EFFECTIVELY CATEGORICAL ABELIAN GROUPS

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Abstract. We study effective categoricity of computable abelian groups of the form $\bigoplus_{i\in\omega} H$, where H is a subgroup of $(Q, +)$. Such groups are called homogeneous completely decomposable. It is well-known that a homogeneous completely decomposable group is computably categorical if and only if its rank is finite.

We study Δ^0_n-categoricity in this class of groups, for $n > 1$. We introduce a new algebraic concept of S-independence which is a generalization of the well-known notion of p-independence. We develop the theory of P-independent sets. We apply these techniques to show that every homogeneous completely decomposable group is Δ^0_3-categorical.

We prove that a homogeneous completely decomposable group of infinite rank is Δ^0_2-categorical if and only if it is isomorphic to the free module over the localization of Z by a computably enumerable set of primes P with the semi-low complement (within the set of all primes).

Finally, we apply these results and techniques to study the complexity of generating bases of computable free modules over localizations of integers, including the free abelian group.

Keywords: abelian groups and modules, computable model theory, effective categoricity

1. Introduction

1.1. **Computable structures and effective categoricity.** Remarkably, the study of effective procedures in group theory pre-dates the clarification of what is meant by a computably process; beginning at least with the work of Max Dehn in 1911 ([8]) who studied word, conjugacy and isomorphisms in finitely presented groups. While the original questions concerned themselves with finitely presented groups, it turned out that they were intrinsically connected with questions about infinite presentations with computable properties. In [22], Graham Higman proved what is now called the Higman Embedding Theorem ([22]) which stated that a finitely generated group could be embedded into a finitely presented one iff it had a computable presentation (in a certain sense).

The current paper is centered in the line of research of effective procedures in computably presented groups. By computable group, we mean groups where the domian is computable and the algebraic operation is computable upon that domain.

Such studies can be generalized to other algebraic structures such as fields, rings, vector spaces and the like, a tradition going back to Grete Herrmann [21], van ver Waerden [44], and explicitly using computability theory, Rabin [40], Maltsev [32] and Frölich and Shepherdson [17].

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More generally, computably presentable algebraic structures are the main objects of study in computable model theory and effective algebra. Recall that for an infinite countable algebraic structure $\mathcal{A}$, a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is called a *computable presentation of* $\mathcal{A}$ if the domain of $\mathcal{B}$ is (coded by) $\mathbb{N}$, and the atomic diagram of $\mathcal{B}$ is a computable set. If a structure has a computable presentation then it is *computably presentable.* In the same way that *isomorphism* is the canonical classification tool in classical algebra, when we take presentations into account, *computable* isomorphism becomes the main tool. Now two presentations are regarded as the same if the agree up to computable isomorphism. However, an infinite computably presentable structure $\mathcal{A}$ may have many of different computable presentations. Such differing presentations reflect differing computational properties. For example, a computable copy of the order type of the natural numbers might have the successor relation computable (as the familiar presentation does), whereas another might have this successor relation non-computable. Such copies cannot be computably isomorphic.

An infinite countable structure $\mathcal{A}$ is *computably categorical* or *autostable* if every two computable presentations of $\mathcal{A}$ have a computable isomorphism between them. This would mean that the computability-theortical properties of every copy are identical. Cantor's back-and-forth argument shows that the dense linear ordering without endpoints forms a computably categorical structure. Computable categoricity is one of the central notions of computable model theory (see [15] or [3]). For certain familiar classes of structures we can characterize computable categoricity by algebraic invariants. For instance, a computably presentable Boolean algebra is computably categorical exactly if it has only finitely many atoms ([19], [29]), a computably presentable linear order is computably categorical if and only if it has only finitely many successive pairs [41], and a computably presentable torsion-free abelian group is computably categorical if and only if its rank is finite ([20], [39]).

Computably categorical structures tend to be quite rare, and it is natural to ask the question of how close to being computably categorical a structure is. As mentioned above, we know that a linear ordering or order type $\mathbb{N}$ is not computably categorical since there is the canonical example where the successor relation is computable, and another where the successor relation is not. But if we are given an oracle for the successor relation, the then structure is computably categorical *relative* to that. The halting problem would be enough to decide whether y is the successor of x in such an ordering. This motivates the followin definition.

We say that a structure $\mathcal{A}$ is Δ^0_n-categorical if every two computable presentations of $\mathcal{A}$ have an isomorphism between them which is computable with oracle $\emptyset^{(n-1)}$, where $\emptyset^{(n-1)}$ is the (n-1)-th iteration of the Halting problem. Once computably categorical structures in a given class are characterized, it is natural to ask which members of this class are Δ^0_2-categorical. Here the situation becomes more complex. There are only few results in this area, most of them are partial. For instance, McCoy [34] characterizes Δ^0_2-categorical linear orders and Boolean algebras under some extra effectiveness conditions. Also it is known that in general Δ^0_{n+1}-categoricity does not imply Δ^0_n-categoricity in the classes of linear orders [2], Boolean algebras [3], abelian p-groups [5], and ordered abelian groups [36].

Our goal will be to give such a higher level classification of effective categoricity for a certain basic class of torsion-free abelian groups.

1.2. **Effective categoricity of torsion-free abelian groups.** We study Δ^0_2-categorical and Δ^0_3-categorical torsion-free abelian groups. Recall that an abelian group is torsion-free if every nonzero element of this group is of infinite order.

Question. *Which computably presentable torsion-free abelian groups are Δ^0_n-categorical, for $n \geq 2$?*

It is not even clear how to build an example of an Δ^0_{n+1}-categorical but not Δ^0_n-categorical torsion-free abelian group, for each $n > 2$. As with the classical theory of torsion-free abelian groups, genral questions about isomorphism classes are either extremely difficult or (in a sense described below) impossible. The main difficulty is the absence of satisfactory invariants for computable torsion-free abelian groups which would characterize these groups up to isomorphism [14]. For instance, Downey and Montalbán showed that the isomorphism problem for computable torsion-free abelian groups is Σ^1_1-complete. To say that a problem is Σ^1_1 means that it can be expressed as $\exists f \forall n R(f(n))$ where here the existential quantification is over *functions*, and R is a computable relation. To say that an isomorphism problem is Σ^1_1-complete means that you cannot make the isomorphism problem any simpler, and hence there are no invariants (like dimension) other than the isomorphism type. Therefore, there cannot be a set of invariants which make the complexity of the problem any simpler.

There are better understood subclasses of the torsion-free abelian groups such as the rank one groups, the additive subgroups of the rationals. As we remind the reader in the next section, these groups have a nice structure theory via Baer's theory of *types* (Baer [4]). This theory can be extended to groups that are of the form $\oplus_i H_i$ where each H_i has rank 1, a class called the *completely decomposable* groups. As is well-known, Baer's theory extends to this class so we would have some hope of understanding the computable algebra in this setting.

For the present paper, we restrict ourselves to a natural subclass, the *homogeneous completely decomposable* groups which are countable direct powers of a subgroup of the rationals. More formally, we consider the groups of the form $\bigoplus_{i\in\omega} H$, where H is an additive subgroup of $(Q, +)$. These groups in the classical setting were first studied by Baer [4]. The class of homogeneous completely decomposable groups of rank ω is certainly the simplest and most well-understood class of torsion-free abelian groups of infinite rank. Note that, from the computability-theoretic point of view, this is the simplest possible non-trivial case we may consider: every torsion-free abelian group of finite rank is computably categorical. As we will see, even in this classically simplest case the complete answer to the problem does not seem to be straightforward.

To understand the effective cagericity of these groups, we will need both new uses of computability theory in the study of torsion-free abelian groups, and some new algebraic structure theory, as described in the next section.

1.3. **A new algebraic notion, and Δ^0_3-categoricity.** To study effective categoricity of homogeneous completely decomposable groups, we introduce a new purely algebraic notion of *S-independence*, where S is a *set* of primes. This is a generalization of the well-known notion of p-independence for a single prime p. In the theory of primary abelian groups, p-independence plays an important role. See Chapter VI of [18] for the theory of p-independent sets and p-basic subgroups. We establish several technical facts about S-independent subsets of homogeneous completely decomposable groups. These facts are of independent interest from the purely algebraic point of view. For instance, Theorem 4.10 essentially shows that S-independence and free modules over a localization of Z play a similar role in the theory of completely decomposable groups as p-independence and p-basic subgroups do in the theory of primary abelian groups.

This paper essentially studies the effective content of S-independence. We will observe that S-independence in general implies linear independence. Effective content of linearly independent sets was studied in the theory of computable vector spaces (see, e.g., [38]). The notion of S-independence seems to be an adequate replacement of linear independence in the case of free modules over a localization of Z (see Lemma 4.4).

We apply the algebraic techniques developed for S-independent sets to establish an upper bound on the complexity of isomorphisms.

Theorem. *Every homogeneous completely decomposable group is Δ^0_3 - categorical.*

This result is sharp: there exist homogeneous completely decomposable groups which are not Δ^0_2-categorical so that we cannot replace Δ^0_3 by Δ^0_2. Also, a homogeneous completely decomposable group of rank ω is never computably categorical (folklore). It is natural to ask for a necessary and sufficient condition for a homogeneous completely decomposable group to be Δ^0_2-categorical. Remarkably, there is a natural condition on the group classifying exactly when this happens.

1.4. **Free modules, semi-low sets, and Δ^0_2-categoricity.** Certain homogeneous completely decomposable groups may be viewed as free modules over localizations of integers by sets of primes. More specifically, let P be a set of primes which is not the set of all primes, and let $Q^{(P)}$ be the additive subgroup of the rationals $(Q, +)$ generated by fractions of the form $\frac{1}{p^m}$, where $p \in P$ and $m \in \omega$. Let G_P be the direct sum of countably many presentations of $Q^{(P)}$: $G_P = \bigoplus_{i\in\omega} Q^{(P)}$. Baer [4] showed that the classical isomorphism of a homogeneous completely decomposable group $\bigoplus_{i\in\omega} H$ is determined by the *characteristic* of H (see Definition 2.4). If the reader is familiar with the concept of *characteristic*, then she or he may observe that a characteristic consisting of only ∞ and 0 correspond to a group of the form G_P. We characterize the case where a computable completely decomposable homogeneous group is Δ^0_2 categorical via a combination of an algebraic (the group must be of the form G_P) and a mild effectiveness consideration (the complement of the corresponding set P is *semi-low*). That is, P must resemble a computable set in the sense that it has a weak guessing procedure for membership, called *semi-lowness.*

We say that a set S is *semi-low* if the set $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable in the Halting problem. As the name suggests (for c.e. sets) this is weaker than being low (meaning that $A' \equiv_T \emptyset'$, since every low c.e. set is one with a semi-low complement, but not conversely (see Soare [42, 43]). Semi-low sets are connected with the ability to give a fastest enumeration of a computably enumerable set as discovered by Soare [42]. In that paper, Soare showed that if **a** is a c.e. degree which is nonlow, then it contains a c.e. set whose complement is not semi-low. Semi-low sets also appear naturally when one studies automorphisms of the lattice $\mathcal{E}$ of computably enumerable sets under the set-theoretical inclusion. Soare (see, e.g., [43], Theorem 1.1 on page 375) showed that if a c.e. set S has a semi-low complement, then the lattice of all c.e. sets is isomorphic to the principle filter

$\mathcal{L}(S)$ of c.e. supersets of S. Furthermore, if a c.e. set S has a semi-low complement, then $\mathcal{L}(A)/\mathcal{F}$ is effectively isomorphic to $\mathcal{E}/\mathcal{F}$, where $\mathcal{F}$ stands for the ideal of finite sets. There exist variations of semi-lowness which appear naturally in the study of lattice-theoretic properties of c.e. sets. We say that a set S is semi-low$_{1.5}$ if $\{e : W_e \cap S \text{ is finite}\}$ is computable in $\emptyset''$. Maass [31] showed that if A is c.e. and coinfinite, then $\mathcal{L}(A)/\mathcal{F}$ is effectively isomorphic to $\mathcal{E}/\mathcal{F}$ if and only if $\overline{A}$ is semi-low$_{1.5}$. For more information about semi-low and semi-low$_{1.5}$ sets see [43]. We mention that a c.e. degree is low if and only if it contains only semi-low$_{1.5}$ c.e. set [11].

