such stages. Any ball of the same priority is either the trace $z = s_5 + 1$ of x or a ball appointed subsequently as a trace for x. Since all such balls w satisfy $w > s_5$ it must be the case that the computations of (7.7) for $s = s_6$ exist at stage s_1. But then x entering B_e can only destroy the B_e side of (7.7) for $s = s_6$ so that the A side remains true at $s = s_1 + 1$ which is a contradiction.

The argument in the case that it is a trace $y \leqslant \varphi(\gamma(p), s_1)$ that enters A at s_1 is similar and we omit it. We should mention that there are two subcases here corresponding to whether y actually passes gate G_e or is appointed as a follower at some gate below gate G_n. $\square$

Corollary 7.24. *There is an r.e. degree* **a** *such that for all r.e. sets* $A \in \mathbf{a}$, A *has the antisplitting property.*

The theorems above are just a small sampling of what is known about the structure of the wtt-degrees within a single Turing degree. Other results of this nature can be found in Downey and Jockusch [27], Ambos-Spies, Cooper and Jockusch [6], and Downey and Stob [37]. A similar direction that might be pursued is to determine what effect of the structure of tt-degrees (or m-degrees) within a single **a** has on the semilattice properties of **a**.

Another direction from which to study the antisplitting property is through the lattice of r.e. sets. We have already seen that hypersimplicity implies the failure of the USP. Perhaps sufficient thinness of complement might imply the antisplitting property. It is quite easy to make a strongly atomic maximal set. Therefore some maximal sets do have the antisplitting property. However it is also possible to construct a maximal set without the antisplitting property.

Theorem 7.25. *There is a maximal set without the antisplitting property.*

Proof. We build such a set M in stages. As in the standard maximal set construction, we let $m_{0,s} < m_{1,s} < \cdots$ enumerate the complement of M_s in increasing order and we have the maximal set requirements

$$\mathbf{N}_e: \lim_s m_{e,s} \text{ exists,}$$

$$\mathbf{P}_e: \overline{M} \text{ has almost constant } e\text{-state.}$$

To insure that M has the antisplitting property, we enumerate pairs (U_e, V_e) of disjoint r.e. sets such that the following requirements are satisfied.

$$\mathbf{R}_e: W_e \equiv_{\mathrm{T}} \emptyset \vee (U_e \leqslant_{\mathrm{T}} W_e \wedge U_e \cup V_e = M \wedge U_e \not\equiv_{\mathrm{T}} \emptyset).$$

The final clause of $\mathbf{R}_e$ is met by meeting

$$\mathbf{R}_{e,i}: \overline{U}_e \neq W_i.$$

The idea is simple. To meet $\mathbf{R}_{e,i}$, we wait until a stage s such that we see $z \in \overline{A}_s$ with $U_{e,s}[z] = \overline{W}_{i,s}[z]$ and such that W_e permits z at stage s. We then enumerate z in A (and U_e at stage s). This meets $\mathbf{R}_{e,i}$ forever and clearly coheres with the other requirements in the usual way. $\square$

8. Mitotic r.e. sets

Almost the opposite of an atomic set is a mitotic set.

Definition 8.1. An r.e. set A is *mitotic* if there is a splitting A_1, A_2 of A such that $A_1 \equiv_T A_2 \equiv_T A$.

Lachlan [60] was the first to show that not all r.e. sets are mitotic but the first systematic investigation of mitotic sets was by Ladner [66,65]. The first basic result results mitoticity to autoreducibility.

Definition 8.2. An r.e. set A is *autoreducible* if there is a functional Φ such that for every x, $\Phi(A \cup \{x\}; x) = A(x)$.

Theorem 8.3. *An r.e. set A is mitotic if and only if A is autoreducible.*

Proof. ($\Rightarrow$) Suppose that A_1, A_2 is a splitting of A such that $A_1 \equiv_T A_2 \equiv_T A$. Suppose that $\Gamma(A_1) = \Delta(A_2) = A$. To decide if $x \in A$ given an oracle for $A \cup \{x\}$ we do the following. Using an oracle for $A \cup \{x\}$ and the fact that A_1, A_2 is a splitting of A, we can enumerate $\Gamma(A_1 - \{x\}; x)$ and $\Delta(A_2 - \{x\}; x)$. Enumerate until a stage such that either $x \in A_s$ or both $\Gamma_s(A_1 - \{x\}; x)$ and $\Delta_s(A_2 - \{x\}; x)$ converge. In the former case of course we answer that $x \in A$. In the latter case, if both computations give the same value, then we output this value since one of the computations must give $A(x)$ (either $A_1 - \{x\} = A_1$ or $A_2 - \{x\} = A_2$). If the computations give different values, then $x \in A$ since both values would be correct if $x \notin A$.

($\Leftarrow$) Suppose that $\Phi(A \cup \{x\}; x) = A(x)$ for all x. Let l be the associated length of agreement function and φ the associated use function. We assume that A and Φ are enumerated so that $l(s+1) > l(s)$ for all s. To enumerate A_1, A_2 we proceed as follows. For each s, let $x_s = (\mu x)[x \in A_{s+1} - A_s]$. Enumerate x_s into A_1 at stage $s+1$ and the remainder of $A_{s+1} - A_s$ into A_2. It is obvious that A_1, A_2 is a splitting of A. Furthermore $A_2 \leqslant_T A_1$ by simple permitting. To see that $A_1 \leqslant_T A_2$ note the following. If $x < l(s)$, and $x \in A_{s+1} - A_s$ then some $y \leqslant \varphi(x, s)$, $y \neq x$ must enter A at stage $s+1$. Thus $A_{1,s}(x) = A_1(x)$ if and only if $A_{2,s+1}[\varphi(x,s)] = A_2[\varphi(x,s)]$. $\square$

Downey noted that the above proof gives the following.

Corollary 8.4. *An r.e. set A is wtt-mitotic if and only if A is wtt-autoreducible.*

Of course all strongly atomic sets are far from being autoreducible so we have already given many constructions of non-mitotic sets. But not all degrees contain non-mitotic sets.

Definition 8.5 (Ladner). An r.e. degree $\mathbf{a}$ is *completely mitotic* if every r.e. set of degree $\mathbf{a}$ is mitotic.

Ladner showed

Theorem 8.6 (Ladner [65]). *There is a completely mitotic r.e. degree. (This set is low_2 but not low by an observation of Ambos-Spies and Fejer [7].)*

Using similar ideas, Downey and Slaman showed

Theorem 8.7 (Downey and Slaman [37]). *There is a promptly simple completely mitotic degree.*

We omit the proofs of Theorems 8.6 and 8.7. The fact that the completely mitotic degree of Downey and Slaman is promptly simple is related to the following result.

Theorem 8.8 (Downey and Slaman [37]). *No low promptly simple degree is completely mitotic.*

The main idea in the proof of Theorem 8.8 is to combine the Robinson trick, prompt permitting, and the technique of Ladner from the next theorem.

Theorem 8.9 (Ladner [66]). *There is a complete nonmitotic set.*

Proof. This follows from Theorem 6.40 but we describe Ladner's original strategy. We construct a set A to be complete and to meet the following requirements for nonautoreducibility.

$$\mathbf{R}_e\colon \Phi_e(A \cup \{x\}; x) \neq A(x) \text{ for some } x.$$

We let $l(e, \cdot)$ be the length of agreement function for $\mathbf{R}_e$. Let f be a 1–1 recursive enumeration of K. To make A complete, we have markers Γ_e. The positions of marker Γ_e at stage s, $\Gamma(e, s)$, will be on an element of A_s. Of course $\Gamma(e, s)$ is increasing in e and nondecreasing in s.

