On Presentations of Algebraic Structures*

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1 Introduction

In view of all the work of Professor Andrea Sorbi in organizing this second COLORET conference, it seemed to me to be especially appropriate to devote part of my short course to presentations of algebraic structures. I will concentrate upon linear orderings and boolean algebras. Andrea is, after all, Professor of Boolean Algebra.

When we learn elementary algebra such as group theory or linear algebra we are introduced to isomorphism as the main classification tool for classifying structures. However, if one views algebraic structures through computational eyes, and asks questions such as "if I am *given* a structure in such an *online* or *computable* manner so that I can perform the basic operations computably, then what else can I perform computably?" For instance, consider the following two questions.

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- Suppose that I am given a field F "computably": That is, the domain is coded by $\mathbb{N}$ so that I have a algorithms to perform the basic field operations. Can I build a computable algebraic closure of F?
- Similarly, suppose that I am given a computable vector space V over $\mathbb{Q}$, can I find a computable basis for V?

The exact meaning of the above questions will be clarified in the paper to follow. But modulo this, Rabin [137] proved that the answer to the first question is yes, *despite the fact that the usual proof does* not *enable us to construct a computable algebraic closure of a computable field.* The usual proof fails because there may be no algorithm for F which decides if a polynomial is irreducible. On the other hand, the answer to the second question is no, and in fact every infinite independent set in V can be as complex as the halting problem. (See Metakides and Nerode [115, 116, 117] and Simpson [167].)

Rabin's Theorem is interesting since it demonstrates that algorithmic considerations can generate new structure theory needed to perform algorithmic tasks. Combinatorial group was generated by algorithmic considerations such as the word problem of Dehn [30], and yet these considerations have provided an engine for new group theory. The point is that pure existence proofs often simply do not provide enough structure theory to enable the construction of algorithms. So computational considerations can often force us to achieve a significantly deeper understanding of the relevant structures than had been hitherto necessary in the "classical" theory.

For the purposes of the present paper, the vector space example is especially interesting. An example of a computable vector space V with no computable basis can be constructed so as to be classically isomorphic to V_∞, the weak direct sum of a countable number of copies of $\mathbb{Q}$. Clearly in the standard copy of V_∞, one can easily find a computable basis, namely $\{e_i : i \in \mathbb{N}\}$, where e_i is the vector with 1 in the i-th position and 0 elsewhere.

The point is that for effective considerations, one must be very careful as to how one presents a structure. Two classically isomorphic structures can have very different algorithmic properties. This basic observation leads to a fundamental area of research where one tries to understand the relationships between classical isomorphism types and "effective" ("computable, etc") isomorphism types; and more generally classical invariants and effective invariants. The theme of this *paper is to study these relationships particularly in the case studies of Boolean Algebras and Linear Orderings.*

There is insufficent room to discuss more than a small fraction of the material. We refer the reader to the long articles of Remmel [148] and the author [35] as well as many other articles in the Handbook of Computable Algebra [49] and the book by Rosenstein [157]. Instead, we will concentrate upon one line of development. *How can we best present an isomorphism type?* We plan to give at least a representative sample of proofs and proof sketches so as to give the reader some inkling of the various techniques used and hopefully inspire the reader to research in this attractive area. To this end, we include some questions which are presently open at the time of the writing of this paper. Furthermore we have put in a large (but by no means complete) bibliography which the reader might find a useful guide to the literature.

2 Basics

Since we will assume Church's Thesis we will not bother with a particular model of computation, and will use the terminology recursive, computable, effective interchangable. We will assume that the reader is familiar with the basics of classical computability theory. The following should be a reminder.

We assume that $\{\varphi_x : x \in \mathbb{N}\}$ is an effective list of all computable (partial) functions[1] from (resp. a subset of) $\mathbb{N}$ to $\mathbb{N}$. We write $\varphi_x^t(y)\downarrow$ if φ_x halts on input y in $\leq t$ steps and we write $\varphi_e(z)\downarrow$ iff $\exists t(\varphi_e^t(z)\downarrow$. We regard languages as coded and consider only subsets of $\mathbb{N}$. For $A, B \subseteq \mathbb{N}$ we say that $A \leq_T B$ if A is B-computable. That is, there is a reduction procedure Φ using B as an oracle computing A from B. We would write $\Phi^B = A$ or $\Phi(B) = A$ and consistently confuse a set with its charactistic function. We let $\{\Phi_e : e \in \mathbb{N}\}$ list all Oracle Turing Machines (Procedures). Naturally we identify φ_e with $\Phi_e^{\emptyset}$. A particular form of T-reduction is an m-reduction. We say that A is m-reducible to B and write $A \leq_m B$ via f iff f is a computable function such that for all x $x \in A$ iff $f(x) \in B$. If f is $1-1$ we say that $A \leq_1 B$. We use $\leq_q$ for $q \in \{m, 1, T\}$ to generate an equivalence relation $\equiv_q$ where $A \equiv_q B$ iff $A \leq_q B$ and $B \leq_q A$. The equivalence classes under $\equiv_q$ are called q-degrees, or simply degrees in the case $q = T$. Degrees measure *equicomputability.*

[1]Often called "(partial) recursive functions." We use the more descriptive term.

We say that a set is computably enumerable if it is finite or the range of a (total) computable function. (This function can be taken to be $1-1$.) We will call a degree computably enumerable iff it contains a computably enumerable set. The canonical computable enumerable sets are $K_0 = \{\langle x, y\rangle : \varphi_x(y)\downarrow\}$ and $K = \{e : \varphi_e(e)\downarrow\}$. Here $\langle\ ,\ \rangle : \mathbb{N}\times\mathbb{N} \mapsto \mathbb{N}$ is a canonical computable bijection (pairing function). K_0 and K are computationally identical since $K_0 \equiv_1 K$ and are called the "halting problem." More generally, we define the jump operator via $A' = \{e : \Phi_e^A(e)\downarrow\}$. A' denotes the halting problem *relativised to* A. Note that $\emptyset' = K$. It is a simple diagonalization to prove that the halting problem is algorithmically unsolvable, or more generally for all A, $A <_T A'$. We often identify the degree of a set X with the corresponding lower case boldface letter $\mathbf{x}$. It was a problem of Post [136] as to whether there were computably enumerable degrees $\mathbf{c}$ with $\mathbf{0} < \mathbf{c} < \mathbf{0}'$. In other words was each computably enumerable problem either decidable or the "halting problem in disguise." This question was settled by Friedberg [58] and Muchnik [127] who proved the existence of intermediate degrees. A beautiful generalization of the Frideberg-Muchnik solution was due to Gerald Sacks [162] who proved that the collection $\mathbf{R}$ of computably enumerable degrees are dense. (That is, in $\mathbf{R}$, if $\mathbf{a} < \mathbf{b}$ then there is a $\mathbf{c}$ (in fact an infinite antichain) with $\mathbf{a} < \mathbf{c} < \mathbf{b}$.)

We write $\mathbf{0}$ for the degree of $\emptyset$ and more generally write $\mathbf{0}^{(n)}$ for the degree of the n-th iterate of the jump operator applied to $\emptyset$. The degrees $\{\mathbf{0}^{(n)} : n \in \mathbb{N}\}$ are called the arithmetical degrees. These degrees are related to arithmetic as follows.

We call a set A Σ_1^0 iff there exists a computable relation R such that $x \in A$ iff $\exists y R(x, y)$. We call B a Π_1^0 set iff $\overline{B} =_{\text{def}} \mathbb{N} - B$ is Σ_1^0, or, equivalently, there is a recursive relation Q such that for all x $x \in B$ iff $\forall y Q(x, y)$. More generally, we call a set C Σ_{n+1}^0 iff there is a Π_n^0 relation D such that for all x, $x \in C$ iff $\exists y D(x, y)$, and similarly for Π_{n+1}^0. Note that X is Σ_n^0 iff there is a computable relation R of $n+1$ variables such that for all y,

$$y \in X \text{ iff } \exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots R(y, x_1, x_2, x_3, x_4, \ldots, x_n).$$

Naturally there are relativizations of the notions, and for instance we say that A is $\Sigma_1^{0,B}$ or just Σ_1^B (Σ_1^0 relative to B) iff there is a B-computable relation R such that $x \in A$ iff $\exists y R(x, y)$. Finally we say that a set E is Δ_n^0 iff it is both Σ_n^0 and Π_n^0. Notice that Δ_1^0 sets are the computable sets. We will need the following.

Theorem 2.1 (Post's Theorem) *(i) For all n a set A is Δ_{n+1}^0 iff $A \leq_T \emptyset^{(n)}$.*

(ii) A set is Σ^0_{n+1} iff A if Σ^0_1 relative to B for some Δ^0_{n+1} set B.
(iii) For $n, m, k \in \mathbb{N}$, all the classes Δ^n_0, Σ^0_k, and Π^0_m are distinct.

For a function f we write "$\lim_s f(s)$ exists" to mean that for almost all x $f(x) = f(x+1)$.

Theorem 2.2 (Shoenfield Limit Lemma) *$A \leq_T \emptyset^{(n)}$ iff there exists a computable function of $n+1$ variables such that for all x,*

$$\lim_{y_1} \lim_{y_2} ... \lim_{y_n} f(x, y_1, ..., y_n)$$

exists and equals $A(x)$.

Finally we will need the following observation whose proof is obvious but whose implications are deep.

Lemma 2.3 (Use Principle) *Suppose that A and B are sets and we have a computation $\Phi^A(x)$ where A and B are identical on all numbers less than or equal to $u(\Phi^A(x))$, the maximum number used (or called) in the computation. Then $\Phi^A(x) = \Phi^B(x)$ via identical computations.*

We will need some more material, but this material will be introduced in context.

3 Degree Spectrums and Degrees

Naturally the first idea one might pursue is to assign a complexity measure to an isomorphism type. First what do we mean by the phrase "degree of a presentation?" Given a structure A its degree will be the degree of its open diagram. Equivalently, the degree will be the joins of the degrees of all the functions and relations. Thus the degree of a linear ordering $(\mathbb{N}, \leq)$ is simply the degree of the ordering relation. The natural guess for a degree for an isomorphism type would be the *least* degree of any presentation in that type. For instance, the degree of dense linear orderings without endpoints is clearly $\mathbf{0}$ since we can computably represent $\mathbb{Q}$. Linda Jean Richter was the first author to systematically study the degrees of structures. (Richter [150]) It turns out that some types have degrees and some do not. One example of a structure with a degree is a finitely presented group. If $G = \langle x_1, ..., x_n | y_1, ..., y_m \rangle$ is finitely presented then its degree is simply the degree of its word problem. Note that if H is isomorphic to G then we only need to look at the images of $\{x_1, ..., x_n\}$ to obtain generators of H and through this, H can compute the open diagram of G. Below are some other examples of structures with nontrivial degrees. We obtain these examples using a preliminary theorem of Richter. This theorem encapsulates the so-called "combination method."

Theorem 3.1 (Richter [150, 151]) *Suppose (i) and (ii) below hold for a degree* $\mathbf{a}$ *and a theory T over a finite language L. Then there is a structure of L whose isomorphism type has degree* $\mathbf{a}$.

(i) There is a infinite computable sequence of finite structures $\{A_i : i \in \mathbb{N}\}$ such that A_i is not embeddable into A_j for $i \neq j$.

(ii) For each $S \leq \omega$, there is a structure A_S such that:
(iia) A_S is a countable structure of T.
(iib) $A_S \leq_T S$.
(iic) A_i is embeddable into A_S iff $i \in S$.

Proof. Fix a degree $\mathbf{a}$ and let D be any set in $\mathbf{a}$. Consider $A_{D \oplus \overline{D}}$. Evidently $A_{D \oplus \overline{D}} \leq_T D$. Now suppose that $B \cong A_{D \oplus \overline{D}}$. Then $i \in D$ iff A_{2i} is embeddable into B, and $i \in \overline{D}$ iff A_{2i+1} is embeddable into B. It follows that $D \leq_T B$ the type of $A_{D \oplus \overline{D}}$ has degree $\mathbf{a}$. □

Theorem 3.1 has many applications. We list a couple below.

Theorem 3.2 (Richter [150, 151, 152]) *Let* $\mathbf{a}$ *be any degree.*

(i) There is an abelian group whose isomorphism type has degree $\mathbf{a}$.

(ii) There is a lattice whose isomorphism type has degree $\mathbf{a}$.

(iii) There is a graph whose isomorphism type has degree $\mathbf{a}$.

Proof. They are all basically similar. We do (i). Consider $A_i = \mathbb{Z}_{p_i}$, the cyclic group of order p_i, where p_i denotes the i-th prime. The structure A_S is the standard direct sum of the A_i. Now we apply the Theorem 3.1. $\square$

At this stage it we should point out that, as with many of these results on structures and their degrees, Theorem 3.2 has several consequences in the degrees. In particular Theorem 3.2 was used by Richter [152] to demonstrate that *If* $\mathbf{0}''' \leq_T \mathbf{a}$ *and* Ψ *is a jump preserving automorphism of the degrees then* $\Psi(\mathbf{a}) = \mathbf{a}$.

On the other hand, some structures do not have degrees. For instance we have the following.

Theorem 3.3 (Richter [150]) *If an order type has a degree, that degree is* $\mathbf{0}$.

To prove Richter's Theorem, Theorem 3.3, we will pass through some preliminary material that is of independent interest in our story. First, since the rationals $\mathbb{Q}$ are dense with their usual ordering, any countable linear ordering presented as $L = \langle B, \leq \rangle$ is isomorphic to a subset of the rationals (with the the standard ordering) or the same degree as L. Furthermore, suppose that we define an ordering to be Σ^0_n presented if L is presented as $\langle \mathbb{N}, \leq \rangle$ where the relation $\leq$ is Σ^0_n. (Similarly Π^0_n, Δ^0_n, and $\mathbf{a}$-presented.) We have the following.

Lemma 3.4 (Feiner [51, 53, 54], essentially) *(i) Every* $\mathbf{a}$*-presented linear ordering is (effectively) isomorphic to a subset of* $\mathbb{Q}$ *(with the usual ordering) of degree* $\mathbf{a}$.
(ii) Furthermore if $L = \langle \mathbb{N}, \leq \rangle$ *is* Σ^0_n *(resp.* Π^0_n, Δ^0_n*)-presented, then* L *is effectively isomorphic to a* Π^0_n *(resp.* Σ^0_n, Δ^0_n*) subordering of* $\mathbb{Q}$.
(iii) In fact, every Π^0_{n+1}*-presented linear ordering is isomorphic to a* Π^0_n *subordering of* $\mathbb{Q}$ *and hence to a* Σ^0_n*-presented linear ordering.*

Proof. We have already observer (i), and (ii) via Post's Theorem. We consider (ii) in the case that L is Σ^0_1-presented, the general case following by relativization and Post's Theorem. Thus we can assume that we are given $L = \langle \mathbb{N}, \leq \rangle$ with $\leq$ a Σ^0_1 relation. Therefore there is a computable relation R such that for each pair x, y,

$$x \leq y \text{ iff } \exists t R(x, y, t).$$

Now we know that for all x, y $x \leq y$ or $y \leq y$ so we can assume that we have enumerations given by R (running $R(x,y,0), R(y,x,0), R(x,y,1), \ldots$ until one is found to hold) so that we are know $x \leq y$ or $y \leq x$ initially. Thus at stage s we will assume that we are given the numbers $0, 1, \ldots, s$ given in some order $L_s = x_0 \leq x_1 \leq \ldots x_s$. The only thing that can later happen is that it might be the case that at stage t, $x_i \leq x_{i+1}$, since $R(x_i, x_{i+1}, s)$ held at some $s \leq t$, but $x_{i+1} \not\leq x_i$ since for all $s \leq t$, $\neg R(x_{i+1}, x_i, s)$. But at stage $t+1$ it might be that $R(x_{i+1}, x_i, t+1)$. Thus at stage $t+1$ we discover that $x_i = x_{i+1}$.

For ease of notation we will write $x_i <_t x_{i+1}$ as a shorthand for the situation at stage t and $x_i =_{t+1} x_{i+1}$ for the situation at stage $t+1$.