It is rather interesting that semi-lowness appears in the characterization of Δ^0_2-categorical abelian groups:

Theorem. *A computable homogeneous completely decomposable group A of rank ω is Δ^0_2 - categorical if and only if A is isomorphic to G_P, where P is a c.e. set of primes such that $\{p : p \text{ prime and } p \notin P\}$ is semi-low.*

In particular, if P is low, then G_P is Δ^0_2 categorical. As far as we know, this is the first application of semi-low sets in effective algebra. Also, the proof of Theorem above is of some technical interest as it splits into several cases depending on the manner by which the type of the group A is enumerated. The flavour of this proof is that of the "limitwise monotonic" proofs in the literature but is a lot more subtle. The method has a number of new ideas which would seem to have other applications.

1.5. **A coding, and further applications.** Note that the map $P \to G_P$ gives an effective coding of a computably enumerable set of primes into a computable abelian group. Furthermore, P defines G_P uniquely up to isomorphism.

Before we pass to the next result, we briefly discuss similar codings of sets into isomorphism types of various classically simple structures. Effective content of such codings have been intensively studied in recent years. In the theory of computable abelian groups, at least two examples of this kind should be mentioned. See [10] for similar examples in the class of linear orders which led to the notions of η-presentable sets and strongly η-presentable sets.

The first example is the coding of a given set of primes S into the abelian group $G(S) = \bigoplus_{p\in S} Q^{(\{p\})}$, where $Q^{(\{p\})}$ was defined above. Khisamiev [25] showed that $G(S)$ has a computable representation with a certain strong basis exactly if the set S belongs to a certain proper subclass of non-hh-immune Σ^0_2-sets. Khisamiev also asked for a necessary and sufficient condition for the group $G(S)$ to have a computable (decidable) presentation. Downey, Goncharov, Knight et al. [12] showed that $G(S)$ has a computable (decidable) presentation if and only if S is Σ^0_3 (Σ^0_2). Although the group is classically simple, the proof is not straightforward.

The second example of this kind is the coding of a given set of natural numbers S into the abelian p-group which is the direct sum of cyclic groups of orders p^s, one component for each s. Khisamiev [24] showed that this group has a computable presentation if and only if the set S has an effective monotonic approximation from below. Such sets are often called limitwise monotonic [26]. Khisamiev built an example of a Δ^0_2 set which has no such a monotonic approximation ([24]; see [26] for an alternate proof). Limitwise monotonic sets have applications in other fields of computable model theory ([26], [23] and [6]), and have connections to degree theory [13]. This example also illustrates that the arithmetical complexity does not always reflect the needed effective properties of abelian groups.

We observe that the following are equivalent: (1) G_P is computably presentable, (2) G_P is computably presentable as a module over $Q^{(P)}$ (to be specified), (3) the set of primes P is computably enumerable. See Proposition 3.6 for the proof. Nonetheless, the complexity of a c.e. set P is reflected in G_P via the complexities of possible isomorphisms between computable presentations of G_P. As a consequence of the main results of the paper, we have:

Theorem. *For a c.e. set P of primes, the group G_P is Δ^0_2-categorical if and only if $\widehat{P} = \{p : p \text{ prime and } p \notin P\}$ is semi-low.*

This gives an characterization of semi-low co-c.e. sets in terms of effective algebra. Using the techniques of the paper one can easily show that the weak jump $H_{\widehat{P}}$ of the complement of P (within the set of all primes) computes some isomorphism between any two computable copies of G_P. It is also not hard to show that $H_{\widehat{P}}$ is indeed the *degree of categoricity* of G_P, for every c.e. P (see [16] for the definition and for more about degrees of categoricity). Although we do not develop this subject any further, we note that this is the first natural example of an algebraic structure having the weak jump of an encoded set as its degree of categoricity. It also follows from our observation and well-known facts about semi-low sets (see, e.g., [43], pp. 72-73) that a c.e. degree is high if and only if it contains a c.e. set of primes P such that the group G_P has two computable copies with an isomorphism between these copies which computes $0''$. This shows we can not improve the upper bound on the complexity of isomorphisms: every homogeneous completely decomposable group is Δ^0_3-categorical, and this is the best we can get even for the groups of the form G_P.

We also apply the main results of the paper to study the complexity of the bases of G_P which generate it as a free module over $Q^{(P)}$. We will see that effective categoricity of G_P can be equivalently reformulated in terms of bases. Our interest is also motivated by the recent results on computable free *non-abelian* groups. More specifically, the computational complexities of sets of generators in free *non-abelian* groups were studied in [7] and [30]. We show:

Theorem. *If a computable presentation of G_P has a Σ^0_2 basis which generates it as a free $Q^{(P)}$-module, then this presentation possesses a Π^0_1 basis which generates it as a free $Q^{(P)}$-module.*

As a consequence of this theorem and the main results of the paper, if $\{p : p \text{ prime and } p \notin P\}$ is semi-low, then G_P has a Π^0_1 basis which generates it as a free $Q^{(P)}$-module. Thus, every computable copy of the free abelian group has a Π^0_1-basis of generators. This is sharp (folklore).

1.6. **The structure of the paper.** First, we give some background on the general theory of computable torsion-free abelian groups. Then we develop a bit of the algebraic theory of S-independent sets. Next, we apply this theory to study effective categoricity of homogeneous completely decomposable groups. We conclude the paper by open problems.

2. Algebraic preliminaries

We use known definitions and facts from computability theory and the theory of abelian groups. Standard references are [43] for computability and [18] for the theory of torsion-free abelian groups. We will see that for our purposes we don't need to use a more complicated two-sorted signature of modules (Proposition 3.6). However, we will use a notation that substitutes the module multiplication (Notation 2.10). Basics of module theory can be found in any classical book on general algebra (see, e.g., [28]).

Definition 2.1 (Linear independence and rank)**.** Elements $g_0, \dots, g_n$ of a torsion-free abelian group G are *linearly independent* if, for all $c_0, \dots, c_n \in Z$, the equality $c_0 g_0 + c_1 g_1 + \ldots + c_n g_n = 0$ implies that $c_0 = c_1 = \ldots = c_n = 0$. An infinite set is *linearly independent* if every finite subset of this set is linearly independent. A maximal linearly independent set is a *basis*. All bases of G have the same cardinality. This cardinality is called the *rank* of G.

We write $A \leqq B$ to denote that A is a subgroup of B. It is not hard to see that a torsion-free abelian group A has rank 1 if and only if $A \leqq \langle Q, + \rangle$.

Definition 2.2 (Direct sum)**.** An abelian group G is the *direct sum* of groups A_i, $i \in I$, written $G = \bigoplus_{i \in I} A_i$, if G can be presented as follows:

(1) The domain consists of infinite sequences $(a_0, a_1, a_2, \dots, a_i, \dots)$, each $a_i \in A_i$, such that the set $\{i : a_i \neq 0\}$ is finite.
(2) The operation $+$ is defined component-wise.

The groups A_i are the *direct summands* or *direct components* of G (with respect to the given decomposition). Note that there may be lots of different ways to decompose the given subgroup. One can check that $G \cong \bigoplus_{i \in I} A_i$, where $A_i \leqq G$, if and only if (1) $G = \sum_{i \in I} A_i$, i.e. $\{A_i : i \in I\}$ generates G, and (2) for all j we have $A_j \cap \sum_{i \in I, i \neq j} A_i = \{0\}$.

We write $k|g$ *in* G (or simply $k|g$ if it is clear from the context which group is considered) and say that k *divides* g *in* G if there exists an element $h \in G$ for which $kh = g$, and we say that h is a k*-root* of g. Note that $k|g$ is simply an abbreviation for the formula $(\exists h)(\underbrace{h + h + \ldots + h}_{k \text{ times}} = g)$ in the signature of abelian groups.

If the group G is torsion-free then every $g \in G$ has at most one k-root, for every $k \neq 0$. Assume there were two distinct k-roots, h_1 and h_2, of an element g. Then $k(h_1 - h_2) = 0$ would imply $h_1 = h_2$, a contradiction.

Definition 2.3 (Pure subgroups and [X])**.** Let G be a torsion-free abelian group. A subgroup A of G is called *pure* if for every $a \in A$ and every n, $n|a$ in G implies $n|a$ in A. For any subset X of G we denote by $[X]$ the least pure subgroup of G that contains X.

For instance, every direct summand of a given group G is pure in G, while the converse is not necessarily the case.

Let us fix the canonical listing of the prime numbers:

$$p_1, p_2, \dots, p_n, \dots.$$

Definition 2.4 (Characteristic and h_p)**.** Suppose G is a torsion-free abelian group. For $g \in G$, $g \neq 0$, and a prime number p, set

$$h_p(g) = \begin{cases} \max\{k : p^k | g \text{ in } G\}, \text{if this maximum exists,} \\ \infty, \text{otherwise.} \end{cases}$$

The sequence $\chi_G(g) = (h_{p_1}(g), h_{p_2}(g), \dots)$ is called the *characteristic* of the element g in G.

Thus, for a torsion-free groups G, a subgroup H of G is a pure subgroup of G if and only if $\chi_H(h) = \chi_G(h)$ for every $h \in H$.

Definition 2.5. Let $\alpha = (k_1, k_2, \dots)$ and $\beta = (l_1, l_2, \dots)$ be two characteristics. Then we write $\alpha \leq \beta$ if $k_i \leq l_i$ for all i, where ∞ is greater than any natural number.

Definition 2.6 (Type)**.** Two characteristics, $\alpha = (k_1, k_2, \dots)$ and $\beta = (l_1, l_2, \dots)$, are *equivalent*, written $\alpha \simeq \beta$, if $k_n \neq l_n$ only for finitely many n, and k_n and l_n are finite for these n. The equivalence classes of this relation are called *types*.

We write $\mathbf{t}(g)$ for the type of an element g. If $G \leq \langle Q, + \rangle$ (equivalently, if G has rank 1) then all non-zero elements of G have equivalent types, by the definition of rank. Hence, we can correctly define the type of G to be $\mathbf{t}(g)$ for a non-zero $g \in G$, and denote it by $\mathbf{t}(G)$. The following theorem classifies torsion-free abelian groups of rank 1:

Theorem 2.7 (Baer [4])**.** *Let G and H be torsion-free abelian groups of rank* 1*. Then G and H are isomorphic if and only if $\mathbf{t}(G) = \mathbf{t}(H)$.*

The next simplest class of torsion-free abelian groups is the class of *homogeneous completely decomposable* groups.

Definition 2.8 (Completely decomposable group)**.** A torsion-free abelian group is called *completely decomposable* if G is a direct sum of groups each having rank 1. A completely decomposable group is *homogeneous* if all its elementary summands are isomorphic.

It is known that any two decompositions of a completely decomposable group into direct summands of rank 1 are isomorphic. Also, two homogeneous completely decomposable groups are isomorphic if and only if these groups have the same type [4]. We will refer to this fact by citing Theorem 2.7 since it is a straightforward consequence of this theorem [18]. For instance, a set of primes P defines the group G_P uniquely up to isomorphism.