Construction

Stage s + 1

Requirement $\mathbf{R}_e$ requires attention at stage $s + 1$ if $l(e,s) > \Gamma(e,s)$ and $e \leqslant f(s)$. Let e be least such that $\mathbf{R}_e$ requires attention at stage s. Enumerate $\Gamma(e,s)$ in A, and define

$$\Gamma(i,s+1) = \begin{cases} \Gamma(s+i,s) & i \geqslant e, \\ \Gamma(i,s) & \text{otherwise.} \end{cases}$$

If no such f requires attention, enumerate $\Gamma(f(s),s)$ in A and define

$$\Gamma(i,s+1) = \begin{cases} \Gamma(s+i,s) & i \geqslant f(s), \\ \Gamma(i,s) & \text{otherwise.} \end{cases}$$

It is easy to see that each requirement $\mathbf{R}_e$ receives attention finitely often and so that $\lim_s \Gamma(e,s)$ exists and that A is complete. $\square$

In fact there are large initial segments of $\mathbf{R}$ containing no completely mitotic degrees as evidenced by the following results.

Theorem 8.10 (Downey and Slaman [35]). *There is a (low$_2$-low) degree* $\mathbf{a}$ *such that for every degree* $\mathbf{b}$ *such that* $0 < \mathbf{b} \leqslant \mathbf{a}$, $\mathbf{b}$ *contains a nonmitotic r.e. set.*

Proof. Downey and Jockusch constructed a strongly atomic 1-topped degree $\mathbf{a}$. All such sets are low$_2$-low and all degrees $\mathbf{b}$ such that $\mathbf{b} \leqslant \mathbf{a}$ are strongly atomic (see Section 6). $\square$

Theorem 8.11 (Downey and Slaman [35]). *If* $\mathbf{a}$ *is nonrecursive, there is an r.e. degree* $\mathbf{b} \leqslant \mathbf{a}$ *such that for every degree* $\mathbf{c} \leqslant \mathbf{b}$, *if* $\mathbf{c}$ *is completely mitotic, then* $\mathbf{c} = \mathbf{0}$.

In view of the above results, it seemed reasonable to conjecture that no low degree is completely mitotic. Indeed, this was conjectured by Cooper, Ladner, and others. However Downey and Slaman also showed

Theorem 8.12 (Downey and Slaman [35]). *There is a low nonzero completely mitotic r.e. degree.*

The proof of Theorem 8.12 is very difficult and bears similarities to Lachlan's proof that the 1–3–1 lattice is embeddable in $\mathbf{R}$. However the technique is sufficiently flexible to show

Theorem 8.13 (Downey and Slaman [35]). *There is a high completely mitotic r.e. degree.*

The constructions for the preceding two theorems yield cappable degrees. Now jump inversion fails for cappable degrees (Shore [92] and Cooper [16]). This the following question is interesting.

Open Question 8.14. Is jump inversion always possible for completely mitotic degrees?

If Question 8.14 has a positive answer, then it is possible that a new construction is needed. Or perhaps, a nonuniform proof involving both the techniques of Theorems 8.11 and 8.8. We do know that there is no interval in **R** consisting entirely of completely mitotic degrees.

Theorem 8.15 (Ingrassia [53]). *The degrees containing nonmitotic r.e. sets are dense in* **R**.

A proof to Theorem 8.15 may also be found in Downey and Slaman [35]. However Downey and Slaman used quite a different technique. Ingrassia derived Theorem 8.15 as a corollary to his work on p-generic sets. Adding the Robinson trick to the Downey and Slaman proof of Theorem 8.15 gives

Theorem 8.16 (Downey and Slaman [35]). *The low completely mitotic degrees are nowhere dense in* **R**.

Probably, the techniques of Shore and Slaman should be able to extend Theorem 8.16 to the low_2 degrees.

The open question remaining from these theorems is the following.

Open Question 8.17. Are the completely mitotic degrees nowhere dense in **R**?

It is natural to extend the notion of mitotic r.e. sets to consider jump classes.

Definition 8.18. An r.e. set A is *jump mitotic* if there is a splitting A_1, A_2 of A such that $A_i' = A'$ for $i = 1, 2$.

Theorem 6.40 implies that there is a complete set which is not jump mitotic. It would be interesting to know which degrees are completely jump mitotic. (Of course all low degrees are and by the above mentioned result, $0'$ is not.)

9. Miscellaneous splitting results

One variation on the results of the last section is to extend the notions there to the wtt-degrees. In particular, we know of no work concerning completely mitotic wtt-degrees. Note that since there are contiguous strongly atomic sets, not all r.e. Turing degrees contain completely mitotic wtt-degrees.

Open Question 9.1. Are the Turing degrees which contain completely mitotic wtt-degrees dense in **R**?

We think that the answer is no. Ladner's construction [65] gives a Turing degree such that every wtt-degree in it is completely mitotic. We do however know a large class of degrees that are not completely wtt-mitotic. This follows from the work of Downey on array nonrecursive sets.

Definition 9.2 (Downey, Jockusch and Stob [28]). A strong array $\{F_e\}_{e \in \omega}$ is a *very strong array* (v.s.a.) if

$$\bigcup_{n \in \omega} F_n = N, \tag{9.1}$$

$$F_n \cap F_m = \emptyset \quad \text{if } n \neq m, \quad \text{and} \tag{9.2}$$

$$0 < |F_n| < |F_{n+1}| \quad \text{for all } n \in \omega. \tag{9.3}$$

Definition 9.3 (Downey, Jockusch and Stob [28]). An r.e. set A is $\{F_e\}_{e \in \omega}$-*nonrecursive* (F-a.n.r.) if

$$(\forall e)(\exists n)[W_e \cap F_n = A \cap F_n]. \tag{9.4}$$

In [28], it is shown that if A is $\{F_e\}_{e \in \omega}$-nonrecursive and $\{G_e\}_{e \in \omega}$ is a very strong array, then there is a set $B \equiv_{\text{wtt}} A$ such that B is $\{G_e\}_{e \in \omega}$-nonrecursive. With this in mind, we have the following definition.

Definition 9.4. An r.e. degree **a** is *array nonrecursive* if for every (some) strong array $\{F_e\}_{e \in \omega}$, there is set A of degree **a** such that A is $\{F_e\}_{e \in \omega}$-nonrecursive.

The array nonrecursive (anr) degrees form a natural subclass of **R** corresponding to the degrees below which certain sorts of multiple permitting arguments can be performed. In [28] it is shown this class is closed upwards, contains a low degree, and includes all non-low$_2$ degrees. However below each nonzero r.e. degree **a**, there is a nonzero degree **b** which is not array nonrecursive. The following results give some relationships between these degrees and notions of mitoticity.

Theorem 9.5 (Downey [26]). (1) *No contiguous r.e. degree is anr.*
(2) *If **a** is anr, then **a** contains an r.e. set A such that the wtt-degree of A is completely mitotic.*

Proof. We do (1); (2) is similar. Let $\{F_e\}_{e \in \omega}$ be a very strong array such that for all n,

$$|F_n| > 2^{|F_{n-1}|},$$

$$m(n) = \min\{x \mid x \in F_n\} > \max\{x \mid x \in F_{n-1}\}.$$

Suppose that C is F-nonrecursive. We will construct $A \equiv_T C$ and a set $B \leqslant_{\text{wtt}} C$ such that we meet the following requirements for all e.

$$\mathbf{R}_e: \quad \Phi_e(A) \neq B.$$

Here Φ_e is a wtt-reduction (with φ_e denoting the corresponding use function). We first construct A (independently of B). Let f be a recursive function enumerating C. We will have a set of markers, Λ_e for $e \in \omega$, such that the position of marker Λ_e at stage s, $\Lambda_{e,s}$, is nondecreasing in s, increasing in e, and marks a member of $\overline{A}_s$.

The construction of A is as follows.

Construction
Stage 0
Set $A_0 = \emptyset$ and let $\Lambda_{e,0} = e$.
Stage $s + 1$
Let $A_{s+1} = A_s \cup \{\Lambda_{f(s),s}\}$. Move the markers $\Lambda_{e,s}$ for $e \geqslant f(s)$ to positions beyond s (preserving their order and the properties mentioned above).