Now here is the idea. At stage s we will assume that we have a partial map $f : L_s \mapsto \mathbb{Q}$. We need to say how to modify the map at stage $s+1$. There are two things to do. First, we must put $s+1$ into L_s to make L_{s+1}. We can do this as described above. Second we must have an image in $\mathbb{Q}$ for $s+1$. Suppose that, without loss of generality, we have $x <_s s+1 <_s y$. The we can use the density of $\mathbb{Q}$ to find a new point z between $f(x)$ and $f(y)$ in $\mathbb{Q}$ with large code (larger than any number seen so far) and define $f(s+1) = z$. Thus we put z into the partial image I_{s+1} Finally we must say what to do with the case that $p <_s q$ yet $p =_{s+1} q$. For whichever of p or q has the larger code in $\mathbb{N}$ (say d) we simply throw away the image $f(d)$ from the image I_{s+1}. Clearly, I is a Π^0_1 set. For the converse direction, simply reverse the reasoning.

Finally we turn to (iii). Again we deal only with the case that L is Π^0_2 presented. Therefore, by (ii), we can regard L as a Σ^0_2 subset of $\mathbb{Q}$. We need to build a Π^0_1 subset B of $\mathbb{Q}$ isomorphic to L. Note that there is a computable relation R such that

$$x \in L \text{ iff } \exists s \forall t R(x, s, t).$$

The reader should think of this expression as saying that a rational x is in the set L only if there is some stage s after which it appears to be in L for all stages t exceeding s. We say that x *appears to be in L at stage t with witness s* iff s is least such that $\forall t' \leq t R(x, s, t')$ and x did not appear to be in L at stage $t-1$ with witness $s' < s$.

Now again we think of both L and the B corresponding to L (built by *us*) as being in a state of formation. When we see an element x appear to be in L (with some witness s), but *not* appear to be in L at the previous stage $t-1$, we use the density of $\mathbb{Q}$ to find a fresh rational z (with code bigger than any seen so far) to be the image of x in B_t, by adding z as in case (ii) to the appropriate place in B_{t-1}. If x leaves L at stage t (in the sense that it appeared to be in L at stage $t-1$ but is not in at stage t), then we delete the stage $t-1$ image of x from B_t forever. $\square$

Notice that this means that the arithmetical hierarchy of order types partially collapses to the collection of Σ_n^0-presented orderings for each $n \in \mathbb{N}$. We will soon show that this hierarchy is proper. Before we do this, we return to Richter's Theorem.

Proof of Richter's Theorem, Theorem 3.3 Let L be a noncomputable subordering of $\mathbb{Q}$. We will build a subordering B of $\mathbb{Q}$ isomorphic to L, and meeting the requirements below.

$$\mathcal{R}_{i,e} : \Phi_e^L = \Phi_i^B = f \text{ implies } f \text{is computable.}$$

Notice that this gives something more than the theorem. Additionally we will build an explicit isomorphism g from L to B. The construction is an "oracle construction" and can be viewed as a basic forcing construction.

Construction, Stage $s+1$

Without loss of generality, we shall assume that L has endpoints and these are in L_{-1}. We will assume that we have L_s containing the first s elements of L. We will assume that for all $x, y \in L_s$ if there are only finitely many elements of L between x and y, then $[x, y] \cap L_s = [x, y] \cap L$. (That is, if there are only finitely many elements between x and y, then they are already present in L_s.) Finally, we are assuming that we have constructed B_s so that $g(L_s) = B_s$ is an order preserving bijection. When we define g on L_{s+1} we will extend g defined at stage s. Finally we will assume that we have a parameter $p(s)$ whose use is described below.

At stage $s+1$ our first task will be to ensure that a_{s+1} is in L_{s+1}. If a_{s+1} is not already in L_s then put it in. Now, if necessary, put further points into L_{s+1} to ensure that for all x, y, if $[x, y]$ is finite in L then $[x, y] \cap L_{s+1} = [x, y] \cap L$. Let the result be L_{s+1}^+. Using the density of $\mathbb{Q}$, find rationals to add to B_s with codes bigger than $p(s)$ to to be the g-images of the points in $Ls + 1^+ - L_s$. Add thses to B_s to make B_{s+1}^+.

We now act to satisfy the requirement $\mathcal{R}_s$. We call a finite set M of rationals $s+1$-*acceptable* if

(i) $M \supseteq B^+_{s+1}$,
(ii) $x \in M - B^+_{s+1}$ implies that (the code of) x is bigger than $p(s)$, and
(iii) For all x, y in B^+_{s+1}, if $[g^{-1}(x), g^{-1}(y)]$ is finite in L, then there is no element of $M - B^+_{s+1}$

The idea of $s+1$-acceptability is that it is possible to extend B^+_{s+1} to M and still be able to pull back to a partial isomorphism in L. We now ask the oracle question :

Does there exist an $s+1$-acceptable M, and a z such that

$$\Phi^L_s(z) \neq \Phi^M_s(z) \downarrow ?$$

If the answer is *no*, then we let $B_{s+1} = B^+_{s+1}$, $L_{s+1} = L^+_{s+1}$, and $p(s+1) = p(s)$.

If the answer is *yes*, then choose one such $s+1$-acceptable extension M of B^+_{s+1}. Let $p(s+1)$ denote the maximum of $p(s)$ and the maximum element queried in the $\Phi^M_s(z)$ computation. Notice that for any set of elements $m_1, ..., m_n \in M - B^+_{s+1}$ if $m_1, ..., m_n$ occur in some successivity $[x, y]$ of B^+_{s+1}, then $[g^{-1}(x), g^{-1}(y)] \cap L$ is infinite. Therefore we can find n elements of L to put into L^+_{s+1} to g-map to the m_i. Therefore it is possible to add to L^+_{s+1} $M - B^+_{s+1}$ many elements of L and hence form L^{++}_{s+1} and thus extend the g-map from L^+_{s+1} to B^+_{s+1} to a map from L^{++}_{s+1} to M. Finally, we form L_{s+1} and B_{s+1} by again adding any new elements to respectively L^{++}_{s+1} and M if necessary, (with codes bigger than $p(s+1)$) to keep the invariant that if $[x, y] \cap L$ is finite and $x, y \in L_{s+1}$, then $[x, y] \cap L_{s+1} = [x, y] \cap L$.

End of Construction.

Finally, we observe that the construction achieves its goals. First $L = \lim_s L_s$ is clear by the first part of stage s. Second, we meet the $\mathcal{R}_s$ at stage $s+1$. The crucial point is that after stage $s+1$ the only way we could *possibly* extend B^+_{s+1} is via acceptable strings. Therefore if no such M, z exist, then no matter how we extend B^+_{s+1} if we have a convergent computation $\Phi^M_s(z)$ and $\Phi^L_s(z) \downarrow$, then they will give the same result. So suppose that $\Phi^B_s = \Phi^L_s$. Then to compute the value of $\Phi^L(z)$ find any $s+1$-acceptable extension M of B^+_{s+1}, (this is a parameter) with $\Phi^M(z) \downarrow$. (Notice that being $s+1$-acceptable is computable from the relevant parameters.) It can only be that

$$\Phi^M_s(z) = \Phi^L_s(z) = \Phi^B(z).$$

Hence $\Phi^L(z)$ is computable. Therefore if the answer is *no* in the construction, stage $s+1$, then we meet $\mathcal{R}_s$. On the other hand, if the answer is *yes*, then we act to create a disagreement at z and, because we protect this disagreement by making $p(s+1)$ large enough to protect it, we ensure that

$$\Phi_s^B(z) = \Phi_s^{B_{s+1}}(z) \neq \Phi_s^L(z).$$

$\square$

Thus we see that no order type has a nontrivial degree. Indeed note that the proof above has the following corollary.

Corollary 3.5 *If L is a noncomputable presentation of a type τ then there is $\widehat{L}$ isomorphic to L such that the degrees of L and $\widehat{L}$ form a minimal pair in the sense that if A is any set with $A \leq_T L, \widehat{L}$, then A is computable.*

Actually, Richter proved a general structure theoretical result which has the linear ordering result as a particular case.

Definition 3.6 (Richter [150, 151]) We define the *computable embedding condition* as follows. Given a structure A, a finite structure B and an embedding $f : B \mapsto A$, define the class $A_{C,f}$ to be

$\{D : D$ is a finite structure extending C embeddable into A via a map extending $f\}$.

Then A satisfies the computable extension property iff for all structures C isomorphic to a finite substructure of A, and for all functions f embedding C into A, the class $A_{C,f}$ is infinite and computable.

Theorem 3.7 (Richter [150, 151]) *For any countable structure A satisfying the computable extension property, there is a countable structure C isomorphic to A such that $degA \cap \deg C = \mathbf{0}$.*

The proof of Theorem 3.7 is similar to that of the Theorem 3.3. Furthermore one can show without too much difficulty that Theorem 3.3 is a consequence of Theorem 3.7.

Theorem 3.3 and Corollary 3.5 lead to the natural question as to the "spectrum" of degrees of a linear ordering (or other structures).

Question 3.8 *What can be said about $S(L) = \{deg(\widehat{L}) : \widehat{L} \cong L\}$ for a fixed ordering L?*

One question from Downey [35] is whether $S(L)$ can have members of all computably enumerable degrees, or all Δ_2^0 degrees, or indeed all degrees if it does not contain $\mathbf{0}$. Steffen Lempp asks if there is any structure which has no computable copies yet has copies of all degrees $\neq \mathbf{0}$. Downey [35] has shown that there exist Σ_0^1-*presented* linear orderings L with the property that they have no computable copy, and yet any Σ_1^0-*presented copy* $\widehat{L}$ of L, $L \leq_T \widehat{L}$. (See Theorem 6.4)

Another example of a structure closely related to orderings is that of Boolean Algebras. while it is possible to prove things directly, it is nice to extract results for boolean algebras from results for linear orderings via the following technical device.

Theorem 3.9 (The Representation Theorem, folklore after Stone) *Every countable boolean algebra is isomorphic to a interval algebra $Intalg(L)$ of a linear ordering of the same degree. (Recall that $Intalg(L)$ is the algebra generated by the finite unions of left closed right open intervals of the ordering L.)*

Proof. This is just an effective version of Stone's theorem. We represent a given B as an algebra of sets. Thus let $B = \cup_s B_s$ with $B_0 = \{0, 1\}$, and $B_{s+1} - B_s = \{b_s\}$. We will define at each stage a subalgebra $\widehat{B_s}$ containing B_s.

Define L_0 the ordering with two points labelled 0 and 1. (Then $Intalg(L_0)$ consists of the two sets $\emptyset$ and $[0, 1)$.) So we have the induced mapping g_0 with $0 \mapsto \emptyset$ and $1 \mapsto [0, 1)$. At stage $s + 1$ we will have a set of $Atoms_s = \{a_{s_1}, ..., a_{s_n}\}$ listing the atoms of the subalgebra of $\widehat{B_s}$ together with the linear ordering $L_s = 0, x_{s_1}, ..., x_{s_n} = 0$ so that $g_s(a_{s_j}) \mapsto [x_{s_{j-1}}, x_{s_j})$ inducing an isomorphism from the subalgebras $\widehat{B_s}$ to $Intalg(L_s)$.

At stage $s + 1$ if b_s is in $\widehat{B_s}$ we need to do nothing. Otherwise, for each $a = a_{s_i}$ such that b_s *splits* a (i.e. both $x \wedge b_s$ and $x \wedge \overline{b_s}$ are nontrivial), add a new point y to L_{s+1} between $x_{s_{j-1}}$ and x_{s_j} to split the interval $[x_{s_{j-1}}, x_{s_j})$ into $[x_{s_{j-1}}, y) \cup [y, x_{s_j})$. Naturally we map $a_{s_j} \wedge b_s$ to one of them, say, $[x_{s_{j-1}}, y)$ and map $a_{s_j} \wedge \overline{b_s}$ to $[y, x_{s_j})$. Note that this generates two new atoms for $\widehat{B_{s+1}}$. And of $Intalg(L_{s+1})$ Let $c_1, ..., c_m$ denotes the atoms of $\widehat{B_{s+1}}$ below b_s. Clearly we have ensured that the induced map $g(b_s) = g(c_1) \cup ... \cup g(c_m)$ works. The results follows. $\square$

The representation theorem, Theorem 3.9, has immediate algorithmic consequences. We summarize some below.

Corollary 3.10 (Feiner [51], essentially) *(i) Every Σ^0_n-(resp. Δ^0_n-, Π^0_n-) presentable boolean algebra is representable as $Intalg(L)$ with L Σ^0_n-(resp. Δ^0_n-, Π^0_n-) presentable. Furthermore every boolean algebra is isomorphic to a subalgebra of the free boolean algebra $Intalg(\mathbb{Q})$ (which we denote where it won't be confusing by $\mathbb{Q}$) of the same degree.*
(ii) Consequently, every Π^0_{n+1}-presentable boolean algebra is isomorphic to a Σ^0_n-presented one.

Proof. (i) is an immediate consequence of Theorem 3.9 and Post's Theorem. (ii) follows by Feiner's Lemma, Lemma 3.4 together with (i). $\square$

Notice that in the proof of the Representation Theorem we get a $1-1$ correspondence between *atoms* in B and *adjacencies* (or *successivities*) in L. One can ask a whole host of questions about the relationships between orderings and successivities, algebras and atoms, and order-types and isomorphism types of $Intalg(L)$. We will close the present section with one example which relies on an important result of Remmel-Vaught.

Theorem 3.11 (Remmel-Vaught [142]) [2] *(i) Suppose that B is any boolean subalgebra of $\mathbb{Q}$ with infinitely many atoms $\{d_i : i \in \mathbb{N}\}$. Let $\widehat{B}$ be the subalgebra of $\mathbb{Q}$ obtained from B splitting each atom of B a finite number of times in $\mathbb{Q}$ (together with B). (Thus the atoms of $\widehat{B}$ would be $\{e_{i_1}, ..., e_{i_{n(i)}} : i \in \mathbb{N}$ where $d_i = e_{i_1} \vee ... \vee e_{i_{n(i)}}\}$.) Then $\widehat{B}$ is isomorphic to B.*
(ii) (Rephrasing (i) in terms of linear orderings). Suppose that L and $\widehat{L}$ are linear orderings with infinitely many adjacencies, and $g : L \mapsto \widehat{L}$ is an orderpreserving embedding such that
(a) if $[x, y]$ is finite in L then $[g(x), g(y)]$ is finite in $\widehat{L}$, and
(b) if $z \in \widehat{L}$ is not in the image of L, then there are x, y in L such that $[x, y]$ is

[2] In his Thesis, Vaught proved this result for atomic boolean algebras and Remmel extended the result to all countable boolean algebras

finite and $z \in [g(x), g(y)]$.
Then $Intalg(L) \cong Intalg(\widehat{L})$.

Proof. We use the form (i). Let $\langle Y \rangle$ denote the subalgebra generated by Y in $\mathbb{Q}$. For a subalgebra X of $\mathbb{Q}$, let $I(X)$ denote the ideal generated by X, and $A(X)$ denote the set of atoms of X. It is relatively easy to see that $B/I(A(B)) \cong \widehat{B}/I(A(\widehat{B}))$.

To demonstrate that B is isomorphic to $\widehat{B}$, let $B = \{b_i : i \in \mathbb{N}\}$, and $\widehat{B} = \{c_i : \subset \in \mathbb{N}\}$. We build the isomorphism in stages, at each stage n specifying a subalgebra B_n of B, a subalgebra $\widehat{B}_n$ of $\widehat{B}$ and a isomorphism $f_n : B_n \mapsto \widehat{B}$. Inductively we suppose that these parameters satisfy the following conditions.

(i) $a \in B_n$ is the union of exactly n atoms of B iff $f_n(a)$ is the union of exactly n atoms of $\widehat{B}$.

(ii) If $a \in B_n$ and $a \notin I(A(B))$ then $f_n(a) \notin I(A(\widehat{B}))$, $f_n(a) \equiv a \bmod I(A(\widehat{B}))$, and $|\{b \in A(B) : b \leq a\}| = |c \in A(\widehat{B}) : c \leq f_n(a)\}|$. (Here and henceforth, $\leq$ means $\leq_{\mathbb{Q}}$.)

Since $0_B = 0_{\widehat{B}} = 0_{\mathbb{Q}}$, and $1_B = 1_{\widehat{B}} = 1_{\mathbb{Q}}$, at stage 0 the identity map can play the role of f_0 and will satisfy (i) and (ii). At stage 1 we do nothing else. Let $n \geq 1$.

