SCHNORR RANDOMNESS

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Abstract. Schnorr randomness is a notion of algorithmic randomness for real numbers closely related to Martin-Löf randomness. After its initial development in the 1970s the notion received considerably less attention than Martin-Löf randomness, but recently interest has increased in a range of randomness concepts. In this article, we explore the properties of Schnorr random reals, and in particular the c.e. Schnorr random reals. We show that there are c.e. reals that are Schnorr random but not Martin-Löf random, and provide a new characterization of Schnorr random real numbers in terms of prefix-free machines. We prove that unlike Martin-Löf random c.e. reals, not all Schnorr random c.e. reals are Turing complete, though all are in high Turing degrees. We use the machine characterization to define a notion of "Schnorr reducibility" which allows us to calibrate the Schnorr complexity of reals. We define the class of "Schnorr trivial" reals, which are ones whose initial segment complexity is identical with the computable reals, and demonstrate that this class has non-computable members.

§1. Introduction. The concern of this paper is the algorithmic randomness of reals, which will be considered as infinite strings. The roots of the study of algorithmic randomness go back to the work of von Mises at the dawn of the 20th century. He suggested that randomness of a sequence $a_1 a_2 \ldots$ could be based on the ability to predict the next bit given the first n bits. Von Mises problem was what would correspond to "acceptable" prediction functions. Of course what was needed was the notion of a computable function. Lacking the language and concepts from computability theory, this work languished until the middle of the 20th century, when it was revisited by Church who suggested a notion of computable randomness[1].

Nevertheless, the real work acceptably clarifying the notion of algorithmic randomness only really came with the work of Solomonoff [16], Kolmogorov [6], Chaitin [3], Levin [10], Martin-Löf [13] and Schnorr [14, 15].

There are three basic approaches taken: measure theoretic, compressibility, and predictability. The first is *measure theoretic.* A real should be random if it avoids all "effectively null" properties. The canonical version

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[1] We refer the reader to Li-Vitanyi [11] and van Lambalgen [9] for a more thorough discussion of the history and evolution of the concept of algorithmic randomness.

of this is due to Martin-Löf [13], and is based on computably enumerable (c.e.) open sets. A c.e. open set is a computably enumerable collection of open intervals with rational endpoints. A *computable collection* of c.e. open sets $\{U_n\}$, $n \in \omega$, is a collection for which there is a single computable function able, given any n, to enumerate the intervals of the set U_n. (We generally restrict our attention to elements and subsets of the interval (0,1).)

DEFINITION 1 (Martin-Löf [13]). *A* Martin-Löf test $\{U_n\}$, $n \in \omega$, *is a computable collection of c.e. open sets such that* $\mu(U_n) \leq 2^{-n}$. *A real number* x withstands *the test* U_n *if* $x \notin \cap_{n\in\omega} U_n$, *and* x *is* Martin-Löf random *if it withstands all Martin-Löf tests.*

Martin-Löf randomness is the most commonly accepted notion of algorithmic randomness. There are several reasons for this. First, if has all the desired "stochastic" properties one would like (see Van Lambalgen [9]). Additionally, it is easy to work with. For instance, there are "universal" Martin-Löf tests: There is a Martin-Löf $\{U_n\}$, $n \in \omega$, such that α is Martin-Löf random iff $\alpha \notin \cap_n U_n$. More importantly, Martin-Löf randomness has other equivalent formulations.

This brings us to the second approach, that of compressibility. We feel that strings should be random iff they are not easy to compress, and hence reals should be random iff all their initial segments are also random. This approach leads to a machine notion of randomness. The Kolmogorov complexity $C(\sigma)$ of a string σ relative to a Turing machine M, is the length of the shortest string τ with $M(\tau) = \sigma$. We call a string random iff $C(\sigma) = |\sigma|$. We'd like to say that a real is random iff all of its initial segments are[2], but this definition is inadequate for infinite strings. Levin [10] and later Chaitin [3] found a remedy for infinite strings via prefix-free machines.

A Turing machine M with domain a subset of Σ^*, is called *prefix free* iff for all σ if $M(\sigma)\downarrow$, then $M(\tau)\uparrow$ for all τ with $\sigma \prec \tau$. (In this paper $\Sigma = \{0,1\}$.) As is well known, prefix free machines are important because the domain of a Turing machine is (effectively) measurable essentially provided that it is prefix free. Thus the notion allows a bridge from effective measure theory to computability. There is a universal prefix free machine M which has the nice property that for all other prefix free machines N, there is a c, such that for all strings σ, $K_M(\sigma) \leq K_N(\sigma) + c$, where K denotes the prefix free Kolmogorov complexity (that is, $K_N(\sigma)$ is $C(\sigma)$ relative to N, *provided* that N is a prefix free machine). Capitalising on

[2]That is there is a constant c such that, for all n, $C(\alpha \upharpoonright n) \geq n - c$. Unfortunately, no real has this property.

this minimality property we fix a universal prefix free machine M, and the subscript on K_M is dropped [3].

DEFINITION 2 (Levin [10], Chaitin [3]). *We call a real* α Chaitin random *iff all of its initial segments are random using prefix free complexity: there is a constant* c *such that, for all* n, $K(\alpha \restriction n) \geq n - c$.

One of the key reasons that Martin-Löf randomness was seen as good was the following theorem of Schnorr.

THEOREM 1 (Schnorr). *A real* α *is Martin-Löf random iff it is Chaitin random.*

The final standard approach used in the study of randomness is via Martingales, which formalize von Mises intuition that randomness should equal unpredictability. A Martingale is a function $f : \Sigma^* \mapsto \mathbb{R}$, such that, for all strings σ,

$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}.$$

We say that a Martingale f succeeds on a real α iff $\lim_s f(\alpha \restriction s) = \infty$. For example suppose that α had the property that every 10th bit was 1. Then, starting with $f(\lambda) = 1$ and keeping $f(\nu) = 1$, until bit 10 where we would have $f(\nu 0) = 0$ and $f(\nu 1) = 2$, etc, we could build a computable Martingale which would succeed on α. Schnorr proved the following: a real x is Martin-Löf random iff no c.e. martingale succeeeds on x. (Here we recall that a real α is c.e. iff it is the limit of a computable (or c.e.) increasing sequence of rationals, and a function $f(\sigma)$ is c.e. iff there is a computable function $\hat{f}(\sigma, s)$ such that for all σ, $(\forall s)\hat{f}(\sigma, s) \leq \hat{f}(\sigma, s+1)$ and $\lim_s \hat{f}(\sigma, s) = f(\sigma)$).

Again we see that there is an equivalence. Schnorr effectivised Ville's observation that null sets are those on which martingales succeed, as Martin-Löf random reals are those on which no c.e. martingales succeed.

We remark that martingales are also important also in that they form the basis of most miniaturizations of randomness and measure to classes like P. (Lutz [12]).

In an important paper [15], Schnorr proved a number of the above results. Additionally, he pointed out a number of deficiencies with the Martin-Löf notion of randomness. To wit, the crucial notion is that a real should be random iff if avoids all *effectively given* null sets. He argued that a Martin-Löf test is more like a *computably enumerably* given set, more akin to the halting problem, than a *computably* given test. Similar deficiencies he argued, can be found in the Martingale equivalence.

To remedy these deficiencies, he proposed the following.

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[3]Here we use C for Kolmogorov complexity and K for prefix free Kolmogorov complexity; sometimes these are denoted K and H respectively.

DEFINITION 3 (Schnorr [14, 15]). *A* Schnorr test *$\{U_n\}$, $n \in \omega$, is a computable collection of c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ and the function $f(n) = \mu(U_n)$ is a computable function of n. A real number x is* Schnorr random *if it withstands all Schnorr tests.*

Schnorr also gave a Martingale characterization of Schnorr randomness, and argued that this notion correctly reflects algorithmic randomness.

While Schnorr's argument has weight, the Schnorr notion of randomness attracted less attention than Martin-Löf randomness. Part of this was because the primary workers in this area found that the Martin-Löf notion was enough for many results. Another important reason, however, was that the notion of Schnorr randomness proved far harder to deal with than Martin-Löf randomness. For instance, there is no universal Schnorr test [4]. Also, many of the basic questions were open. For instance, as we will see, there was no known machine characterization of Schnorr's notion of randomness, and this was seen as a significant obstacle to the notion's development.