It is evident that A so constructed satisfies $A \equiv_T C$. The set A so constructed is sometimes called the (really, a) *kick set* of C.

We now construct B to meet the requirements $\mathbf{R}_e$ and to satisfy $B \leqslant_{\text{wtt}} C$. We will meet the latter condition by guaranteeing that x may be permitted to enter B at stage $s + 1$ only if $B[m(x)]$ changes at stage $s + 1$. We will devote ω^{e+1} to $\mathbf{R}_e$. Let $l(e, \cdot)$ denote the length of agreement function for $\Phi_e(A) = B$. We will also enumerate certain auxiliary sets D_e for $\mathbf{R}_e$. We describe the sequence of events for one possible witness x to $\mathbf{R}_e$. Suppose that $x \geqslant e$.

(1) Wait until C permits x at $s + 1$ and $l(e, t) \geqslant \langle e + 1, \langle x, x \rangle \rangle$ for some $t \leqslant s$.
(2) Wait for $t \geqslant s$ such that $l(e, t) \geqslant \langle e + 1, \langle x, x \rangle \rangle$. Enumerate into D_e at stage t (if necessary) to make $F_y \cap C_t \neq F_y \cap D_{e,t+1}$ for all y, $e \leqslant y \leqslant x$.
(3) Wait for a stage $u \geqslant t$ such that C permits $m(x)$ but such that $C_t[x] = C_u[x]$. Enumerate $\langle e + 1, \langle 0, x \rangle \rangle$ into B at stage $u + 1$. This temporarily satisfies $\mathbf{R}_e$, because since C permitted x at stage $s + 1$ (Step 1), we

have that for all $y \geqslant x$, $A_{y,t} > s > \varphi_e(A; \langle e + 1, \langle x, x \rangle \rangle)$ because we are dealing with wtt-reductions. Therefore, we have that $A_{u+1}[s] = A_t[s]$ but $B_{u+1}[\langle e + 1, \langle 0, x \rangle \rangle] \neq B_t[\langle e + 1, \langle 0, x \rangle \rangle]$ causing a disagreement at $\langle e + 1, \langle 0, x \rangle \rangle$. (On the next time through this cycle, we use $\langle e + 1, \langle 1, x \rangle \rangle$ instead of $\langle e + 1, \langle 0, x \rangle \rangle$, and so on.)

(4) If there is a stage $w > t$ such that C permits x then we return to step 2 (whether or not step 3 occurs). There can only be x such injuries so that for some $i \leqslant x$ we get a disagreement on $\langle e + 1, \langle i, x \rangle \rangle$. Furthermore, only x members of F_x can be used.

The point of the kicking procedure in the construction of A is that, if there is a coding of some $A_{i,s}$ which ruins the disagreement we have constructed above, then we move all markers $A_{j,s}$ $j \geqslant i$ to positions which can never later injure the disagreements. Furthermore, the enumeration of i into C does not interfere with the disagreement $D_{e,s} \cap F_m \neq C_u \cap F_m$ for $m \leqslant k$ such that $i \in F_m$. This means that we cannot "use up" all of a set F_m as there cannot be more than $|F_m|$ injuries. $\square$

There are a number of interesting splitting theorems for anr sets. For example we have the following.

Theorem 9.6 (Downey, Jockusch and Stob [28]). *For every array nonrecursive set A there is a splitting A_1, A_2 of A such that each of A_1 and A_2 is anr.*

Proof. Suppose that A is $\{F_e\}_{e \in \omega}$-nonrecursive. For each $e \in \omega$ and $i \in \{1, 2\}$ we have the requirement

$$\mathbf{R}_{e,i} : (\exists n)[W_e \cap F_n = A_i \cap F_n].$$

To meet $\mathbf{R}_{e,i}$ we will enumerate a certain set $V_{e,i}$ and use the fact that

$$(\overset{\infty}{\exists} n)[V_{e,i} \cap F_n = A \cap F_n]. \tag{9.5}$$

During the course of the construction, we will reserve certain n for $\mathbf{R}_{e,i}$. Each n may be reserved for at most one requirement $\mathbf{R}_{e,i}$ at any one stage, but the reservation may be cancelled at a later stage for the purpose of reserving n for a requirement of higher priority. (The intention of these reservations is that there will be some n which is reserved for $\mathbf{R}_{e,i}$ and for which $W_e \cap F_n = A \cap F_n$.) The priority order of the requirements $\mathbf{R}_{e,i}$ is in order of increasing $\langle e, i \rangle$.

Construction

Stage $s + 1$

Step 1. For each $x \in A_{s+1} - A_s$ let n be the integer such that $x \in F_n$. If n is reserved for the requirement $\mathbf{R}_{e,i}$, then enumerate x in A_i. If n is not reserved for any requirement, enumerate x in A_1.

Step 2. For each x and e, if $x \in W_{e,s+1} - W_{e,s}$, $x \in F_n$, and n is reserved for a requirement $\mathbf{R}_{e,i}$, then enumerate x in $V_{e,i}$.

Step 3. $\mathbf{R}_{e,i}$ *requires attention* at stage $s + 1$ if

$$(\forall n)\,[n \text{ is reserved for } \mathbf{R}_{e,i} \Rightarrow W_{e,s} \cap F_n \neq A_{i,s} \cap F_n],\quad \text{and} \tag{9.6}$$

$$(\exists n)\,[A_s \cap F_n = \emptyset \text{ and } n \text{ is not reserved}$$
$$\text{for any } \mathbf{R}_{f,j} \text{ such that } \langle f, j \rangle \leqslant \langle e, i \rangle\,]. \tag{9.7}$$

If such a pair e, i exists, choose the pair such that $\langle e, i \rangle$ is least and let n be the least integer satisfying (9.7) for e, i. Perform the following actions for these fixed e, i, n. Reserve n for $\mathbf{R}_{e,i}$. Cancel any other reservation of n. Enumerate all of $W_{e,s+1} \cap F_n$ into $V_{e,i}$. This ends the construction.

Lemma 9.7. *If n is reserved for $\mathbf{R}_{e,i}$, and that reservation is never cancelled, then $W_e \cap F_n = V_{e,i} \cap F_n$ and $A_i \cap F_n = A \cap F_n$.*

Proof. The first clause of the conclusion is by steps (2) and (3) of the construction. To see that $A_i \cap F_n = A \cap F_n$, notice that at the stage that n is first reserved for $\mathbf{R}_{e,i}$, $A_{i,s} \cap F_n = A_s \cap F_n$ $(= \emptyset)$ by (9.7). Step (1) guarantees that this equality is maintained for all later stages. $\quad\square$

Lemma 9.8. *If $V_{e,i} \cap F_n \neq \emptyset$, then n is reserved for $\mathbf{R}_{e,i}$ or some requirement of higher priority at cofinitely many stages.*

Lemma 9.9. *Each requirement $\mathbf{R}_{e,i}$ receives attention only finitely often and is satisfied.*

Proof. Given e, i, let s_0 be such that if $\langle f, j \rangle < \langle e, i \rangle$, $\mathbf{R}_{f,j}$ does not receive attention after s_0. By (9.5), there are infinitely many n such that $V_{e,i} \cap F_n = A \cap F_n$. Let n be any such n which is not reserved for $\mathbf{R}_{f,j}$ for any $\langle f, j \rangle < \langle e, i \rangle$. There are two cases.

Case (i): n is reserved for $\mathbf{R}_{e,i}$ at some stage of the construction. Then by Lemma 9.7, $W_e \cap F_n = V_{e,i} \cap F_n = A \cap F_n = A_i \cap F_n$. Thus $\mathbf{R}_{e,i}$ is satisfied. Let s_1 be a stage such that $W_{e,s_1} \cap F_n = W_e \cap F_n$ and $A_{i,s} \cap F_n = A_i \cap F_n$. Then by 9.6, $\mathbf{R}_{e,i}$ never receives attention after stage s_1.