Definition 2.9. Suppose G is a torsion-free abelian group, g is an element of G, and $n|g$ some n. If $r = \frac{m}{n}$ then we denote by rg the (unique) element mh such that $nh = g$.

Notation 2.10. Let G be an abelian group and $A \subseteq G$. Suppose $\{r_a : a \in A\}$ is a set of (rational) indices. If we write $\sum_{a \in A} r_a a$ then we assume that $r_a a \neq 0$ for at most finitely many $a \in A$, and every element $r_a a$ is well-defined in G, according to Definition 2.9. We will use this convention without explicit reference to it.

Now suppose $R \leqq \langle Q, + \rangle$, and $A \subseteq G$. We denote by $(A)_R$ the subgroup of G (if this subgroup exists) generated by $A \subset G$ over $R \leqq Q$, i.e. $(A)_R = \{\sum_{a \in A} r_a a : r_a \in R\}$.

Finally, for $R \leqq Q$ and $a \in G$, we denote by Ra the subgroup $(\{a\})_R$ of G.

Let $R \leqq Q$. If a set $A \leqq G$ is linearly independent then every element of $(A)_R$ has the unique presentation $\sum_{a \in A} r_a a$. Otherwise we would have $\sum_{a \in A} r_a a = 0$ for some set of rational indices $\{r_a : a \in A\}$, and thus $m \sum_{a \in A} r_a a = \sum_{a \in A} m r_a a = 0$, for some integer m such that $m r_a \in Z$ for all $a \in A$, contrary to our hypothesis. Therefore, $(A)_R = \bigoplus_{a \in A} Ra$ for every linearly independent set A.

3. Computable abelian groups and modules

The notion of a c.e. characteristic is one of the central notions of computable abelian group theory.

Definition 3.1. Let $\alpha = (h_i)_{i\in\omega}$, where $h_i \in \omega \cup \{\infty\}$ for each i, be a characteristic. We say that α is c.e. if the set $\{\langle i,j\rangle : j \leq h_{p_i}, h_{p_i} > 0\}$ is c.e. (see [37]). This is the same as saying that there is a non-decreasing uniform computable approximation $h_{i,s}$ such that $h_i = \sup_s h_{i,s}$, for every i. Observe that this is a type-invariant property. Thus, a type $\mathbf{f}$ is c.e. if α is c.e., for every α in $\mathbf{f}$ (equivalently, for some α in $\mathbf{f}$).

Theorem 3.2 below was rediscovered several times by various mathematicians including Knight, Downey, and others (see, e.g., [9]).

Theorem 3.2 (Mal′tsev [33]). *Let G be a torsion-free abelian group of rank 1. Then the following are equivalent:*

(1) The group G has a computable presentation.

(2) The type $\mathbf{t}(G)$ is c.e.

(3) The group G is isomorphic to a c.e. additive subgroup R of a computable presentation of the rationals $(Q, +, \times)$. Furthermore, we may assume that $1 \in R$.

Furthermore, each c.e. type corresponds to some computably presented subgroup of the rationals. See [37] for a proof. If a group G is homogeneous completely decomposable then $\mathbf{t}(G)$ is also well-defined. The (1) $\leftrightarrow$ (2) part of Theorem 3.2 can be easily generalized to the class of homogeneous completely decomposable groups:

Proposition 3.3. *A homogeneous completely decomposable group G has a computable presentation if and only if $\mathbf{t}(G)$ is c.e.*

See [37] for more details.

Definition 3.4. We say that C is a computable presentation of a module M over a ring R if

(1) the ring R is isomorphic to a c.e. subring R_1 of a computable ring R_2,

(2) C is a computable presentation of M as an abelian group, and

(3) there is a computable function $f : R_2 \to C$ which maps (r, m) to $r \cdot m \in C$, for every $m \in C$ and $r \in R_1$.

Recall that $Q^{(P)}$ is the subgroup of the rationals $(Q, +)$ generated by the set of fractions $\{\frac{1}{p^k} : k \in \omega \text{ and } p \in P\}$.

Remark 3.5. According to Definition 2.9, for every $r = \frac{m}{n} \in Q^{(P)}$ and a an element of the *group* G_P, the element $ra \in G_P$ is definable by a formula $\Phi_r(x, a) \leftrightharpoons mx = na$ in the language of abelian groups (recall that mx and na are abbreviations).

Proposition 3.6. *The following are equivalent:*

(1) *P is c.e.*

(2) *$Q^{(P)}$ is a c.e. subring of a computable presentation of $(Q, +, \times)$.*

(3) *G_P is computably presentable as an abelian group.*

(4) *G_P is computably presentable as a module over $Q^{(P)}$.*

Proof. The implications (1) $\rightarrow$ (2) and (2) $\rightarrow$ (3) are obvious.

(3) $\rightarrow$ (4). By Proposition 3.3, the characteristic α of G_P is c.e. By Theorem 3.2, $Q^{(P)}$ is isomorphic to a c.e. additive subgroup A of $(Q, +, \times)$. Observe that $Q^{(P)}$ may be considered as a c.e. *subring* of Q, because we can ensure that $1 \in A$. It remains to observe that for each element $g \in G_P$ and each rational $r \in Q_P$, the element rg can be found effectively and uniformly.

(4) $\rightarrow$ (1). Pick an element g of G_P which is divisible by a prime p if and only if $p \in P$. Thus, $p \in P$ if and only if $(\exists x \in G_P)[px = g]$, proving that P is c.e.

□

Remark 3.7. Actually we have shown that *every computable presentation of G_P is already a computable presentation of G_P as a module over $Q^{(P)}$.*

Lemma 3.8. *For a c.e. set of primes P, the following are equivalent:*

(1) *Every computable presentation of the group G_P has a Σ^0_n basis which generates this presentation as a module over $Q^{(P)}$.*
(2) *The group G_P is Δ^0_n-categorical.*
(3) *The $Q^{(P)}$-module G_P is Δ^0_n-categorical.*

Proof. By Proposition 3.6, the *ring* $Q^{(P)}$ is a c.e. subring of a computable presentation of $(Q, +, \times)$.

(1) $\rightarrow$ (2). Let A and B be computable presentations of the group G_P. Both A and B have Σ^0_n bases which generate these groups over $Q^{(P)}$. We map these bases one into another using $0'$. By Remark 3.5, we can extend this map to an isomorphism effectively, using the c.e. subringring $Q^{(P)}$ of Q.

(2) $\rightarrow$ (3). Observe that every computable *group-isomorphism* between to computable module-presentations of G_P is already a computable *module-isomorphism.*

(3) $\rightarrow$ (1). Pick a computable presentation H of G_P such that the basis which generates H over $Q^{(P)}$ is computable. If G_P is Δ^0_n-categorical then every computable presentation of G_P has a Σ^0_n basis which is the image of the computable one in H.

□

Thus, from the computability-theoretic point of view, G_P may be alternatively considered as an abelian group or a $Q^{(P)}$-module.

4. S-independence and excellent S-bases.

The notion of p-independence (for a single prime p) is a fundamental concept in abelian group theory (see [18], Chapter VI). We introduce a certain generalization of p-independence to *sets* of primes:

Definition 4.1 (S-independence and excellent bases)**.** Let S be a set of primes, and let G be a torsion-free abelian group. If $S \neq \emptyset$, then we say that elements $b_1, \ldots, b_k$ of G are S-independent in G if $p | \sum_{i \in \{1,\ldots,k\}} m_i b_i$ in G implies $\bigwedge_{i \in \{1,\ldots,k\}} p | m_i$, for all integers $m_1, \ldots, m_k$ and $p \in S$. If $S = \emptyset$, then we say that elements are S-independent if they are simply linearly independent.

Every maximal S-independent subset of G is said to be an *S-basis* of G. We say that an S-basis is *excellent* if it is a maximal linearly independent subset of G.

It is easy to check that S-independence in general implies linear independence. However, an S-basis does not have to be excellent. Lemma 35.1 in [18] implies that *the free abelian group of rank ω contains a $\{p\}$-basis which is not excellent.*

The main reason why we introduce the notion of S-independence is reflected in the example and the lemma below.

Example 4.2. Let Z^2 be the free abelian group of rank 2, and let e_1 and e_2 be such that $Z^2 = Ze_1 \oplus Ze_2$. Suppose that we need to *test*, given a pair of elements g_1 and g_2, if $Zg_1 + Zg_2 = Z^2$. That is, we wish to be able to say "no" if g_1 and g_2 do not generate Z^2. If g_1 and g_2 together generate the group, then $\{g_1, g_2\}$ should be linearly independent. But this is not sufficient: suppose that $g_1 = 2e_0 + e_1$ and $g_2 = e_1$; then $2|g_1 - g_2$, but the element $h = \frac{g_1 - g_2}{2}$ is not in the span of $\{g_1, g_2\}$.

Now we make each Z-component of Z^2 infinitely divisible by 2 and consider the group $Q^{(2)}e_1 \oplus Q^{(2)}e_2$. Note that $2|g_1 - g_2$ in $Q^{(2)}e_1 \oplus Q^{(2)}e_2$, but it is not a problem: it is easy to check that $\{g_1, g_2\}$ generates $Q^{(2)}e_1 \oplus Q^{(2)}e_2$ over $Q^{(2)}$. In contrast, the elements $h_1 = 3e_0 + e_1$ and $h_2 = e_1$ fail to generate $Q^{(2)}e_1 \oplus Q^{(2)}e_2$ over $Q^{(2)}$.

More generally, in $Q^{(P)}e_1 \oplus Q^{(P)}e_2$, *the existence of p-roots for $p \in P$ can not be used to test if two given elements generate the whole group over $Q^{(P)}$ or not.*

Notation 4.3. In this section P stands for a set of primes and $\widehat{P}$ for the complement of P within the set of *all* primes:

$$\widehat{P} = \{p : p \text{ is prime and } p \notin P\}.$$

Lemma 4.4. *Suppose $G \cong \bigoplus_{i \in I} Q^{(P)}$, and let $B \subseteq G$. Then B is an excellent $\widehat{P}$-basis of G if and only if B generates G as a free module over $Q^{(P)}$.*

Let $\mathcal{P}$ be the set of *all* primes. Then $\widehat{\mathcal{P}} = \emptyset$. Recall that $\emptyset$-independence is simply linear independence, and $G_{\mathcal{P}} \cong D(\omega) = \bigoplus_{i \in \omega} Q$. It is well-known that every maximal linearly independent set generates the vector space $D(\omega)$ over Q. If $P = \emptyset$ then $G_\emptyset \cong FA(\omega) = \bigoplus_{i \in \omega} Z$ is the free abelian group of the rank ω. As a consequence of the lemma, every excellent $\mathcal{P}$-basis of $FA(\omega)$ generates $FA(\omega)$ as a free abelian group.

Proof. $(\Rightarrow)$. Let B be an excellent $\widehat{P}$-basis of G. Suppose $g \in G$. By our assumption, B is a basis of G. Therefore, there exist integers m and m_b, $b \in B$, such that $mg = \sum_b m_b b$. Suppose $m = pm'$ for some $p \in \widehat{P}$. By Definition 4.1, $p|m_b$ for all $b \in B$. Therefore, without loss of generality, we can assume that $(m, p) = 1$, for every $p \in \widehat{P}$. By the definition of G, we have:

$$g = \sum_b \frac{m_b}{m} b \in (B)_{Q^{(P)}} \leqq G.$$

The set B is linearly independent, therefore $(B)_{Q^{(P)}} = \bigoplus_{b \in B} Q^{(P)} b$ (see the discussion after Notation 2.10). We have $g \in (B)_{Q^{(P)}} \leqq G$ for every $g \in G$. Thus, $G = (B)_{Q^{(P)}}$.