In this paper, we will begin a systematic study of Schnorr's notion of randomness. Our first contribution is to provide a machine definition of Schnorr randomness, the existence of which has been a longstanding open question (See e.g. Ambos-Spies and Kucera [1], Ambos-Spies and Mayorodomo [2], etc). These are in terms of a new class of machines miniaturizations of which should be relevant to resource bounded complexity. Specifically, a prefix free machine M is called *computable* iff

$$\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}$$

is a computable real. In Section 2 we prove

THEOREM 5. *A real α is Schnorr random iff for all computable machines M, there is a constant c such that, for all n, $K_M(\alpha \upharpoonright n) \geq n - c$.*

We also establish some other characterizations of Schnorr randomness in terms of Solovay tests. Next we turn to trying to understand the class of Schnorr random reals and, in particular, the class of Schnorr random c.e. reals. For any prefix free machine, the quantity $\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}$ is a typical c.e. real. The most famous such c.e. real is Chaitin's Omega, the halting probability:

$$\Omega_M = \sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}$$

where now M is a universal prefix free machine. In the context of algorithmic randomness, computably enumerable reals occupy the same

[4]This follows from the contradiction that a universal Schnorr test $\{S_n\}$, $n \in \omega$, would contain all computable reals in its null set, but on the other hand there is a computable real in the complement of every Schnorr test's null set (Schnorr [14]).

distinguished place that computably enumerable sets do in the study of decision problems.

Schnorr [14] showed that there is a Schnorr random real that is not Martin-Löf random using a series of results based on Martingale characterizations of randomness. Another argument proving the existence of such a real, this time using a kind of forcing within a universal Martin-Löf test, can be found in van Lambalgen [9]. We demonstrate that such reals can be c.e. This is by no means easy since determining whether a Martin-Löf test is a Schnorr test is a Π^0_2 question [5], and this is difficult to use in an effective construction. Generally this leads to a infinite injury argument as in Theorem 7 below, or or some equivalent rearrangement of Π_2 requirements as in the proof of the following:

THEOREM 6. *There is a Schnorr random c.e. real which is not Martin-Löf random.*

Since the class of Schnorr random c.e. reals differs from the class of c.e. Martin-Löf random reals, it is natural to try to understand them. All Martin-Löf random c.e. reals have the same Turing degree as the halting problem (Kučera [7]). We prove the following.

THEOREM 7. (i) *All Schnorr random c.e. reals have high Turing degree. That is, if α is Schnorr random, then $\alpha' \equiv_T \emptyset''$.*

(ii) *There are Schnorr random c.e. reals α such that α does not have complete Turing degree.*

Of course a consequence of Theorem 7 is another proof that Schnorr random c.e. reals need not be Martin-Löf random. We believe that the degrees of Schnorr random c.e. reals are precisely the high c.e. degrees. It is known that below each high c.e. degree there is a Schnorr random c.e. real ([5]).

The machine characterization of Martin-Löf randomness allows one to calibrate randomness. That is, we say that $\alpha \leq_H \beta$ iff there is a constant c such that, for all n,

$$K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + c.$$

Thus, by the work of Solovay [17], Kučera [7] and of Kučera and Slaman [8], we know that a c.e. real α is random iff $\beta \leq_H \alpha$ for all c.e. reals β.

Inspired by this we can calibrate the complexity of c.e. reals in terms of their Schnorr complexity. That is, we say that $\alpha \leq_{Sch} \beta$ iff for all *computable* machines M, there is a constant c and computable machine $\hat{M}$ such that for all n,

$$K_{\hat{M}}(\alpha \upharpoonright n) \leq K_M(\beta \upharpoonright n) + c.$$

[5]A test set U_n has measure $f(n)$ iff $(\forall i)(\exists s)\ (f(n) - \mu(U_n[s])) < 2^{-i}$, where $U_n[s]$ is a stage-s approximation to the c.e. set U_n.

Clearly a real α is Schnorr random if $\beta \leq_{Sch} \alpha$ for all (c.e.) reals β.

Solovay proved the remarkable fact that there are noncomputable reals α with $\alpha \leq_H \omega = \{1^n : n \in \omega\}$. We call these H-trivial reals. This has been recently improved by Downey, Hirschfeldt, Nies and Stephan to construct c.e. sets with this property, and to show that such reals form an ideal in the c.e. reals. We prove the following.

THEOREM 8. *There are c.e. noncomputable reals α such that $\alpha \leq_{Sch} \omega$.*

Again, the proof is significantly more complex than the corresponding proof for $\leq_H$. Downey et. al. [4] proved that all H-trivial reals are Turing incomplete, and indeed they are not of high degree.

§2. Schnorr Randomness. Schnorr [14] showed that there is a Schnorr random real that is not Martin-Löf random. We use a priority argument approach to show that this result also holds for c.e. reals.

In the proofs we often use the following notation: for any finite binary string σ denote by $[\sigma]$ the set of all infinite extensions of σ, equivalently this is the closed interval $[0.\sigma, (0.\sigma) + 2^{-|\sigma|}] \subseteq [0, 1]$. Also we make use of the following technical result which allows us to simplify the class of Schnorr tests we need to consider.

LEMMA 2. *If $f, g : \omega \to \mathbb{R}$ are non-increasing computable functions with limit 0 and V_n is a (Martin-Löf) test such that $(\forall n)\mu(V_n) = f(n)$, then from an index for V_n we can effectively find a test U_n such that:*

$$\cap_{n\in\omega} V_n = \cap_{n\in\omega} U_n \text{ and } (\forall n)\mu(U_n) = g(n)$$

The proof of this involves manipulation of the c.e. sets to show that the following three steps can be accomplished: (i) choose increasing sequences n_i, and m_i such that such that $f(n_i) > g(m_i) > f(n_{i+1})$, (ii) build U_{m_i} as a superset of $V_{n_{i+1}}$, by adding elements of V_{n_i} but without adding any element to the null set, and (iii) define sets of the appropriate size (measure) between U_{m_i} and $U_{m_{i+1}}$.

The implication of the lemma is that we can choose to deal with Schnorr tests having a particular computable function for the measure of test sets $g(n) = \mu(U_n)$ and still produce all the Schnorr null sets. Henceforth we will assume that all Schnorr tests satisfy not only $\mu(U_n) \leq 2^{-n}$ but in fact $\mu(U_n) = 2^{-n}$.

There is no effective enumeration of all Schnorr tests, as such an enumeration could be used to immediately produce a Universal Schnorr test, which Schnorr showed does not exist [14]. As a result, in many of the proofs below we use an effective enumeration of the Martin-Löf tests, and monitor whether or not each one appears to be a Schnorr test, that is, whether say, V_n^m, the n^{th} c.e. set from the m^{th} Martin-Löf test, appears to satisfy $\mu(V_n^m) = 2^{-n}$.

Before turning to the machine characterisation of Schnorr randomness we show there is another characterisation analogous to one provided by Solovay for Martin-Löf randomness.

DEFINITION 4 (Solovay [17]). *We say that a real x is* Solovay random *iff for all computable collections of c.e. sets U_n, $n \in \omega$, such that $\Sigma_n \mu(U_n) < \infty$, x is in only finitely many U_i.*

Solovay [17] showed that a real is Solovay random iff it is Martin-Löf random. This notion, as with many connected with Martin-Löf randomness, can be directly related to Schnorr randomness if the right way to "increase the effectivity" can be found. (The following definition is equivalent to a definition in terms of martingales mentioned in Wang [18].)

DEFINITION 5. *A* total Solovay test *is a computable collection of c.e. open sets V_i: $i \in \omega$, such that the sum $\Sigma_{i=0}^{\infty} \mu(V_i)$ is finite and a computable real. A real α passes a total Solovay test if $\alpha \in V_i$ for at most finitely many V_i.*

The proof of the theorem below follows that same path as the proof of Solovay's result that Martin-Löf randomness is equivalent to Solovay randomness.

THEOREM 3. *y is Schnorr random iff y passes all total Solovay tests.*

Proof ($\leftarrow$) Suppose y is not Schnorr random, so it fails some Schnorr test $\{U_n\}_{n \in \omega}$. The infinite sum of the measures of these sets is computable, and y is in infinitely many of them so fails the total Solovay test represented by U_n.

($\rightarrow$) Suppose y is Schnorr random. Let $\{U_n\}_{n \in \omega}$ be an arbitrary total Solovay test. We note that $f(n) = \mu(U_n)$ is a computable function, since each $\mu(U_n)$ is left computable and their sum is bounded and computable. Define a c.e. open set $V_k = \{y \in (0, 1) : y \in U_n$ for at least 2^k of the $U_n\}$.