Case (ii): n is never reserved for $\mathbf{R}_{e,i}$. Then by Lemma 9.8, $V_{e,i} \cap F_n = \emptyset$. Thus $A \cap F_n = \emptyset$. Thus (9.7) applies to n at cofinitely many stages of the construction. Since n is never reserved for $\mathbf{R}_{e,i}$, it must be that $\mathbf{R}_{e,i}$ receives attention only finitely often and that at cofinitely many stages of the construction 9.6 fails. This implies the existence of m such that $W_e \cap F_m = A_i \cap F_m$ and hence that the requirement is satisfied. $\quad\square$

It is clear that the construction above can be combined with requirements to make A_1 and A_2 low. Thus we have the following corollary which was first proved (directly) by Cameron Smith.

Corollary 9.10. *For every array nonrecursive degree **a** there is an array nonrecursive degree* **b** $<$ **a** *such that* **b** *is low.*

It is not true that if A is a.n.r. and A is the disjoint union of sets A_1 and A_2, then at least one of A_1 or A_2 is anr. However this result is true up to degree. In fact we have the stronger result of the next theorem. We first need a definition.

Definition 9.11. If $\{E_e\}_{e\in\omega}$ is a strong array and A an r.e. set with a given enumeration, then A *E-permits* n *at stage* $s + 1$ if

$$(\exists z \leqslant y)(\exists x \leqslant \max(E_z))[x \in A_{s+1} - A_s].$$

Theorem 9.12. *Suppose that* $A \leqslant_{\mathrm{wtt}} A_1 \oplus A_2$ *and that* A *is array nonrecursive. Then there are r.e. sets* B_1 *and* B_2 *such that* $B_i \leqslant_{\mathrm{wtt}} A_i$ *and one of* B_1 *or* B_2 *is array nonrecursive.*

Proof. Let $\{F_e\}_{e\in\omega}$ and $\{E_e\}_{e\in\omega}$ be very strong arrays such that $|E_n| > 2|F_{\langle i,n\rangle}|$ for every i and $n > i$. We first show that we may assume that A is E-nonrecursive and $A = A_1 \cup A_2$. To see this we first notice that since A is array nonrecursive, the wtt-degree of A contains an array nonrecursive set $\hat{A}$. This follows from Corollary 2.9 of [28]. We next rely on Lachlan's lemma, Lemma 5.15. Applying the lemma with $B = \hat{A}$ gives sets $\hat{A}_1$ and $\hat{A}_2$ such that $\hat{A} = \hat{A}_1 \cup \hat{A}_2$ and $\hat{A}_i \leqslant_{\mathrm{wtt}} A_i$. The sets B_i which result from the proof of the theorem satisfy $B_i \leqslant_{\mathrm{wtt}} \hat{A}_i$ and thus $B_i \leqslant_{\mathrm{wtt}} A_i$. We shall also assume that A, A_1, and A_2 are enumerated so that

$$A_s = A_{1,s} \cup A_{2,s}. \tag{9.8}$$

We will meet the following requirements for every $e, j \in \omega$:

$$\mathbf{R}_{e,j}: \quad (\exists n)[W_e \cap F_n = B_1 \cap F_n \text{ or } W_j \cap F_n = B_2 \cap F_n].$$

(These requirements suffice to make one of B_1 or B_2 F-nonrecursive since if e is such that there is no n with $W_e \cap F_n = B_1 \cap F_n$ then the satisfaction of $\mathbf{R}_{e,j}$ for all $j \in \omega$ implies that B_1 is F-nonrecursive.) We will reserve the sets $F_{\langle i,0\rangle}, F_{\langle i,1\rangle}, \ldots$ for requirement $\mathbf{R}_{e,j}$ where $i = \langle e, j\rangle$. We will use the fact that A is a.n.r. by enumerating r.e. sets V_i and assuming that

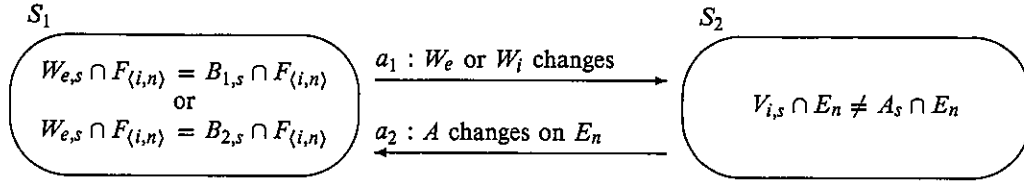$$(\overset{\infty}{\exists} n)[V_i \cap E_n = A \cap E_n].$$

Fig. 5. State diagram of the construction.

To insure that $B_i \leq_{\text{wtt}} A_i$ we will use permitting as follows. We allow $y \in F_{\langle i,n \rangle}$ to enter B_1 (B_2) at stage $s + 1$ only A_1 (A_2) E-permits n at stage $s + 1$.

Fix e and j and let $i = \langle e, j \rangle$. Requirement $\mathbf{R}_{e,j}$ is split into the following subrequirements for all $n > \langle e, j \rangle$.

$$\mathbf{R}_{e,j,n} : \quad V_i \cap E_n = A \cap E_n \Rightarrow [W_e \cap F_{\langle i,n \rangle} = B_1 \cap F_{\langle i,n \rangle} \text{ or } \\ W_j \cap F_{\langle i,n \rangle} = B_2 \cap F_{\langle i,n \rangle}].$$

We describe the construction for $\mathbf{R}_{e,j,n}$ as a two-state automaton. We say that $\mathbf{R}_{e,j,n}$ is in state S_1 at stage s if the condition for state S_1 in Figure 5 holds. Otherwise $\mathbf{R}_{e,j,n}$ is in state S_2 at stage s and the construction guarantees that if this happens, the condition in the diagram for state S_2 holds. In order to accomplish this, the action corresponding to arrow a_1 is the following. If $\mathbf{R}_{e,j,n}$ is in state S_1 at stage s but not at stage $s + 1$, we enumerate an element of E_n into V_i if necessary to cause the condition of state S_2 to hold. Since this happens only if an element of $F_{\langle i,n \rangle}$ is enumerated in W_e or W_j at stage $s + 1$, this action need only be performed at most $2|F_{\langle i,n \rangle}|$ many times. Since $|E_n| > 2|F_{\langle i,n \rangle}|$ if $n > i$, we will be able to perform this action. Similarly, if s is such that the condition of state S_2 holds at s but fails at $s + 1$, we must be able to ensure that the condition of state S_1 holds at stage $s + 1$. For such an s, it must be the case that an element of E_n is enumerated into A at stage $s + 1$, and hence by (9.8), that element is enumerated in either A_0 or A_1 at stage $s + 1$. By our condition on permitting, this allows us to enumerate elements of $F_{\langle i,n \rangle}$ into either B_0 or B_1 at stage $s + 1$, thereby guaranteeing that $\mathbf{R}_{e,j,n}$ is in state S_1 at stage $s + 1$.

Construction

Stage $s + 1$

Step 1. (Arrow a_2) For every triple e, j, n such that $\langle e, j \rangle < n$, if $W_{e,s} \cap F_{\langle\langle e,j \rangle,n\rangle} \neq B_{1,s} \cap F_{\langle\langle e,j \rangle,n\rangle}$ and A_1 E-permits n at stage $s + 1$, enumerate all of $W_{e,s+1} \cap F_{\langle\langle e,j \rangle,n\rangle}$ into B_1 and similarly for W_j, A_2 and B_2 in place of W_e, A_1 and B_1.