$(\Leftarrow)$. Let $G = \bigoplus_{b \in B} Q^{(P)} b$ for some $B \subseteq G$, and $ph = \sum_{b \in B} m_b b$, where m_b is integer for every $b \in B$, and $p \in \widehat{P}$. We have $h \in G_P$ and thus $h = \sum_{b \in B} h_b$, where $h_b \in Q^{(P)} b$ for each $b \in B$ (recall that $h_b = 0$ for a.e. b).

Therefore $ph = p \sum_{b \in B} h_b = \sum_{b \in B} ph_b = \sum_b m_b b$, and $ph_b = m_b b$ for every b (by the uniqueness of the decomposition of an element). Each direct component of G

in the considered decomposition has the form $Q^{(P)}b$. In other words, the element b plays the role of $\mathbf{1}$ in the corresponding $Q^{(P)}$-component of this decomposition. Now recall that $p \notin P$. Thus, $m_b \neq 0$ implies $p|m_b$ for every b, by the definition of $Q^{(P)}$.

□

In later proofs we will have to approximate an *excellent* basis stage-by-stage, using a certain oracle. Recall that not every maximal $\widehat{P}$-independent set is an excellent basis of G_P. Therefore, we need to show that, for a given finite $\widehat{P}$-independent subset B of G_P and an element $g \in G_P$, there exists a finite extension B_1 of B such that B_1 is $\widehat{P}$-independent and the element g is contained in the $Q^{(P)}$-span of B_1.

Proposition 4.5. *Suppose $B \subset G_P$ is a finite $\widehat{P}$-independent subset of G_P. For every $g \in G_P$ there exists a finite $\widehat{P}$-independent set $B^\star \subset G_P$ such that $B \subseteq B^\star$ and $g \in (B^\star)_{Q^{(P)}}$.*

Proof. Pick $\{e_i : i \in \omega\} \subseteq G_P$ such that $G_P = \bigoplus_{i\in\omega} Q^{(P)}e_i$. Let $\{e_0, e_1, \dots, e_n\}$ be such that both $B = \{b_0, \dots, b_k\}$ and g are contained in $(\{e_0, e_1, \dots, e_n\})_{Q^{(P)}}$. We may assume $k < n$.

Lemma 4.6. *Suppose $B = \{b_0, \dots, b_k\} \subseteq \bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i$, is a linearly independent set. There exists a set $C = \{c_0, \dots, c_n\} \subseteq \bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i$, and coefficients $r_0, \dots, r_k \in Q^{(P)}$ such that*

(1) $\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i, = \bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}c_i$, and

(2) $(\{r_0c_0, \dots, r_kc_k\})_{Q^{(P)}} = (B)_{Q^{(P)}}$.

Proof. It is a special case of a well-known fact ([28], Theorem 7.8) which holds in general for every finitely generated module over a principle ideal domain (note that $Q^{(P)}$ is a principle ideal domain). □

We show that if B is $\widehat{P}$-independent (not merely linearly independent) then we can set $B^\star = \{b_0, \dots, b_k\} \cup \{c_{k+1}, \dots, c_n\}$, where $C = \{c_0, \dots, c_n\}$ is the set from Lemma 4.6. Suppose $p|\sum_{0\le i\le k} n_ib_i + \sum_{k+1\le i\le n} n_ic_i$ for a prime $p \in \widehat{P}$. We have

$$\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i = \bigoplus_{1\le i\le k} Q^{(P)}c_i \oplus \bigoplus_{k+1\le i\le n} Q^{(P)}c_i,$$

and $\sum_{1\le i\le k} n_ib_i \in \bigoplus_{1\le i\le k} Q^{(P)}c_i$. By the purity of direct components, we have $p|\sum_{1\le i\le k} n_ib_i$ within $\bigoplus_{1\le i\le k} Q^{(P)}c_i$ and $p|\sum_{k+1\le i\le n} n_ic_i$ within $\bigoplus_{k+1\le i\le n} Q^{(P)}c_i$. But the former implies $p|n_i$ for all $1 \le i \le k$ by our assumption, and the latter implies $p|n_i$ for all $k+1 \le i \le n$ by the choice of C and Lemma 4.4.

The set $B^\star$ is actually an excellent $\widehat{P}$-basis of $\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i$, since the cardinality of $B^\star$ is $n+1 = rk(\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i)$. Therefore, the set $B^\star = \{b_0, \dots, b_k\} \cup \{c_{k+1}, \dots, c_n\}$ is a $\widehat{P}$-independent set with the needed properties.

□

Suppose G is a torsion-free abelian group, and $a, b \in G$. Recall that $\chi(a) \le \chi(b)$ iff $h_p(a) \le h_p(b)$ for all p. In other words, $p^k|a$ implies $p^k|b$, for all $k \in \omega$ and every prime p.

Definition 4.7. Let G be a torsion-free abelian group. For a given characteristic α, let $G[\alpha] = \{g \in G : \alpha \leq \chi(g)\}$.

We have $h_p(a) = h_p(-a)$ and $\inf(h_p(a), h_p(b)) \leq h_p(a+b)$, for all p. Furthermore, $\chi(0) \geq \alpha$, for every characteristic α. Therefore, $G[\alpha]$ is a subgroup of G.

Definition 4.8. Let $\alpha = (h_1, h_2, \ldots)$. Then $Q(\alpha)$ is the subgroup of $(Q, +)$ generated by elements of the form $1/p_k^x$ where $x \leq h_k$.

Example 4.9. *Let $\alpha = (\infty, 1, \infty, 1, \ldots, h_{2k} = 1, h_{2k+1} = \infty, \ldots)$. Consider*

$$\beta = (\infty, 2, \infty, 0, \infty, 1, \ldots) = \alpha + (0, 1, 0, -1, 0 \ldots).$$

By Definition 2.6, $\beta \cong \alpha$. Consider the group $H = Q(\alpha)$. We have $1 \in Q(\alpha)$ and $\chi(1) = \alpha$ within $Q(\alpha)$. Note that the characteristic of $a = 3/7$ in $H(\alpha)$ is β. Observe that a/p_{2k+1}^j belongs to $H[\beta]$, for every $k, j \in \omega$. In contrast, a/p_{2k} does not belong to $H[\beta]$. Indeed, the characteristic of $a/13$ in H is $(\infty, 2, \infty, 0, \infty, 0, \ldots)$ and

$$(\infty, 2, \infty, 0, \infty, 0, \ldots) \not\geq \beta = (\infty, 2, \infty, 0, \infty, 1, \ldots).$$

Recall that the type is an equivalence class of characteristics. Thus, the type of $H \leqq Q$ is simply the type of any nonzero element of H. We are ready to state and prove the main result of this section.

Theorem 4.10. *Let $\mathcal{G} = \bigoplus_{i \in \omega} H$, where $H \leqq Q$, $\mathbf{t}(H) = \mathbf{f}$ and α is of type $\mathbf{f}$. Then $\mathcal{G}[\alpha] \cong G_P$, where $P = \{p : h_p = \infty \text{ in } \alpha\}$. Furthermore, if B is an excellent $\widehat{P}$-basis of $\mathcal{G}[\alpha]$, then $\mathcal{G}$ is generated by B over $Q(\alpha)$.*

Informally, this theorem says that each homogeneous completely decomposable group of rank ω has a subgroup isomorphic to G_P, for some P. Furthermore, every excellent $\widehat{P}$-basis of this subgroup generates the whole group G over a certain rational subgroup $Q(\alpha)$ taken as a domain of coefficients. The group $Q(\alpha)$ is not necessarily a ring (recall Notation 2.10). The idea of the technical proof below was essentially illustrated in Example 4.9.

Proof. We prove that $\mathcal{G}[\alpha] \cong G_P$.

Let g_i be the element of the i'th presentation of H in the decomposition $\mathcal{G} = \bigoplus_{i \in \omega} H$ such that $\chi(g_i) = \alpha$. The collection $\{g_i : i \in \omega\}$ is a basis of $\mathcal{G}$. Therefore, $\{g_i : i \in \omega\}$ is a basis of $\mathcal{G}[\alpha]$. By the definition of P, $(\{g_i : i \in \omega\})_{Q^{(P)}}$ is a subgroup of $\mathcal{G}[\alpha]$. Furthermore, since $\{g_i : i \in \omega\}$ is linearly independent, $(\{g_i : i \in \omega\})_{Q^{(P)}} \cong \bigoplus_{i \in \omega} Q^{(P)} g_i$. Thus, we have

$$\bigoplus_{i \in \omega} Q^{(P)} g_i \subseteq \mathcal{G}[\alpha].$$

We are going to show that every element $g \in G_\alpha$ is generated by $\{g_i : i \in \omega\}$ over $Q^{(P)}$. This will imply $\mathcal{G}[\alpha] \cong G_P$.

Pick any nonzero $g \in \mathcal{G}[\alpha]$. The set $\{g_i : i \in \omega\}$ is a basis of $\mathcal{G}[\alpha]$, therefore $ng = \sum_{i \in \omega} m_i g_i$ for some integers n and m_i, $i \in \omega$. Since direct components are pure, $n | \sum_{i \in I} m_i g_i$ implies $n | m_i g_i$ for every $i \in \omega$, and $g = \sum_{i \in I} \frac{m_i}{n} g_i$. After reductions we have $g = \sum_{i \in I} \frac{m_i'}{n_i} g_i$, where $\frac{m_i'}{n_i}$ is irreducible. It suffices to show that $\frac{m_i'}{n_i} \in Q^{(P)}$.

Assume there is i such that $\frac{m'_i}{n_i} \notin Q^{(P)}$. Equivalently, for some $p \in \widehat{P}$, we have $m'_i \neq 0$ and $n_i = pn'_i$, where n'_i is an integer (recall that $\frac{m'_i}{n_i}$ is irreducible).

We have $h_p(\frac{m'_i}{n_i}g_i) = h_p(\frac{m'_i}{n'_i}\frac{g_i}{p}) \leq h_p(\frac{g_i}{p})$, since m'_i is not divisible by p. But $h_p(\frac{g_i}{p}) < h_p(g_i)$ (recall that $h_p(g_i)$ is finite). It is straightforward from the definitions of h_p that $h_p(g) = \min\{h_p(\frac{m'_i}{n_i}g_i) : i \in I, m_i \neq 0\}$, since each g_i belongs to a separate direct component of $\mathcal{G}$. Therefore $h_p(g) \leq h_p(\frac{m'_i}{n_i}g_i) < h_p(g_i)$. But $\chi(g_i) = \alpha$. Thus, $\chi(g) \ngeq \alpha$ and $g \notin \mathcal{G}[\alpha]$, and this contradicts our choice of g. This shows that $\mathcal{G}[\alpha] \cong G_P$.

We show that if B is an excellent $\widehat{P}$-basis of $\mathcal{G}[\alpha]$, then $\mathcal{G} = (B)_{Q(\alpha)}$ (recall Notation 2.10).

For every $b \in B$ consider the minimal pure subgroup which contains b (recall that we denote this group by $[b]$, see Definition 2.3). Consider $\langle B \rangle = \sum_{b\in B}[b] \leqq \mathcal{G}$. In fact $\langle B \rangle = \bigoplus_{b\in B}[b]$, because B is linearly independent within $\mathcal{G}[\alpha]$ and, therefore, within $\mathcal{G}$ as well.