Now $\mu(V_n) < 2^{-n}$ and furthermore $g(n) = \mu(V_n)$ is a computable function of n, as to determine $\mu(V_n)$ to within ϵ we enumerate U_0 till its measure is within $\epsilon 2^{-2}$ of its final value, U_1 to within $\epsilon 2^{-\frac{5}{2}}$, and U_n to within $\epsilon 2^{-\frac{n-4}{2}}$, up to the point where $2^{-n'} < \epsilon 2^{-\frac{n'-4}{2}}$ (we can ignore U_m for $m > n'$). Most (in the sense of at least 'final measure-ϵ') of the elements of V_k are already in V_k defined as in terms of being in at least 2^k of these approximations to each U_n.

As y is Schnorr random, $y \notin \cap_n V_n$ so y is in only finitely many U_i, y passes the total Solovay test. $\dashv$

The following theorem provides the final tool needed to establish the machine characterisation of Schnorr randomness in terms of the *computable machines* defined in the introduction.

THEOREM 4 (Kraft, see [11]). (i) *If A is prefix free then $\Sigma_{n \in A} 2^{-|n|} \leq 1$*

(ii) *(sometimes called Kraft-Chaitin) Let $d_1, d_2, ...$ be a collection of lengths, possibly with repititions. Then $\Sigma_i 2^{-d_i} \leq 1$ iff there is a prefix-free set A with members σ_i and σ_i has length d_i. Furthermore, from the sequence d_i we can effectively enumerate such a set A.*

The proof of part (i) uses the map f which takes σ to the real interval $[\sigma]$ with the right end-point removed, as for a prefix free set the images under f of any two strings in the set do not intersect. Notice that $\mu(f(\sigma)) = 2^{-|\sigma|}$. The infinite sum is equal to the sum of $\mu(f(\sigma))$ and so is less than $\mu([0, 1)) = 1$. An important implication of (ii) is that if we are given an effective enumeration of length-string pairs $\langle d_i, \sigma_i \rangle$ then, provided $\Sigma_i 2^{-d_i} \leq 1$, we can build a prefix-free machine M and a collection of strings τ_i with $|\tau_i| = d_i$ and $M(\tau_i) = \sigma_i$. Now we can prove that the machine characterisation of Schnorr randomness is correct:

THEOREM 5. *A real is Schnorr random iff for every computable machine M,*

$$(\exists d)(\forall n) K_M(x \upharpoonright n) \geq n - d$$

Proof (*Only If* direction) Suppose z is Schnorr random. For any f given by a computable machine M, suppose for the sake of contradiction that $(n - K_M(z \upharpoonright n))$ is unbounded as a function of n. Let $R = \Sigma_{x \in dom(f)} 2^{-|x|}$.

Define $U_k = \{x : \exists n K_M(x \upharpoonright n) \leq n - k\}$. If $\mu(U_k) > \delta$ then there exists a prefix-free subset of $2^{<\omega}$, $\{x_1, ..., x_n\}$, such that $\Sigma_{j=1}^n 2^{-|x_j|} > \delta$ and $K_M(x_j) \leq |x_j| - k$ for all $j = 1, 2, ..., n$, in each case via a p_j with $f(p_j) = x_j$ and $|p_j| \leq |x_j| - k$. We notice that:

$$\Sigma_{j=1}^n 2^{-|p_j|} \geq 2^k \Sigma_{j=1}^n 2^{-|x_j|} > \delta 2^k$$

Now as $\delta 2^k < R$ we have $\delta < R 2^{-k}$, and considering this for all $\delta > 0$ we have $\mu(U_k) \leq R 2^{-k}$. Furthermore $\mu(U_k)$ is a computable function of k as to approximate $\mu(U_k)$ to within ϵ we need only enumerate the strings of the domain of f in order of increasing length $y_1, y_2, ..., y_t$ until $R - \Sigma_{j=1}^t 2^{-|y_j|} < \epsilon 2^k$. We can then determine all possible p_j relevant to the definition of U_k, except some that may provide extra x_i with the sum of $2^{-|x_i|}$ less than ϵ. From some point on the U_k form a Schnorr test giving the contradiction, since $z \in \cap_{k \in \omega} U_k$. Hence $(n - K_M(z \upharpoonright n))$ must be bounded for any such f: $(\exists d) n - K_M(z \upharpoonright n) \leq d$, so $K_M(x \upharpoonright n) \geq n - d$.
(*If* direction) Suppose z is not Schnorr random. Let U_k be a Schnorr test such that $z \in \cap_k U_k$, $U_{k+1} \subset U_k$, and $\mu(U_k) = 2^{-k}$. Represent each U_k as a union of extensions $[\sigma_{k,i}]$ of a prefix-free set $\{\sigma_{k,i} : i \in \omega\}$, such that $g(\langle k, i \rangle) = \sigma_{k,i}$ is a computable function from ω to $2^{<\omega}$. Note that $\Sigma_{a \in \omega} 2^{-|g(a)|} = 2$.

Since $\mu(U_{2n+2}) = 2^{-2n-2}$ we have, for all $i \in \omega$, $|\sigma_{2n+2,i}| \geq (2n+2)$. Consider the collection of 'lengths' $(|\sigma_{2n+2,i}| - n)$ for $n, i \in \omega$. $\Sigma_{n,i\in\omega} 2^{-(|\sigma_{2n+2,i}|-n)} = \Sigma_{n\in\omega}(2^n \Sigma_{i\in\omega} 2^{-|\sigma_{2n+2,i}|}) = \Sigma_n 2^n 2^{-2n-2} = \frac{1}{2}$. We wish to map a string of length $|\sigma_{2n+2,i}| - n$ to the string $\sigma_{2n+2,i}$ for each $i \in \omega$, $n \in \omega$; and by Kraft-Chaitin there is a prefix-free machine M that does precisely this. The partial computable function f defined by M satisfies $\Sigma_{x\in dom(f)} 2^{-|x|} = \frac{1}{2}$. Since $\forall n \exists i\ z \in [\sigma_{2n+2,i}]$, we have $(n - K_M(z\upharpoonright n))$ is unbounded. $\dashv$

The following theorem is a consequence of the fact that there is an incomplete Schnorr random c.e. real, but the direct proof below is considerably simpler than that of Theorem 7.

THEOREM 6. *There is a Schnorr random c.e real which is not Martin-Löf random.*

Proof The following uses a priority construction arranged on a tree. The 'true path' through the tree always moves leftward, so approaches a limit as typical of finite injury arguments, rather than the true outcomes being defined as a lim inf. This is essentially because Π^0_2 conditions have been split into countably many Σ^0_1 conditions distributed down the tree.

Let V^m_n be the n^{th} set from the m^{th} Martin-Löf test. We select a single set from each test: specifically let $Z_n = V^n_{2n+2}$. Note that $\mu(Z_n) \leq 2^{-2n-2}$ and that if V^m is a Schnorr test then $\mu(Z_n) = 2^{-2n-2}$. We construct a Schnorr random real x by ensuring that if the measure of Z_n is precisely 2^{-2n-2} then $x \notin Z_n$.

We will define a collection of closed sets X_n, let $x_n = \min(X_n)$, and the c.e. real constructed will be $x = \lim_n x_n$.

Requirements

R_i : $(\forall n \leq i)[\mu(Z_n) > 2^{-2n-2}(1 - 2^{-2i-4}) \rightarrow \mu(X_n \cap Z_n) < 2^{-2n-2}2^{-2i-4}]$

These requirements will be satisfied in such a way that if a set Z_n is from a Schnorr test then not only will its intersection with X_i have measure 0, but the intersection will be empty.

The Tree

Each node of length i has a strategy working to satisfy R_i. The outcome of an R_i strategy at stage s is the number of sets $Z_0, Z_1, ..., Z_j$ that have exceeded the proportion $1 - 2^{-2i-4}$ of their maximum possible measure. Formally this is:

$$|\{n :\ 0 \leq n \leq i,\ \mu(Z_n[s])/(2^{-2n-2}) > 1 - 2^{-2i-4}\}|$$

The tree $T \subset \omega^{<\omega}$ used in the construction has finitely many branches below each node. In fact a node of length l has outcomes $\{0, 1, ..., l+1\}$ with the ordering $l+1 < l < ... < 0$.

At each stage s we define an apparent true path θ_s, and this moves leftward only. The limit of the apparent true path is the true path, and codes how close test sets Z_n are to the maximum measure possible, 2^{-2n-2}.

As the outcome of a strategy for R_i depends only on i, we define $\theta_s(i)$ directly (for $i \leq s$) as the outcome of R_i described above.

CONSTRUCTION

For the tree construction we will define a set X_i at a node of length i; each set X_i will be a finite union of closed intervals and have measure 2^{-i-1}.

