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CLASSIFICATION OF DEGREE CLASSES ASSOCIATED WITH R.E. SUBSPACES

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In this article we show that it is possible to completely classify the degrees of r.e. bases of r.e. vector spaces in terms of weak truth table degrees. The ideas extend to classify the degrees of complements and splittings. Several ramifications of the classification are discussed, together with an analysis of the structure of the degrees of pairs of r.e. summands of r.e. spaces.

1. Introduction

In many areas of algebra and model theory, one underlying principle is to encode a structure by some simpler set and/or structure. The phenomena we have in mind are typified by bases in vector spaces, abelian groups, etc, indiscernables in models, and to a lesser extent Frechet sequences in boolean algebras, Ulm invariants etc. The central question we shall analyse in this paper is the manner and extent to which algorithmic complexity (e.g. T-degrees) of such sets and structures is interrelated with the algorithmic complexity of the structures associated with them.

To be specific, in the present work we shall investigate the relationship between the degree of a vector space and the degrees of its bases, although (as we later observe) our results hold in a considerably wider setting. Following Metakides and Nerode [29] we analyse $L(V_\infty)$, the lattice of r.e. subspaces of V_∞, an $\aleph_0$-dimensional computable vector space over a computable field for which we have an algorithm to determine linear dependence (cf. [29]). For $V \in L(V_\infty)$ there seem three natural degrees associated with V.

Class 1. The degrees of r.e. bases of V.

Class 2. The degrees of halves of splittings of V.

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Namely the degrees of $W \in L(V_\infty)$ for which there exists $W' \in L(V_\infty)$ with $W \oplus W' = V$.

Class 3. The degrees of pairs of splittings of V.

Quite a number of *ad hoc* results have been established in the literature regarding class 1 and class 2. For example Dekker [8] showed that V has a recursive basis, and Remmel [34] showed that V has an r.e. basis B of degree $\deg(V)$ and bases of infinitely many r.e. degrees. Remmel then asked if every $V \in L(V_\infty)$ has bases in each degree $\leq \deg(V)$. Here we interpret 'degree' by Turing degree. Remmel defined $V \in L(V_\infty)$ to have the *universal basis property* (UBP) if V had this property and *non-BUP* otherwise. When viewed correctly this naturally leads to a notation investigated by Lerman and Remmel [27, 28] for $L(\omega)$ the lattice of r.e. sets, called the *universal splitting property* (USP). Here an r.e. set A has USP if for all r.e. degrees $\mathbf{a}_1 \leq \deg(A)$, there exists an r.e. splitting $A_1 \sqcup A_2 = A$ of A (that is, $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = A$) with A_1 of degree $\mathbf{a}_1$. It turned out that USP and non-USP r.e. sets both exist and have many interesting properties. Following these papers several related notions together with extensions of the [27, 28] results were analysed. A summary of a portion of this material is presented in [14]. In [14], we, together with L. Welch, began what we felt was a systematic study of the classes 1, 2, and 3. Three results that we feel are highpoints of the theory so far are:

Theorem [14]. *Let* $\mathbf{a}$ *be an r.e. degree (actually weak truth table degree) and* $V \in L(V_\infty)$. *Then* $\mathbf{a}$ *is the degree of an r.e. basis of* V *iff* $\mathbf{a}$ *is the degree of an r.e. summand of* V *iff* $\mathbf{a}$ *is the degree and dependence degree of an r.e. summand of* V.

Theorem [14]. *There exist* $W_1, W_2, V \in L(V_\infty)$ *with* $W_1 \oplus W_2 = V$ *but for which if* $Q, R \in L(V_\infty)$ *and* $Q \oplus R = V$ *then if* $Q \equiv_T W_1$, $\deg(Q) \vee \deg(R) \neq \deg(V)$.

Theorem [16]. *There exists a nonzero r.e. degree* $\mathbf{a}$ *such that if* $V \in L(V_\infty)$ *and* V *has degree* $\mathbf{a}$, *then there exists a degree* $\mathbf{b}$ *with* $\mathbf{0} < \mathbf{b} < \mathbf{a}$ *such that if* B *is an r.e. basis of* V *and* $\deg(B) \leq \mathbf{b}$, *then* B *is recursive* (V *has the 'antibasis property'*).