By our choice, $b \in \mathcal{G}[\alpha]$. Thus, $\chi(b) \geq \alpha$ within $\mathcal{G}$. We show that in fact $\chi(b) = \alpha$. Assume $\chi(b) > \alpha$. We have $b = pa$ for some $a \in \mathcal{G}[\alpha]$ and $p \in \widehat{P}$. But B is $\widehat{P}$-independent. This contradicts the fact that $p|1 \cdot b$ and 1 is evidently not divisible by p. Therefore, we have

$$[b] = Q(\alpha)b.$$

It remains to prove that $\mathcal{G} \subseteq \langle B \rangle$. Pick any nonzero $g \in \mathcal{G}$. There exist integers m and n such that $(m, n) = 1$ and $\chi(\frac{m}{n}g) = \alpha$. To see this we use the fact that $\chi(g) \in \mathbf{f}$. It is enough to make only finitely many changes to $\chi(g)$ to make it equivalent to α.

Equivalently, $\frac{m}{n}g \in \mathcal{G}[\alpha]$. We have $\frac{m}{n}g = \sum_{b\in B, r_b\in Q^{(P)}} r_b b$, by Lemma 4.4. By our assumption, $\chi(b) = \chi(\frac{m}{n}g) = \alpha$, for every $b \in B$. Obviously, $m|\frac{m}{n}g$ in G. Therefore, by the definition of α and B, we have $m|b$ *in* $Q(\alpha)b$. Thus, there exist $x_b \in [b] = Q(\alpha)b$ such that $mx_b = b$. We can set $g = \sum_{b\in B} nr_b x_b$, where $nr_b x_b \in [b]$. This shows $\mathcal{G} = (B)_{Q(\alpha)}$. □

5. Effective content of S-independence, and Δ^0_3-categoricity.

Theorem 5.1. *Every computably presentable homogeneous completely decomposable torsion-free abelian group is Δ^0_3-categorical.*

The proof of the Theorem 5.1 is based on the lemma below. The proof of this lemma uses Theorem 4.10. The proof of Theorem 5.1 was sketched in [35].

Lemma 5.2. *Let $\mathcal{G} = \bigoplus_{i\in\omega} H$, where $H \leqq Q$, the type $\mathbf{t}(H)$ is $\mathbf{f}$, and α is a characteristic of type $\mathbf{f}$. Let G_1 and G_2 be computable presentations of $\mathcal{G}$. Suppose that both $G_1[\alpha]$ and $G_2[\alpha]$ have Σ^0_n excellent $\widehat{P}$-bases. Then there exists an Δ^0_n isomorphism from G_1 onto G_2.*

We first prove Theorem 5.1, and then prove Lemma 5.2. We need to show that a given homogeneous completely decomposable group satisfies the hypothesis of Lemma 5.2 with $n = 3$.

Proof of Theorem 5.1. Let G be a computable presentation of $\mathcal{G} \cong \bigoplus_{i\in\omega} H$, where $H \le Q$. Let α be a characteristic of type $\mathbf{t}(H)$ and $P = \{p : h_p = \infty \text{ in } \alpha\}$. By Theorem 4.10 and Lemma 5.2, it suffices to construct a excellent $\widehat{P}$-basis of $G[\alpha]$ which is Σ^0_3.

We are building $C = \bigcup_n C_n$. Assume that we are given C_{n-1}. At step n of the procedure, we do the following:

1. Pick the n-th element g_n of $G[\alpha]$.

2. Find an extension C_n of C_{n-1} in $G[\alpha]$ such that (a) C_n is a finite $\widehat{P}$-independent set, and (b) $C_n \cup \{g_n\}$ is linearly dependent.

Let $G = \bigoplus_{i\in I} Re_i$, where $\chi(e_i) = \alpha$ and $R \cong H$. Observe that at stage n of the procedure we have $g_n \cup C_{n-1} \subset (\{e_0, \dots, e_k\})_{Q^{(P)}}$, for some k. By Proposition 4.5, the needed extension denoted by C_n can be found.

It suffices to check that the construction is effective relative to $0''$. We use computable infinitary formulas in the proofs of the claims below. See [3] for a background on computable infinitary formulas.

By Theorem 4.10, we have $G[\alpha] \cong G_P$, where $P = \{p : p^\infty | h\}$ is a Π^0_2 set of primes.

Claim 5.3. *The group $G[\alpha]$ is c.e. in $0''$.*

Proof. Pick any $h \in G$ with $\chi(h) = \alpha$. By its definition, for every $g \in G$, the property $\chi(g) \ge \alpha$ is equivalent to

$$\bigwedge_{p-\text{prime}} \bigwedge_{k\in\omega} ((\exists x)p^k x = h \to (\exists y)p^k y = g).$$

Therefore, the group $G[\alpha]$ is a Π^0_2-subgroup of G. $\square$

Claim 5.4. *There is a $0''$-computable procedure which decides if a given finite set $B \subseteq G[\alpha]$ is $\widehat{P}$-independent, uniformly in the index of B.*

Proof. It suffices to show that the property "B is a $\widehat{P}$-independent set in $G[\alpha]$" can be expressed by a Π^0_2 infinitary computable formula in the signature of abelian groups with parameters elements from B.

Note that in general $P \in \Pi^0_2$. By Claim 5.3, the group $G[\alpha]$ is a Π^0_2-subgroup of G. Thus, the condition "B is a $\widehat{P}$-independent set in $G[\alpha]$" seems to be merely Π^0_3:

$$\bigwedge_{\overline{m}\in Z^{<\infty}} \bigwedge_{p-\text{prime}} ([p \notin P \wedge (\exists x)(x \in G[\alpha] \wedge px = \sum_{b\in B_n} m_b b)] \to \bigwedge_b p|m_b).$$

The idea is to substitute the Σ^0_3 formula $(\exists x)(x \in G[\alpha] \wedge px = \sum_{b\in B_n} m_b b)$ by an equivalent Σ^0_2 one, using a non-uniform parameter $h \in G$ such that $\chi(h) = \alpha$. More specifically, we are going to show that for every $p \notin P$, the formula

$$(\exists x)(x \in G[\alpha] \wedge px = \sum_{b\in B_n} m_b b)$$

is equivalent to

$$(\exists k)(\exists y \in G)(h_p < k \wedge p^k y = \sum_{b\in B_n} m_b b),$$

where h_p is the p-th component of α, and $h_p < k \Leftrightarrow^\neg (h_p \ge k) \Leftrightarrow^\neg (\exists h_1)(p^k h_1 = h)$.

Suppose there is $x \in G[\alpha]$ such that $px = \sum_{b \in B_n} m_b b$. Since $h_p(x) \geq h_p$, we have $p^{h_p} y = x$ and $p^{h_p+1} y = px$, for some $y \in G$, so we can set $k = h_p + 1$. For the converse, suppose there exist such k and y. Then $px = p^k y$ for $x = p^{k-1} y$. We have $k > h_p$, and therefore $(k-1) \geq h_p$. But $h_p(x) \geq (k-1)$ because $x = p^{k-1} y$ is divisible by $k-1$, and thus $h_p(x) \geq h_p$. The characteristic of x differs from the characteristic of y only at the position for the prime p. Thus, for every $q \neq p$,

$$h_q(x) = h_q(p^k y) = h_q(\sum_{b \in B_n} m_b b)) \geq h_q,$$

since $\sum_{b \in B_n} m_b b \in G[\alpha]$. Therefore, $\chi(x) \geq \alpha$ and $x \in G[\alpha]$.

□

By Claim 5.3 and Claim 5.4, the procedure is computable relative to $0''$. This establishes the theorem.

□

Proof of Lemma 5.2. Recall that G_1 and G_2 are computable presentations of $\mathcal{G}$ such that both $G_1[\alpha]$ and $G_2[\alpha]$ have Σ^0_n excellent $\widehat{P}$-bases. We need to show that there exists an Δ^0_n isomorphism from G_1 onto G_2. Let B_1 and B_2 be excellent $\widehat{P}$-bases of G_1 and G_2, respectively.

Observe that the group $Q(\alpha)$ is isomorphic to a c.e. additive subgroup R of $(Q, +, \times)$. Furthermore, we may assume that $1 \in R$. To see this pick h with $\chi(h) = \alpha$ non-uniformly, and then apply Theorem 2.7 to the group $[h]$. By Theorem 4.10, we have

$$G_1 = \bigoplus_{b \in B_1} Rb \cong G_2 = \bigoplus_{b' \in B_2} Rb'.$$

To build a Δ^0_n isomorphism from G_1 to G_2 first define the map from B_1 onto B_2 using a standard back-and-forth argument. Then extend it to the whole G_1 using the fact that $r \cdot b$ can be found effectively and uniformly, for every $r \in R$ and $b \in B_1$.

□

By Proposition 3.6 and Remark 3.7, "computable presentation of G_P" can be equivalently understood as "computable presentation of the group G_P" or "computable presentation of the $Q^{(P)}$-module G_P". Before we turn to a more detailed study of Δ^0_2-categorical completely decomposable groups, we prove a fact about excellent $\widehat{P}$-bases of the group G_P which is of an independent interest for us:

Theorem 5.5. *If a computable presentation of G_P has a Σ^0_2 basis which generates it as a free $Q^{(P)}$-module, then this presentation possesses a Π^0_1 basis which generates it as a free $Q^{(P)}$-module.*

Proof. Recall that, by Lemma 4.4, a basis generates G_P as a free $Q^{(P)}$-module if and only if this basis is an excellent $\widehat{P}$-basis. The proof of the theorem is based on Lemma 4.4 and the short technical lemma below.

Lemma 5.6. *Suppose $\{e_i : i \in \omega\} \subset G_P$ is such that $G_P = \bigoplus_{i \in \omega} Q^{(P)} e_i$, and suppose $\{b_1, \dots, b_k\} \subset G_P \setminus \{0\}$. For any integer $m, k \neq 0$, the set $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent if and only if $B_m = \{e_0, b_1, \dots, b_{k-1}, b_k + m e_0\}$ is $\widehat{P}$-independent. Furthermore, $(B)_{Q^{(P)}} = (B_m)_{Q^{(P)}}$, for every m.*

Note that for the second part of Lemma 5.6 *we do not assume that B is $\widehat{P}$-independent.*

Proof of Lemma 5.6. Suppose $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent. We show that $B_m = \{e_0, b_1, \dots, b_{k-1}, b_k + me_0\}$ is $\widehat{P}$-independent as well.

Pick an arbitrary $p \in \widehat{P}$. Suppose that p divides $g = n_0e_0 + \sum_{1\leq i\leq k-1} n_ib_i + n_k(b_k+me_0) = (n_0+n_km)e_0+\sum_{1\leq i\leq k} n_ib_i$. Recall that the set $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent. Therefore, $p|n_i$, for every $1 \leq i \leq k$. As a consequence, p divides $n_0e_0 = g - n_kme_0 - \sum_{1\leq i\leq k} n_ib_i$. By our assumption on the element e_0, we have $p|n_0$.

□

Suppose that $E = \{e_0, e_1, \dots\}$ is a Σ^0_2 excellent $\widehat{P}$-basis of $G = \bigoplus_{i\in\omega} Q^{(P)}e_i = \{g_0 = 0, g_1, \dots\}$ which is a computable group. We fix a computable relation R such that $x \in C$ if and only if $(\exists^{<\infty}y)R(x, y)$. We build a co-c.e set of elements B such that the following requirements are met:

R_0: $e_0 \in B$;

R_j: if $g_j = e_k$ for some k then B contains exactly one element of the form $(e_k + me_0)$.