Stage 0:

If the R_0 outcome is 0 then $X_0 = [\frac{1}{8}, \frac{5}{8}]$. If the outcome is 1 then $X_0 \subseteq [\frac{1}{8}, \frac{7}{8}] - Z_0[s]$, X_0 is defined as far to the left as possible, that is, the left endpoints of the intervals making up X_0 are the least possible, and X_0 has measure a half.

Stage $s > 0$: Proceed down the apparent true path defining X_i at a node of length i for each $i \leq s$ (X_i could therefore be described as an approximation $X_i[s]$ to the final form of the set).

For each $n \leq i$, let $\hat{Z}_n = Z_n[s]$ if $\mu(Z_n[s]) > 2^{-2n-2}(1-2^{-2i-4})$, $\hat{Z}_n = \emptyset$ otherwise ($\hat{Z}_n$ depends on i as well as n). Using intervals as far to the left as possible define $X_i \subset X_{i-1} - \cup_{n \leq i} \hat{Z}_n$.

This completes the construction of x. There is always enough of X_{i-1} available to define X_i as

$$\begin{aligned}
\mu(X_{i-1} - (\cup_{n=0}^{i} \hat{Z}_n)) &\geq \mu(X_{i-1} - (Z_i \cup \cup_{n=0}^{i-1}(\hat{Z}_n))) \\
&\geq \mu(X_{i-1}) - [2^{-2i-2} + \Sigma_{n=0}^{i-1} 2^{-2i-2}] \\
&\geq 2^{-i} - [2^{-2i-2} + i2^{-2i-2}], \text{ and as } ix \leq 2^{i-1}x \\
&\geq 2^{-i} - [2^{-2i-2} + 2^{-i-3}] \\
&\geq 2^{-i}[1 - (2^{-2} + 2^{-3})] \\
&> 2^{-i-1}
\end{aligned}$$

The "inductive condition" that $\mu(X_{i-1} \cap \hat{Z}_n) < 2^{-2i-2}$ is given by the requirement at the node above that X_{i-1} intersects with $Z_n[s]$ on a set of measure at most $2^{-2n-2}2^{-2(i-1)-4}$.

The c.e. real x is Schnorr Random as as if V^m is a Schnorr test then $x \notin Z_m$ and $x \notin \cap_{n \in \omega} V_n^m$. To see that x is not Martin-Löf random we describe a Martin-Löf test U_n such that $x \in \cap_n U_n$.

Although the number of nodes in the tree increases exponentially as we proceed down the tree, the number of different definitions given for X_n is bounded by a polynomial. Consider X_0: it has at most two definitions, as only when $\mu(Z_0[s])$ passes a certain threshold does it change. X_1 may change twice because of Z_0 (two different thresholds) and once because of Z_1. In general X_n may *change* $1+2+...+(n-1) = \frac{1}{2}(n+1)(n+2)$ times, so if we let X_n^* be the union of all sets X_n defined in the construction, then for all n, $\mu(X_n^*) \leq p(n)2^{-n}$ where $p(n) = 1 + \frac{1}{2}(n+1)(n+2)$. Hence we can effectively choose U'_n, a subsequence of X^*_n, such that $\mu(U'_n) \leq 2^{-n-1}$. This is a computable collection of c.e. *closed* sets but can be converted into a collection of c.e. open sets $\{U_n\}$, $n \in \omega$, $U_n \supseteq U'_n$, and $\mu(U_n) \leq 2^{-n}$. Then $x \in \cap_n U_n$ shows that x is not Martin-Löf random.

$\dashv$

We use the Solovay-type characterisation of Schnorr randomness, as well as a direct test-set argument, to prove

THEOREM 7. (i) *All Schnorr random c.e. reals have high Turing degree. That is, if α is Schnorr random, then $\alpha' \equiv_T \emptyset''$.*
(ii) *There are Schnorr random c.e. reals α such that α does not have complete Turing degree.*

Proof Part (i): Given an arbitrary c.e. real α we show using the limit lemma and the Solovay-type characterisation of Schnorr randomness that if α is Schnorr random then *Tot*, the index set of total recursive functions, is Turing reducible to α'.

Let α_s be a strictly increasing c.e. sequence of rationals with limit α, a Schnorr random real. Let T be a c.e. set such that $\langle i, j\rangle \in T$ iff $|W_i| > j$. We show that there exists a computable reduction $\hat{\Gamma}$ with $\hat{\Gamma}^{\alpha'} = Tot$, by constructing Γ such that $\Gamma^{\alpha}(i,k) = T(\langle i,k\rangle)$.

We define a single reduction Γ, but define for each $i \in \omega$ a total Solovay test: a collection of open rational intervals $Q^i = \{Q^i_0, Q^i_1, Q^i_2, ...\}$.

Proceed in stages $s \in \omega$. At each stage s there are two main steps: defining an axiom for the least argument $\langle i,k\rangle$ not yet dealt with, and checking whether α_s has changed on the use of previously defined axioms.

- Find the least pair $\langle i,k\rangle$ for which Γ has no axiom. Set $\Gamma^{\alpha_s}(i,k) = T_s(\langle i,k\rangle)$ with use $\gamma(i,k) = k$. If this value is 1, that is $\langle i,k\rangle$ has already entered T_s, then add the interval $(0, 2^{-k})$ to Q^i (these intervals are added simply to make it easy to show later that Q^i is a *total* Solovay test).
- Consider all Γ axioms that have just been "injured" by a change from α_{s-1} to α_s. Reset Γ on the injured arguments i,k to the current value $T_s(\langle i,k\rangle)$ with use $\gamma(i,k) = k$. If $\langle i,k\rangle$ has just entered T_s, then add the interval $(0, 2^{-k})$ to Q^i. If $\Gamma^{\alpha_s}(i,k) = 0$ for a pair i,k, but $T_s(\langle i,k\rangle)$ has changed from 0 to 1, then add the interval $(\alpha_s{\restriction}\langle i,k\rangle, \alpha_s{\restriction}\langle i,k\rangle + 2^{-k})$ to the test Q^i (these are the really crucial intervals in relation to α).

If $i \notin Tot$ then we will define $\Gamma^{\alpha}(i,k)$ to be zero for all but finitely many k, and the limit in k exists and is correct, equal to 0.

Each Solovay test Q^i contains either finitely many intervals or, if $Tot(i) = 1$, an interval of width 2^{-k} for each $k \in \omega$. In either case it is a total Solovay test.

If $i \in Tot$ then as α is Schnorr random it will move out of all but finitely many of our sets $\{Q_n^i\}_{n\in\omega}$ so we will get the chance to reset $\Gamma(\langle i,k\rangle)$ to 1 if necessary in all but finitely many cases. Hence Γ will have the correct limit in k.

$$(\forall i)\ \lim_k \Gamma^\alpha(i,k) = \lim_k T(\langle i,k\rangle) = Tot(i)$$

This concludes the proof of part (i)

Part (ii): this proof is quite different in character; it is an infinite injury priority argument in which we use an enumeration of all Martin-Löf tests and act on those that appear to be Schnorr tests.

We construct a c.e. real α and a c.e. set C such that α is Schnorr random but $C \not\leq_T \alpha$.

Let V_k^e denote the k^{th} test set of the e^{th} Martin-Löf test in some effective enumeration of all such tests. For each test V^e we have $(\forall k)\mu(V_k^e) \leq 2^{-k}$. There is no effective enumeration of all Schnorr tests. In particular the set $\{e : (\forall k)\mu(V_k^e) = 2^{-k}\}$ is not computably enumerable, therefore during the construction we continuously monitor Martin-Löf test sets V_k^e to see if they appear to be part of a Schnorr test. This monitoring consists of checking whether an approximation to V_k^e at stage s, $V_k^e[s]$, has measure greater than $2^{-k}(1-2^{-n})$ for increasingly large n. Note that V_k^e has measure precisely 2^{-k} iff it satisfies the following Π_2 condition: $(\forall n)(\exists s)\ \mu(V_k^e[s]) > 2^{-k}(1-2^{-n})$.

To be Schnorr random α must satisfy $(\forall e)[((\forall k)\mu(V_k^e) = 2^{-k}) \rightarrow \alpha \notin \cap_{k\in\omega} V_k^e]$, but we will construct α to satisfy the stronger condition $(\forall e)(\exists k)[\mu(V_k^e) = 2^{-k} \rightarrow \alpha \notin V_k^e]$. Without loss of generality we may assume that no set V_k^e is enumerated in finitely many stages, that is $(\forall s)V_k^e[s] \neq V_k^e$. This property is technically useful in the construction.