Step 2. (Arrow a_1) For each triple e, j, n, if

(1) $W_{e,s+1} \cap F_{\langle\langle e,j \rangle,n\rangle} \neq B_{1,s+1} \cap F_{\langle\langle e,j \rangle,n\rangle}$, and

(2) $W_{j,s+1} \cap F_{\langle\langle e,j \rangle,n\rangle} \neq B_{2,s+1} \cap F_{\langle\langle e,j \rangle,n\rangle}$, but

(3) $W_{e,s} \cap F_{\langle\langle e,j \rangle, n \rangle} = B_{1,s} \cap F_{\langle\langle e,j \rangle, n \rangle}$ or $W_{j,s} \cap F_{\langle\langle e,j \rangle, n \rangle} = B_{2,s} \cap F_{\langle\langle e,j \rangle, n \rangle}$,
then enumerate one element of $E_n - V_{\langle e,j \rangle, s}$, if necessary, into $V_{\langle e,j \rangle}$ so that $V_{\langle e,j \rangle, s+1} \cap E_n \neq A_{s+1} \cap E_n$. (Such an element will exist by the construction.) This ends the construction.

It is not difficult to prove the following lemmas.

Lemma 9.13. $B_1 \leqslant_{\mathrm{wtt}} A_1; B_2 \leqslant_{\mathrm{wtt}} A_2$.

Lemma 9.14. *For every* e, j, $\mathbf{R}_{e,j}$ *is satisfied.* $\square$

The following corollary follows directly from the Theorem and Corollary 2.8 of [28].

Corollary 9.15. *Suppose that* $A \leqslant_{\mathrm{wtt}} A_1 \oplus A_2$ *and that* A *is array nonrecursive. Then the weak-truth-table degree of either* A_1 *or* A_2 *contains an array nonrecursive set.*

An immediate consequence of the preceding corollary is the following.

Corollary 9.16. *The array recursive wtt-degrees form an ideal in the uppersemilattice of r.e. wtt-degrees.*

Proof. By the corollary, the array recursive wtt-degrees are closed under join. By Corollary 2.8 of [28], the array recursive wtt-degrees are closed downward. $\square$

The analogue of Corollary 9.15 and hence of Corollary 9.16 is not available for the Turing degrees as we now show in Theorem 9.17.

Theorem 9.17. *There are r.e. degrees* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *such that* $\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{0}'$ *and* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *are array recursive.*

Proof. Fix a v.s.a. $\{F_e\}_{e \in \omega}$ such that $|F_n| > 2^{n^2}$ for all $n \in \omega$. We construct sets A_1 and A_2 of array recursive degree by showing that every set recursive in either is not F-a.n.r. To do this we will enumerate sets V_e and U_e so that for every e and $n > e$ the following requirements are satisfied.

$$\mathbf{R}_{e,n}: \quad \Phi_e(A_1) = B_e \Rightarrow V_e \cap F_n \neq B_e \cap F_n,$$

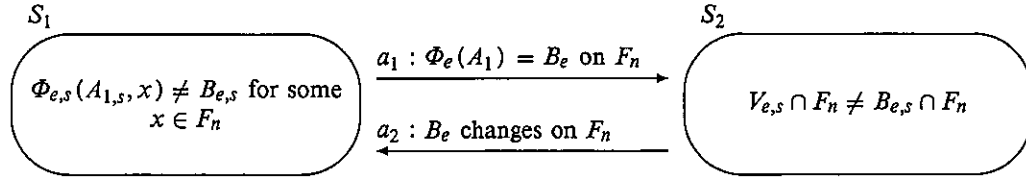$$\mathbf{Q}_{e,n}: \quad \Phi_e(A_2) = B_e \Rightarrow U_e \cap F_n \neq B_e \cap F_n.$$

Fig. 6. State diagram of the construction.

Here $(\Phi_e, B_e)_{e \in \omega}$ enumerates all pairs (Φ, B) of reductions Φ and r.e.sets B. To guarantee that $K \leqslant_T A_1 \oplus A_2$, we will define a recursive function $\gamma : \omega^2 \to \omega$ such that

$$\lim_s \gamma(x, s) \text{ exists,} \tag{9.9}$$

$$\gamma(x, s + 1) \neq \gamma(x, s) \text{ only if}$$

$$(\exists y \leqslant \gamma(x, s))[y \in A_{1,s+1} - A_{1,s} \text{ or } y \in A_{2,s+1} - A_{2,s}], \tag{9.10}$$

if $x \in K_{s+1} - K_s$ then

$$(\exists y \leqslant \gamma(x, s))[y \in A_{1,s+1} - A_{1,s} \text{ or } y \in A_{2,s+1} - A_{2,s}]. \tag{9.11}$$

The existence of such a function γ implies that $K \leqslant_T A_1 \oplus A_2$; the fact that γ depends on s makes this a Turing reduction rather than a weak-truth-table reduction which is prohibited by Theorem 9.12. We define $\gamma(x, 0) = x$ for all $x \in \omega$.

The two-state automaton corresponding to requirement $R_{e,n}$ is in Fig. 6.

Arrow a_1 is traversed at any stage $s + 1$ such that $\Phi_{e,s+1}(A_{1,s+1}, x) = B_{e,s+1}(x)$ for all $x \in F_n$. At this stage, we enumerate as usual into V_e to cause $V_{e,s+1} \cap F_n \neq B_{e,s+1} \cap F_n$. We also take further action to attempt to preserve all the computations $\Phi_{e,s+1}(A_{1,s+1}, x)$ for $x \in F_n$. Suppose that it is possible to preserve these computations forever and suppose there is a stage $t + 1 > s + 1$ at which the condition of state S_2 fails. This implies that an integer $x \in F_n$ is enumerated in B_e at stage $t + 1$. But then we have that $\Phi_{e,t+1}(A_{1,t+1}) = \Phi_{e,s+1}(A_{1,s+1}) = B_{e,s+1} \neq B_{e,t+1}$ and this disagreement is preserved forever. Thus requirement $R_{e,n}$ remains in state S_1 forever and is satisfied. The bound on $|F_n|$ above reflects the fact that in taking action a_1 we will not always be able to preserve all computations because of the requirements for coding K. We will ensure that the action a_1 is injured fewer that 2^{n^2} times and thus that arrow a_1 requires traversal at most 2^{n^2} times.

Construction

Stage $s + 1$

Step 1. Let n be the least element of $K_{s+1} - K_s$. Enumerate $\gamma(n, s)$ into A_1. Define $\gamma(y, s + 1) = \gamma(y + s, s)$ for all $y \geqslant n$.

Step 2. (Arrow a_1) Requirement $\mathbf{R}_{e,n}$ ($\mathbf{Q}_{e,n}$) *requires attention* at stage $s + 1$ if

$$\Phi_{e,s+1}(A_{1,s+1}, x) = B_{e,s+1}(x)$$

$$(\Phi_{e,s+1}(A_{2,s+1}, x) = B_{e,s+1}(x)) \quad \text{for all } x \in F_n, \quad \text{and} \tag{9.12}$$

$$V_{e,s} \cap F_n = B_{e,s+1} \cap F_n \qquad (U_{e,s} \cap F_n = B_{e,s+1} \cap F_n). \tag{9.13}$$

Let n be least and e least for n such that either $\mathbf{R}_{e,n}$ or $\mathbf{Q}_{e,n}$ requires attention. If $\mathbf{R}_{e,n}$ requires attention do the following. Let u be the maximum element of A_1 used in the computations mentioned in (9.12). If $\gamma(n,s) \leqslant u$, enumerate $\gamma(n,s)$ into A_2 and define $\gamma(y, s + 1) = \gamma(y + s, s)$ for all $y \geqslant n$. (By the usual conventions on the use function of a computation, $\gamma(y, s + 1) > u$ for all $y \geqslant n$. Thus this step has the effect of clearing the computations of (9.12) of lower priority markers.) Also, choose $z \in F_n - V_{e,s}$ (such will exist) and enumerate $z \in V_e$. If instead $\mathbf{Q}_{e,n}$ requires attention but $\mathbf{R}_{e,n}$ does not, attend to $\mathbf{Q}_{e,n}$ just as $\mathbf{R}_{e,n}$ but with U_e, A_1, and A_2 in place of V_e, A_2, and A_1 respectively. This ends the construction.