There is no priority order on the requirements. All strategies in the construction will share the same global restraint (to be defined). We first show that if all the requirements are met, then the set B is an excellent $\widehat{P}$-basis of G. Assume R_j is met, for every j. It follows that for every k there exists m such that $e_k+me_0 \in B$. Also, if B contains two elements of the form $e_k + me_0$ and $e_k + ne_0$, then necessarily $n = m$. It remains to show that B is an excellent $\widehat{P}$-basis of G. Note that, if B is not $\widehat{P}$-independent, then there is a finite subset B_0 of B which is not $\widehat{P}$-independent. By (a multiple application of) Lemma 5.6, this contradicts the choice of $E = \{e_0, e_1, \dots, \}$. It remains to apply the second part of Lemma 5.6 and see that the $Q^{(P)}$-spans of B and E coinside.

Strategy for R_0:

Permanently put a restraint onto e_0.

Strategy for R_j, $j > 0$:

If R_j currently has no witness then pick a witness c_j which is equal to $g_j + me_0$, where m is the least such that $g_j + me_0$ is not restrained and is not yet enumerated into $\overline{B}$. Declare c_j restrained (thus, our current guess is: $c_j \in B$). If c_j is the n^{th} element of the group, $c_j = g_n$, then enumerate each g_x with $x < n$ into $\overline{B}$ unless g_x is already in $\overline{B}$ or is restrained.

If, at a later stage, a fresh y is found such that $R(g_j, y)$ holds, then enumerate $g_j + me_0$ into $\overline{B}$, and initialize R_j by making c_j undefined.

Construction.

Stage s. Let R_j, $j \leq s$, act according to their instructions.

End of construction.

Observe that B consists of elements which eventually become forever restrained by strategies. Also note that each element of the group can be restrained at most once. Thus, the set $\overline{B}$ is c.e.

To see why R_j is met note that the requirement eventually puts a permanent restraint on its witness $g_j + me_0$ if an only if $(\exists^{<\infty} y)R(g_j, y)$. This is the same as saying that $g_j = e_k$, for some k.

This finishes the proof of Theorem 5.5.

□

6. Semi-low sets, and Δ^0_2-categoricity.

Recall that a set A is semi-low if the set $H_A = \{e : W_e \cap A \neq \emptyset\} = \{e : W_e \nsubseteq \overline{A}\}$ is computable in $\emptyset'$.

Theorem 6.1. *A computably presentable completely decomposable abelian group G is Δ^0_2-categorical if and only if G is isomorphic to G_P where $\widehat{P}$ is semi-low.*

The proof of this theorem is split into several parts. Each part corresponds to a different hypothesis on the isomorphism type of G. Different cases will need different techniques and strategies.

Proof. We need the following technical notion:

Definition 6.2. Let $\alpha = (h_i)_{i\in\omega}$ be a sequence where $h_i \in \omega \cup \{\infty\}$ for each i (in other words, let α be a characteristic). Also, suppose that there is a non-decreasing uniform computable approximation $h_{i,s}$ such that $h_i = \sup_s h_{i,s}$, for every i (in other words, the characteristic is c.e., see Definition 3.1).

We say that α has *a computable setting time* if there is a (total) computable function $\psi : \omega \to \omega$ such that

$$h_i = \begin{cases} h_{i,\psi(i)}, & \text{if } h_i \text{ is finite,} \\ \infty, & \text{otherwise,} \end{cases}$$

for every i. We also say that ψ is a computable setting time for $(h_{i,s})_{i,s\in\omega}$.

This is the same as saying that, given i, there exists an effective (and uniform) way to compute a stage s after which the approximation of h_i either does not increase, or increases and tends to infinity. Note that this is the property of a characteristic, not the property of some specific computable approximation. Indeed, given an approximation of α having a computable setting time, we can define a computable setting time for any other computable approximation of α. Furthermore, as can be easily seen, this is a type-invariant property. Thus, we can also speak of types having computable setting times.

If a homogeneous completely decomposable group G of type $\mathbf{f}$ is computable, then $\mathbf{f}$ is c.e. (see Proposition 3.3). Suppose that G is a computable homogeneous completely decomposable group of type $\mathbf{f}$, and let $\alpha = (h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. We consider the cases:

(1) The type $\mathbf{f}$ of G has no computable setting time. In this case G is not Δ^0_2-categorical by Proposition 6.4. Observe that if $\mathbf{f}$ has no computable setting time then the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ has to be infinite (see, e.g., Proposition 3.6). Thus, G can not be isomorphic to G_P, for a set of primes P.
(2) The type $\mathbf{f}$ of G has a computable setting time, $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is empty (finite), and the set $\{i : h_i = 0\}$ is semi-low. In other words,

the group G is isomorphic to G_P with $\widehat{P}$ semi-low. In this case G is Δ^0_2-categorical, by Proposition 6.3 below.

(3) The type $\mathbf{f}$ of G has a computable setting time, the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is empty (finite), and the set $\{i : h_i = 0\}$ is not semi-low. Here G is again isomorphic to G_P, but in this case G is not Δ^0_2-categorical, by Proposition 6.5 below.

(4) The type $\mathbf{f}$ of G has a computable setting time, and the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is infinite and not semi-low. As in the above case[1], G is not Δ^0_2-categorical, by Proposition 6.5.

(5) The type $\mathbf{f}$ of G has a computable setting time, and the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is infinite and semi-low. The group is not Δ^0_2-categorical, by Proposition 6.6 below.

We first discuss why case (3) and case (4) above can be collapsed into one case. First, define $Inf(\alpha) = \{i : h_i = \infty\}$ and $V = \{i : 0 < \psi(i) < \infty\}$, where ψ is a computable setting time for α. Note that V is c.e. Evidently, $\overline{Inf(\alpha)} = Fin(\alpha) \cup \{i : h_i = 0\}$ and $Fin(\alpha) = \overline{Inf(\alpha)} \cap V$. We claim that "$Fin(\alpha)$ is not semi-low" implies "$\overline{Inf(\alpha)}$ is not semi-low". We assume that $\overline{Inf(\alpha)}$ is semi-low and observe that $\{e : W_e \cap Fin(\alpha) \neq \emptyset\} = \{e : W_e \cap V \cap \overline{Inf(\alpha)} \neq \emptyset\} = \{e : W_{s(e)} \cap \overline{Inf(\alpha)} \neq \emptyset\}$ for a computable function s. Therefore, $H_{Fin(\alpha)} \leq_m H_{\overline{Inf(\alpha)}} \leq_T \emptyset'$, as required.

Therefore, cases (3) and (4) are both collapsed into

(3′) If $\mathbf{f}$ has a computable setting time and $\overline{Inf(\alpha)}$ is not semi-low, then G is not Δ^0_2-categorical.

Now we state and prove the propositions which cover all the cases above.

Recall that, by Proposition 3.6, the group G_P has a computable presentation as a group (module) if and only if P is c.e.

Proposition 6.3. *If $\widehat{P}$ is semi-low (and co-c.e.) then G_P is Δ^0_2-categorical.*

Proof. The proof may be viewed as a simpler version of the proof of Theorem 5.1. Let $G = \{g_0 = 0, g_1, \ldots\}$ be a computable copy of G_P. By Lemma 3.8, it is enough to build a Σ^0_2 excellent $\widehat{P}$-basis of G.

We are building $C = \bigcup_n C_n$. Assume that we are given C_{n-1}. At stage n of the construction, we do the following:

1. Pick the n-th element g_n of G.
2. Find an extension C_n of C_{n-1} in G such that (a) C_n is a finite $\widehat{P}$-independent set, and (b) $C_n \cup \{g_n\}$ is linearly dependent.

The algebraic part of the verification is the same as in Theorem 5.1 (and is actually simpler). Thus, it is enough to show that (a) in (2) above can be checked effectively and uniformly in $\emptyset'$. Given a finite set F of elements of G, define a c.e. set V consisting of primes which could potentially witness that F is $\widehat{P}$-dependent:

$$V = \Big\{p : \bigvee_{\overline{m} \in Z^{card(F)}} \Big[p | (\sum_{g \in F} m_g g) \wedge (\bigvee_{g \in F} p \not| m_g)\Big]\Big\}.$$

[1]We distinguish these two cases only because these cases correspond to (algebraically) different types of groups. We discuss a bit later why these cases are essentially not different.

The c.e. index of V can be obtained uniformly from the index of F. It can be easily seen from the definition of $\widehat{P}$-independence that

$$V \cap \widehat{P} = \emptyset \text{ if and only if } F \text{ is } \widehat{P}\text{-independent.}$$

By our assumption on $\widehat{P}$, this can be decided effectively in $\emptyset'$.

□

Proposition 6.4. *Suppose that the type* $\mathbf{f}$ *of a computably presentable* $G = \bigoplus_{i\in\omega} H$ *has no computable setting time. Then* G *is not* Δ^0_2*-categorical.*

We first give an informal description of the proof. Let $\mathbf{f}$ be the type of G, and let $\alpha = (h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. We build two computable groups, A and B, both isomorphic to G. The group A is a "nice" copy of G. The group B is a "bad" copy of G in which the e^{th} elementary direct component is used to defeat the e^{th} potential Δ^0_2-isomorphism from B onto A.

The strategy can be roughly described as follows. Wait for the e^{th} potential isomorphism to converge on some specifically chosen element b_e from the e^{th} component of B. Pick a large j such that, if the e^{th} potential isomorphism is indeed an isomorphism, the characteristic $\chi(b_e) = (d_i)_{i\in\omega}$ of b_e and the characteristic $\alpha = (h_i)_{i\in\omega}$ have to be equal starting from the j^{th} position. We will see that such a number j can be effectively chosen (we use that A is "nice").

We make $d_{k,s} = h_{k,s} - 1$ for the least $k \geq j$ such that $h_{k,s} > 0$. We also attempt to define a computable setting time for α. Thus, we declare that $h_{k,s}$ is either a final value of h_k, or $h_k = \lim_t h_{k,t} = \infty$. If $h_{k,s} = \lim_t h_{k,t}$ then we win (unless the e^{th} potential Δ^0_2-isomorphism changes). Otherwise, if $h_{k,s_0} > h_{k,s}$, for some $s_0 > s$, then we set $d_{k,s_0} = h_{k,s_0} - 1$. We also pick another position $k_0 > k$ in which $h_{k_0,s_0} > d_{k,s_0}$. We declare all current values in α between k and k_0 to be "final" (including k_0), as we did for k. Then repeat the argument for k_0 (the only difference is that the next position k_1 may be chosen between k and k_0, and k_1 is picked only if both h_k and h_{k_0} increase), etc.

If, at a later stage, we see a new computation of the e^{th} potential Δ^0_2-isomorphism, then we (1) make the characteristic of b_e equivalent to α at every position they currently differ, and (2) repeat the above strategy, starting from a fresh large position j_0 in the characteristic of b_e.

The only "dangerous" situation we should worry about is:

$$\text{Each time we pick a position } k_i, \text{ we have } h_{k_i} = \lim_t h_{k_i,t} = \infty.$$

But this would imply that α has a computable setting time, contradicting the choice of $\mathbf{f}$. The groups A and B are both isomorphic to G by Theorem 2.7.

We give formal details below.

Proof of Proposition 6.4. It suffices to build two computable presentations, A and B, of the group $G = \bigoplus_{i\in\omega} H$, and meet the requirements:

R_e : If $\lim_t \Phi_{e,t}(b_e, t)$ exists, then $\lim_t \Phi_{e,t}(x, t)$ is not an isomorphism from B to A.

Without loss of generality, we may assume that $\Phi_{e,t}(x, t)$ is defined for every e and t. The construction is injury-free, thus we do not really need any priority order on the strategies.