Stage 2n. Assume that at stage $2n-1$ we have (i) and (ii) and additionally, $\langle\{b_0, ..., b_{n-1}\}\rangle \subseteq B_{2n-1}$, and $\langle\{c_0, ..., c_{n-1}\}\rangle \subseteq \widehat{B}_{2n-1}$. Let $a_0, ..., a_s$ list $A(B_n)$ so that $f_{2n-1}(a_0), ..., f_{2n-1}(a_s)$ list $A(\widehat{B})$. If $b_n \in B_{2n-1}$, do nothing. Otherwise, let $B_{2n} = \langle B_n \cup \{b_n\}\rangle$. We can renumber the list of atoms so that the atoms of B_{2n} are $a_0 \wedge b_n, a_0 - b_n, ..., a_j \wedge b_n, a_j - b_n, a_{j+1}, a_{j+2}, ..., a_s$. (This is essentially Stone's process.) We define f_{2n} on the $A(B_{2n}$ and then extending via the natural induced map. For i with $j+1 \leq i \leq s$, let $f_{2n}(a_i) = f_{2n-1}(a_i)$. For the remaining i with $0 \leq i \leq j$, we define $f_{2n}(a_i \wedge b_n)$ and $f_{2n}(a_i - b_n)$ depending on one of the cases below.

Case 1. *a_i is the union of exactly m atoms of B.* Then for some k, $a_i \wedge b_n$ is the union of k atoms of B. Now $f_{2n-1}(a_i)$ is the union of exactly m atoms of $\widehat{B}$. Thus we let c be such that c is the union of exactly k atoms $\leq f_{2n-1}(a_i)$, and define $f_{2n}(a_\wedge b_n) = c$ and $f_{2n}(a_i - b_n) = f_{2n-1}(a_i) - c$.

Case 2.$a_i \notin I(A(B))$. By hypothesis, $a_i \equiv f_{2n-1}(a_i) \bmod I(A(B))$. Thus $a_i \wedge b_n \equiv f_{2n-1}(a_i) \wedge b - n \bmod I(A(B))$ and $a_i - b_n \equiv f_{2n-1}(a_i) - b_n \bmod I(A(B))$. There are now 3 subcases.

Subcase 2a. $|\{b \in A(B) : b \leq a_i \wedge b_n\}| = |\{b \in A(B) : b \leq a_i - b_n\}| = \infty$. Then we can let $f_{2n}(a_i \wedge b_n) = f_{2n-1} \wedge b_n$, $f_{2n}(a_i - b_n) = f_{2n-1} - b_n$.

Subcase 2b. $|\{b \in A(B) : b \leq a_i \wedge b_n\}| = m < \infty$, *and* $|\{b \in A(B) : b \leq a_i - b_n\}| = k < \infty$. Then there exactly $m = k$ atoms in $\widehat{B}$ below $f_{2n-1}(a_i)$, say, $c_{i_1}, ..., c_{i_{m+k}}$. Then let $d = [f_{2n-1}(a_i) \wedge b_n - \wedge_{j=1}^{m+k} c_{i_j}] \wedge \wedge_{j=1}^{m} c_{i_j}$. Then d has exactly m atoms of $\widehat{B}$ under it, and $d \equiv f_{2n-1} \wedge b_n \bmod I(A(\widehat{B}))$. The we let $f_{2n}(a_i \wedge b_n) = d$ and $f_{2n}(a_i - b_n) = f_{2n-1}(a_i) - d$.

Subcase 2c.*Exactly one of $a_i - b_n$ of $a_i \wedge b_n$ has infinitely many atoms under it.* Without loss suppose that $|\{b \in A(B) : b \leq a_i \wedge b_n\}| = m < \infty$. Since $a_i \wedge b_n \equiv f_{2n-1}(a_i) \wedge b_n \bmod I(A(\widehat{B}))$, it follows that $f_{2n-1}(a_i) \wedge b_n$ has only finitely many atoms under it, say $c_{i_1}, ..., c_{i_k}$. By renumbering, let $g_1, ..., g_m$ denote the first m atoms under $f_{2n-1}(a_i)$. Let $d = [f_{2n-1}(a_i) \wedge b_n - \wedge_{j=1}^{k} c_{i_j}] \wedge \wedge_{j=1}^{k} g_k$. Then d has exactly m atoms of $\widehat{B}$ under it and $d \equiv f_{2n-1}(a_i) \wedge b_n \bmod I(A(\widehat{B}))$. We can now define $f_{2n}(a_i \wedge b_n) = d$ and $f_{2n}(a_i - b_n) = f_{2n-1}(a_i) - d$.

Finally, we let $\widehat{B}_{2n} = \langle \{f_{2n}(x) : x \in A(B_{2n})\} \rangle$. At odd stages we do essentially the same thing except we use f^{-1} and go from $\widehat{B}_{2n+1}$ back to B_{2n+1}. In this way one can easily see that $\cup_n f_n$ defines an isomorphism from B to $\widehat{B}$. □

In his thesis, [177], Thurber proved an extension to the Remmel-Vaught Theorem for higher levels of the Frechet sequence. Specifically, Thurber replaced "atoms" in the above by "n-atoms" which are the atoms of the n-th Cantor-Bendixson derivative of the boolean algebra.

To finish this section here is one application of the Remmel-Vaught Theorem.

Theorem 3.12 (Remmel [142]) *Every computable boolean algebra B with infinitely many atoms is isomorphic to a computable boolean algebra $\widehat{B}$ where the set of atoms is immune. (That is, has no infinite computably enumerable subset.)*[3]

Proof. We suppose that we are given B as $Intalg(L)$ with L a computable subset of $\mathbb{Q}$. We build $\widehat{L}$ and an embedding $L \mapsto \widehat{L}$ as in Theorem 3.11 (ii) above. We

[3]Actually, the set of atoms can be made hyperimmune.

let W_e denote the e-th computably enumerable set which it is convenient to this of as a set of pairs. We must meet the requirements

$$\mathcal{R}_e : W_e \subseteq \widehat{L} \wedge |W_e| = \infty \to W_e \text{ contains a nonsuccessivity of } \widehat{L}.$$

We also have some negative requirements $\mathcal{N}_{\langle x,y \rangle}$ which say that if $[x, y]$ is an adjacency of L then $[x, y]$ is finite in $\widehat{L}$. The way we meet $\mathcal{R}_e$ and the way the positive and negative requirements interact is straightforward. To meet $\mathcal{R}_e$ we wait for a stage where we see some $\widehat{L_s}$-adjacency $[p, q] \in W_{e,s}$. We meet $\mathcal{R}_e$ forever by splitting $[p, q]$ in $\widehat{L_{s+1}}$ simply by adding a new point r between p and q. Note that should L eventually make some $g(x) = p$ and $g(y) = q$ and then enumerate a point z between p and q we can make $g(z) = r$. On the other hand we will not allow a new point between p and q if it injures some $\mathcal{N}_{\langle w,v \rangle}$ of higher priority. Injury can only occur if (i) $[v, w]$ is a L_s-adjacency, and (ii) $p, q \in [g(v), g(w)]$ in $\widehat{L_s}$.

Now either $[v, w]$ really is a L-adjacency, in which case if W_e is really infinite we will meet it somewhere else, or it will get split at some stage s and hence we will be free to meet $\mathcal{R}_e$ as we desire. The rest is a standard application of the finite injury method. $\square$

4 Feiner's Linear Order Theorem and Watnik's Theorem

In the last section we have seen that the arithmetical hierarchy of presentations of order types and isomorphism types of linear orderings and boolean algebras is of the form

$$\Delta_1^0 \subseteq \Sigma_1^0 \subseteq \Sigma_2^0 \subseteq \ldots$$

In the same way that we ask if a group has a solvable word problem it is natural to ask if the hierarchy collapses. Using very ingenious techniques Feiner [51, 52, 53] proved that they did not. The arguments to follow are particularly creative since they relied on "coding at very high levels" which had not previously been seen. The boolean algebra argument of Section 5.1 is particularly ingenious and is a precursor to the so-called $\mathbf{0}^\omega$ worker method of Harrington, Ash, Knight, Lempp-Lerman etc (see e.g. [68, 2, 3, 93, 99]). We begin with the linear ordering result.

Theorem 4.1 (Feiner [51, 53]) *For each n there is a Σ^0_{n+1}-presented linear ordering not isomorphic to a Σ^0_n-presented one.*

Proof. Here is Feiner's argument. We can concentrate upon $n = 0$ and make only a Δ^0_2 presented linear ordering not isomorphic to a computable one by Lemma 3.4. (The result will then follow by Post's Theorem, relativization and Theorem 3.4.)

If L is any linear ordering then notice that the (complete) *block* relation $B(L)$ which says that "L has a block of n-elements" is Σ^L_3. More precisely,

$$n \in B(L) \text{ iff } \exists x_1 < \ldots < x_n([x_i, x_{i+1}] \text{ adjacent and } x_1, x_n \text{ limit points})$$

is clearly Σ^L_3. Thus, by relativization, it suffices to argue that, given any Σ^0_3 set V there is a computable linear ordering L with $B(L) = V$.

Let $y \in V$ iff $\exists x \forall s \exists t R(y, x, s, t)$. We construct L in stages. At stage 0 we let L_0 be the sum of $\mathbb{N}$ many copies of $\mathbb{Z}$. So

$$L_0 = \mathbb{Z} + \mathbb{Z} + \ldots + \mathbb{Z} + \ldots$$

For each x, we regard the x-th copy of $\mathbb{Z}$ as being split up as $-\mathbb{N} + 1 + 2 + 3 + 4 + \ldots + y + \ldots$ where i denotes a block of i elements. For convenience, we label the right point of $-\mathbb{N}$ as 0 and we regard 0 as not being in V.

At each stage of the construction we shall add new points. These will either be labelled as z points or q points. We always put in z points and only put in q points depending on what V is currently telling us. These points are put in as follows.

If there were no V around the construction would *always* put a a z point at the exterior of a $\mathbb{Z}$ block being built around each n-block. This mysterious sentence means the following.

At stage 0, for each copy of $\mathbb{Z}$, we'd put a z point after the point labelled 0 as well as each side of the block labelled n for $n \in \mathbb{N}$. Thus the block in copy x of $\mathbb{Z}$ would become a block of $y + 2$ elements $z(y)z$. The whole of the x-th copy of $\mathbb{Z}$ at the end of stage 0 would appear as

$$-\mathbb{N} + z(1)z + z(2)z + z(3)z + \ldots (\cong \mathbb{Z}).$$

Here the notation (y) denotes a block of y points. For convenience we denote the i-th z-points placed by z_i

In general, without the q points whose role is yet to be revealed, at stage $s+1$ we'd continue this process by adding a z point to the end of the y block for each y, x as well as 0. Thus at stage $s+1$, the picture would go from $z_s z_{s-1} ... z_0 (y) z_0 z_1 ... z_s$ to $z_{s+1} z_s z_{s-1} ... z_0 (y) z_0 z_1 ... z_s z_{s+1}$. Note that the z_{s+1}'s are placed on the *outside*.

Notice that if there were no q action, and this is all we did, then after $\mathbb{N}$ steps, we'd replace the x-th copy of $\mathbb{Z}$ by $\mathbb{N}$ copies of $\mathbb{Z}$ added together, since around each y we simply extend to a infinite $\mathbb{Z}$ block $.....zzzzz(y)zzzzz.....$

Finally, we say how V affects all of this. If we do so at all, we place a q points into a y block *between the rightmost z of the left group (z_0) and the leftmost z of the right group (z_0) and y.* If we place only one then it would look like $....zzzzzzq(y)qzzzzzz....$ Again, we label a q point by q_i if it is one of the i-th pair placed. If, at stage s we have place already n q points and we decide to place one more then we place them respectively right and left of the current q points. That is, $z...zq_0...q_n(y)q_n...q_0z...z$ would go to $zz....zq_0...q_nq_{n+1}(y)q_{n+1}q_n...q_0z...zz$ at the next stage.

There are two crucial observations regarding this process.

- Notice that if we place infinitely many q points into the $\mathbb{Z}$ block being built around y then at then end of the construction we would have built

  $$\mathbb{Z} + y + \mathbb{Z} \text{ as } ...zzzqqqq...(y)...qqqqzzz....$$

  In that case $y \in B(L)$.

- On the other hand if only finitely many q points are placed then we will have built

  $$\mathbb{Z} \text{ as } ...zzzq_0...q_n(y)q_n...q_0zzz...$$

  (In this case, while it might very well be that $y \in B(L)$, *it won't be that the x-th copy of $\mathbb{Z}$ will contribute y to $B(L)$.*)

To complete the construction, we need only say how $R(y, x, s, t)$ controls the placement of the q points. (Remember at stage s we *always* place z points.) The idea is that the y block in the x-th copy of $\mathbb{Z}$ at the beginning of the construction corresponds to the guess that $\forall s \exists t R(y, x, s, t)$. Then it is easy. For each x, y we ensure that at stage u that there are at s q points around y in the x-th copy of $\mathbb{N}$ iff

$$s \leq u \text{ and } \forall s' \leq s \exists t' \leq s R(y, x, t', s').$$

Notice that in this way, we place infinitely many points to separate (y) iff $\forall s \exists t R(y, x, s, t)$ as required. More generally, since $y \in V$ iff $(\exists x)(\forall s)(\exists t) R(x, y, x, s, t)$, we see that

$$y \in V \text{ iff } (\exists x)[y \text{ is a block built from the } x\text{-th copy of } \mathbb{Z}.$$

$\square$

There have been many subsequent variations of the coding technique above. (See for instance Moses [122].) We mention one more. One might ask if one need finite blocks in order to code. A linear ordering is called *discrete* if each point has both a predecessor and a successor. Thus a linear ordering L is discrete iff $L \cong \mathbb{Z}.\widehat{L}$ for some linear ordering $\widehat{L}$. (Recall that for orderings A and B, $A.B$ is the result of replacing each point of B by a copy of A.)

Theorem 4.2 (Moses [122]) *For each n there are discrete linear orderings which are Σ^0_{n+1} presentable but not Σ^0_n presentable.*

Actually Theorem 4.2 follows from an often rediscovered Theorem of Watnik [183].

Theorem 4.3 (Watnik [183], Downey [33], Ash, Knight, Jockusch [6]) *Suppose that L is any linear ordering presented with $L \leq_T A''$. Then there is a A-computable presentation of $\mathbb{Z}.L$. (Here $\mathbb{Z}.L$ denotes the replacement of each point of L by a copy of Z.)*

Proof of Theorem 4.2. Choose L to be $\emptyset'''$ presentable but not $\emptyset''$ presentable and apply Watnik's Theorem 4.3. $\square$

Proof of Watnik's Theorem. Using Feiner's Theorem, we can suppose that we are given L as a Π^0_2 subset of $\mathbb{Q}$. We prove that $\mathbb{Z}.L$ is computably presentable and the result will then follow by relativization. As we construe L as a Π^0_2 subset of $\mathbb{Q}$, we have a computable relation R such that for all x,

$$x \in L \text{ iff } \forall s \exists t R(x, s, t).$$

Let $A(x,s)$ denote $\max\{s' \leq s : \forall s'' \leq s' \exists t'' \leq s R(x,s'',t'')\}$. We say that x *appears to be in L at stage s* iff $A(x,s+1) > A(x,s)$. We write $x \in L_s$ if x appears to be in L at stage s. Notice that

$$x \in L \text{ iff } \exists^\infty s(x \in L_s).$$

It is an observation of Jockusch [77] that we can enumerate L so that for all finite sets F,

$$F \subseteq L \text{ iff } \exists^\infty s F \subseteq L_s.$$

(That is the members of F appear *together* in L infinitely often.) We will assume that we are so given L. The actual argument is quite involved and uses a Π^0_2 priority argument. Thus we will only provide a sketch and refer the reader to the sources Watnik [183] and Ash, Jockusch, Knight [6] as well as Downey [35] for details.

As with the last construction we will, at each stage put z points on the board. However we only do so around "golden points of the correct guess." When we first place a point in the ordering as an incarnation of some q we believe to be in L, we colour it gold, and attach to it a *guess* which will be the set of $\{i : i \leq q$ (in the $\mathbb{N}$ ordering) and $i \in L_s\}$. Guesses are ordered lexicographically by $\leq_L$. We denote guesses by $\sigma, \tau, ...$ A golden point can lose its colour and then it becomes a z point. No point becomes golden once it is a z point.