Lemma 9.18. *For every $e, n \in \omega$ such that $n > e$, requirements $\mathbf{R}_{e,n}$ and $\mathbf{Q}_{e,n}$ receive attention at most 2^{n^2} times and are satisfied.*

Proof. We assume the lemma is true for all pairs e', n' such that $n' < n$ or $n' = n, e' < e$ and give the proof for $\mathbf{R}_{e,n}$. The proof for $\mathbf{Q}_{e,n}$ is identical. Suppose that $\mathbf{R}_{e,n}$ receives attention at stage $s + 1$ and there is $z \in F_n - V_{e,s}$. Then $V_{e,s+1} \cap F_n \neq B_{e,s+1} \cap F_n$. Furthermore, by (9.12) $\Phi_{e,s+1}(A_{1,s+1}, x) = B_{e,s+1}(x)$ for all $x \in F_n$ so that if these computations are never injured, either $V_e \cap F_n \neq B_e \cap F_n$ or $\Phi_e(A_1) \neq B_e$ and $\mathbf{R}_{e,n}$ never requires attention after stage $s + 1$. Now by the definition of $\gamma(y, s + 1)$ for $y \geqslant n$, the computation in (9.12) can be injured at a later stage $t + 1$ only if $\gamma(y, t + 1) = \gamma(y, s)$ enters A_0 for some $y < n$. This happens only if such a number y enters K at stage $t + 1$ or because a requirement $\mathbf{R}_{e',y}$ or $\mathbf{Q}_{e',y}$ for some e' such that $e' < y < n$ receives attention at stage $t + 1$. Therefore there can be at most $n + \sum_{0 < y < n} 2^{y^2}$ many stages $s + 1$ at which $\mathbf{R}_{e,n}$ receives attention and is later injured. Thus $\mathbf{R}_{e,n}$ receives attention at most $1 + n + \sum_{0 < y < n} 2^{y^2} \leqslant 2^{n^2}$ times. Since $|F_n| > 2^{n^2}$, $F_n - V_e \neq \emptyset$. Thus, if $\Phi_e(A_1) = B_e$, $\mathbf{R}_{e,n}$ will receive attention enough times to enumerate V_e to make $V_e \cap F_n \neq B_e \cap F_n$. $\square$

Lemma 9.19. $K \leqslant_{\mathrm{T}} A_1 \oplus A_2$.

Proof. The definition of γ satisfies (9.11) by step (1) of the construction. (9.10) is satisfied since $\gamma(y, s) \neq \gamma(y, s + 1)$ only if some $\gamma(n, s)$ for $n \leqslant y$ is enumerated in either A_1 or A_2 at stage $s + 1$, and γ is increasing in its first

argument. To see that (9.9) is satisfied, note that $\gamma(y, s+1) \neq \gamma(y, s)$ only if some $n \leqslant y$ enters K at stage $s+1$ or some requirement $\mathbf{R}_{e,n}$ or $\mathbf{Q}_{e,n}$ receives attention for some $n < y$. Because of Lemma 9.18, there are only finitely many such stages and thus (9.9) is satisfied. $\square$

Another variation on the concept of mitoticity is atomicity, defined but not explored in Downey and Welch [41].

Definition 9.20 ([41]). A set A is *atomic* if for every splitting A_1, A_2 of A, $A_1 \leqslant_T A_2$ implies that $A_1 \equiv_T \emptyset$.

Several of the results of Section 7 stated for strongly atomic sets hold as well for atomic sets. For example, all high atomic sets have the antisplitting property (same proof as Theorem 7.5).

Downey and Welch hoped that atomic sets might be useful in studying noncappable degrees in the same way that strongly atomic sets are for studying cappable degrees. The first indication that this program might fail was the following result of Ingrassia.

Theorem 9.21 (Ingrassia [52]). *There is no complete atomic set.*

Ingrassia suggested that Theorem 9.21 might be extended to all noncappable (promptly simple) degrees. This is the case as we shall now prove.

Theorem 9.22. *If* $\mathbf{a}$ *is promptly simple then* $\mathbf{a}$ *contains no atomic set.*

Proof. Let A be a set of promptly simple degree. We must construct a splitting A_1, A_2, such that $\emptyset <_T A_1 \leqslant_T A_2$. The requirements to make A_1 nonrecursive are the usual ones:

$$\mathbf{R}_e: \overline{A}_1 \neq W_e.$$

For the sake of $\mathbf{R}_e$ we will enumerate auxiliary sets U_e and V_e. We will assume that the indices of U_e and V_e, $f(e)$ and $g(e)$ satisfy $W_{f(e),s} \cap U_{e,\text{at}s} = \emptyset$ and $W_{g(e),s} \cap V_{e,\text{at}s} = \emptyset$. (This is the Slowdown Lemma of Soare, [97, p. 284].) Since A is of promptly simple degree, we may also assume ([97, p. 284]) that we are given an enumeration of A and a recursive function p which satisfies

$$W_e \text{ infinite } \Rightarrow (\overset{\infty}{\exists} x)(\exists s)[x \in W_{e,\text{at}s} \wedge A_s[x] \neq A_{p(s)}[x]].$$

We describe only the basic module for one requirement $\mathbf{R}_e$; the argument is finite injury. Unless otherwise directed, numbers entering A are enumerated into A_2. Let $a_{0,s} < a_{1,s} < \cdots$ enumerate $\overline{A}_s$. We wait for a stage s such that for some unrestrained i, we have that $A_{1,s}[a_{i,s}] = \overline{W}_{e,s}[a_{i,s}]$. We then enumerate

$a_{i,s}$ into U_e and wait to see if A promptly permits $a_{i,s}$. (That is $a_{i,s} \in A_{p(t)}$ where t is such that $a_{i,s} \in W_{f(e),t}$.) If not, we abandon $a_{i,s}$ and move on to $a_{i+1,s}$. If so, then there is a least $j \leqslant i$ such that $a_{j,s} \in A_{p(t)}$. At stage $p(t)$ we enumerate $a_{j-1,p(t)} = a_{j-1,s}$ into V_e and similarly see if A promptly permits $a_{j,p(t)}$. If not, we then enumerate $a_{j,s}$ into $A_{2,p(t)}$. Otherwise we enumerate $a_{j,s}$ into $A_{1,p(t)}$. Obviously $A_1 \leqslant_T A_2$ by simple permitting. The construction succeeds since the promptness condition guarantees that we get infinitely many V_e permissions.   □

Ingrassia has also claimed the following theorem for which we supply our own proof.

Theorem 9.23 (Ingrassia). *There is a nonzero r.e. degree* **a** *such that if* **b** $\leqslant$ **a**, *then* **b** *contains no atomic set.*

Proof. This can be proved by modifying the Lachlan nonbounding construction [63]. We will follow the approach of Soare [97] and we assume the reader is familiar with it. We only describe the basic module. We construct A to meet the following requirements.

$\mathbf{R}_e$: $\overline{A} \neq W_e$,

$\mathbf{Q}_e$: $(\Phi_e(A) = W_e \wedge W_e \not\equiv_T \emptyset) \Rightarrow$ (there is a splitting U_e, V_e of A
$\qquad\qquad\qquad\qquad\qquad\qquad$ with $U_e \leqslant_T V_e \wedge (\forall i)\mathbf{Q}_{e,i}$),

$\mathbf{Q}_{e,i}$: $\overline{U}_e \neq W_e$.