In the following, we enumerate $A = \bigoplus_{n\in\omega} Ha_n$ and $B = \bigoplus_{e\in\omega} C_e b_e$ in such a way that the sets $\{a_n : n \in \omega\}$ and $\{b_e : e \in \omega\}$ are computable. The element b_e is a witness for the R_e strategy. Let $(h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. We make sure $\chi(a_n) = (h_i)_{i\in\omega}$, for every n, while the characteristic $\chi(b_e) = (d(e)_i)_{i\in\omega}$ of b_e will be merely equivalent to $(h_i)_{i\in\omega}$, for each e (thus, $C_e \cong H$, for each e).

Given a computable copy of G, define a computable approximation $(h_{i,s})_{i,s\in\omega}$ of $(h_i)_{i\in\omega}$ such that (1) $h_{i,s} \leq h_{i,s+1}$, and (2) $h_i = \lim_s h_{i,s}$, for every i and s.

For every e, the strategy for R_e defines its own computable function g_e which[2] is an attempt to define a computable setting time for $(h_i)_{i\in\omega}$. To define g_e the strategy uses the sequence $(k_{e,i})_{i\in\omega}$.

Strategy for R_e.

If (at a stage s of the construction) the parameter $k_{e,0}$ is undefined then:

1. Compute $\Phi_{e,s}(b_e, s)$. Since this moment, the strategy is always waiting for a later stage t such that $\Phi_{e,t}(b_e, t) \neq \Phi_{e,s}(b_e, s)$. As soon as such a stage is found, R_e initializes by making all its parameters undefined and also making $d(e)_{j,t} = h_{j,t}$ for every j we have ever used so far.

2. Let $a \in A$ be such that $a = \Phi_{e,s}(b_e, s)$. Find integers c_n and c such that $ca = \sum_n c_n a_n$. Let j be a fresh large index such that (1) the prime p_j does not occur in the decompositions of the coefficients c and c_n, (2) $h_{j,s} > 0$, and (3) $d(e)_{j,s} < h_{j,s}$.

3. Once j is found[3], declare $g_e(j) = h_{j,s}$. Since this moment, make sure $d(e)_j = h_j - 1$ by redefining $d(e)_{j,t}$ at later stages if needed, unless the strategy initializes. Set $k_{e,0} = j$, and proceed.

Now assume that the parameters $k_{e,0}, \ldots k_{e,y}$ have already been defined by the strategy. We also assume that $g_e(i)$ has already been defined for each i such that $k_{e,0} \leq i \leq max\{k_{e,x} : 0 \leq x \leq y\}$. Assume also that $k_{e,y}$ was first defined at stage $u < s$. Then do the following:

I. Wait for a stage $t \geq s$ (of the construction) such that either (a) $h_{i,t} > h_{i,s}$ for some i such that $k_{e,0} \leq i \leq max\{k_{e,x} : 0 \leq x \leq y\}$ and $i \notin \{k_{e,0}, \ldots k_{e,y}\}$, or (b) $h_{i,u} < h_{i,t}$ for each $i \in \{k_{e,0}, \ldots k_{e,y}\}$. While waiting, make $d(e)_{j,r} = h_{j,r}$ (r is the current stage of the construction), where $j \leq r$ and $j \notin \{k_{e,0}, \ldots k_{e,y}\}$.

II. If (a) holds for some i, then set $k_{e,(y+1)} = i$. If (b) holds, then let i be a fresh large index such that (1) $h_{i,t} > 0$, and (2) $d(e)_{i,t} < h_{i,t}$, and set $k_{e,(y+1)} = i$. In this case also define $g_e(j)$ to be equal to the current value of h_j (namely, $h_{i,t}$), for each $j \in [max\{k_{e,x} : 0 \leq x \leq y\}, k_{e,(y+1)}]$. Then proceed to III.

III. Since this moment, make sure $d(e)_j = h_i - 1$ (where $i = k_{e,(y+1)}$) by redefining $d(e)_i$ at later stages if needed, unless the strategy initializes.

End of strategy.

Construction.

At stage 0, start enumerating A and B as free abelian groups over $\{a_n\}_{n\in\omega}$ and $\{b_e\}_{k\in\omega}$, respectively. Initialize R_e, for all e.

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[2]Since it will be clear from the construction at which stage g_e is defined (if ever), we omit the extra index t in $g_{e,t}$ and write simply g_e. We omit the index t for parameters $k_{e,i,t}$ as well.

[3]We may assume that at stage s such an index j can be found, otherwise we re-define the approximation $(h_{i,s})_{i,s\in\omega}$ during the construction making it "faster".

At stage s, let strategies R_e, $e \leq s$, act according to their instructions. If R_e acted at the previous stage, then return to its instructions at the position it was left at the previous stage.

Make sure $\chi(a_n) = (h_{i,s})_{i\in\omega}$ in A_s for every n, and $\chi(b_e) = (d(e)_{i,s})_{i\in\omega}$ in B_s for every e, by making a_n and b_e divisible by corresponding powers of primes.

End of construction.

Verification.

For each e, the following cases are possible:

(1) $\lim_s \Phi_{e,s}(b_e, s)$ does not exist. In this case the strategy initializes infinitely often. By the way the strategy is initialized, the characteristic of b_e is identical to α.

(2) $\lim_s \Phi_{e,s}(b_e, s)$ exists and is equal to $\Phi_{e,l}(b_e, l)$. By the way the function g_e is defined, its domain can not be co-finite. For if it were defined on a co-finite set, then α would have a computable setting time. Therefore, there is a parameter $k_{e,y}$ such that the $k_{e,y}^{th}$ position in α is finite. Thus, the strategy ensures $\lim_s \Phi_{e,s}(b_e, s)$ is not an isomorphism since the characteristic of b_e and α differ at $k_{e,y}^{th}$ position. Therefore, α differs from $\chi(b_e)$ in at most finitely many positions, and the differences are finitary.

In both cases, we have $\chi(b_e)$ equivalent to α. By Theorem 2.7, $A \cong B \cong G$.

□

Recall that cases (3) and (4) are both reduced to:

Proposition 6.5. *Let G be computable homogeneous completely decomposable abelian group of type $\mathbf{f}$, and suppose $\alpha = (\sup_s h_{i,s})_{i\in\omega}$ in $\mathbf{f}$ has computable setting time ψ. Furthermore, suppose $\overline{Inf(\alpha)}$ is not semi-low. Then G is not Δ^0_2-categorical.*

The idea of the proof can be roughly described as follows. We build two computable groups, A and B, both isomorphic to G. The group A is a "nice" copy of G. The group $B = \bigoplus_{e\in\omega}\bigoplus_{n\in\omega} C_{e,n}b_{e,n}$ is a "bad" copy of G in which the e^{th} direct component is used to defeat the e^{th} potential Δ^0_2-isomorphism from B onto A.

Recall that $Inf(\alpha)$ is a c.e. set. Given e, we attempt to define a functional $\Psi(e, n, s)$ such that $H_{\overline{Inf(\alpha)}}(n) = \lim_s \Psi(e, n, s)$. For every n, we pick an element $b_{e,n}$ in B and attempt to destroy the e^{th} potential Δ^0_2-isomorphism from B to A. We start by setting $\Psi(e, n, 0) = 0$. We wait for j to appear in $W_{n,s} \setminus Inf(\alpha)_s$. If we never see such a j, then our attempt to define $\Psi(e, n, s)$ is successful. If we find such a j, make $b_{e,n}$ divisible by a large power of p_j destroying the potential isomorphism (this power depends on our current guess on the isomorphic image of $b_{e,n}$ in A). We will set $\Psi(e, n, t) = 1$ only if the e^{th} potential isomorphism changes on $b_{e,n}$ at a later stage t. We make $\Psi(e, n, r) = 0$ as soon as j enters $Inf(\alpha)$, and then we start waiting for a new fresh number to show up in $W_n \setminus Inf(\alpha)$. If we see such a number then we repeat the above strategy with this number in place of j.

Our attempt to define $\Psi(e, n, s)$ necessarily fails for at least one index n. Therefore, the e^{th} potential isomorphism will be defeated at the element $b_{e,n}$. Algebra is sorted out using Theorem 2.7.

Note that the algebraic strategy above differs from the one we used in Proposition 6.4. More specifically, we make elements divisible instead of keeping elements non-divisible. This strategy could not be used in Proposition 6.4, because it would not be consistent with the infinitary outcome (the case when the e^{th} potential isomorphism changes infinitely often). We will see that this is not a problem here.

Proof of Proposition 6.5. We build to computable copies of G by stages. Recall that the first copy $A = \bigoplus_i Ha_i$ is a "nice" copy with $\chi(a_i) = \alpha$, for every i. The second ("bad") copy $B = \bigoplus_{e\in\omega} \bigoplus_{n\in\omega} C_{e,n}b_{e,n}$ is built in such a way that $\chi(b_{e,n})$ is equivalent to α, for every e and n.

As in Proposition 6.4, it suffices to meet the requirements:

R_e : If $\lim_t \Phi_{e,t}(x,t)$ exists for every x, then $\lim_t \Phi_{e,t}(x,t)$ is not an isomorphism from B to A.

Without loss of generality, we may assume that $\Phi_{e,t}(x,t)$ is defined for every e and t. The strategy for R_e initially defines a computable operator $\Psi(n,s)$ such that $\Psi(n) = \lim_s \Psi_s(j,s)$ (if it exists) attempts to witness $H_{\overline{Inf(\alpha)}} \leq_T \emptyset'$. More specifically, we attempt to make sure that Ψ is total and $\Psi(n) = 0$ iff $W_n \subseteq Inf(\alpha)$. If we succeeded, this would imply $H_{\overline{Inf(\alpha)}} = \{n : W_n \not\subseteq Inf(\alpha)\} \leq_T \emptyset'$, contradicting the hypothesis. We split R_e into substrategies $R_{e,n}$, $n \in \omega$:

Strategy for $R_{e,n}$.

In the following, we write I in place of $Inf(\alpha)$. We permanently assign the element $b_{e,n}$ to $R_{e,n}$. Suppose that the strategy becomes active first time at stage s of the construction. Then:

(1) Start by setting $\Psi_s(n,s) = 0$ (we may suppose that $\Psi_j(n,j) = 0$, for every $j < s$). At a later stage t, we define $\Psi_t(n,t)$ to be equal to $\Psi_{t-1}(n,t-1)$, unless we have a specific instruction not to do so.
(2) Wait for a stage $t > s$ and a prime $p \in W_{n,t} \setminus I_t$.
(3) We see $p = p_j$ with $j \in W_{n,t} \setminus I_t$ at a later stage t. Find $a \in A_t$ such that $a = \Phi_{e,t}(b_e,t)$ (recall that the enumeration of A is controlled by us). Find integers c_n and c such that $ca = \sum_n c_n a_n$. Let k be a fresh large natural number such that (i) the prime $p = p_j$ has power at most $[k/2]$ in the decompositions of the coefficients c and c_n, and (ii) $h_{j,\psi(j)} < [k/2]$, where ψ is the computable setting time. Note that (i) and (ii) imply k is so large that p^k does not divide $a = \Phi_{e,t}(b_{e,n},t)$ within A, unless $j \in I_t$. Make $b_{e,n}$ divisible by p^k within B.