These results suggest that any general relationships concerning these degree classes would appear quite complicated. Furthermore, results like Ash and Downey's [5] (that given any $V \in L(V_\infty)$ there exist complemented $D_1, D_2 \in L(V_\infty)$ with $D_1 \oplus D_2 = V$) mean that even to prove results like an analogue of Sacks' splitting requires that we use new techniques (the technical point being that we need not only preserve agreements but create disagreements as in some α-recursion theory arguments). Thus we were a little pessimatic that any classification relating to class 1 and so class 2 might be possible.

However, our first result is to show that we can give a complete classification of class 1 in terms of *weak truth table* (W-) *degrees* rather than T-degrees. Namely,

Theorem. *Let V be an r.e. subspace of W-degree* $\mathbf{V}$. *Then an r.e. W-degree* $\mathbf{A}$ *is the W-degree of an r.e. basis of V iff* $\mathbf{A}$ *is the W-degree of Q and $D(Q)$ for some r.e. summand Q of V. (Here $D(Q)$ denotes the set of tuples dependent over Q as in* [29].)

Thus, in particular, classes 1 and 2 are *exactly* the W-degrees below $\mathbf{V}$. This somewhat (to us at least!) surprising result provides a powerful unifying result for much of the earlier theory. Moreover, when linked with known structural results of $\mathbf{W}$ (the r.e. weak truth table degrees), structural interactions of $\mathbf{W}$ and $\mathbf{R}$ (the r.e. T-degrees) and fine analysis of results from [14] and other places, this theorem provides a host of existence and classification results. For example, appealing to some results of Ladner and Sasso [26], we see that we obtain:

Corollary. *There exist completely UBP r.e. degrees. Namely, there exist r.e. degrees* $\mathbf{a}$ *such that every r.e. subspace of degree* $\mathbf{a}$ *has the universal basis property.*

If we combine this with the results from [14] we see that there are r.e. degrees $\mathbf{a}$ such that every subspace of degree $\mathbf{a}$ having the universal splitting property for summands. We end Section 3 with some results towards a classification of the degrees containing r.e. spaces with the antibasis property. For example, using the classification, together with some results of Ambos-Spies et al. [3] we can show that no promptly simple degree contains an r.e. space with the antibasis property.

In Section 4, we turn to class 3. Namely, what are the r.e. degrees realized as *pairs* of summands? At present, we cannot see any reasonable results along the above lines. For r.e. sets, if $A = A_1 \sqcup A_2$ then $\deg(A_1) \vee \deg(A_2) = \deg(A)$. The first real indication of the complexity of class 3 was the result from [14] that this formula fails badly for $l(V_\infty)$. Using the classification of Section 3, the results from [17], together with a new type of r.e. subspace ('hyperatomic'), we shall establish the existence of what we feel is a very surprising type of space, showing the full complexity of class 4. This space is defined as follows: Let V, W, $Q \in L(V_\infty)$, with $V \oplus W = Q$. By the results of Ash and Downey [5] it is possible that for any V, $W \equiv_T D(W) \equiv_T \emptyset$. We shall show that this is the best possible result even if $V <_T Q$.

Definition. Let $V \in L(V_\infty)$ and $\mathbf{0} < \mathbf{a} < \deg(V)$. We shall say V has *singularity* $\mathbf{a}$ if:

(i) There exists W, $Q \in L(V_\infty)$ with $W \oplus Q = V$ and $\deg(W) = \mathbf{a}$.

(ii) Whenever $W' \oplus Q' = V$ and $W' \equiv_T W$, then $D(Q') \equiv_T \emptyset$.

Thus $\mathbf{a}$ is singular for V if and only co-summands of subspaces of degree $\mathbf{a}$ are decidable. We feel that the existence of such spaces gives a spectacular example of the failure of the 'join property' for r.e. spaces. Our result here is:

Theorem. *There exists an r.e. degree* $\mathbf{a} \neq \mathbf{0}$ *such that if* $\mathbf{0} < \mathbf{b} < \mathbf{a}$ *is r.e., then* $\mathbf{b}$ *contains an r.e. subspace with a singularity.*

On the other hand, not every r.e. degree contains such r.e. subspaces. Indeed, we define an r.e. space V to have the *strong universal decomposition property* (SUDP) if given any r.e. degrees $\mathbf{a}_1, \mathbf{a}_2 \leq \deg(V)$ there exist $V_1, V_2 \in L(V_\infty)$ with $V_1 \oplus V_2 = V$ and $\deg(V_i) = \mathbf{a}_i$ for $i = 1, 2$. We show:

Theorem. *There exists a completely SUDP r.e. degree. Namely an r.e. degree* $\mathbf{a} \neq \mathbf{0}$ *such that each r.e. space of degree* $\mathbf{a}$ *is SUDP.*

On the other hand we prove that the degrees containing non-SUDP r.e. spaces is dense in $\mathbf{R}$. To do this we prove the degrees containing nonmitotic r.e. subspaces are dense. This result was originally proved in an earlier draft by a direct construction. Subsequently we found the following coding theorem to obtain this and several other results.