- We shall say that a *guess σ appears wrong at stage s* if $F \subseteq L_s$ and $\sigma \not\leq_L F$. That is, σ appears wrong at stage s if it believes that some rational r with lower code is *not* in L and yet $r \in L_s$. Note that we don't say that σ appears wrong if $\sigma \leq_L F$ despite the fact that σ might believe some $r' \in L$ yet $r' \notin L_s$. This is because it is still consistent that *later* $r' \in L_t$ might occur. If x is a golden element and it has guess σ which appears wrong at stage s, then we remove the colour of x and consider it henceforth a z point.

- We say that a guess σ *appears correct* at stage s if $L_s \upharpoonright |\sigma| = \sigma$ That is σ codes precisely the elements that appear to be in L_s, (up to $|\sigma|$). Notice that at stage s there can be many strings that neither appear wrong nor appear correct. If τ is a guess in this category then we sat that τ appears consistent at stage s.

Here's the basic idea. Suppose that we have 5 points to worry about, $\{r_1, ...r_5\}$. Suppose that, for instance, their $\mathbb{Q}$ ordering is $r_1 < r_5 < r_4 < r_3 < r_2$ but their code $\mathbb{N}$ ordering is as given (i.e. $r_1, ..., r_5$.). We need to worry which is in L. That is we need to figure out $L \cap \{r_1, ..., r_5\}$. Without loss of generality, $r_1 \in L$. We will have a point on the board x_1 corresponding to r_1. Thus around x_1 we will build a copy of $\mathbb{Z}$. If we suppose that r_1 has code 0 and is hence the least member of $\mathbb{Q}$ then note that x_1 guesses nothing. We can suppose that $r_1 \in L_s$ for all s. Now at each stage we try to put a z point around x_1 and at the end as with the previous construction, we will build (more or less) $...zzzzzzx_1zzzzzz...$ This is not quite correct as we now see.

Now r_i must guess which of $\{r_j : j < i\}$ are in L. So suppose that at some stage s the first of these elements to appear to be in L with r_1 is r_3. It would have a guess $\nu = \{r_1, r_3\}$ since at this stage only r_1 and r_3 appear to be in L. Hence at this stage we would put down a golden point $x_3(\nu, s)$ corresponding to "this is a version of r_3 guessing that only r_1 is a point with lower code in L." The idea is that if nothing interferes with this golden point then at each stage that ν looks correct, we will put a z point around it.

Thus at stage $s-1$ we might have

$$z_n...z_1x_1z_1...z_n.$$

At stage s we would have

$$z_{n+1}z_n...z_1x_1z_1...z_nz_{n+1}z_1x_3(\nu, s)z_1.$$

At the next 7 stages it may be that the only member of L appears to be r_1. It may be that $r_3 \notin L$ in fact and it might *never* again appear to be in L. We should be careful not to build between x_1 and $x_3(\nu, s)$ unless the guess looks good. The principal idea is that while x_3 looks consistent but not correct we will attach it to someone who looks correct with smaller number. In this case we would attach it to x_1, and while it looks consistent but not correct regard it as adhering to x_1. In the 7 stage situation described we would get

$$z_{n+8}...z_{n+1}z_n...z_1x_1z_1...z_{n+1}z_1x_3(\nu, s)z_1z_2...z_8.$$

That is we have placed the 7 new z points *right* of $x_3(\nu, s)$'s potential $\mathbb{Z}$ block. If at the $s+8$'th stage we again see that ν looks correct, then we would place the new z points, which we denote by z', in the appropriate place to support this belief.

$$z'z_{n+8}...z_1x_1z_1...z_{n+1}z'z'z_1x_3(\nu, s)z_1...z_8z'.$$

Notice that if

- ν never looks wrong after stage s when we first define $x_3(\nu, s)$ and $\{r_1, r_3\} \subset L$, then we will successfully put $\mathbb{Z}$ blocks around the points x_1 and $x_3(\nu, s)$.

- If ν does not later look wrong but $\{r_1, r_3\} \not\subset L$ then we will not remove $x_3(\nu, s)$'s colour *but we will from some stage stop putting points between x_1 and $x_3(\nu, s)$. Therefore we will only build a $\mathbb{Z}$ block around x_1 provided we are not silly enough to put other points between x_1 and $x_3(\nu, s)$.*

The second point is important. If we see the point r_5 which is $\mathbb{Q}$-between r_1 and r_3 appear to be in L_t *it is only legal to put it between x_1 and $x_3(\nu, s)$ if it's guess agrees with ν, on r_1, r_2, and r_3.* This device ensures that if we never cancel $x_3(\nu, s)$'s colour then we put infinitely many points between x_1 and $x_3(\nu, s)$ iff $\{r_1, r_3\} = L \cap \{r_1, r_3\}$. But the point is that a version of r_5 guessing $\tau \not\leq_L \nu$ would build right of $x_3(\nu, s)$'s potential $\mathbb{Z}$ block. Thus while we are consistent with but not believing ν then we regard $x_3(\nu, s)$ as part of x_1's $\mathbb{Z}$ block. Or, to put it another way if $\tau \not\leq_L \sigma$ and τ does not agree with σ then τ views anything with guess σ as a z point.

Finally we need to say what happens to wrong guesses. Now suppose that in the above situation, we reach a stage where we see $x_2 \in L_t$. (So $t > s$.) Now we know that $x_3(\nu, s)$ looks incorrect. At stage t, we would cancel the $x_3(\nu, s)$ label and regard it henceforth as an uncoloured z point. For instance if it happened at the next stage after the placement of the z' points above we would have the situation

$$zz'z_{n+8}...z_1x_1z_1...z_{n+1}z'z'z_1\widehat{z}z_1...z_8z'zzx_3(\tau, t)z.$$

Here we are supposing that the guess correct at stage this stage t is τ that believes that only r_1 and r_2 are in L, and we are denoting by $\widehat{z}$ the z point that was previously $x_3(\nu, s)$. Now it is okay for us to later put down a golden r_3 point guessing ν but this would need to be of the form $x_3(\nu, t')$ for $t' > t$ and *it would need to be right of the block of points above.*

The above is a sketch of the main ideas of the proof. The point is that there is some σ which does code the correct members of $\{r_1, ..., r_{|\sigma|}\} \cap L$ and after some stage all permanently golden points ith guesses correct infinitely often will agree with σ. It is only the true points of L which give such golden points and it is only such golden points that get $\mathbb{Z}$'s built around them. The formal details are to arrange the strategies on a "strategy tree" and do a an argument as above (Downey [33, 35]), to use a "worker argument" Ash, Knight Jockusch [6], or use other combinatorial infinitary methods Watnik [183] to coordinate the strategies. We refer the reader to this material for further details. $\square$

We remark that using induction/worker arguments one can extend the above.

Theorem 4.4 (Ash, Knight, Jockusch [6], see also Roy and Watnik [161]) *(i) Suppose that α is a computable ordinal, and L is presentable computably in $\emptyset^{(2\alpha)}$. Then there is a computable presentation of $\mathbb{Z}^{\alpha}.L$.*
(ii) Similarly, there is a computable presentation of $\mathbb{N}^{\alpha}.L$.
(iii) Furthermore, the procedures above are uniform.

5 Feiner's Boolean Algebra Theorem and Thurber's Approach

The technique of the previous section demonstrably fails for boolean algebras. For suppose that we attempted to use $Intalg(L)$ where L is constructed as in Theorem 4.1. Then notice that $IntalL \cong Intalg(\widehat{L})$ where $\widehat{L}$ replaces each n block of L by one of size 2. (Here we are using Remmel-Vaught). It is then not difficult to build a computable copy of $Intalg(L)$ by a back and forth argument combined with a priority argument, again using the Remmel-Vaught technique.

Nevertheless, Feiner proved that there are Σ_1^0 presentable boolean algebras not isomorphic to computable ones.

Theorem 5.1 (Feiner [51, 52]) *For each n there exist Σ_{n+1}^0 presentable boolean algebras not isomorphic to Σ_n^0 presentable boolean algebras.*

Feiner's proof of Theorem 5.1 again needed an invariant combined with a coding argument. However, instead of coding some arithmetical set into B Feiner used the very clever idea of coding *using the whole arithmetical hierarchy.* In the same way that the proof of theorem 4.1 is a primitive infinite injury priority argument, Feiner's argument can be construed as a sort of primitive $\mathbf{0}^{\omega}$ "worker" argument, a 'la Harrington [68], Knight [93], Ash [2], etc. Several other authors, notably Goncharov and Remmel have used Feiner's technique. But it is fair to say that the first real simplification to Feiner's machinery was due to John Thurber in his PhD Thesis (Thurber [177]). In this section we will sketch Thurber's approach. We will assume that the reader is familiar with computable ordinals, Kleene's $\mathcal{O}$, and the hyperarithmetical hierarchy.

Feiner's key idea is to define a measure of complexity of sets computable from $\mathbf{0}^\omega$. Under this notion of complexity, for a boolean algebra B, Feiner considered the complexity of the set

$$\mathcal{S}_B = \{B \models \Gamma_n\},$$

where $\{\Gamma_n : n \in \mathbb{N}\}$ is a family of infinitary sentences in the language of boolean algebras.

Feiner's main idea is that if B is computable then $\mathcal{S}_B$ has a certain complexity under the measure mentioned above, whereas it is possible to construct a Σ^0_1 presented algebra $\widehat{B}$ where $\mathcal{S}_{\widehat{B}}$ must have a different complexity than that of a computable algebra.

In fact, Thurber and independently Jockusch and Soare [80] each proved a converse to Feiner's Theorem by demonstrating that for any set S of this complexity, there is a computable boolean algebra B such that $S = \mathcal{S}_B$, and through a bit relativization and further analysis, there is a similar relationship between Σ^0_1 presented boolean algebras and sets of related and slightly higher complexity.

The proof of Feiner's Theorem then boils down to proving the purely recursion theoretical fact that there exist sets of this complexity which are of strictly higher complexity than those yielded by computable algebras. This fact is not too difficult to establish. Following Thurber, We will, in fact, generalize the results mentioned above, by defining a family $\{\Gamma_{\alpha_n}\}_{n\in\mathbb{N}}$ of computable infinitary sentences and extending the measure of complexity for an arbitrary computable sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of

For this section it is convenient to use the traditional notation of $\{e\}$ for the e-th Turing procedure. We will write $f = \{e\}^{\Delta^0_\alpha}$ to indicate that f is computable in some Δ^0_α complete set. This allows the following definition which is critical to our work.

Definition 5.2 (i) Let f be a function and $\{\alpha_n\}$ a computable sequence of ordinals. We say f is $\Delta^0_{<\alpha_n>}$ if there is some e such that, for all n, $f(n) = \{e\}^{\Delta^0_{\alpha_n}}(n)$.
(ii) For $S \subseteq \mathbb{N}$, we will say S is $\Sigma^0_{<\alpha_n>}$ if, for some e, $S = W_e^{<\alpha_n>} = \{n : \{e\}^{\Delta^0_{\alpha_n}}(n)\downarrow\}$.

Clearly if S is $\Sigma^0_{<\alpha_n>}$, then S is $\Delta^0_{<\alpha_n+1>}$ [4]

For a boolean algebra B and $x, y \in B$, define $x =_1 y$ iff x and y differ by finitely many atoms. Then the n-th Cantor-Bendixson derivative of B, denoted by $D(B)$, is defined via $D^0(B) = B$, $D^{(n+1)}(B) = D^{(n)}/_{=_1}$ and for limit n, $D^(n)(B) = \cap_{\lambda<n} D^(\lambda)(B)$. Finally recall that an element x of B is *atomic* if for each $y \leq x$, there is some z such that $z \leq y$ and z is an atom.

For a boolean algebra B, $a \in B$ and α, let $Atom_\alpha(a)$ say a contains infinitely many distinct α-atomic elements. (i.e. infinitely many elements no two of which are identified in the (β)-th derivative, for any $\beta < \alpha$.)

Let $\gamma_\alpha(a)$ say $Atom_\alpha(a)$ holds and for all $y \leq a$, if y is atomic then $Atom_\alpha(y)$ does not hold. Clearly, if $\gamma_\alpha(a)$ holds then a is not atomic[5]. Finally we are in a position to define the Feiner sentence.

Definition 5.3 (The Feiner Sentences) We define Γ_α be the sentence saying $\exists x \gamma_\alpha(x)$.

The decisive structural Lemma is the following.

Lemma 5.4 (Feiner [51, 52], Thurber [177]) *(a) If $B \cong \widehat{B}$ then for every α, $B \models \Gamma_\alpha$ if and only if $\widehat{B} \models \Gamma_\alpha$.*
(b) If $a = b \vee c$ where b is atomic and c is atomless, then $\gamma_\alpha(a)$ fails.
(c) Intal$(\omega^\alpha + \mathbb{Q}) \models \Gamma_\alpha$.
(d) Let $X \subseteq N$ and let $\{\alpha_n\}$ be a sequence of ordinals. Then Intal$(\Sigma_{i \in X}(\omega^{\alpha_i} + \mathbb{Q})) \models \Gamma_\beta$ if and only if $\beta = \alpha_i$ for some $i \in X$.

Thurber's reinterpretation and extension of Feiner's material for computable ordinals yields the following.

Lemma 5.5 (Thurber [177]) *Let B be a computable boolean algebra. Let $\{\alpha_n\}$ be a computable sequence of ordinals. Let $S = \{n : \alpha_n \geq 3 \text{ and } B \models \Gamma_{\alpha_n}\}$. Then S is $\Sigma^0_{<2\alpha_n+2>}$. (And hence S is $\Delta^0_{<2\alpha_n+3>}$).*

[4] With this definition, Thurber noted the following correspondence with the work of Feiner. $S \sim (a, b)$ if and only if S is $\Delta_{<a+nb+1>}$.

[5] As Thurber observes, the formula γ_α is an effort to describe the $Intalg(\mathbb{N}^\alpha + \mathbb{Q})$.

Basically, the proof of Lemma 5.5 comes from the Lemma 5.4 and the fact that there is a uniformly effective procedure to assigning recursive $\Sigma^0_{2\alpha+2}$ formulae to describe Γ_α.

Finally we are in a position to describe the construction of a Σ^0_1 presented boolean algebra not isomorphic to a computable one. We need to construct a set S such that S is not $\Sigma^0_{<2\alpha_n+2>}$ and build an Δ^0_2 boolean algebra $\widehat{B}$ such that $S = \{n : \widehat{B} \models \Gamma_{\alpha_n}\}$. Lemmas 5.4 and 5.5 assure us that $\widehat{B}$ has no computable copy. To do this we prove a converse to Lemma 5.5.

Theorem 5.6 (Thurber [177], after Feiner [51, 52]) *Let $\{\alpha_n\}$ be an increasing computable sequence of ordinals, and $S \subseteq \mathbb{N}$ such that S is $\Sigma^0_{<2\alpha_n+2>}$. Then there is a computable boolean algebra B such that $S = \{n : B \models \Gamma_{\alpha_n}\}$.*

Proof. We follow Thurber [177]. Let e be such that $S = W_e^{<2\alpha_n+2>}$. For a fixed n, we can now build a particular computable ordering. By the relativised limit lemma, from the index e we can find a function f, $\{0,1\}$-valued, such that for fixed $n, f(n,s)$ is $\Delta^0_{2\alpha_n+1}$ and f has the feature that $n \in S$ if and only if $f(n,s) = 1$ for all but finitely many s, or equivalently, $n \notin S$ if and only if $f(n,s) = 0$ for infinitely many s. Using f we can build a $\Delta^0_{2\alpha_n+1}$ ordering $\widehat{B}_n$ such that $\widehat{B}_n \cong k$ for some $k \in \mathbb{N}$ if $n \in S$ and $B_n \cong 1 + \mathbb{Q}$ if $n \notin S$. We do this by fixing a first element a_0 and then working toward density in the $\mathbb{Q}$-piece whenever $f(n,s) = 0$. When $f(n,s) = 1$ we don't do anything. (Here we use the method of Theorem 6.6.) By Theorem 4.4, there is a computable ordering B_n with
(i) $B_n \cong \mathbb{N}^{\alpha_n} \cdot k$ if $n \in S$ and
(ii) $B_n \cong \mathbb{N}^{\alpha_n} + \mathbb{N}^{\alpha_n} \cdot \mathbb{Q}$ if $n \notin S$.