Naturally we must show there is an increasing sequence of rationals α_s with limit α, but because it is of critical importance to the construction how much of the unit interval is still available for α_t at stages $t > s$ (not eliminated by test sets or other restrictions at stage s) we define a set $X_s \subset (0,1)$ of such "available space" at each stage. Each set X_s is a finite collection of closed intervals with rational end points, and α_s is the least element of X_s.

The construction will start with X_0 defined as $[\frac{1}{4}, \frac{3}{4}]$, or any closed interval with rational endpoints contained in $(0,1)$. At stage s, we build the set X_{s+1} by removing open intervals from X_s so that $X_{s+1} \subseteq X_s$, and $\alpha_{s+1} \geq \alpha_s$.

Requirements

$R_e\ :\ (\exists k)[\mu(V_k^e) = 2^{-k} \rightarrow V_k^e \cap (\cap_{s\in\omega} X_s) = \emptyset]$

$N_i\ :\ \Phi_i^\alpha \neq C$

Strategy for a single requirement

R_e: Pick $k \in \omega$ such that $2^{-k} < \mu(X_0)$ and ensure that if $\mu(V_k^e) = 2^{-k}$ then $(\cap_{s\in\omega} X_s) \cap V_k^e = \emptyset$.

This is achieved by requiring $X_{s+1} \subseteq X_s \cap \overline{V_k^e[s]}$ at every stage s in which $\mu(V_k^e[s]) > 2^{-k}(1 - 2^{-n})$ for some $n \leq s$ larger than any $n < s$ for which the condition was true at an earlier stage. Call such a stage s a *growth stage* for V_k^e. Thus $\mu(V_k^e) = 2^{-k}$ iff V_k^e has infinitely many growth stages.

At stage s, R_e has outcome ∞ if s is a growth stage for V_k^e, it has outcome f (finite) otherwise.

N_i: A Friedberg-Muchnik strategy.

1. Pick a witness z, targeted for C, larger than any witness previously chosen.
2. Wait for $\Phi_i^{\alpha_s}(z) \downarrow= 0$
3. Enumerate z into C and preserve σ, the use of the convergent computation and an initial segment of α_s, by setting $X_{s+1} \subseteq [\sigma] \cap X_s$.

Outcome: N_i has outcome w (wait) until it completes step 3, if ever, at which point it changes permanently to outcome s (stop).

Combining Two Requirements

All that is required to combine two test set requirements R_a and R_b is to choose k_a and k_b large so that $V_{k_a}^a$ and $V_{k_b}^b$ are small compared to $\mu(X_0)$. The precise condition is $\mu(X_0) - (2^{-k_a} + 2^{-k_b}) > 0$. This ensures that X_{s+1} is never empty, though all elements of $V_{k_a}^a$ and $V_{k_b}^b$ may eventually be excluded from some set X_{s+1}.

The preservation strategies for two negative requirements N_i and N_j, as defined above, do not conflict.

The interaction of two requirements R_e and N_i illustrates the main conflict in the construction. Let V_k^e be the test set R_e is monitoring, and suppose it does have measure 2^{-k}. At some stage s, N_i may wish to preserve the use σ of a computation, where σ is an initial segment of α_s. This means, in effect, continuing the rest of the construction within the cone $[\sigma]$ of infinite extensions of σ. (Equivalently $[\sigma]$ is the closed interval $[0.\sigma, 0.\sigma + 2^{-|\sigma|}]$.) The restriction is a problem if $[\sigma]$ is small relative to the effect that R_e may have in the future via V_k^e. Formally, if $\mu([\sigma] \cap X_s) \leq 2^{-k} - \mu(V_k^e[s])$ then setting $X_{s+1} \subseteq X_s \cap [\sigma]$ may result in a situation where eventually V_k^e eliminates all of X_{s+1}. That is, we risk the possibility that $X_{s+1} \cap \overline{V_k^e} = \emptyset$.

If we consider N_i to be of higher priority than R_e we can handle the situation by requiring R_e to choose a new and much larger $k' > k$ when N_i wishes to preserve a computation. This new k' must satisfy $2^{-k'} < \mu([\sigma] \cap X_s)$ so that it is impossible for the new set $V_{k'}^e$ to eliminate all of $X_{s+1} = [\sigma] \cap X_s$. Subsequently in the construction we work with $V_{k'}^e$ and ignore V_k^e, though it has already had a partial effect since $X_s \subseteq \overline{V_k^e[s-1]}$, and we do not attempt to reverse this. Switching from V_k^e to a smaller test set $V_{k'}^e$ a finite number of times is an *essential modification to* R_e for the full construction below.

If R_e is of higher priority than N_i then we must modify the N_i strategy. N_i will preserve a use σ if, and only if, it has space within $[\sigma]$ to do so considering the future impact of V_k^e. Otherwise N_i will destroy its own computation by increasing α_s by at least $2^{-|\sigma|}$. *Modified strategy for* N_i:

1. Pick a witness z, targeted for C, larger than any witness previously chosen. Choose a rational $r > 0$ such that $r < \frac{1}{2}(\mu(X_0) - 2^{-k})$.
2. Wait for $\Phi_i^{\alpha_s}(z) \downarrow = 0$ and $2^{-k} - \mu(V_k^e[s]) < r$. Let σ be the use of the computation.
3. (a) If $\mu([\sigma] \cap X_s) \leq 2^{-k} - \mu(V_k^e[s])$ then destroy the computation by setting $X_{s+1} \subseteq X_s - [0.\sigma, 0.\sigma + 2^{-|\sigma|})$. Reset r to half its current value. Return to step 2.
   (b) If $\mu([\sigma] \cap X_s) > 2^{-k} - \mu(V_k^e[s])$ then enumerate z into C and preserve σ by setting $X_{s+1} \subseteq [\sigma] \cap X_s$. Stop.

The outcome of N_i is w until it completes step $3(b)$, when it changes to s. If N_i destroys its computation infinitely often the limit r on measure removed from the sequence X_s halves each time. Thus the total loss because of N_i is less than twice the original r value, so less than $\mu(X_0) - 2^{-k}$. This means that the combined effects of R_e and N_i do not result in any set X_{s+1} being empty, whatever the long term behaviour of N_i.

With this modified strategy two negative requirements N_i and N_j may now conflict: one attempting to preserve a use whilst the other attempts to increase α_s. A negative requirement attempting to move α_s can be handled in much the same way as a test set requirement R_e, the details will be described in the full construction.

Combining infinitely many requirements

The interactions between strategies naturally increase in complexity when we attempt to satisfy all $\{R_e\}_{e\in\omega}$ and $\{N_i\}_{i\in\omega}$ simultaneously. There are four main elements to managing the interaction:

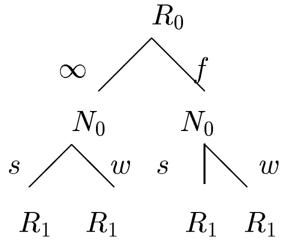
- Choose the test sets V_k^e such that the sum of the measures of all test sets mentioned in the construction is finite, and less than $\mu(X_0)$.
- Modify the strategy for N_i to take into account all other requirements, including negative requirements that may move α_s. Specifically, alter the inequalities involving r and the condition for choosing 3(a) or 3(b). The basic idea for N_i will remain unchanged: preserve the computation if this is feasible, otherwise destroy the convergent computation by moving α_s.
- Distinguish between the effect N_i has on X_{s+1} when it has outcome w (these effects increase α_s so must be permanent) and the restrictions made when it has outcome s. The ceiling for the construction of

$0.\sigma + 2^{-|\sigma|} = \max([\sigma])$ at step 3(b) may turn out to be temporary if N_i is later injured by another requirement.

- Use a tree of strategies so that each strategy has a guess about the outcome of higher priority requirements.

Tree of strategies.

To combine all the requirements we use a binary tree $T \subset \{\infty, f, s, w\}^{<\omega}$ with strategies for R_e at level $2e$ and for N_i at level $2i+1$. We order the outcomes $\infty < f < s < w$, so the tree is as follows:



Denote by R_γ the strategy for R_e at node γ, where $|\gamma| = 2e$. Similarly N_γ is a strategy for the requirement N_i if $|\gamma| = 2i+1$.