We will have a length of agreement function $l(e,s)$ associated with $\Phi_e(A) = W_e$. As in Soare, we shall have a pair of restraint functions, $r_1(e,s)$ and $r_2(e,s)$. The basic module has the following steps.

Step 1. For the sake of $\mathbf{Q}_{e,i}$ choose a candidate x_0 and wait for a stage such that $l(e,s_0) > x_0$ and $\overline{Q}_{e,s_0}[x_0] = W_{i,s_0}[x_0]$.

Step 2. At s, we open a 1-gap by setting $r_1(e,s_0) = 0$. We wait for a stage $s_1 > s_0$ such that $l(e,s_1) > x_0$.

Step 3. At s_1 we have either

Successful closure: $W_{e,s_1}[x_0] \neq W_{e,s_0}[x_0]$. In this case we open a 2-gap by setting $r_2(e,s_1) = 0$ as well. In this case we wait for a stage s_2 such that $l(e,s_2) > s_1$.

Unsuccessful closure: Otherwise we reset $r_1(e,s_1) = s_1$ and choose a new candidate $x_1 > s_1$.

Step 4. At the closing of the 2-gap, we again have either

Successful closure: $W_{e,s_2}[x_0] \neq W_{e,s_1}[x_0]$. In this case two or more numbers entered A since step 1 and we can enumerate the minimum of these in V_e and the rest in U_e meeting $\mathbf{Q}_{e,i}$ forever.

Unsuccessful closure: Otherwise we reset the restraint functions $r_1(e, s_2) = r_2(e, s_2) = s_2$ and choose a new candidate x_1.

The above construction works for exactly the same reasons as does the Lachlan nonbounding construction as described by Soare. $\square$

We close this section by showing that not all atomic sets are strongly atomic.

Theorem 9.24. *There is an atomic set A which is not strongly atomic.*

Proof. We construct a nonrecursive set A, a splitting A_1, A_2 of A, and a nonrecursive set $D \leqslant_T A_1, A_2$. This ensures that A is not strongly atomic. To guarantee that A is atomic, we let $\langle U_e, V_e, \Phi_e \rangle$ enumerate triples consisting of a pair of disjoint r.e. sets and a functional and we meet the following requirements.

$\mathbf{N}_e$: U_e, V_e is a splitting of A $\wedge \Phi_e(U_e) = V_e \Rightarrow V_e$ is recursive,

$\mathbf{R}_e$: $D \neq \overline{W}_e$.

We will let $l(e, s)$ be a length of agreement function for $\mathbf{N}_e$ defined as follows.

$$l(e, s) = \max\{x \mid (\forall y < x)[\Phi_{e,s}(U_{e,s}; y) = V_{e,s}(y)$$

$$\wedge (\forall z < u(\Phi_{e,s}(U_{e,s}; y)))[U_{e,s} \cup V_{e,s}(z) = A_s(z)]]\}.$$

Using $l(e, s)$, we define e-expansionary stages as usual. The construction is on a tree. We describe the strategy for meeting one requirement $\mathbf{R}_e$ at a node τ in the presence of one requirement $\mathbf{N}_j$ at node σ of higher priority. The steps are as follows.

Step 1. Choose a fresh follower x targeted for A_1 and D with $x + 1$ targeted for A_2.

Step 2. At a τ-stage t such that $x \in W_{e,t}$, enumerate x into A_1. Choose a fresh number $y > x$ as the new A_1-trace for x.

Step 3. At the next σ-stage $u > t$, enumerate $x + 1$ in A_2 and choose a new number $z > u$ as the new A_2-trace for x.

Step 4. At the next σ-stage, enumerate y into A_1, z into A_2 and x into D. $\cdot$

Why does this work? It of course meets the requirement $\mathbf{R}_e$ and guarantees that $D \leqslant_T A_1, A_2$. It meets $\mathbf{N}_j$ as follows. To compute $V_e[n]$ recursively, we find the least σ-stage s such that $l(e, s) > n$ and such that we are not at step 3 or 4 above for any τ. (We arrange that this happens infinitely often by cancellation at each stage that τ receives attention.) We claim that $V_{e,s}[n] = V_e[n]$.

To see this, the only way that $V_e[n]$ could change after stage s is that some number $w \leqslant n$ enters A after stage s. By assumption on s, such numbers must

be of form x or $x + 1$ for follower x of some $\tau \supset \sigma$. Now we only put one such number into A between σ-stages until both numbers we put into A exceed s and so exceed n. Thus if $w \leqslant n$ enters A after s, then w enters U_e and not V_e. Therefore V_e does not change after all.

It remains to be seen that the strategies for several requirements N_j cohere. Suppose that N_i and N_j are such that $i < j$, σ_i is devoted to N_i, σ_j is devoted to N_j, and $\sigma_i \subset \sigma_j$. The problem is this. To keep σ_j happy, σ_j requires us in steps 3 and 4 to wait until σ_j-stages to enumerate into A. Every σ_j-stage is of course a σ_i stage but not conversely. Now to meet σ_i, we must have infinitely many stages where we allow its length of agreement to increase. σ_i cannot delay forever. What we do is this. Suppose that we have x following R_e at τ, such that $\tau \supset \sigma_j \supset \sigma_i$. At step 3 for σ_i, we cannot really enumerate $x + 1$ into A_1 until the next σ_j-stage. Perhaps there are infinitely many σ_j stages and we eventually allow $l(j, r) > y$. Now at the next σ_j-stage, we cannot now safely enumerate both z and y into A since perhaps $z < \Phi_{e(\sigma_i),s}(Q_{e(\sigma_i),s}; y))$. Our solution is this. The basic module gives us a way of enumerating two numbers into A in the presence of one N_e without any injury. What we do is build inductive strategies based on the basic module to solve the problem at the "next level". In this case we would modify steps 3 and 4 as follows (for $\tau \supset \sigma_j \supset \sigma_i$).

Step 3. At the next σ_j-stage $u > t$, enumerate $x + 1$ in A_2 and choose a new number $z > u$ as the new A_2-trace for x.

Step 4. At the next σ_j-stage, enumerate y into A_1, z into A_2 and x into D.

Now note that after step 3 we have finished the effect of σ_j in the sense that y and z can be safely added without injury to σ_j. Our only problem is with σ_i alone now. Thus we have reduce the problem to put two numbers z and y into A in the presence of one N_j. However we know how to do this; we use the basic module. $\square$

It is not difficult to modify the above construction to prove the following.

Theorem 9.25. *There is an atomic r.e. set A such that for all $B \equiv_{\text{wtt}} A$, B is not strongly atomic.*

The remaining open question is however

Open Question 9.26. Is there an atomic set A such that for all $B \equiv_{\text{T}} A$, B is not strongly atomic?

10. d.r.e. degrees

One natural place to look for splitting theorems analogous to those for the r.e. degrees is in the d.r.e. degrees.

Definition 10.1. A set D is called *d.r.e.* if there are r.e. sets U and V such that $D = U - V$.

More generally, a set X is *n-r.e.* if there is a recursive approximation $\{X_s\}_{s \in \omega}$ to X such that for every x, $|\{s \mid X_s(x) \neq X_{s+1}(x)\}| \leqslant n$. Of course, r.e. is the same as 1-r.e., and d.r.e. is the same as 2-r.e.

A degree $\mathbf{d}$ is *properly n-r.e.* if $\mathbf{d}$ contains an n-r.e. set but no $(n-1)$-r.e. set.

In this section we briefly describe the status of splitting theorems for d.r.e. degrees without giving many details. Early results suggested that d.r.e. sets enjoy many of the same properties that r.e. sets have. For example, there are no minimal d.r.e. degrees. In addition, the following result was noted by Lachlan, Cooper and others.