We now let $B = \Sigma_{n\in\mathbb{N}}(B_n + \mathbb{Q}) \cong \Sigma_{n\in\mathbb{N}}(\mathbb{N}^{\alpha_n} \cdot \widehat{B}_n + \mathbb{Q})$. B is computable as the sequence $\{B_n\}$ is uniformly computable. To show that B is the desired ordering it suffices to prove the following two lemmas. The first says that, for a given n, the boolean algebra generated by the subordering $B_n + \mathbb{Q}$ is related to the sentence Γ_{α_n} in the way that we want and the second says that this is the only part of the ordering that matters.

Lemma 5.7 *Intal*$(B_n + \mathbb{Q}) \models \Gamma_{\alpha_n}$ *if and only if* $B_n \cong \mathbb{N}^{\alpha_n} \cdot k$.

Proof. By Lemma 5.4, we need only prove the "only if" part. Suppose that $B_n \cong \mathbb{N}^{\alpha_n}+\mathbb{N}^{\alpha_n}\cdot\mathbb{Q}$. Let $D = Intal(B_n + \mathbb{Q})$. Then $D \cong \mathrm{Intal}(\mathbb{N}^{\alpha_n} + \mathbb{N}^{\alpha_n} \cdot \mathbb{Q} + \mathbb{Q})$. Suppose that $D \models \Gamma_{\alpha_n}$ and let $x = [a, b)$ be an interval of D such that $\gamma_{\alpha_n}(x)$ holds. As x is not atomic we can assume that a is in the $\mathbb{N}^{\alpha_n} \cdot \mathbb{Q}$ part of the ordering and b is in the $\mathbb{Q}$ part of the ordering. Let $c \in [a, b)$ be such that c is in some $\mathbb{N}^{\alpha_n}$-piece to the right of the $\mathbb{N}^{\alpha_n}$-piece in which a lies. There is a dense set of copies of $\mathbb{N}^{\alpha_n}$-pieces between a and c; $[a, c)$ is atomic and $Atom_{\alpha_n}([a, c))$ holds, contradicting $\gamma_{\alpha_n}(x)$. □

Thurber make the following observation which is used in the proof of the next Lemma. Thurber observed that the argument above shows the following. If x contains any subinterval of type $\mathbb{N}^{\alpha_n} \cdot \mathbb{Q} + \mathbb{Q}$ for any n, then x contains an atomic interval $[a, b)$ where a and b are in different copies of $\mathbb{N}^{\alpha_n}$, with densely many copies between them. This means that the α_nth derivative collapses $[a, b)$ to a dense piece and higher derivatives are trivial on this piece. Thus $Atom_\beta([a, b))$ holds for all β and $\gamma_\beta(x)$ fails.

Lemma 5.8 *If $B = \Sigma_{n\in\mathbb{N}}(B_n+\mathbb{Q})$, then $Intal(B) \models \Gamma_{\alpha_n}$ if and only if $Intal(B_n + \mathbb{Q}) \models \Gamma_{\alpha_n}$.*

Proof. Assume that there is some interval $x \in Intal(B)$ such that $\gamma_n(x)$ holds. By Lemma 5.4 (b) it is enough to show that for any $m \neq n$ and for any interval $y \in Intal(B_m + \mathbb{Q})$, $\gamma_{\alpha_n}(y)$ cannot hold. First, suppose that $\alpha_m < \alpha_n$. Then $Intalg(B_m + \mathbb{Q}) \cong Intal(\mathbb{N}^{\alpha_m} \cdot k + \mathbb{Q})$ or $Intalg(B_m + \mathbb{Q}) \cong Intal(\mathbb{N}^{\alpha_m} + \mathbb{N}^{\alpha_m} \cdot \mathbb{Q} + \mathbb{Q})$. In the first case, for any such $y \in Intal(B_m + \mathbb{Q})$, its clear that $Atom_{\alpha_n}(y)$ fails, as the k copies of $\mathbb{N}^{\alpha_m}$ have collapsed to at most k elements in the α_mth derivative. In the second case, since $\gamma_{\alpha_n}(y)$ implies that y is not atomic, by the remark following the proof of Lemmalll1, y contains an atomic element u such that $Atom_\beta(u)$ holds for all β, so $\gamma_{\alpha_n}(y)$ fails. Now suppose that $\alpha_m > \alpha_n$. Again there are two cases. In the first case, for any β and $y \in Intal(B_m + \mathbb{Q}), \gamma_\beta(y)$ implies that y is an interval of type $\mathbb{N}^{\alpha_m} + \mathbb{Q}$, whence for any $\beta < \alpha_m$, y contains an atomic interval u for which $Atom_\beta(u)$ holds, thus $\gamma_\beta(y)$ fails. In the second case, as before, if y is not atomic then y contains an atomic interval u such that $Atom_\beta(u)$ holds for all β, and $\gamma_{\alpha_n}(y)$ fails. □

Again there are many relativizations and corollaries. Together with Lemma 3.4, the above yields the following.

Lemma 5.9 (Feiner [51, 52], Thurber [177]) *Let B be an Σ^0_1 presented boolean* *algebra and* $\{\alpha_n\}$ *a computable sequence of ordinals. Let* $S = \{n : \alpha_n \geq 3 \text{and} B \models \Gamma_{\alpha_n}\}$*. Then* S *is* $\Sigma^0_{<2\alpha_n+p>}$*, with* $p = 3$ *for* $\alpha_n < \mathbb{N}$ *and* $p = 2$ *for* $\alpha_n \geq \mathbb{N}$.

Theorem 5.10 (Thurber [177], after Feiner [51, 52]) *Let* $S \subseteq \omega$ *be such that* S *is* $\Sigma^0_{<2\alpha_n+3>}$*. There is a a* Σ^0_1 *presented boolean algebra* B *such that* $S = \{n : B \models \Gamma_{\alpha_n}\}$.

Proof of Feiner's Theorem, concluded In view of the above, to complete the proof we only need a set S which is $\Sigma^0_{<2\alpha_n+3>}$ but not $\Delta^0_{<2\alpha_n+3>}$. This is just an iterated relativized version of the halting problem (or Post's Theorem). $\square$

As we remarked earlier, Feiner's Technique has found numerous other applications. We refer the reader to Remmel [139, 142, 148] and Goncharov [63]. We give one illustration.

Theorem 5.11 (Goncharov [63]) *There is a computable boolean algebra* B *such that any boolean algebra isomorphic to* B *has noncomputable atoms.*

Actually, Goncharov did the following. We call a structure *decidable* if we can decide all first order sentences for it. (That is its diagram is computable.) Goncharov constructed a computable boolean algebra not isomorphic to a decidble one. However, Tarski proved that the theory of boolean algebras admits elimination of quantifiers in terms of the predicates $\{At_n(x) : n \in \mathbb{N}\}$ which say that x is the join of n atoms. Now if the atoms of a boolean algebra are computable these predicates are all uniformly decidable. Hence, there would be a decidable structure isomorphic to the given one if there was one with computable atoms.

Again we remark that Feiner's technique of coding into all levels of the Frechet sequence (i.e. the algebras generated by the Cantor-Bendixson process) is the only way known to generate arithmetically presented boolean algebras not isomorphic to computable ones. This leads to the following very interesting question.

Question 5.12 *Is there an (arithmetical) boolean algebra of finite Cantor-Bendixson rank not isomorphic to a computable one?*

6 Lowly Orderings

At this stage we return to Feiner's Theorem on orderings. Recall that this result proved that there were Σ_1^0 presented linear orderings not isomorphic to computable ones by constructing from a given Σ_4^0 complete set F a Σ_1^0 presented linear ordering L such that the collection of finite blocks $B(L)$ was equal to F. Then since $B(L)$ is an *algebraic* invariant, it must be that if $\widehat{L} \cong L$ then $B(L) = B(\widehat{L})$ and hence, in particular, $\widehat{L}$ must have $\Sigma_3^{\widehat{L}}$ of degree at least $\mathbf{0}''''$, the degree of a Σ_4^0 complete set.

The key observation is that Feiner's construction proves a lot more. By the last sentence of the preceding paragraph, in fact $\widehat{L} \cong L$ implies that $\widehat{L}''' \geq_T \emptyset''''$. Recall that a computably enumerable degree is called low_n ($high_n$) iff $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ (resp. $\mathbf{a}^{(n)} = \mathbf{0}^{n+1}$). In fact, one easily sees that Feiner's coding can only construct linear orderings not isomorphic to low$_2$ ones. The boolean algebra case is even more interesting since the technique only constructs boolean algebras which are not isomorphic to any low$_n$ boolean algebra for any $n \in \mathbb{N}$.

Thus from a purely technical viewpoint, one can ask if this is a necessary limitation. A related idea is due to Jockusch. Since there are no proper nontrivial degrees for order types, perhaps this is because we need enough definability to get some sort of coding in the structures. Thinking in this vein prompted Jockusch to propose the following definition.

Definition 6.1 *(i) We say that (an isomorphism type of a structure) A has* jump degree $\mathbf{a}$ *if* $\mathbf{a}$ *is least in* $\{deg(B)' : B \cong A\}$. *We say that this jump degree is* proper *if A has no degree.*
(ii) More generally, we say that A has n-th jump degree $\mathbf{a}$ *if* $\mathbf{a}$ *is least in* $\{deg(B)^{(n)} : B \cong A\}$. *This is proper if A has no $n-1$-st jump degree.*

Richter proved that linear orderings and boolean algebras have no nontrivial degrees. Julia Knight proved the same for jump degrees.

Theorem 6.2 (Knight [92]) *If an order type or isomorphism type of a boolean algebra has a jump degree then that jump degree is* $\mathbf{0}'$.

The proof of Knight's theorem is a forcing argument along the lines of, but more difficult than, Richter's Theorem. Due to space restrictions we omit it. Naturally, two question suggest themselves.

- Can the Knight's Theorem be extended to n-jumps?
- Does Knight's Theorem imply that all low linear orderings and boolean algebras are isomorphic to computable ones?

We begin with the second question first. For linear orderings, the second question was solved by Jockusch and Soare [79] who directly constructed a low linear ordering not isomorphic to a computable one. The argument is very interesting since *it does not use any sort of coding, but uses a priority argument to diagonalize against* classical *isomorphisms.* While there are some earlier precursors to this idea such as the work of Watnik [183] and Lerman and Rosenstein [102] the Jockusch-Soare argument is the first to directly use priority methods to achieve classical noninvariance.

Theorem 6.3 (Jockusch and Soare [79]) *Let* $\mathbf{a}$ *be any nonzero computably enumerable degree.*
(i) Then there is a $\mathbf{a}$ *presentable linear ordering* L *not isomorphic to a computable one. This ordering can be chosen to be* Σ_1^0 *and of degree* $\mathbf{a}$.
(ii) In particular, if $\mathbf{a}$ *is low then* L *has proper jump degree* $\mathbf{0}'$.

Proof. We sketch the proof. As we will see, it suffices to show that, given a nonzero computably enumerable degree $\mathbf{a}$, we can construct a presentation L computable in $\mathbf{a}$ with the desired properties. The result will then follow from Theorem 6.4. We build L as a Π_1^0 subordering of $\mathbb{Q}$ to satisfy the requirements below.

$$\mathcal{R}_e : W_e \ncong L.$$

Without the machinery to keep $L \leq_T A$ (for $A \in \mathbf{a}$), the argument goes as follows.

For convenience, we first discuss the strategies in the situation where we suppress the "permitting" which we use to keep the degree down.

For a single $\mathcal{R}_e$, we will try to defeat W_e. We do so in infinitely many stages. Initially we begin to enumerate an ω^* (i.e. $-\mathbb{N}$) sequence into L. Thus at stage n, we will have enumerated

$$a_n < \ldots < a_0$$

Now at some stage s, W_e must respond with at least 2 points $b_1 < b_0$, lest *our* ordering be infinite and *his* ordering be finite.

Our action is to now delete the points $a_n, ..., a_1$ and add a new point a'_1 *right* of a_0. Then between a_0 and a'_1 we will begin to build a new $-\mathbb{N}$ sequence. A typical situation would be

$$a_0 < a'_m < ... < a'_1.$$

Now the situation is that *our* linear ordering has the property that *every* point *except* a_0 has innthe limit infinitely many points to the left. So our opponent must respond with another point $b_2 < b_0$. At this stage we repeat out initial action, except with a'_1 in place of a_0. Namely we will delete $a'_m, ...a'_2$ and create a new point a''_2 right of a'_1 and begin a new $-\mathbb{N}$ sequence between a'_1 and a''_2. We continue inductively.

Now if our opponent *fails* to respond then we will build an ordering of type $k + -\mathbb{N}$, yet his ordering has a point b_0 with only $k+1$ points left of it. On the other hand if our opponent always responds then *we* will build an ordering of type $\mathbb{N}$ and yet *his* will contain a point b_0 with infinitely many point left of it. In either case W_e cannot be isomorphic to our ordering.

To combine strategies, Jockusch and Soare invented the clever idea of using *separators*. Out idea is to satisfy divide the ordering we construct into intervals $\{I_e : e \in \mathbb{N}\}$ separated by separators S_e to be described, so that L looks like

$$I_0 + S_0 + I_1 + S_1 + I_2 + S_2 + ...$$

We will use the interval I_e to satisfy $\mathcal{R}_e$ more or less as in the basic module.

The n-th separator will have order type $1 + \mathbb{Q} + 1 + n + 1 + \mathbb{Q} + 1$ and can be coded by the 4 points representing the 1's in the separator. The key information we need to meet the $\mathcal{R}_e$ in interval I_e is what our opponent is doing between the $e-1$'th and e'th separator. The crucial observation is that the predicate "a, b, c, d code the n-th separator in W_e," is a Π^0_2 predicate. Therefore we can represent separators via *guesses* as with Watnik's Theorem, and meet all the $\mathcal{R}_e$'s on the "true path" of a Π^0_2 priority argument.

Finally, to achieve downward degree control we add standard permitting. By this we mean that we assume that we are given a computably enumerable noncomputable set B and a 1-1 enumeration $f(\mathbb{N}) \mapsto B$. Then we are allowed to delete a point a from L_s (as above) only when $a > f(s)$. So we cut back as bets we can. This is perfectly standard and creates no especial difficulties. As usual one argues that if we fail to meet some $\mathcal{R}_e$ then B is recursive. We refer the reader to [79] for further details. $\square$.

The proof sketched above only gives $L \leq_T B$. To get $L \equiv_T B$ Jockusch and Soare used the following.

Theorem 6.4 (Jockusch and Soare [79], after Knight [92]) *Let B be and set with a linear ordering $L \leq_T B$. Then there is a linear ordering $B \equiv_T \widehat{L}$ such that $\widehat{L} \cong L$.*

Proof. To code B into $\widehat{L}$ we ensure that, for all n, $a_{2n} <_{\widehat{L}} a_{2n+1}$ iff $n \in B$. Regard the domains of L and $\widehat{L}$ as $\mathbb{N}$. We specify the ordering $\leq_{\widehat{L}}$ via an an isomorphism $f : L \mapsto \widehat{L}$. First label L so that for all k, $2k <_L 2k+1$ using the L-oracle. Now f simply switches the labels $2k$ and $2k+1$ iff $k \notin B$. $\square$

The Jockusch-Soare technique of separators is quite powerful and has found other applications. Downey used it to prove that if we only look at Σ^0_1 presentations of orderings the nontrivial degrees are possible. (Downey [35], Theorem 8.7) and Downey and Moses [45] used more intricate methods to prove that there are computable linear orderings L such that for all linear orderings $\widehat{L} \cong L$, the set of adjacencies of $\widehat{L}$ computes the halting problem. (In the terminology of Ash-Nerode [9], $Adj(L)$ is "intrinsically complete.") We refer the reader to Downey [35] for these and other applications.

Using Δ^0_2 permitting, Seetapun and independently Downey (both unpublished) observed that the Jockusch-Soare Theorem can be made to work in and degree below $\mathbf{0}'$. Thus *if* $\mathbf{a}$ *is* Δ^0_2 *and nonzero, then* $\mathbf{a}$ *contains a linear ordering not isomorphic to a computable one.* Julia Knight inproved this as follows:

Theorem 6.5 (Knight (unpublished)) *If* $\mathbf{a} \neq \mathbf{0}$ *is any degree then* $\mathbf{a}$ *contains a linear ordering not isomorphic to a computable ordering.*

To prove Theorem 6.5 we need the following result of Downey and Knight. It is a 1-jump version of Watnik's Theorem.