   Wait for one of the two things to happen:

   I. (I changes first). We see $j \in I_u$ at a later stage $u > t$, and $\Phi_{e,v}(b_{e,n},v) = \Phi_{e,t}(b_{e,n},t)$ for each $v \in (t,u]$. We return to (2) with u in place of s.
   II. (Φ_e changes first). We see $\Phi_{e,u}(b_{e,n},u) \neq \Phi_{e,t}(b_{e,n},t)$ at a later stage $u > t$, and $j \in W_{n,v} \setminus I_v$ for each $v \in (t,u]$. Then we set $\Psi_u(n,u) = 1$ and start waiting for a stage $w > u$ such that $j \in I_u$. If such a stage w is found, then we set $\Psi_w(n,w) = 0$ and go to (2) with w in place of s (and we do nothing, otherwise).

End of strategy.

Construction.

At stage 0, start enumerating A and B as free abelian groups over $\{a_i\}_{i\in\omega}$ and $\{b_{e,n}\}_{e,n\in\omega}$.

At stage s, let strategies $R_{e,n}$, $e,n \leq s$, act according to their instructions. If $R_{e,n}$ acted at the previous stage, then return to its instruction at the position it was left at the previous stage.

Make sure $\chi(a_i) = \alpha = (h_j)_{j\in\omega}$ in A for every i. For every $e, n \in \omega$, make $\chi_j(b_{e,n}) = h_j$ in B for every j except at most one position, according to the instructions of $R_{e,n}$. We do so by making a_i and $b_{e,n}$ divisible by corresponding powers of primes.

End of construction.

Verification.

By Theorem 2.7, $A \cong B \cong G$. Assume that $\lim_s \Phi_{e,s}(b_{e,n}, s)$ exists for every n. Given n, consider the cases:

- $R_{e,n}$ eventually waits forever at substage (2). Then $\lim_s \Phi(n, s) = 0$ and $W_n \subseteq I$. Thus, we have a correct guess about $H_{\overline{Inf(\alpha)}}$.
- $R_{e,n}$ visits substage (I) of (3) infinitely often. Then $\lim_s \Phi(n, s) = 0$ and $W_n \subseteq I$, and we again have a correct guess about $H_{\overline{Inf(\alpha)}}$.
- $R_{e,n}$ eventually waits forever at substage (3). Then $x_{e,n}$ witnesses that $\lim_s \Phi_{e,s}(x_{e,n}, s)$ is not an isomorphism from B to A.

There should be at least one n for which $\lim_s \Phi(n, s) \neq H_{\overline{Inf(\alpha)}}(n)$. Therefore, for at least one n, the strategy $R_{e,n}$ eventually waits forever at substage (3). Thus, R_e is met.

$\square$

Proposition 6.6. *If the type* $\mathbf{f}$ *of a computable homogeneous completely decomposable group* G *has a computable setting time, and* $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ *is infinite and semi-low for* α *of type* $\mathbf{f}$*, then* G *is not* Δ^0_2*-categorical.*

The idea is to combine the algebraic strategy from Proposition 6.4 and the guessing procedure based on the hypothesis $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is semi-low. As before, we are building two computable copies, A and B, of G. Imagine we have p_j with a (large) $j \in Fin(\alpha)$. To destroy the e^{th} potential Δ^0_2-isomorphism, we make the witness b_e in B *not divisible* by p, as we did in Proposition 6.4. We may have to pick another prime, due to the isomorphism change.

We note that the algebraic strategy from Proposition 6.5 would not succeed. If $\lim_s \Phi_{e,s}(b_e, s)$ does not exist, then B would not be isomorphic to G. Indeed, we we would have to make b_e divisible by infinitely many extra primes.

It remains to guess for which primes p_j we have $j \in Fin(\alpha)$. Each strategy defines its own sequence of c.e. sets and tests if a c.e. set from the sequence intersects $Fin(\alpha)$. Since the construction is effective an uniform, we may assume that the indexes of these c.e. sets are listed by a computable function, and the index of this function is given ahead of time.

We give all details in the formal proof below.

Proof. Let Ψ be a computable function such that $Fin(\alpha) \cap W_n = \lim_s \Psi(n, s)$. As in the proof of Proposition 6.4, we are building two computable copies, $A = \bigoplus_{n\in\omega} Ha_n$ and $B = \bigoplus_{e\in\omega} C_e b_e$, of G. We make sure $\chi(a_n) = \alpha$ and $\chi(b_e) = (d(e))_{i\in\omega} \simeq \alpha$, for every n and e. The requirements are:

R_e : If $\lim_t \Phi_{e,t}(b_e, t)$ exists, then $\lim_t \Phi_{e,t}(x, t)$ is not an isomorphism from B to A.

Without loss of generality, we may assume that $\Phi_{e,t}(x, t)$ is defined for every e and t.

The strategy for R_e.

Suppose we have a computation $\Phi_{e,s}(b_e, s)$ such that $\Phi_{e,s}(b_e, s) \neq \Phi_{e,s-1}(b_e, s-1)$ or $s = 0$. We do the following substeps:

(1) Make $\chi(b_e) = (d(e))_{i\in\omega}$ and α equal at all positions they are currently defined.
(2) Begin enumerating $W_{g(e,s)}$ by setting $W_{g(e,s)} = \emptyset$ first.
(3) Wait for a stage u such that $\Psi(g(e, s), u) = 0$.
(4) Let $a \in A$ be such that $a = \Phi_{e,s}(b_e, s)$. Find integers c_n and c such that $ca = \sum_n c_n a_n$. Let j be a fresh large index[4], such that (1) the prime p_j does not occur in the decompositions of the coefficients c and c_n, (2) $h_{j,\psi(j)} > 0$, and (3) $d(e)_{j,s} < h_{j,s}$.
(5) Enumerate j into $W_{g(e,s)}$. Since this moment, make sure $d(e)_j = h_j - 1$ by redefining $d(e)_{j,t}$ at later stages if needed, unless $\Phi_{e,s}(b_e, s) \neq \Phi_{e,t}(b_e, t)$ at a later stage t. We *restrain* the element b_e by not allowing the construction to add more roots to b_e.
(6) We wait for one of the following three things to happen:
    I. $\Phi_{e,s}(b_e, s) \neq \Phi_{e,t}(b_e, t)$ at a later stage t. Then declare b_e not restrained and restart the strategy with t in place of s.
    II. The index j enters the c.e. set $Inf(\alpha)$ at stage $s > t$ (thus, $h_j = \infty$). We return to (5) with $j + 1$ in place of j (we keep b_e restrained).
    III. $\Psi(g(e, s), t) = 1$ (thus, we believe that $W_{g(e,s)} \cap Fin(\alpha) \neq \emptyset$). We remove the restraint from the element b_e (that is, we allow the construction to make $\alpha_i = d(e)_i$ for every $i \notin W_{g(e,s)}$.) If at a later stage r we see $W_{g(e,s),r} \subseteq Inf(\alpha)_r$, then wait for a stage $w \geq r$ such that $\Psi(g(e, s), w) = 0$. Then return to (4).

End of strategy.

Construction.

At stage 0, start enumerating A and B as free abelian groups over $\{a_n\}_{n\in\omega}$ and $\{b_e\}_{k\in\omega}$, respectively.

At stage s, let strategies R_e, $e \leq s$, act according to their instructions. If R_e acted at the previous stage, then return to its instruction at the position it was left at the previous stage.

Make sure $\chi(a_n) = (h_{i,s})_{i\in\omega}$ in A_s for every n, and $(h_{i,s})_{i\in\omega} = (d(e)_{i,s})_{i\in\omega}$ in B_s for every e which is not restrained, unless R_e keeps $h_{i,s} - 1 = d(e)_{i,s}$. We do so by making a_n and b_e divisible by corresponding powers of primes.

End of construction.

Verification.

Assume that $\lim_t \Phi_{e,t}(b_e, t)$ exists. Let s be a stage such that $\Phi_{e,s}(b_e, s) = \lim_t \Phi_{e,t}(b_e, t)$. Let u be a stage such that $\lim_t \Psi(g(e, s), t) = \Psi(g(e, s), u)$. It remains to show that $\Psi(g(e, s), u) = 1$, the rest is clear. If $\Psi(g(e, s), u) = 0$, then the strategy finds a new index j and starts a new loop at stages (5) and (6). The set $Fin(\alpha)$ is infinite, therefore $\{j, j + 1, \ldots, j + k\} \cap Fin(\alpha) \neq \emptyset$ for some $k \geq 0$. But the construction ensures $\{j, j + 1, \ldots, j + k\} \subseteq W_{g(e,s)}$, contradiction. Thus, $\Psi(g(e, s), u) = 1$, and the diagonalization is successful.

[4]We may assume that at stage s such an index j can be found, otherwise we re-define the approximation $(h_{i,s})_{i,s\in\omega}$ during the construction making it “faster”.

As in Proposition 6.5, the algebraic part of the verification can be easily derived from Theorem 2.7. It is important that we remove the restraint from b_e at substage (III) of (6).

□

This concludes the proof of Theorem 6.1.

□

Corollary 6.7. *For a c.e. set P, the following are equivalent:*

(1) *G_P has a Σ^0_2 excellent $\widehat{P}$-basis;*
(2) *G_P has a Σ^0_2-basis as a free $Q^{(P)}$-module;*
(3) *G_P has a Π^0_1-basis as a free $Q^{(P)}$-module;*
(4) *G_P is Δ^0_2-categorical;*
(5) *$\widehat{P}$ is semi-low.*

Proof. The proof is a combination of Theorem 6.1, Theorem 5.5, and Lemma 3.8. □

Corollary 6.8. *Each computable copy of the free abelian group of rank ω has a Π^0_1 set of free generators.*

Proof. The free abelian group can be viewed as the free Z-module. It remains to apply Theorem 5.5 and Theorem 6.1 with $\widehat{P}$ the set of all primes. □

7. Concluding Remarks and Open Questions

The notion of S-independence seems to be a natural generalization of linear independence to the case of free modules.

Problem 7.1. *Study the effective content of S-independence.*

Note that the effective content of p-independent sets (for a single prime p) seem to be unstudied. As we mentioned in the introduction, p-independent sets play an important role in the theory of primary abelian groups. It would be interesting to develop the effective theory of S-independent sets and (excellent) S-bases.

Problem 7.2. *For every n build a computable presentable completely decomposable group which is not Δ^0_n-categorical.*

We expect that such groups exist. These groups can not be homogeneous for $n \geq 4$. As a consequence of the main construction in [1], such examples exist in the class of computable torsion-free abelian groups. Nonetheless, these examples are not completely decomposable.

Problem 7.3. *Extend the results of the paper to other classes of completely decomposable abelian groups.*

We expect that if the collection of types is computable and well-founded as a partial order, then it requires at most one or two extra jumps to build an isomorphism. Is it sharp?

Problem 7.4. *What is the complexity of the index set of all computable completely decomposable groups?*

We mention that this index set belongs Σ_1^1, since a countable torsion-free abelian group is completely decomposable if and only if every finite set of elements of this group is contained in a direct summand of finite rank [18].

The theory of completely decomposable groups is an example of a beautiful and nontrivial mathematical theory having a number of pleasant results, especially in the countable case.

Problem 7.5. *Study the reverse mathematics of completely decomposable abelian groups.*

Limitwise monotonic sets were mentioned in the introduction. Recently the notion of a *limitwise monotonic sequence* proved to be useful in computable model theory [27]. Note that a c.e. characteristic can be viewed as a limitwise monotonic sequence in $(\omega \cup \{\omega\})^\omega$.

Problem 7.6. *Study limitwise monotonic sequences in $(\omega \cup \{\omega\})^\omega$ having a computable setting time (see Definition 6.2). Do they have another applications in computable model theory?*

We also expect that the results of the paper have analogs for modules over computably presentable principle ideal domains.

Problem 7.7. *Extend the results of the paper to modules over computable rings.*

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