Theorem. *Let* R *be an r.e. subset of a recursive basis of* V_∞. *Suppose* $V \oplus W = (R)^*$ *for r.e. subspaces* V, W. *Then there exist r.e. disjoint sets* C, D *with* $C \cup D = R$ *and such that* $V \leq_W C$ *and* $W \leq_W D$.

Several other related topics are discussed in Section 4, the final one we'd like to mention is that a classification similar to that of Section 3 can be established for the degrees of r.e. complements of fully co-r.e. subspaces, following the ideas of [13].

2. Terminology

In many ways, we feel that this paper would be best read after reading (or as a companion to) [14] and we strongly recommend the reader to read Sections 3 and 4 of [14], plus the statements of results from its later sections. For this reason we shall keep the notation and terminology exactly as it was there, together with the following from some later papers.

We denote T-functionals by upper Greek letters ($\Phi, \Gamma, \ldots$) and W-functionals by those with hats ($\hat{\Phi}, \hat{\Gamma}, \ldots$). For the latter the corresponding use function is the corresponding lower case Greek letter ($\phi, \gamma, \ldots$). All degrees, sets, etc. will be r.e. unless specifically stated othewise. We denote (r.e.) sets by upper case Roman letters ($A, B, C, \ldots$) with ($\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$) their T-degrees and ($\mathbb{A}, \mathbb{B}, \ldots$) their W-degrees. We let $\cup$ ($\cap$) denote the *appropriate* sup (inf) operator. Thus, for example, $A \cap B$ means A intersect B and $\mathbb{A} \cap \mathbb{B}$ means the W-inf (if any) of the W-degrees $\mathbb{A}$ and $\mathbb{B}$. We denote set splitting by $\sqcup$. Finally, if P is a property of r.e. sets we shall say $\mathbf{a}$ is *completely* P if every r.e. set of degree $\mathbf{a}$ has property

P. Since we shall be using so much outside material, it seems more reasonable to spread out most of the remaining definitions etc. throughout the paper. For any yet unexplained terminology the reader should consult [14, §2] or [40]. We remark that we use explicit Gödel numbering only when we think some confusion might exist. When used explicitly, this is denoted by #. Finally, we remark that as with most linear algebra our principal tool is the *Steinitz exchange lemma*: if $x \in (A \cup \{y\})^* - (A)^*$ then $y \in (A \cup \{x\})^*$.

3. Classification of the degrees of bases

The main result of this section is to prove the classification of Theorem (3.10). Using this we shall be able to deduce several quite surprising theorems, such as the existence of completely UBP r.e. degees. We need the following lemma:

(3.1) **Lemma.** *Let $V \in L(V_\infty)$ and B be an r.e. basis of V. Let Q be an r.e. set with $Q \leq_W B$. Then there exists an r.e. basis R of V with $R \equiv_W Q$.*

Proof. Let V, B and Q satisfy the hypotheses of (3.1). We suppose Q is not recrusive since Dekker [8] has shown any $V \in L(V_\infty)$ has a recursive basis. Now, let D be any infinite recursive subset of B. We first shall construct a recursive basis $N = \{n_0, n_1, \ldots\}$ of $(D)^*$ as follows:

Construction 1. Let $D = \{d_0, d_1, \ldots\}$ with $d_0 \neq 0$.

Stage 0. Set $i_0 = 0$, $n_0 = d_0$ and $N_0 = \{d_0\}$.