At each stage $s \in \omega$ we will proceed in substages $t \leq s$. At substage t the strategy at $(\theta_s \restriction t)$ on the tree is eligible to act, where $|\theta_s| = s$ and θ_s is defined inductively: $(\theta_s \restriction 0) = \lambda$, if $(\theta_s \restriction t)$ is defined and the strategy at $(\theta_s \restriction t)$ has outcome $O \in \{\infty, f, s, w\}$, then $(\theta_s \restriction (t+1)) = (\theta_s \restriction t)^\frown \langle O \rangle$. We refer to θ_s as the *apparent true path* at stage s. Ultimately the *true path* $TP \in \{\infty, f, s, w\}^\omega$ is the left most path visited infinitely often, $TP(n) = \liminf_s \theta_{n+s}(n)$

As before we have a sequence of closed sets X_s representing the part of the unit interval available for α_t, $t > s$. Let $\alpha_s = \min(X_s)$ and $\alpha = \lim_s \alpha_s$. The set X_{s+1} will still exclude approximations $V_k^e[s]$ to any test set monitored by R_e which appears to have measure 2^{-k}. X_{s+1} will also exclude any interval $[\sigma]$, where σ is the use of a computation some strategy N_γ wishes to destroy.

However if a strategy N_γ wishes to preserve a use σ, then instead of setting $X_{s+1} \subseteq [\sigma] \cap X_s$ the strategy will instead define a "ceiling value" of $c_\gamma = 0.\sigma + 2^{-|\sigma|}$ to be applied to the rest of the construction. (Nominally $c_\gamma = 1$, for all strategies N_γ, unless c_γ is specifically redefined.) Let C_s be the least value c_γ currently defined for any strategy N_γ at the *beginning* of stage s, so $0 < C_s \leq 1$. The construction will in future work within $Y_{s+1} = X_{s+1} \cap [0, C_{s+1}]$.

The reason for this change is that N_γ may be *initialized* by another strategy, and we will wish to disregard the current value for c_γ thereafter, but not disregard X_s.

Initializing N_γ means (i) setting $c_\gamma = 1$, (ii) canceling the follower z, and (iii) returning the strategy to step 1. (N_γ will have outcome w when next visited, even if previously it had outcome s.)

The strategy R_γ, $|\gamma| = 2e$ also has a new parameter for the full construction, d_γ, a specification of which test sets should be considered "small" by R_γ. Initially $d_\gamma = 1$ for all strategies R_γ, but this value will be reduced for strategies visited during the construction. Whenever R_γ chooses a value k to specify its test set V_k^e, it will ensure $2^{-k} < d_\gamma$

Initializing R_γ means canceling its choice of test set; initialization does *not* change d_γ, nor does it reduce the minimum n for which $s \geq n$ may be a growth stage for this strategy.

Notation: let γ^- denote the predecessor of a non-empty string $\gamma \in T$, that is $\gamma^- = \gamma\restriction(|\gamma| - 1)$. Let $T_R \subset T$ be the strings in T of even length (working on $\{R_e\}_{e\in\omega}$ strategies). Let $T_N \subset T$ be the strings of odd length (working on strategies $\{N_i\}_{i\in\omega}$). The use of substages means we need to define X_s^t and Y_s^t which are analogous to X_s and Y_s but represent the situation at the beginning of substage t of stage $s \geq t$. So, $X_s^t \subseteq X_s$ is the result of modification of X_s at substages $t < s$, that is it excludes the intervals eliminated from X_s by earlier substages. $X_s^0 = X_s$, and X_{s+1} will be X_s^s modified by the final substage of stage s. Similarly C_s^t is defined as the least current value c_γ excluding those for γ to the right of $(\theta_s\restriction(t-1))$ (they will be reset to 1 by initialization at the end of stage s). At the beginning of substage t let $Y_s^t = X_s^t \cap [0, C_s^t]$.

We wish to ensure that if a strategy N_γ has outcome w infinitely often, and moves α_s infinitely often, then the total effect (in terms of reduction of measure of the sets X_s) is no greater that $\frac{1}{2}d_{\gamma^-}$. The intention is that the total effect of all negative strategies with true outcome w is no more than the total effect of all strategies R_γ visited in the construction. Thus if $2(\Sigma_{\gamma\in T_R} d_\gamma) < \mu(X_0)$ we will never face the possibility that $\mu(X_s) = \emptyset$.

Whereas the test set strategies R_δ are fairly passive in their actions, a negative requirement N_γ must, whenever visited, consider the future impact of all strategies above and to the right of γ. These are the strategies that could undermine an attempt by N_γ to preserve a computation, but would never actually initialize N_γ.

To monitor the potential effects of strategies at γ^-, $(\gamma^-)^-$ etc, we define for all strategies N_γ on the apparent true path θ_s a quantity A_γ representing an upper limit on the possible measure that can in future be removed from X_s by the strategies above N_γ (without N_γ being initialized). The strategies above N_γ are $R_\lambda, N_{(\gamma\restriction 1)}, R_{(\gamma\restriction 2)}, ..., N_{(\gamma^-)^-}$, and R_{γ^-}. The definition is inductive on the length of γ.

If $|\gamma| = 1$ there are two cases:

- $A_{\langle\infty\rangle} = 2^{-k} - \mu(V_k^0)$, where V_k^0 is the current R_λ choice of test set.

- $A_{\langle f\rangle} = 0$, as the negative strategy below this finite outcome believes that V_k^0 will have no further growth stages (otherwise $N_{\langle f\rangle}$ will be initialized).

If $|\gamma| = 2i+1$, $i > 0$, let $B = \min\{r_{(\gamma^-)^-}, A_{(\gamma^-)^-}\}$, and let V_k^e be the test set currently chosen by R_{γ^-}.

- If $\gamma = ((\gamma^-)^-)^\frown\langle s\infty\rangle$, then $A_\gamma = A_{(\gamma^-)^-} + (2^{-k} - \mu(V_k^e[s]))$
- If $\gamma = ((\gamma^-)^-)^\frown\langle sf\rangle$, then $A_\gamma = A_{(\gamma^-)^-}$
- If $\gamma = ((\gamma^-)^-)^\frown\langle w\infty\rangle$, then $A_\gamma = A_{(\gamma^-)^-} + 2B + (2^{-k_{\gamma^-}} - \mu(V_k^e[s]))$
- If $\gamma = ((\gamma^-)^-)^\frown\langle wf\rangle$, then $A_\gamma = A_{(\gamma^-)^-} + 2B$

Full strategies for the tree construction

Strategy R_γ, $|\gamma| = 2e$.

1. If R_γ has not chosen k since the start of the construction, or since R_γ was last initialized, then (i) choose k larger than any previous k and such that $2^{-k} < d_\gamma$ and (ii) reduce d_δ to $\frac{1}{16}(d_\gamma)$ for each of the four requirements R_δ with $|\delta| = |\gamma| + 2$ (if this is not a reduction for some d_δ then leave it unchanged at the lower value).
2. If s is a growth stage for V_k^e then set $X_{s+1} \subseteq X_s \cap \overline{V_k^e[s]}$.

Strategy N_γ, $|\gamma| = 2i+1$.

1. Choose large z. Choose a rational r, $0 < r < \frac{1}{4}d_{\gamma^-}$. Modify d_δ for all δ above and to the right of γ so that their sum is less than $\frac{1}{3}(\frac{1}{4}d_{(\gamma^-)} - r)$.
2. Wait for $\Phi_i^{\alpha_s}(z)\downarrow= 0$ and $A_\gamma < r$. Let σ be the use of the computation.
3. (a) If $\mu([\sigma] \cap Y_s^t) \leq A_\gamma$ then destroy the computation by setting $X_{s+1} \subseteq X_s - [0.\sigma, 0.\sigma + 2^{-|\sigma|})$. Reset r to half its current value, or the lesser value of $\frac{1}{2}A_\gamma$ if $A_\gamma > 0$. Return to step 2.
   (b) If $\mu([\sigma] \cap Y_s^t) > A_\gamma$ then preserve the computation by setting $c_\gamma = \sigma + 2^{-|\sigma|}$. Reset d_δ for δ above, below and to the right of γ so that the total of all such d_δ is less than $\frac{1}{3}(\mu([\sigma] \cap Y_s^t) - A_\gamma)$. Stop.

Modifying d_δ, $\delta \subset \gamma$, for the sake of N_γ does *not* affect the current choice of test set for the (higher priority) strategy R_γ. It will only have an effect later if R_γ is initialized and then revisited. However this reduction in d_δ will influence $N_{\delta^\frown\langle f\rangle}$ if this is currently to the right of θ_s and is visited at a later stage.

Limiting the sum of all d_δ in a specified tree is relatively simple because of the action of step (1) of each strategy R_δ. It resets four $d_{\delta'}$ values below it to $\frac{1}{16}d_\delta$, and the cumulative effect of this as we proceed down paths in this part of the tree is to ensure the sum never exceeds $d_\delta(1 + \frac{1}{4} + (\frac{1}{4})^2 + \ldots) = \frac{4}{3}d_\delta$. So, if there are n values d_δ currently less than 1 in a specified part of the tree, and we wish to limit the sum of these values and all $d_{\delta'}$ below them to q, we simply reduce the n values to have sum less than

$3q/4$. In the construction we generally wish to allow for N-strategy action of equal magnitude, so we would reduce the sum of the n values to less than $3q/8$. The initial value of r is chosen in step 1 so that each strategy N_η has an effect in terms of measure of *less than a half* of d_{η^-}, thus limiting the potential effect of the two N strategies below d_{η^-} to d_{η^-}.