Theorem 6.6 (Downey and Knight [37]) *Let L be an ordering. Define $\gamma(A) = (\mathbb{Q} + 2 + \mathbb{Q}).L$ (i.e. replace each point of L by a copy of $\mathbb{Q} + 2 + \mathbb{Q}$). Then for any set A, L is presentable computably in A' iff $\gamma(L)$ is presentable computably in A. In particular, L is Σ^0_{n+1} presentable iff $\gamma(L)$ is Σ^0_n presentable.*

Proof. We consider the case $n = 0$ and then relativise. Note that if L is a Π^0_1 subset of $\mathbb{Q}$ then there is a computable relation R such that

$$x \in L \text{ iff } \forall s R(x, s).$$

Like Feiner's Theorem, we begin with a copy of some ordering and add points. This time we begin with $\gamma(\mathbb{Q})$. Identify the x-th rational r_x with "$(\mathbb{Q}+2+\mathbb{Q}).r_x$" in the obvious way. We make the 2 corresponding to r_x dense after stage s if we see $\neg R(x, s)$ hold. □

Proof of Theorem 6.5. Let C be given. We show how to construct an ordering computable from C not isomorphic to a computable one. The proof splits into 3 cases.

Case 1. $\emptyset'' <_T C''$. This case uses some work of Ash, Jockusch and Knight [6]. (or of Lerman [97]) Define the $\sigma(X)$ to be the "shuffle sum" of X consisting of densely many copies of $\mathbb{N}$ and of n for $n \in X$. Then it turns out that $\sigma(C'' \oplus \overline{C''})$ can be chosen to have degree X (using a proof similar to Watnik's Theorem, and cannot be isomorphic to a computable linear ordering since $\emptyset'' <_T C''$. (See Ash, Jockusch, Knight [6] or Downey [35], Section 8, for details.)

Case 2. *$C \leq_T \emptyset'$ and is noncomputable.* This is the Downey-Seetapun Theorem.

Case 3. *$C'' \equiv_T \emptyset''$ and $\emptyset' <_T C'$.* Using Case 2 in relativized form, we see that C' contains a linear ordering A not isomorphic to a computable one. Then $\gamma(A)$ can be chosen to have degree C and cannot be isomorphic to a computable ordering. □

Before we leave the Jockusch-Soare technique we quote a related question of Mat Raw.

Question 6.7 (Raw [138]) *Is there a computably presented linear ordering of classical type $A + \omega^*$ where A is not computably presentable?*

Raw's question is of some interest since it seems to point at the limits of known coding techniques and of the Jockusch-Soare separator technique. There is related material on computable self embeddings in Downey[35]. We finish the present section by quoting an interesting question from Downey [35].

Question 6.8 *Is there a property P of order types which guarantees that if L is low and $P(L)$ then L is isomorphic to a computable linear ordering?*

We remark that some results are known. For instance, Downey and Moses [45] proved that any low discrete linear ordering is isomorphic to a computable one.

7 Lowly Boolean Algebras

Again if we analyse the Feiner construction we get a lot more than simply the construction of a Σ_1^0 boolean algebra not isomorphic to a computable one. We get the following theorem.

Theorem 7.1 (Consequence of Feiner's Theorem) *There exist Σ_1^0 presented boolean algebras not isomorphic to a low_n boolean algebra for any n.*

In view of this result, the Jockusch-Soare Theorem raises the natural question of whether the analogue of Theorem 6.3 holds for boolean algebras. The answer is no.

Theorem 7.2 (Downey and Jockusch [36]) *Suppose that B is a low boolean algebra. Then B is isomorphic to a computable boolean algebra.*

Proof. We use the Remmel-Vaught Theorem. Clearly the result holds if B has only finitely many atoms. Thus suppose $B = Intalg(L)$ such that L is low with infinitely many adjacencies. By the lowness of L we can suppose that the Adjacency Relation $A(x, y)$ is computable from $\mathbf{0}'$. By the limit lemma we may assume that $L \subset \mathbb{Q}$ is given by an enumeration $L = \lim_s L_s$ such that for all x, y

(i) $x \in L \rightarrow (\exists s)(\forall t > s)[x \in L_t]$.

(ii) $x \notin L \rightarrow (\exists s)(\forall t > s)[x \notin L_t]$

(iii) $x, y \in L \rightarrow (\exists s)(\forall t > s)[A(x, y)$ iff $A_t(x, y)]$.

Furthermore, we can suppose that $|(L_s - L_{s+1}) \cup (L_{s+1} - L_s)| = 1$. Suppose that we have constructed a computable subset B_s of $\mathbb{Q}$ and a isotone embedding $f_s : L_s \mapsto B_s$. Consider the situation at stage $s + 1$. What can happen is that a new point appears in L_{s+1} or an old one disappears. If a new point p enters, then without loss of generality it is between two points x and y of L_s. Now if there is already some point q between $f_s(x)$ and $f(y)$ not in the image of L_s, then for the q with the smallest code set $f_{s+1}(p) = q$. If there is no such point q, then pick a rational r with a code bigger than $s + 1$ $\mathbb{Q}$-between $f_s(x)$ and $f_s(y)$ and add it to B_s, defining $f_{s+1}(p) = r$. Finally, if some point p leaves L_s, then we must arrange that some $w \in Im(L_s)$ is no longer the the image of some x. The obvious candidate is $f_s(x)$, but a little care is needed so that f settles down. What we do is that for any point added to L_s since the stage x last appeared in L, if we can rearrange the maps on these elements (only) to keep $f_s(x)$ in the image of L_{s+1} then do so. One argues that if r is not between the image of an adjacency of L then this strategy eventually ensures that r is in the image of L. The result now follows from the special properties of the approximations and the Remmel-Vaught Theorem. □

Again the technique above has found several other applications. In the last section we saw that there were linear orderings such that every computable copy had a Turing complete set of successivities. Downey [34] used an infinite injury extension of the argument above to prove the following.

Theorem 7.3 (Downey [34]) *If B is a computable boolean algebra then there is a computable boolean algebra $\widehat{B} \cong B$ such that the set of atoms of $\widehat{B}$ are Turing incomplete.*

Theorem 7.3 should be compared with Theorem 5.11 whose proof actually demonstrates that there exist boolean algebras such that the set of atoms was intrinsically nonlow$_n$ for all $n \in \mathbb{N}$. A possibly easy question is to ask the following.

Question 7.4 *Can a boolean algebra have atom which are intrinsically* $high_n$ *for some* n*? What are the possible degree spectrums of such atoms sets?*

John Thurber used an interesting argument to extend Theorem 7.2.

Theorem 7.5 (Thurber [177]) *Every* low_2 *boolean algebra is isomorphic to a computable one.*

There seem to be subtantial difficulties in extending Theorems 7.2 and 7.5 to even the low_3 case.

Conjecture 7.6 (Downey) *Every* low_n *boolean algebra is isomorphic to a computable one.*

There is really not too much evidence to support this conjecture. I believe that the low_3 case is the critical one. In any case new ideas would certainly seem to be needed here.

There are a constellation of other questions but these are best left to the next section where they will be more in context.

8 Jump Degrees

We return to the theme of n-jump degrees. We have seen that there are no nontrivial proper degrees for order types (Richter's Theorem) and that the only nontrivial proper jump degree for a linear ordering is $\mathbf{0}'$ (Knight [82] + Jockusch-Soare). What happens for $n \geq 2$? The answer is perhaps a little surprising.

Theorem 8.1 *Let* $n \geq 2$ *be a computable ordinal.*
*(i) (*Ash, Jockusch, Knight [6]*) If* $\mathbf{0}^{(n)} < \mathbf{a}$ *then there is a linear ordering of proper* n*-th jump degree* $\mathbf{a}$.
*(ii) (*Downey and Knight [37]*) There are linear orderings of proper* n*-th jump degree* $\mathbf{0}^{(n)}$.

Proof Sketch. The proofs are, for the most part, to involved to give in any detail. For the cases where n is not finite then we use so-called "worker" arguments. To prove (ii) for finite n, we proceed as follows. Using the Jockusch-Soare construction, let $D(X)$ be low over X and not isomorphic to any X-presentable linear ordering. Applying this to $X = \emptyset'$, we get an $\emptyset''$ presentable linear ordering low over $\emptyset'$ not isomorphic to a $\emptyset'$ presentable ordering. Now apply Theorem 6.6 to $D = D(\emptyset')$. Then $L = \gamma(D)$ is $\emptyset'$ presentable, and we claim it has proper 2-jump degree $\mathbf{0}''$. Clearly the 2-jump degree of L is $\mathbf{0}''$. Suppose that L had proper 1-jump degree, say $\mathbf{b}$. Let $\widehat{L} \cong L$ with $\widehat{L}' \in \mathbf{b}$. Applying Theorem 6.6 backwards, there must be some $\widehat{D} \in \mathbf{b}$ with $\gamma(\widehat{D}) = \widehat{L}$. But then it can only be that $\widehat{D} \cong D$. Therefore, D has degree $\mathbf{b}$. For $n > 2$ one can use inductive arguments.

Finally for (i) and finite n, Ash, Jockusch and Knight use results on n-generic sets combined with iterated versions of Watnik's Theorem such as Theorem 4.4. The details are quite involved so we refer the reader to either [6] or Downey [35], Section 8. $\square$.

For boolean algebras, the situation is equally interesting. Again the details are beyond the scope of this paper.

Theorem 8.2 (Jockusch and Soare [80]) *(i) For each* $\mathbf{a} \geq \mathbf{0}^{\omega}$ *there is a boolean algebra* B *with proper* ω*-th jump degree* $\mathbf{a}$.
(ii) For finite n and degree $\mathbf{b} > \mathbf{0}^{(n)}$, *there is no boolean algebra which has proper* n*-th jump degree* $\mathbf{b}$. *Hence the only possible proper* n*-th jump degrees are* $\mathbf{0}^{n}$ *for* $n \geq 3$, *and no boolean algebra can have* n*-th jump degree above* $\mathbf{0}^{(n)}$.

Proof. For (i), Jockusch and Soare used Feiner type coding arguments, and generic sets, combined with Macintyre's ω jump inversion. ([107]) For (ii), they resorted to structural analysis. We refer the readers to [80] for details. $\square$

The theorems above highlight the low$_n$ question, and the question of whether there are boolean algebras of finite Cantor-Bendixson rank which are not isomorphic to computable ones.

9 Beyond Computable Presentations

Having looked at general questions about the class of all structures, one can look more deeply into the relationship of the structure theory and the degrees in the type. Furthermore, one can look for best presentations within the class of objects that are computably presentable. Again these topics are intermingled and there are many interesting results, indeed, far to many for us to give more than a glimpse.

To begin with one can ask does a particular structure have a presentation that is nice algebraically and algorithmically? This is a theme pursured by Remmel [139, 142, 143, 148] for boolean algebras. Let $\widehat{\mathbb{N}}$ denote the boolean algebra of finite and cofinite subsets of $\mathbb{N}$, $\widehat{\mathbb{C}}$ denote an effective presentation of $Intalg(1 + \mathbb{N}.\mathbb{Q})$, $\widehat{\mathbb{H}}$ denote an effective presentation of $Intalg(\mathbb{N} + \mathbb{Q})$ and $\widehat{\mathbb{G}}$ denote a an effective presentation of $Intalg(1 + (\mathbb{N} + \mathbb{Q})\mathbb{Q})$. These algebras played a central role in many of Remmel's results in [139, 142, 143, 147, 148]. The relevant structure theorem is the following. For a boolean algebra B and $x \in B$ let $(B|x)$ (or sometimes $B|x$) denote the algebra $\{z \in B : z \leq_B x\}$, with the naturally restricted operations of B.

Theorem 9.1 (Remmel [143]) *(i) Suppose that B is any countable infinite boolean algebra. Then there is some $x \in B$ such that $B|x$ is (classically) isomorphic to one of the 3 boolean algebras $\mathbb{Q}, \widehat{\mathbb{N}}, \widehat{\mathbb{C}}$.*
(ii) Futhermore, if B has infinitely many atoms, then there is some x in B such that $B|x$ is of the 4 boolean algebras $\widehat{\mathbb{N}}, \widehat{\mathbb{C}}, \widehat{\mathbb{H}}, \widehat{\mathbb{G}}$.

The proof of theorem 9.1 is a case by case back and forth argument and is not difficult. In the case there is more information about the computability structure of the given B we can say quite a bit more. Let $At(B)$ denote the set of atoms of B, let $Atl(B)$ denote the set of atomless elements of B, let $[X]$ denote the ideal generated by X, and let $\cong_c$ denote *computable* isomorphism. We have the following.

Theorem 9.2 (Remmel [139, 142, 143]) *Suppose that B is an infinite computable boolean algebra. Then*
(i) If B is atomless then $B \cong_c \mathbb{Q}$.
(ii) If B is atomic and $B/[At(B)] \cong \{0,1\}$ with $[At(B)]$ computable then $B \cong_c \widehat{\mathbb{N}}$.

(iii) If B is atomic and $B/[At(B)] \cong \widehat{\mathbb{Q}}$ and both $[At(B)]$ and $At(B)$ are computable, then $B \cong_c \widehat{\mathbb{C}}$.
(iv) Suppose that $At(B)$ is infinite, B has no infinite atomic elements, and for all $z \in B$, either $At(B|z)$ or $At(B|-z)$ is finite. Furthermore, suppose that $At(B)$ and $Atl(B)$ are computable. Then $B \cong_c \widehat{\mathbb{H}}$.
(v) Suppose that $At(B)$ is infinite, B has no infinite atomic elements, and for all $z \in B$ such that $At(B|z)$ is infinite there exists $x \leq_B z$ such that both $At(B|x)$ and $At(B|z-x)$ are infinite. Furthermore, suppose that the following sets are all computable: $At(B), Atl(B), [At(B)], \{z \in B : At(B|x) \text{ is computable}\}$. Then $B \cong_c \widehat{\mathbb{H}}$.

Again the details are not difficult but involve the careful checking that the relevant back and forth arguments can be effectivized. Notice that we have carefully distinguished between the notions of classical isomorphism and computable isomorphism. We remark in passing that there are many analyses both structure theoretical and algebraic relating the two. Again the literature is vast. We point out two results that are directly related to our story, and prove one to give the reader a flavour of such results. We call a computable structure A *computably categorical*[6] if every computable structure B classically isomorphic to A is computably isomorphic to A.

Theorem 9.3 *(i)*(**Remmel [144]**) *A computable linear ordering is computably categorical iff it has a finite number of adjacencies.*
(ii) (**Goncharov [63], La Roche [98]**) *A computable boolean algebra is computably categorical iff it has a finite number of atoms.*

Proof. (i) Is proven by a finite injury priority argument. We refer the reader to Remmel [144] or Downey [35] for details.
(ii) To prove (ii), we clearly need only prove that if B has infinitely many atoms then it is not computably categorical. By Theorem 3.12, B has a computable copy with the set of atoms immune. By Theorem 9.1, there is some $x \in B$ such that $B|x$ is isomorphic to one of $\mathbb{N}, \widehat{\mathbb{H}}, \widehat{\mathbb{C}}$, or $\widehat{\mathbb{G}}$. Therefore since $B \cong_c (B|x) \times (B|-x)$ for any $x \in B$, B is isomorphic to one of $\mathbb{N} \times (B|-x)$, $\widehat{\mathbb{H}} \times (B|-x)$, $\widehat{\mathbb{C}} \times (B|-x)$, or $\widehat{\mathbb{G}} \times (B|-x)$. Each of these is a computable boolean algebra with a nonimmume

[6]Other terms have been used for this notion. The original one predominantly used in the russian literature is *autostable*. The term recursively or computably categorical seems to be the one that is now most widely accepted.

set of atoms. The result now follows since immunity is preserved by computable bijections. □

We remark that the theorem above can be obtained by model theoretical methods. In particular both results are particular instances of a general result of Dzgoev and Goncharov [48] on computable categoricity of *branching* structures. There are many other results on computable categoricity of general structures. We mention one more.

Theorem 9.4 (Cholak, Goncharov, Khoussainov, and Shore [26]) *There is a computable structure A and an element x of A such that A is computably categorical yet $\langle A, x\rangle$ is not computably categorical.*

Related to Theorem 9.4 and earlier work of Goncharov, is the following unpublished result of Goncharov. *If a structure has computable copies with a Δ^0_2 isomorphism between them then they have a computable isomorphism between them.* The computebel structure A of Theorem 9.4 has two computable copies. Therefore the isomorphism between the two copies is not computable from $\mathbf{0}'$, but by the proof can be seen to be computable from $\mathbf{0}''$. The following question is open.