Stage $s + 1$. If $d_s \in \{n_0, \ldots, n_{i_s}\}^*$ do nothing save to set $i_{s+1} = i_s$ and $N_{s+1} = N_s$. If $d_s \notin \{n_0, \ldots, n_{i_s}\}^*$, find the least $t > s$ with $\{d_s + d_t, d_t\}$ independent over N_s and such that for all $k \leq \max\{2s+2, \#(n_{i_s})\}$, $k < \#(d_s + d_t)$, $\#(d_t)$ and $k < \#(k + d_s + d_t)$, $\#(d_t + k)$. Now let

$$n_{i_s+1} = \min\{d_s + d_t, d_t\} \quad \text{and} \quad n_{i_s+2} = \max\{d_s + d_t, d_t\}.$$

Set $i_{s+1} = i_s + 2$, and $N_{s+1} = \{n_j : j \leq i_{s+1}\}$. □End of Construction 1

Since $n_{i_{s+1}} \neq n_{i_s}$ implies $\forall t > s\ (n_t > s)$, it follows that N is recursive, and by exchange $(N)^* = (D)^*$ so N is a basis. Let $g(\omega) = N$ list N in order of magnitude. The property we need for N is the following.

$$\forall z\, \forall y \leq z\, [\#(y) < \#(y + g(z))]. \tag{3.2}$$

We prove (3.2) by induction. Evidently $g(x) = n_x$ as $\#(n_{i_s}) < \#(d_s + d_t)$, $\#(d_t)$. Thus $g(z)$ is defined at some stage t with $\frac{1}{2}z \leq t$. For $z = 0$, (3.2) holds since the only vector with Gödel number 0 is $\vec{0}$, and $d_0 = g(0) \neq \vec{0}$. Assume (3.2) for $2z$. We consider $2z + 1$ and $2z + 2$. Assume $g(2z)$ is defined at stage q. Let $\hat{q} > q$ be the stage where $g(2z + 1)$, $g(2z + 2)$ are defined (after stage 0, the g's are always defined two at a time). Now as $\hat{q} \geq z$ we have chosen d_t to ensure that for $k \leq \max\{2\hat{q}+1\}, \#(n_{i_{\hat{q}}})\}$. $k \leq \#(k + d_{\hat{q}} + d_q)$, $\#(d_q + k)$. By definition of N and the fact that $\hat{t} \geq z$, this means for all $y \leq 2z + 2$,

$$\#(y) \leq \#(y + g(2z+2)), \quad \text{and}$$
$$\#(y) \leq \#(y + g(2z+1)), \quad \text{giving (3.2).}$$

Now, let $M = B - D$. We have $M \equiv_W B$ and $M \cup N$ is an r.e. basis of V. We assume, without loss, that we are given a W-reduction $Q \leq_W M$ via $\hat{\Gamma}(M) = Q$ with a recursive use function γ such that

(3.3) $\quad x < y$ implies $\gamma(x) < \gamma(y)$.

We may suppose we are given enumerations $M = \bigcup_s M_s$, $Q = \bigcup_s Q_s$ and $\hat{\Gamma}$, γ sufficiently fast that the following properties hold:

(3.4) $\quad$ If we define $l(s) = \max\{x : \forall y < x\, (\hat{\Gamma}_s(M_s; x) = Q_s(x))\}$ then $\forall s\, (l(s+1) > l(s))$,

(3.5) $\quad \exists y \in Q_{s+1} - Q_s$, with $y < l(s)$, and (hence)

(3.6) $\quad \exists z \in M_{s+1} - M_s$.

In our next construction we shall build r.e. independent sets $H = \bigcup_s H_s$ and $G = \bigcup_s G_s$ in stages.

Construction 2

Stage 0. Set $H_0 = M_0$ and $G_0 = \emptyset$.

Stage $s+1$. By (3.5) and (3.6) above, find $y(s) = \mu y\, (y \in Q_{s+1} - Q_s)$ and $z(s) = \mu z\, (z \in M_{s+1} - M_s)$. Define

$$H_{s+1} = H_s \cup \{g(\gamma(y(s))) + z(s)\} \quad \text{and}$$
$$G_{s+1} = G_s \cup (M_{s+1} - (M_s \cup \{z(s)\})). \qquad \square\text{End of Construction 2}$$

It is obvious by exchange that $H \cup G \cup N$ is a basis of V. We claim that $H \equiv_W Q$.

(3.7) **Lemma.** $H \leq_W Q$.