The results of strategy N_γ step 3(b) put Y_s^{t+1} and the future scale of action on X_s (below and to the right of the current true path) in a similar relationship to that of $\mu(X_0)$ and the whole construction at stage 0; the combined effect of R and N requirements is less than $2/3$ of the measure initially available for our c.e. real.

This relationship is the purpose of much of the technical detail of the construction. In the case of step 3(a) the amount of measure lost to Y_s^t is less than A_γ, which is turn is less than r. Each time this occurs the value of r halves, so the total effect is less than twice the initial r, and the construction is set up to deal with N_γ having this effect (this is why initially $r < \frac{1}{4}d_{\gamma^-}$). In the case of step 3(b) the measure remaining after the computation is preserved is $\mu([\sigma] \cap Y_s^t)$. Many d_δ values are reduced so that while the requirements above N_γ may use A_γ of this amount, the R requirements to the right and below N_γ use less than a third of the remainder, i.e. less than $\frac{1}{3}(\mu([\sigma] \cap Y_s^t) - A_\gamma)$; the N requirements to the right and below N_γ are limited to the same amount. Thus the R and N requirements combined cannot use all of Y_s^t.

CONSTRUCTION

At stage 0

Set $X_0 = [\frac{1}{4}, \frac{3}{4}]$, and since $C_0 = 1$ this implies $Y_0 = X_0 \cap [0, C_0] = X_0$. Set $d_\lambda = \frac{1}{8}$ and allow R_λ (the unique strategy for R_0) to act.

At stage $s > 0$

(A) Substages 0 to s: Proceed down the apparent true path θ_s allowing one strategy of length t to act at each substage $t \leq s$.

(B) Initialize all strategies to the right of θ_s. Find the left-most string β of even length such that R_β has never been visited, but $R_{(\beta^-)^-}$ has been visited. Modify the values d_δ on and to the right of θ_s so that their sum is less that $\frac{1}{3}d_\beta$. Halve the value of d_β.

END OF CONSTRUCTION

Much of the work is accomplished in a standard way by the tree: we expect each strategy on the true path TP so satisfy its requirement, and each strategy on the true path is initialised only finitely often, by the definition of TP and the fact that only strategies to the right of the apparent true path θ_s are initialised at stage s.

VERIFICATION

It follows from the construction that $(\forall s)X_{s+1} \subseteq X_s \subseteq [\frac{1}{4}, \frac{3}{4}]$, therefore, provided none of the sets is empty, the least elements $\alpha_s = \min(X_s)$ form an increasing sequence with limit α a c.e. real no larger than $\frac{3}{4}$.

Lemma 1 For all $s \in \omega$, $\mu(X_s) > 0$.

Proof $\mu(X_0) -$ (total R_e effect) $-$ (total N_i effect)
$\geq \mu(X_0) - (\Sigma_{\gamma \in T_R} d_\gamma + \Sigma_{\gamma \in T_N} \frac{1}{2} d_{\gamma^-})$
$\geq \frac{1}{2} - (\frac{1}{6} + \frac{1}{6}) > 0$

To see that $\Sigma_{\gamma \in T_R} d_\gamma \leq \frac{1}{6}$ we note that $d_\lambda \leq \frac{1}{8}$, and that every strategy R_γ visited reduces d_δ for the four R strategies immediately below it to a sixteenth of the value of d_γ. So the sum is less that $\frac{1}{8}(1 + 4\frac{1}{16} + 4^2(\frac{1}{16})^2 + \ldots) = \frac{1}{6}$.

It is for the sake of this property that d_β and many d_δ were modified at the end of each stage s of the construction: these modifications ensure that injury to the right of the apparent true path does not result in the total effect of requirements initialised and later visited exceeding the initial estimate of $\frac{8}{3} d_\lambda$.

Lemma 2 For all s, $Y_s \neq \emptyset$.

Proof Since $X_s \neq \emptyset$ the only problem for Y_s is if some value c_γ is set too low. However when c_γ is set by N_γ at set 3(b) the strategy has checked that there is enough space (measure) in $\mu([\sigma] \cap Y_s^t)$ to continue the construction. Unless N_γ is initialised then no strategy can ever increase α_s up to c_γ. This is because without N_γ initialisation the effect of strategies above is limited to $A_\gamma < \mu([\sigma] \cap Y_s^t)$, and the effect of R strategies below N_γ or to the right of θ_s is restricted to $\frac{1}{3}(\mu([\sigma] \cap Y_s^t) - A_\gamma)$, as is the effect of the N strategies below or to the right.

Lemma 3 For all $\gamma \in T_N$, if $\gamma \subset TP$ then $A_\gamma \to 0$ as $s \to \infty$

Proof Strategies on the true path have the correct guess about the final outcome of the requirements above, and by induction on the length of $\gamma \subset TP$, each element in the definition of A_γ tends to zero as required. That is, $A_{(\gamma^-)^-} \to 0$, $B \to 0$, and $(2^{-k} - \mu(V_k^e)) \to 0$ if γ^- has true outcome ∞.

Lemma 4 Each strategy $\gamma \subset TP$ satisfies its requirement.

Proof Proceed by induction on the length of γ, and consider each strategy after the last occasion on which it was initialised.

Case 1: $\gamma \in T_R$.

R_e is satisfied because k is now fixed for R_γ and if R_γ has true outcome ∞ then $(\cap_{s \in \omega} X_s) \subseteq \overline{V_k^e}$.

Case 2: $\gamma \in T_N$.

N_γ satisfies N_i, $|\gamma| = 2i + 1$. Since $A_\gamma \to 0$ the strategy never becomes stuck at step 2 if $\Phi_i^{\alpha_s}(z) \downarrow = 0$. So if diagonalisation seems necessary N_γ will complete either 3(a) or 3(b) successfully. If 3(a) is chosen infinitely often then $\Phi_i^\alpha(z) \uparrow$ as the use of the computation on (fixed) z changes infinitely often. $\dashv$

The result below on Schnorr trivial reals is stated in terms of computable machines, and is proved by 'building' the required computable machines. There is no effective enumeration of computable machines, just as there is no effective enumeration of Schnorr tests. Hence we need to use an enumeration of all prefix-free machines and monitor approximations to the Π^0_2 condition that the machine is a computable machine. This produces an infinite injury tree construction.

DEFINITION 6. *The* size $\nu(M)$ *of a machine* M *is a property of its domain:* $\nu(M) = \Sigma_{\tau \in dom(M)} 2^{-|\tau|}$. *If the sum is not convergent we say* M *has infinite size.*

Remark The size of any prefix-free machine is between 0 and 1, and is often referred to as the halting probability. Any machine F with finite size is equivalent to a machine P with the same range and size between 0 and 1 in the sense that

$$(\exists c \in \omega)(\forall \tau \in ra(F)) \;\; K_P(\tau) = K_F(\tau) + c$$

The machine P can be obtained simply by letting $P(1^c\sigma) = F(\sigma) = \tau$ where $\nu(F) < 2^c$. Furthermore, by the effective version of Kraft-Chaitin the machine P can be constructed to be prefix-free using not $1^c\sigma$ but a string of *length* $c + |\sigma|$ in the domain of P, for each σ in the domain of F.

Finally, if $\nu(F)$ is computable and τ_0 is in the range of F then P can be prefix-free and have size 1. The domain size of P can be increased by $1 - 2^{-c}\nu(F)$, also a computable number, by adding axioms of the form $\langle \gamma, \tau_0 \rangle$ for a c.e. set of γ of the appropriate length. We can ensure that $K_P(\tau_0) \not< K_F(\tau_0) + c$ by requiring for all γ that $|\gamma| \geq K_F(\tau_0) + c$.

THEOREM 8. *There are c.e. non-computable reals* α *such that* $\alpha \leq_{Sch} \omega$. *That is, for every computable machine* M *there is a computable machine* M' *satisfying* $(\forall n) K_{M'}(\alpha \restriction n) \leq K_M(1^n) + c_M$.