Theorem 10.2. *If D is d.r.e. and nonrecursive, then there is a nonrecursive r.e. set $A \leqslant_T D$.*

Proof. Let U and V be r.e. sets with $D = U - V$ d.r.e. If $U - V$ is r.e. then D is r.e. and hence $A = D$ suffices. Suppose then that $U - V$ is not r.e. Let f be a 1–1 recursive enumeration of U. Define $A = \{s \mid f(s) \in V\}$. We have that $A \leqslant_T D$. Furthermore, A is not recursive since A recursive implies $U - V$ is r.e. □

Note that Theorem 10.2 immediately reveals some complexity in the d.r.e. degrees since, for example, a construction of a minimal pair of d.r.e. degrees immediately yields a minimal pair of r.e. degrees. The proof of Theorem 10.2 is not uniform. That is, it does not determine the index for A uniformly in indices for U and V. This nonuniformity is necessary as can be seen from the following theorem.

Theorem 10.3. *There is no recursive function g and functional Φ such that if $W_e - W_i$ is nonrecursive then $W_{g(e,i)}$ is nonrecursive and $\Phi^{W_e - W_i} = W_{g(e,i)}$.*

Proof. Given V r.e. and functional Φ, we construct a d.r.e. set D such that the following requirements are satisfied.

$$\mathbf{R}: \quad \Phi(D) \neq V \text{ or } V \text{ is recursive,}$$

$$\mathbf{P}_e: \quad D \neq \overline{W}_e.$$

These requirements (together with the recursion theorem) are easily seen to be enough to show that the desired uniformity does not exist.

Let $l(s)$ be the length of agreement function for $\Phi(D) = V$ and $u(s)$ be the use (of D_s) is establishing this agreement. The strategy for $\mathbf{R}$ is that of preserving this length of agreement. The strategy for a requirement $\mathbf{P}_i$ (in the presence of a higher priority $\mathbf{R}_e$) is as follows. We choose a witness for $\mathbf{P}_i$ and wait until $x \in W_{i,s}$. In this case we wish to place x in D and so win $\mathbf{P}_i$ forever. This conflicts with the strategy of preserving the length of agreement for $\mathbf{R}_e$. Nevertheless, we place x in D anyway. Now either the length of agreement is never restored (in which case we win $\mathbf{R}$ outright) or at some stage $t > s$ we have that $l(t) > l(s)$. At stage t either $V_t[l(s)] = V_s[l(s)]$ or not. If so, there was no harm in placing x in D; V was "preserved" anyway. On the other hand, if $V_t[l(s)] \neq V_s[l(s)]$, we remove x from D at stage t and thereby restore the left-hand-side of the computation that existed at stage s. Namely, we have that for all $\Phi_t[D_t][l(s)] = \Phi_s[D_s][l(s)] = V_s[l(s)] \neq V_t[l(s)]$. We can now preserve this disagreement forever and so win $\mathbf{R}$. Of course we have now lost our success on witness x for $\mathbf{P}_i$, but we need only choose a new witness for $\mathbf{P}_i$ and that witness can ignore $\mathbf{R}$. $\square$

Ishmuchametov has extended Theorem 10.2 to show that for every n-r.e. degree a, there are degrees $0 < \mathbf{a}_1 \leqslant \mathbf{a}_2 \leqslant \cdots \leqslant \mathbf{a}_{n-1} \leqslant \mathbf{a}$ such that $\mathbf{a}_i$ is properly i-r.e. for every i.

Splittings of d.r.e. sets do not have the same nice degree theoretic properties as do splittings of r.e. sets. In particular, if D_1, D_2 is a d.r.e. splitting of the d.r.e. set D, it is not necessarily the case that $D \equiv_T D_1 \oplus D_2$. (Of course we still have that $D \leqslant_T D_1 \oplus D_2$. For example suppose that B, A are r.e. sets, $B \subseteq A$. Then B and $A - B$ is a splitting of the (d)r.e. set A but it is not necessarily the case that $A - B \leqslant_T A$. Thus most theorems about splittings in the d.r.e.degrees are degree theoretical rather than set theoretical results.

The analogue of Sacks Splitting Theorem was proved by Cooper.

Theorem 10.4 (Cooper [18]). *Suppose that* $\mathbf{d} > \mathbf{0}$ *is n-r.e. Then there are n-r.e. degrees* $\mathbf{a} \mid \mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{d}$.

Proof (sketch). We describe briefly the case $n = 2$. We need to construct d.r.e. sets D_1 and D_2 so that $D \equiv_T D_1 \oplus D_2$ and such that the following Sacks requirements are satisfied.

$$\mathbf{R}_{e,i}: \Phi_e(D_i) \neq D_{2-i}.$$

The idea as in the r.e. case is to pursue the Sacks strategy of preserving lengths of agreement. Suppose that D is d.r.e. and, without loss of generality, not of r.e. degree. Injury will occur as in the usual construction and also owing

to the fact that D is d.r.e. rather than r.e. Matters are arranged so that the injuries are essentially co-r.e. The idea is then that if a requirement is not met, then D is of the same degree as the injury set, a co-r.e. set and this is the desired contradiction. The possibility of co-r.e. injury sets yields a new outcome, that of unbounded use, so the construction becomes a Π_2 argument as opposed to a finite injury argument. □

Note that there is a nonuniformity in the proof of Theorem 10.4 and we do not know if it can be removed. Cooper has shown that, in fact, all low$_2$ d.r.e. degrees split over all lesser ones. This suggests the following question.

Open Question 10.5. Are the low$_2$ d.r.e. degrees elementarily equivalent (as an upper semilattice) to the low$_2$ degrees?

There are differences between the structure of **R** and that of **D**, the upper-semilattice of d.r.e. degrees. A notable one is the following degree-theoretic splitting theorem of Downey.

Theorem 10.6 (Downey [22]). *There are incomparable d.r.e. degrees* **c** *and* **d** *such that* **c** $\cup$ **d** $= 0'$ *and* **c** $\cap$ **d** $= 0$.

Proof (sketch). The basic idea is the following. We code K into both d.r.e. sets C and D rather than just one as one might expect. This of course interferes with the minimal pair strategy. To meet the minimal pair requirements, we note that if we destroy both sides of an agreement in an infimum requirement, we can force a disagreement by extracting numbers from the d.r.e. set on just one side of the computation thereby returning to the original value. □

Downey and Haught have used the ideas of the proof of Theorem 10.6 to show that all finite lattices can be embedded into the wtt-degrees below $0'$.

Another difference between **R** and **D** not related to splittings is the following result.

Theorem 10.7 (Cooper, Harrington, Lachlan, Lempp, and Soare [15]). $0'$ *is a minimal cover in the d.r.e. degrees.*

We close this section by mentioning Cooper's solution to an old question of Rogers — that of the definability of the jump. This result relies on a nonsplitting theorem for the degrees.

Definition 10.8 (Cooper [17]). A degree **d** is *unsplittable over* **a** *avoiding* **b** if **b** $\not\leqslant$ **a**, **a** $\leqslant$ **d**, **b** $\leqslant$ **d** and if d_1 and d_2 are two degrees such that **a** $\leqslant d_1, d_2$ and $d_1 \cup d_2 = $ **d**, then **b** $\leqslant d_1$ or **b** $\leqslant d_2$.

Using a $0'''$ argument, Cooper proved the following.

Theorem 10.9 (Cooper [17]). *There is a d.r.e. degree* **d** *and degrees* **a** *and* **b**, *such that* **d** *is unsplittable over* **a** *avoiding* **b**.

Using this result and some work of Jockusch and Shore [57]. Cooper proves

Theorem 10.10 (Cooper [17]). $0'$ *is definable in the r.e. degrees as the greatest degree satisfying the formula* $\neg (\exists \mathbf{a}, \mathbf{b}) [\mathbf{x} \cup \mathbf{a}$ *is unsplittable over* $\mathbf{b}]$.

Corollary 10.11. *The jump operator is definable in the Turing degrees.*

Corollary 10.12. *The r.e. degrees are definable in the Turing degrees.*

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