Question 9.5 *Is there a structure A with only 2 computable copies A_1 and A_2 such that the isomorphism between A_1 and A_2 is not computable from $\mathbf{0}''$? What about not arithmetical, etc?*

The reader might note that our proof of Theorem 9.3 (ii) gives a little more. It says that we can distinguish a relation $R(X)$ that says "x is an atom," and find another copy of B where the relation cannot be computably mapped to the realization of R in the new structure. This is an instance of the a general class of investigations devoted to studying *intrinsic* properties of relations on computable structures. Again we can only give a flavour of these results. Ash and Nerode [9] studied the problem of when a relation R on a structure A is *intrinsically computably enumerable.* We say that R is intrinsically computably enumerable on R if for any computable B isomorphic to A has R computably enumerable. Ash and Nerode proved that, subject to certain computability conditions, a relation R on A is intrinsically computably enumerable iff it can be described by a "computable enumerable infinitary formula over A." Again this results has generated much research. In our context, we have seen that it is possibly to construct computable linear orderings whose adjacency relation is "intrinsically complete (in Turing degree)." (Downey and Moses [45])

Returning to orderings and algebras, another natural approach is to look at increased levels of effectivity. Earlier we met the notion of a *decidable* structure as one whose whole diagram is computable. That is, given any sentence we can effectively compute its validity in the structure. Along similar lines one can define a structure to be n-computable if we can decide effectively an n quantifier sentences. Sometimes the two concepts are more closely related than the obvious fact that A is decidable iff it is uniformly n computable for all $n \in \mathbb{N}$. For instance, the old result of Langford [97], shows the following. Recall that L is called discrete iff $L \cong \mathbb{Z}.\widehat{L}$ for some $\widehat{L}$.

Theorem 9.6 (Langford [97]) *A discrete linear ordering is decidable iff it is* $1-$ *computable.*

The proof of Langford's Theorem uses quantifier elimination. We illustrate this technique in the next result which completely classifies $1-$computability for computable linear orderings.

Theorem 9.7 (Moses [122, 123]) *A computable linear ordering is* $1-$*computable iff the collection of adjacencies of* L *is a computable set.*

Proof Sketch. First one shows that if φ is a quantifier free sentence on variables $x_1, ..., x_n$ then φ is equivalent to a disjunction of formulae of the form $\nu_1 \wedge ... \wedge \nu_n$ where each ν_i is of the form $x_{\pi(i)} < x_{\pi(i+1)}$ or $x_{\pi(i)} = x_{\pi(i+1)}$ for some permutation π of $\{1, ..., n\}$. We call this the standard form of φ. It follows that an existential formulae γ is equivalent to a disjunction of the form

$$\exists \overline{x} \psi_1(\overline{x}, \overline{y}), \vee ... \vee \exists \overline{x} \psi_m(\overline{x}, \overline{y}),$$

where each ψ_i is in standard form. Thus to se if γ is satisfiable, we need only look at each $\exists \overline{x} \psi_i(\overline{x}, \overline{y}) = \gamma_i$, say.

It is easy to see that each γ_i is equivalent to a finite set of statements of the form "$y_i = y_j$", $y_i < y_j$, "there are at least $k(\leq n)$ elements between y_i and

y_j," "there are at least k elements $> y_i$," and "there are at least k elements $< y_i$." These are decidable iff the collection of succesivities is computable. $\square$

We remark that Moses [126] has constructed for each n an n-computable not isomorphic to a $n+1$-computable one. Chisholm and Moses have also proven the following.

Theorem 9.8 (Chisholm and Moses [27]) *There exists a linear ordering $\widehat{L}$ which is n-computable for each $n \in \mathbb{N}$ but $\widehat{L}$ not isomorphic to a decidable linear ordering.*

Here are some related questions.

Question 9.9 *(i) Are there n computable boolean algebras not isomorphic to $n+1$ computable ones for each $n \in \mathbb{N}$? What about a boolean algebra which is n computable for each $n \in \mathbb{N}$ but is not isomorphic to a decidable one?*
(ii) For each $n \in \mathbb{N}$, is there a finitely presented group which is n-computable but not $n+1$ computable? Can we construct a nondecidable finitely presented group which is n-computable for each $n \in \mathbb{N}$?

Proof Sketch of Theorem 9.8. We briefly sketch the proof of Theorem 9.8 as it is not yet available. For structures P and Q we will write $P \equiv_{\Sigma^0_n} Q$ to mean that P and Q satisfy the same Σ^0_n sentences. The idea is to build a sequence

$$L_0, L_1, L_2, ...$$

of decidable linear orderings with limit $\widehat{L}$ so that for each n and each $m > n$, we ensure that for all $\overline{a} \in \widehat{L}$ we can effectively find a $\overline{b} \in L_m$ such that $\langle \widehat{L}, \overline{a} \rangle \equiv_{\Sigma^0_n} \langle L_m, \overline{b} \rangle$ and conversely. This will ensure that $\widehat{L}$ is n-computable by the decidability of L_n and playing an Ehrenfeucht-Fraisse game between $\widehat{L}$ and L_n.

To ensure that $\widehat{L}$ has no decidable copy, we will ensure that L_n is one of L^+_n or L^-_n. (As usual, we are building all the orderings in stages and at each stage they are finite.) We begin by building $L_n = L^-_m$ and only change when we see an opportunity to diagonalize. We will have a specific sentence φ_n used to distinguish between L^+_n and L^-_n. $L^+_n \models \varphi_n$ and $L^-_n \not\models \varphi_n$. We ensure that the theory of $\widehat{L}$ is not computable and hence $\widehat{L}$ cannot have a computable copy. Thus the process and the switching is simply controlled by an opponent trying to enumerate the theory of $\widehat{L}$ and we wait till the opponent enumerates $\neg\varphi_n$ to switch. The argument is finite injury.

Let T_1,...,T_k be orderings. Then

$$\sigma(T_1, ..., T_k)$$

denotes the *shuffle sum* of $T_1, ..., T_k$ which is defined by taking a copy of $\mathbb{Q}$ and densely putting copies of each of the T_i's in. (So between T_i and T_j there is always a copy of T_q for any i, j, q.)

Now $L_0 = \sigma(1, 2, 3)$. For sequence $1, ..., k$, let $1, ..., \widehat{j}, ..., k$ denote the subsequence obtained by leaving out j. We define $L_1^0 = \sigma(1, 2, 3)$, and define $L_1^1 = \sigma(2, 3)$, $L_1^2 = \sigma(1, 3)$, and $L_1^3 = \sigma(1, 2)$. Finally define

$$L_1^- = \sigma(L_1^0, L_1^1, L_1^2) \text{ and } L_1^+ = \sigma(L_1^0, L_1^1, L_1^2, L_1^3).$$

More generally, we will know that

$$L_{n-1} = \sigma(T_1, ..., T_k).$$

We will let $L_n^0 = L_{n-1}$, and

$$L_n^j = \sigma(T_1, ..., \widehat{T_j}, ..., T_k).$$

Then, as above,

$$L_n^+ = \sigma(L_n^0, ..., L_n^{k-1}) \text{ and } L_n^- = \sigma(L_n^0, ...L_n^k).$$

Notice that in a stage by stage construction we can switch from L_n^- to L_n^+ easily. Also note that each L_n^j misses an L_{n-1}^p for some p. (This is what the T_i's are.) However, each L_{n+1}^w has all the $L_{n_1}^p$'s appearing in arbitary order. One now proves by induction that each L_n^j with $n > 1$ satisfies a certain Π_{n+1} sentence φ_n^j that is not satisfied by any L_n^k for $k \neq j$ nor any shuffle of them. (For instance, φ_1^j says that there is no maximal block of size j.) Second one proves by induction that all the L_n^p are $\equiv_{\Sigma_{n-1}^0}$ to each other or any shuffle of each other. The result will then follow by the diagonalization of the construction, and these two facts. $\square$

Corollary 9.10 (Chisholm and Moses [27]) *There is a structure that is intrinsically n-computable for each $n \in \mathbb{N}$ (i.e. each computable structure isomorphic to it is n-computable) yet is not isomorphic to a decidable structure.*

Proof. Take the ordering $\widehat{L}$ above and add constants c_i for each member of the ordering. $\square$

Returning to orderings and algebras, there are very many other results studying the relationships between specific types and presentation degrees. We mention Goncharov [62] studying superatomic boolean algebras, and Moses [122, 123] who study types of linear orderings with computable successivities. We only quote three further results in this vein. These come from analysing boolean algebras model theoretically. Recall that a structure $\mathcal{A}$ is called *countably prime* if it is countable and $\mathcal{A}$ is elementarily embeddable into every countable model of $Th(\mathcal{A})$. $\mathcal{A}$ is called *countably saturated* if it is countable and ω-saturated, and finally $\mathcal{A}$ is called *countably homogeneous* if for any n and $\{x_1, ..., x_n\}, \{y-1, ..., y_n\} \subset \mathcal{A}$, whenever $\langle \mathcal{A}, x_1, ..., x_n \rangle$ is elementarily equivalent to $\mathcal{A}, y_1, ..., y_n \rangle$, and $a \in \mathcal{A}$, then there exists a $b \in \mathcal{A}$, such that $\langle \mathcal{A}, x_1, ..., x_n, a \rangle$ is elementarily equivalent to $\langle \mathcal{A}, y_1, ..., y_n, b \rangle$.

Theorem 9.11 *(i)* **(Mead [113])** *Every countable prime boolean algebra is isomorphic to a decidable one.*
(ii) **(Morozov [120])** *Every countably saturated boolean algebra is isomorphic to a decidable boolean algebra.*
(iii) **(Morozov [119])** *Define a sequence of ideals of a boolean algebra B as follows. Let $E_1 = E(B)$ be the ideal generated by the atomless and atomic elements of B. More generally, let E_{i+1} denote $\Pi^{-1}(E(B^{(i)}))$ where $B^{(i)} = B/E_i$ and $\Pi : B \mapsto B^{(i)}$ is the canonical map. Then if $\mathcal{A}$ is countably homogeneous and $E_r(\mathcal{A})$ is trivial for some $r \in \mathbb{N}$ then $\mathcal{A}$ is isomorphic to a decidable algebra.*

We finish this section by looking even further into decidability conditions. Here we are even more brief.

We could now ask what happens if we look at computably presentable structures from a complexity theoretical point of view. Now we are interested in the resources we need to perform tasks and we concentrate on time. We assume that the reader is at least familiar with the rudiments of complexity theory. Thus we could, for instance, define a ordering to be P-time if the relation $x \leq y$ is can be decided in time polynomial in $|x| + |y|$ where we usually assume that $x, y \in \{0,1\}^*$. Similarly for other structures. Now the universe becomes very important. For instance, a simple padding argument can be used to show the following.

Theorem 9.12 (Folklore, see e.g. Nerode and Remmal [128],Cenzer and Remmel [23]) *Every linear ordering and boolean algebra is effectively isomorphic to a P-time algebra where the universe A is a P-time subset of $\{0,1\}^*$ upon which $\leq_A$ is P-time.*

However, the theorem is false if we demand that the universe be *all of* $\{0,1\}^*$. Nevertheless using priority arguments, we can show the following.

Theorem 9.13 *(i)* **(Gregorieff [67])** *Every computable linear ordering is isomorphic to one presentable in $DTIME(n)$ and $DSPACE(\log(n))$ with universe $\{0,1\}^*$.*
(ii) **(Cenzer and Remmel [23])** *Every computable boolean algebra is isomorphic to one presentable in $DTIME(n)$ and $DSPACE(\log(n))$ with universe $\{0,1\}^*$.*

The proofs of thses facts are highly nontrivial. It is possible to deduce the second from the first with some work. We also remark that the results are not true for structures in general. It can be that one can have computable copies but none are isomorphic to copies with universe $\{0,1\}^*$ low in the polynomial hierarchy. We refer the reader to Nerode and Remmel [128] and Cenzer and Remmel [23] for more details. We remark that one could go even further and ask that the structures be *automatically* presented. Thus we assume that we have an automaton M such that $x \leq y$ iff M accepts $\langle x, y\rangle$. This turns out to be interesting and natural. We refer the reader to the forthcoming book of Khoussainov and Nerode for more details. (Khoussainov and Nerode [41, 42])

Finally we mention one further question. In the same way that the Ulm invariants classify abelian groups, Ketonen [38] provided a set of invariants for classifying countable abelian groups. the problem is the following.

Question 9.14 *What is the effective content/proof theoretical strength of Ketonen's Theorem.*

10 A Case Study: Torsion Free Abelian Groups.

There are many other structures which we could perform similar analyses to the above. A good area here would be countable ordered groups. We pick one for a short case study. On the surface the one we pick would appear to be fairly innocuous. Torsion free abelian groups and we will mainly concentrate on rank 1. We remark that this is of some interest since groups have function symbol, whereas orderings are only relational structures. Thus one would expect the results for groups to have quite a different flavour than the ones for orderings.

Recall that a group p $(G, ., =)$ is said to have torsion if some element has finite order. (i.e. $x^n = 1$ for some n) and the group is a torsion group if all elements have finite order. We remark that in abelian groups with torsion one can use the Ulm sequence to code a great deal of information. This leads to a well behave degree and jump degree theory analysed in Lin [105, 106], Smith [168], Goncharov [65], and, for jump degrees, by Oates [132]. However, the structure theory and the computability theory become more subtle if we assume that the group is torsion free.

We begin with the arithmetical hierarchy of isomorphism types of presentations of such groups. In the linear ordering case, the hierarchy collapsed to the Σ_n^0 presentable ones. Immediately we see our first significant difference.

Theorem 10.1 (Khisamiev [82]) *If $G = (G, ., =)$ is a Σ_1 presented torsion-free group then G is isomorphic to a computably presented group.*

In the classical language of combinatorial group theory, every recursively presentable torsion free abelian group is isomorphic to one with a solvable word problem.

We note that Khisamiev's Theorem, Theorem 10.1, solves affirmatively a question of Baumslag, Dyer and Miller [10]. The reader should note that the proof of Theorem 10.1 relativizes to show that if G is torsion free abelian and Σ_{n+1} presented, then G is isomorphic to a Δ_{n+1} presented group. In the galley proofs of Khisamiev [82], Khisamiev claimed that Theorem 10.1 could be extended to show that if G is Σ_{n+1} presented, then G is isomorphic to a Π_n^0 presented group. A published proof of this result has not yet appeared, although one will appear in the Handbook of Computable Algebra. Certainly, the claim is correct as we will also prove below.

Theorem 10.2 (Khisamiev, unpubl.) *If G as above is Σ_{n+1}^0 presented then G is isomorphic to a Π_n^0 presented group.*

Thus for torsion free abelian groups, the hierarchy looks like

$$\Delta_1^0 \subseteq \Pi_1^0 \subseteq \Pi_2^0 \subseteq \dots.$$

Later we will give coding proofs that all containments are proper.

Finally the proofs of Theorems 10.1 and 10.2 also establish the following.

Theorem 10.3 (Khisamiev [82])

i) If G is Σ_1^0 presented then G is embeddable as computable subgroup of $V_\infty = \oplus_{n\in\mathbb{N}}\mathbb{Q}$ (Where V_∞ denotes a computable direct sum of countably many copies of the additive group of the rationals $(\mathbb{Q}, +, =)$.)

ii) Similarly if G is Σ_{n+1}^0 presented then G is embeddable as a Π_n^0 subgroup of V_∞.

Proof. To get Theorem 10.1, suppose that G is a given Σ_1^0 presented group. Then $G = F/H$ with F the standard free abelian group on $x_0, x_1, \dots$ and H an computably enumerable normal free subgroup of F. We regard G as an additive group. Note that as G is torsion free, if $nx = a$ then $x \in H$ iff $a \in H$. This fact can be easily used to show that if G has a finite basis, then the given presentation is, in fact, Δ_1^0.

So suppose that G has infinite rank. The obvious idea is to define the map on a basis of G and extend to all of G in the usual. Unfortunately by (e.g.) Metakides and Nerode [116], G may have no computable basis (even if it is a vector space!). Thus the trouble with our strategy is in identifying the basis. Again the key idea is to change the map from time to time, but eventually stabilize.