Proof We represent α by the set A. By the *remark* on machine sizes it is sufficient to consider only prefix-free machines M_i of size 1 when constructing A, since if the conclusion fails for any computable machine then it fails for a computable machine of size 1. We will construct for each such M_i a computable machine M'_i. Since we do not have an effective enumeration of the prefix-free machines with size 1, we must use an enumeration $\{M_i\}_{i\in\omega}$ of all prefix-free machines and monitor the size of $M_i[s]$, the stage-s approximation to M_i, to see if it appears to be approaching 1.

Requirements

$R_i \; : \; (\exists c_i \in \omega)(\forall n) \;\; K_{M'_i}(A \restriction n) \leq K_{M_i}(1^n) + c_i$

$P_e \; : \; A \neq \overline{W_e}$

Strategy for a single requirement

R_i: Every time an axiom of the form $\langle \sigma, 1^n \rangle$ enters M_i we put an axiom for $\langle \sigma, A \restriction n \rangle$ into M'_i.

P_e: Choose a follower $x = \langle e, m \rangle$. Ensure $A(x) = W_e(x)$ by putting x into A iff x enters W_e.

Combining several requirements

The prefix-free machines M_i' that we build do not conflict since there is a separate one for each i, and the positive requirements P_e also never injure each other as their witnesses are quite distinct, though the single set A is global to the construction.

The main conflict lies in the question of how a machine M_i' can deal with a requirement P_e putting its follower into A, and in particular into $A\restriction n$, *after* M_i' has already enumerated an axiom $\langle\sigma, A\restriction m\rangle$ for some $m \geq n$. For M_i' dealing with a single follower x the answer is simple: when an axiom $\langle\sigma, 1^n\rangle$ enters M_i we put two axioms into M_i': $\langle\sigma^\frown\langle 0\rangle, (A-\{x\})\restriction n\rangle$ and $\langle\sigma^\frown\langle 1\rangle, (A\cup\{x\})\restriction n\rangle$ into M_i. The axioms for M_i' are always enumerated in such pairs, even before x enters A, since M_i' does not have to correctly predict the final form of A. It needs only to ensure that, up to a constant, it compresses any possible $A\restriction n$ as well as M_i compresses 1^n.

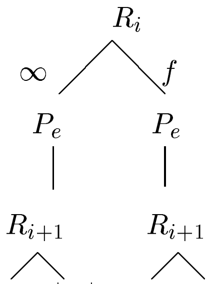
This approach could be modified for any finite number of followers simply by adding longer and longer strings to σ, but in the case where we have infinitely many followers to deal with a slightly different approach is required. We will use a tree of strategies to simultaneously overcome the problem of not knowing in advance whether $\nu(M_i) = 1$ or $\nu(M_i) < 1$, and the need for each computable machine constructed to deal with infinitely many followers entering A.

Consider the construction of a machine M_i' with finitely many followers of higher priority above it in the tree, and infinitely many followers to be chosen for lower priority requirements below. The strategy will be to delay any action (enumeration of axioms) until the k followers of higher priority are chosen. M_i' will then add 2^k axioms of the form $\langle\tau, A^*\restriction n\rangle$ for each axiom $\langle\sigma, 1^n\rangle$ that has entered M_i, where $|\tau| = |\sigma| + k + 1$, and A^* is one of the 2^k possible versions of A formed by the inclusion and exclusion the k followers of higher priority. If n is less than the least of the k followers then there will be repetition in the strings $A^*\restriction n$ but we nevertheless enumerate 2^k axioms to ensure that the size of M_i' is ultimately a computable real.

The followers below M_i' will be dealt with in a different way. Call $1 - \nu(M_i[s])$ the *size of the tail* of M_i; the size of the part of $\mathrm{dom}(M_i)$ not yet enumerated. We assume that $\nu(M_i) = 1$ for the remainder of the strategy. As s increases the size of the tail of M_i tends to zero, and so does the size of the tail of M_i', since $M_i'[s]$ is (so far) precisely half the size of $M_i[s]$. We choose the first follower y of lower priority than M_i' at a stage s such that the size of the tail of M_i' is expected to be less than 2^{-2} and ensure that y is large enough not to change A up to the greatest length so far specified by $M_i'[s]$. This means that although in later stages we double the number of axioms M_i' enumerates in response to an M_i axiom (to account for $y \in A$ and $y \notin A$) we increase the eventual size of M_i' by less than 2^{-2}. In choosing the second follower of lower priority we wait until y as above is chosen, and until the 'doubled' tail of M_i' is itself of size less than 2^{-3}. This continues for all followers below so that the total increase in the size of the domain of M_i' is less than $2^{-2} + 2^{-3} + 2^{-4} + \ldots = \frac{1}{2}$. Thus the final size of M_i' will be less than 1.

Tree of Strategies

We use a subset of a binary tree, $T \subset \{\infty, f\}^{<\omega}$.



During the construction, at stage s, we proceed down the tree in substages $t \leq s$ working on strategies at nodes of length t. The requirements R_i have outcome ∞ or f according to whether $1 - \nu(M_i[x]) < 2^{-n}$ for an $n \leq s$ greater than at any previous visit to R_i, or not. The requirements P_e have only one outcome. They are present on the tree only to choose followers appropriate to their guesses about the size of machines above, and to terminate substages early if machines above with infinite outcome do not yet have sufficiently small tails.

The true path and initialisation

We define an apparent true path θ_s at stage s. We initialise to the right of it. Initialisation of P_e means the follower is cancelled: if it has not yet been enumerated into A it never will be. A new follower must be chosen if this strategy is visited at a later stage. Initialisation of R_e means we abandon the current version of M_i' being constructed, and if R_e is visited again we begin the construction of a new machine M_i'.

Full tree strategies

Strategy R_γ, $|\gamma| = 2i$.

A stage s is *expansionary* for M_i iff $1 - \nu(M_i[s]) < 2^{-n}$ for some $n \leq s$ greater than at any previous stage. When 2^k axioms are added to M_i' they represent all versions of A including and excluding k followers, where k is the number of followers above M_i' and also those chosen below when the tail of M_i' was sufficiently small.

1. Wait for all P_δ, $\delta \subset \gamma$, to choose their followers
2. If s is expansionary for M_i then add 2^k M_i' axioms $\langle \tau, A_s^* \restriction n \rangle$ for each M_i axiom $\langle \sigma, 1^n \rangle$ not yet dealt with. (Repeat this step)

Strategy P_δ, $|\delta| = 2e + 1$.

When P_δ waits for a machine M_i' to have a small tail, *small* means less than $2^{\langle s_1, |\delta| \rangle}$, where s_1 is the first stage at which P_δ was visited (or the first since its last initialisation if it has been initialised). We use the stage number as well as $|\delta|$ in this definition as we want the total effect of *all* strategies at this level of the tree to be bounded by $2^{-|\delta|}$.

1. Wait till machines M_i' for strategies R_γ, $\gamma \subset \delta$, which have outcome ∞, have small tails. Terminate substages if any do not.
2. Choose a follower $x = \langle e, m\rangle$ larger than any previous follower and larger than n for any axiom $\langle \sigma, 1^n\rangle$ so far enumerated in any machine M_i. Request R_γ above to double their tails to accommodate the new follower x. Terminate substages.
3. If x has entered the W_e then enumerate x in A. (Repeat this step)

CONSTRUCTION

At stage s proceed down the apparent true path θ_s in substages $t \leq s$ allowing one strategy of length t to act at each substage. Substages may be terminated early by a strategy P_δ on the apparent true path, in which case proceed immediately to stage $s+1$.

VERIFICATION

Lemma: True path $TP = \liminf \theta_s$ is infinite

Proof: Any strategy P_δ on the true path terminates substages only finitely often after it is initialised for the last time, as eventually all machines above δ on the tree, and with infinite outcome, will have "small tails."

Lemma: Each strategy on TP satisfies its requirement

Proof:

Case 1: P_δ. There is a single strategy for P_e on the true path, and it has infinitely many opportunities to diagonalise if needed.

Case 2: R_γ builds a machine M_i'. This is a computable machine as the c.e. sequence of increases to its domain size (above a half) are less than decreasing powers of two, hence $\nu(M_i')$ is a computable number. That is, $\nu(M_i) = 2^{-1} + b_1 + b_2 + ...$, where $b_1 < 2^{-2}, b_2 < 2^{-3}, b_3 < 2^{-4}$ etc. M_i' has axioms to compress $A\restriction n$ no matter which of the followers above and below R_γ are enumerated into A. Those to the right of TP, but not below R_γ, do not disrupt the axioms of M_i': if M_i' is visited after such a follower is chosen but before it is enumerated then the strategy will be initialised and the follower cancelled. If such a follower x is chosen and enumerated into A without any visit to M_i' in the interim then that follower will be larger than the longest initial segment of A so far specified in an M_i' axiom. $\dashv$

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