Thus at stage s, we will have defined a partial apparent basis

$$b_0^s, \dots, b_{n(s)}^s$$

upon which we will have defined our approximation to the isomorphism g on numbers $\leq s$.

The final isomorhism will be to a computable group R with a computable basis, which will be a subgroup of V_∞. (This is actually automatic if R has a computable basis by Downey–Kurtz [43]).

Hence at stage s, we will have defined

$$g_s(b_i^s) = r_i^s \quad \text{with} \quad r_i^s \in V_\infty,$$

and for numbers $y \le s$ if $y = \Sigma \alpha_i b_i^s$ then we define $g_s(y) = \Sigma \alpha_i r_i^s$.

Now as we pass from stage s to stage $s+1$, the problem is that b_i^s may become dependent on $\{b_0^s, \ldots, b_{i-1}^s\}$ mod H_{s+1} with say, $b_i^s = \sum_{j<i} \beta_j b_j^s$. We are then forced to change the map g on b_i^s to map $g_{s+1}(b_i^s) = \sum_{j<i} \beta_j g_s(b_j^s)$. However, we have previously defined

i) $r_i^s \in ra\ g$, and

ii) r_i^s independent of $\{r_0^s, \ldots, r_{i-1}^s\}$.

- Furthermore, we have *promised that (i) and (ii) will be maintained forever.*

Now if G were a vector space (i.e., divisible), but of infinite dimension (rank), then we'd simply chose the *least* $p \in G_u$ we see (for $u \ge s+1$), independent of $\{b_0^s, \ldots, b_{i-1}^s\}$ mod H_{s+1} which exists and define $b_i^{s+1} = p$ so that $g_{s+1}(p) = r_i^s$. Then as G has infinite rank, there is some (least) number q really independent of $\{b_0^s, \ldots, b_{i-1}^s\}$ mod H, this process will clearly ensure that $(\forall i)\ (\exists q)\ (g(q) = r_i^s)$.

Of course we are not dealing with a vector space, but the idea essentially succeeds. We now only need a little more care. We must note that if G is *not* a vector space, we must actually take steps to ensure that all the relevant multiples are in.

The point is that at some stage $\le s$ we may have, say, some z in $ra(g_s)$ with $mz = r_i^s$ for some m.

We must ensure that not only is p independent of $\{b_0^s, \ldots, b_{i-1}^s\}$ mod H_{s+1}, but also $1/np$ is in G.

The solution is straightforward. Choose some large number t (bigger than all numbers previously seen) and set $r_i^{s+1} = 1/tr_i^s$, then choose the least p as before, but now map $g_{s+1}(p) = r_i^{s+1}$ with $p = b_i^{s+1}$. This choice will guarantee that any equation previously declared to have a solution will continue to do so. Again choice of p means that $\lim_s b_i^s = b_i$ exists (as G has infinite rank) and for all x, $\lim_s g_s(x) = g(x)$ exists, and $\lim_s g^{-1}(x)$ exists. □

We turn to Theorem 10.2. By the above in relativized form, we see that if G is Σ_{n+1} presented then $G \cong H$ with G a Δ^0_{n+1} presented group. Again, by relativization and Post's Theorem, to get Theorem 10.2 we only need to show that if G is Δ^0_2 presented then G is isomorphic to a Π^0_1–group.

Thus, suppose G is Δ^0_2. We will again modify the above. For the cognoscenti, we work in the "true stages" of the construction. We regard $=$ as given by $=_s$ and the limit lemma. Thus $x =_s y$ and $x =_{s+1} y$ for a fixed pair x, y can only occur finitely often.

The idea is as follows. Suppose that at stage s_0 we had $g_{s_0}(x) = a$, $g_{s_0}(y) = b$ with x and y appearing independent at stage s. Suppose at some stage $s_1 > s_0$ we see $x =_{s_1} 3y$ (say). As in the previous construction, we'd need to map another $p = p_{s_i}$ to b. This suffices in the previous that even later we may have $x \neq_{s_2} 3y$ for some $s_2 > s_1$. The idea would then be to *return* to our old map $x \to a$, $y \to b$ and any new equalities enumerated based on the *guess* that $x = 3y$ in G would be abandoned and we'd find new pre–images.

The problem is actually a little trickier than above. Note that it might be possible that at all stages s there is some $n(s)$ with $x =_s n(s)y$ (mod H_s) get x is independent of y (mod H) (as $u(s)$ has not limit). To overcome this problem we need "nondeficiency stages". Thus as above, if we see a stage s_3 where for all $t < s_1$, $x \neq_{s_3} tg$¡ then we will declare x to be independent of y at stage s_3 *even if* there are other supposed $n(s_3)$ with $x =_{s_3} n(s_3)y$ by *changing the enumeration of* $=_{s_3}$. The final isomorphism $f_s(y)$ is now given as the value that it assumes infinitely often (this is a Π^0_2 isomorphism). □

One notes that the proof of Theorem 10.2 given here and the one given in Khisamiev [82] are not uniform in the sense that we need to know the rank of G to perform the construction. This is in fact necessary since we can easily use the recursion theorem (see Soare [171]) to prove

Theorem 10.4 *There is no algorithm that uniformly takes a nontrivial Σ^0_1 presented torsion free abelian group to a Δ^0_1 subset of V_∞.*

Proof. Briefly to see that 10.4 holds we must build $A = W_i = W_{i(e)}$ to try to meet the requirement

R_e : If φ_e is total then either $W_{\varphi_c(i)}$ is not isomorphic to A or $W_{\varphi_e(i)} \not\leq V_\infty$.

To meet R_e we start enumerating A a 2–dimensional space with basis $\{e_1, e_2\}$ and at stage s enumerate $\lambda_1 e_1 + \lambda_2 e_2$, etc. for $\lambda_1 \lambda_2 \leq s$. Now we wait till two independent vectors occur in $W_{\varphi_e(i)} \subseteq V_\infty$. When this occurs, we declare e_i to equal e_2 in A (so it is one dimensional). Now if we apply the recursion theorem we delete e on some i. $\square$

In spite of Theorem 10.4, we can resurrect some uniformity in the proof of Theorem 10.1.

Theorem 10.5 *There is a uniform procedure which takes a nontrivial Σ^0_1 presented torsion free abelian G to a $\Delta^0_1 H$ isomorphic to G.*

Proof sketch. The change in the proof of Theorem 10.1 is as follows. At stage s we need to deal with finding that b^s_i is no longer independent of $\{b^s_0, \ldots, b^s_{i-1}\}$. Now we will not guarantee that r^s_i is independent (forever) of $\{r^s_0, \ldots, r^s_{i-1}\}$ only that if we believe r^s_i independent of $\{r^s_0, \ldots, r^s_{i-1}\}$ *at stage s, then for all $\lambda_i \leq s$ we have in $g(G) = R$, $r^s_i \neq \sum_{j<i} \lambda_g r^s_i$.*

So now, if we see b^s_i dependent on $\{b^s_0, \ldots, b^s_{i-1}\}$ we will make r^s_i dependent on $r^s_0, \ldots, r^s_{i-1}$ but will do so via very large scalar. This again can be done consistently. $\square$

It is also possible to extend Theorem 10.2 similarly.

Now we will turn to realizing degrees and jump degrees, etc. We begin by showing the hierarchy for torsion free abelian groups is infinite. In particular, there are torsion free abelian Π^0_1 presented groups G not Δ^0_1 presentable. This was originally an observation of Khisamiev [82] after Khisamiev and Khisamiev [83]). We give a much simpler proof than the complex Khisamiev and Khisamiev one, and show that G can be chosen to have rank 1. (The rank of a group is defined below.) We then explore the notion of jump degree for rank 1 groups.

We remind the reader of some definitions, etc. from Fuchs [58]. Every countable (X–presentable) torsion free abelian group is isomorphic on an (X–presented) subgroup of V_∞. If G is isomorphic to a subgroup of $\mathbb{Q}^n$, then the least such n is called the rank of the group. The simplest and best understood of the torsion free abelian groups are those of rank 1. These are isomorphic to a subgroup of $(\mathbb{Q}, +, =)$.

Thinking in terms of presentations, again we get the the Σ_n^0/Π_n^0-inversion. That is, every Σ_n^0 (Π_n^0) torsion free abelian group is isomorphic to a $\P^0 i_n$ (Σ_n^0) subgroup of $(\mathbb{Q}, +, =)$.

Consequently we really need only consider subgroups of $(\mathbb{Q}, +, =)$ as with the ordering case. We will differentiate between the two notations by talking of a Σ_n^0–presented group and a Σ_n subgroup of Q. Let $p_1, p_2, \ldots$ be the primes in increasing order. The p–height $h_p(a)$ of $a \in G$, a subgroup of $\mathbb{Q}$, is defined via

$$h_p(a) = k \text{ if } k \text{ is greatest with } p^k \mid a \text{ (in } G)$$

and

$$h_p(a) = \infty \text{ if } p^k \mid a \text{ for all } a$$

and the above, for $a \in G$ we define the *characteristic* a, $\chi(a)$ via

$$\chi(a) = (h_{p_1}(a), h_{p_2}(a), \ldots).$$

It is not difficult to see that as G has rank 1, if $a, b \in G$ with $a, b \neq 0$ then $\chi(a)$ and $\chi(b)$ are equivalent in the sense that they differ only finitely often. We write $\chi(a) =^* \chi(B)$. We can thus define the *type of* $\chi(G)$ to be the $=^*$ equivalence class of $\chi(a)$ for any $a \neq 0$ in G. The classical theorem of Baer is

Theorem 10.6 (Baer, see e.g. Fuchs [58])) *If G and H have rank 1 then $G \cong H$ iff G and H have the same type.*

Proof. The proof of Theorem 10.6 is easy. In the nontrivial direction take $g \in G$ and $h \in H$ with $\chi(g) =^* \chi(b)$. From this it follows that $mx = ng$ is solvable in G iff $mx = nh$ is solvable to H. The isomorphism is given by $x \mapsto y$. □

We remark that since, for all $x \in G$, such m, n exist, the isomorphism given by the proof above is effective relative to the presentations. We also note that this means torsion free abelian groups of rank 1 are computably categorical in the sense that $G \cong H$ implies $G \cong_c H$, where $\cong_c$ denotes computable isomorphism. The same is in fact true of finite rank torsion free abelian groups: The work of (e.g.) Nurtazin [131], Goncharov [65], Ash–Nerode [9] or direct constructions show the following.

Theorem 10.7 *A computable torsion free abelian group is computably categorical iff it has finite rank.*

Actually, Theorem 10.7 again follows from the Dzgoev-Goncharov [48] work on branching structures.

Let G be a rank 1 subgroup of $(\mathbb{Q}, +, =)$. Define the *standard type of G* $S(G)$ via

$$S(G) = \{\langle i, j\rangle : j \leq \text{ the } i\text{–th member of } \chi(G)\}$$

for some fixed $a \in G$. Note that if G is X–presented, then $S(G)$ is Σ_1^X. Let A be Σ_2^X. Let $\widehat{A} = \{\langle x, 0\rangle, \langle y, 1\rangle : x \in \mathbb{N}, y \in A\}$. Note that $A \equiv_m \widehat{A}$. Then it is easy to construct an X–presentable G with $S(G) = \widehat{A}$. (In fact given any finite type sequence $T = (a_0, a_1, ...)$, with $a_i < \infty$ for all i, which is given by a function which is Σ_1^X there is a X-presentable group $G(T)$ with this type sequence.) It is now easy to see

Theorem 10.8 *There is a rank 1* Π_{n+1}^0*–presented torsion free abelian group* G *not isomorphic to a* Π_n^0*–presented group.*

Proof. We do the case $n = 0$ and then relativize. Choose A to be Σ_2^0–complete and construct a $\emptyset'$–presentable G with $S(G) = \widehat{A}$. Then by our earlier work we can suppose G is Π_1 and as $S(G)$ is $\Sigma_2^0 - \Sigma_1^0$, by Baer's theorem, G cannot be computably presentable lest $S(G)$ be Σ_1^0. $\square$

We remark that Khisamiev and Khismaiev [83] construct a Π_1^0 computable torsion free abelian group F with a computable basis and a maximal divisible subgroup N of F with F/N in $\Pi_1^0 - \Delta_1^0$. Their construction is much more intricate. We also remark that there is related work by Friedman, Simpson and Smith [57].

Now that we know that the hierarchy is proper, we turn to questions on degrees of presentations. Using a fairly simple argument, Julia Knight [unpubl.] showed that there exist torsion free abelian groups of arbitrary degree. We give the following improvement to rank one groups:

Theorem 10.9 (Downey, after Knight) *Let* **a** *be any degree. Then there exists a torsion free abelian group* G *of rank one with degree* **a**.

Proof. Let S be any set of degree $\mathbf{a}$. Note that if A is any set with $S \oplus \overline{S}$ Σ_1^A then $S \oplus \overline{S} \leq_T A$. Let $G = G(S)$ be the rank one group defined via the type sequence $(a_n)_{n \in N}$ such that $a_n = 1$ if $n \in S \oplus \overline{S}$ and $a_n = 0$ otherwise.

Clearly there is a presentation of G computable in S. Conversely, suppose $H \cong G$. Then H as type sequence $(a_n)_{n \in N}$. The type sequence of H is Σ_1^H, and hence $S \oplus \overline{S}$ is Σ_1^H. But then $S \leq_T H$. $\square$

It remains to investigate if all torsion free abelian groups possess degrees. As we will see, this is not the case, even for rank one groups. In fact even for 1-degrees, we can't say to much much even if we guarantee that the type sequence contains only finite entries. We define a group G has *finite type* if for all i, $|N^{(i)} \cap S(G)| < \infty$. So for instance, the group of Theorem 10.9 has finite type. That is, G has no elements of infinite height. Note that if G has finite type and is X–presentable, then $S(G) \leq_T X'$, and conversely there is some $H \cong G$ with $H' \equiv_T \emptyset' \oplus S(G)$. These observations lead to the following observation.

Observation 10.10 (Downey and Jockusch) *Every torsion free abelian group of finite type and of rank 1 has 1-degree iff the following question has a positive solution.*

- Open Question. *Is it true that for all sets M, the collection of degrees $\{deg(B') : M$ is computably enumerable in $B\}$ has a least element?*

Downey and Jockusch conjecture that the question above has a negative solution. Naturally there is a 2-jump Σ_2^0 version of the observation above for general rank one torsion free abelian groups.

We now see that some G have proper 1–degree and hence not all groups (even of finite type) have degree.

Theorem 10.11 (Downey and Jockusch) *(i) There is a rank 1 G (of finite type) with proper* 1*–degree* $\mathbf{0}'$.
(ii) More generally for every degree $\mathbf{a} \geq \mathbf{0}'$ *there is a torsion free abelian group of proper 1-degree* $\mathbf{a}$.
(iii) Similarly, for all degrees $\mathbf{b} \geq \mathbf{0}''$ *there is a torsion free abelian group of rank 1 with proper 2-degree* $\mathbf{b}$.

Proof. We only do (i) and (iii), since (ii) uses the same argument plus a result on 1-generic sets. Let Q be 1-generic an low. There is a set of the form $\widehat{A}$ as above which is Σ_1^Q but not Σ_1^0. Take a group G of degree $\deg(Q)$ with standard type $\widehat{A}$. Then G cannot be computably presentable lest $\widehat{A}$ be Σ_1^0.
For (iii) choose Q this time to be 2-generic with double jump $\mathbf{b}$. Notice that the type of a rank one group is Π_2^G. Take a type sequence χ which is Π_2^Q so that the collection of sets X with $\Pi_2^X = \chi$ has no least jump. (See Ash, Jockusch and Knight [6] or Downey [35] for this sort of argument.) $\square$

At this point we close, but finish with the remark that it is not clear what n-jump degrees proper are possible for (rank one) torsion free abelian groups. Also an exact analysis of, say, pure, or divisible subgroups would seem to be of some interest. We also remark that there are a whole complex of questions regarding orderable (and hence torsion free) groups. For instance the following seem are open.

Question 10.12 (Downey and Kurtz [43]) *(i) Is every orderable computably presented group isomorphic to a a computably ordered computably presented group? (ii) Every countable ordered group is has a presentation as a ordered version of a free group factored by a convex normal subgroup.* (Neumann [129]). *What is the effective content of this theorem?*

We refer the reader to Kokorin and Kopytov [95] for more of the model theory of ordered groups.

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