Proof. If $s > n+1$ and $n \in H_{s+1} - H_s$, then $n = g(\gamma(y(s))) + z(s)$ with $z(s) < \gamma(y(s))$ by (3.3), (3.4) and (3.5). By (3.2) we know $y(s) < \gamma(y(s)) < g(\gamma(y(s))) + z(s)$. Consequently, $n \in H_{s+1} - H_s$ only in response to some number $<n$ entering $Q_{s+1} - Q_s$. Thus to decide if $n \in H$ or not find the least stage $s > n+1$ with $Q_s[n+1] = Q[n+1]$. Then $n \in H$ iff $n \in H_{s+1}$. $\square$

(3.8) **Lemma.** $Q \leq_W H$.

Proof. Let n be given. Find the least stage $t > n+1$ such that $H_t[y] = H[y]$ for all $y \leq \#(q_1 + q_2)$ where $q_1, q_2 \leq g(\gamma(n)) + 1$. We claim $Q[n] = Q_t[n]$. Otherwise let s be the least stage $\geq t$ with $Q_{s+1}[n] \neq Q_t[n]$. Then $H_{s+1} = H_s \cup \{g(\gamma(y(s))) + z(s)\}$ where $y(s) \leq n$ and since $l(s) > n$, $z(s) < \gamma(y(s))$. But γ and g are nonotone. Thus by (3.2), $g(\gamma(y(s))) \leq g(\gamma(n))$ and $z(s) < g(\gamma(n))$. This specifically contradicts choice of t. Hence $Q \leq_W H$. □

(3.9) **Conclusion of the proof of (3.1).** We may conclude the proof of (3.1) as follows. We now have an r.e. basis $G \cup H \cup N$ of V with G, H and N r.e., $N \equiv_T \emptyset$ and $H \equiv_W Q$. Apply Dekker's result [8] to $[G]^*$ to find a recursive basis T of $(G)^*$ (or apply Construction 1). Then $R = H \cup T \cup N$ is an r.e. basis of V, and $R \equiv_W H \equiv_W Q$ as required. □

This lemma gives the classification result:

(3.10) **Theorem.** *Let $V \in L(V_\infty)$ and B be any r.e. set. Then $\mathbf{B}$ is the W-degree of an r.e. basis iff $\mathbf{B} \leq \mathbf{V}$.*

Proof. Certainly, if B is an r.e. basis of V, then $B \leq_W V$; to decide if $x \in B$ ask if $x \in V$?, if yes enumerate B until $x \in (B_s)^*$. Then $x \in B$ iff $x \in B_s$.

For the reverse direction as we observed in [14] the natural basis B given by Remmel's [34] process has the property that $B \equiv_W V$. To be specific: let $A = \{a_0, a_1, \ldots\}$ be any r.e. basis of V. Then we build B as follows: Stage 0, $B_0 = \{a_0\}$; at stage $s+1$ find $y(s) = \mu y\,(y \in (A_{s+1})^* - (A_s)^*)$. Set $B_{s+1} = B_s \cup \{y(s)\}$. It is easy to demonstrate that $V \leq_W B$: to see if $z \in V$ or not find the least s such that $B_s[z] = B[z]$. Then $z \in V$ iff $z \in (B_s)^*$. Now apply (3.1) to B. □

Now recall from [27, 28] that $\mathbf{a}$ is *completely UWP* if given any $A \in \mathbf{a}$ and $\mathbf{b} \leq \mathbf{a}$ there exists $B \in \mathbf{b}$ such that $B \leq_W A$. Here UWP refers to the *universal weak truth table reduction property*. We have the following corollaries to the classification result:

(3.11) **Corollary.** *Let $\mathbf{a}$ be any r.e degree. Then $\mathbf{a}$ is completely UWP iff $\mathbf{a}$ is completely UBP.*

(3.12) **Corollary.** *Let $\mathbf{a} \notin \mathbf{0}$. Then there exists $\mathbf{0} < \mathbf{b} \leq \mathbf{a}$ such that $\mathbf{b}$ is completely UBP.*

Proof. By Ladner and Sasso [26], every nonzero r.e. degree $\mathbf{a} \neq \mathbf{0}$ has a contiguous predecessor $\mathbf{b} \neq 0$. Here we recall that $\mathbf{b}$ is contiguous if given any $C, D \in \mathbf{b}$, $C \equiv_W D$. It is obvious that $\mathbf{b}$ is completely UWP. The result follows by (3.11). □

We remark that (3.12) contrasts very strongly with the analogous USP situation for r.e. sets. For USP we have:

(3.13) **Theorem.** (i) (Downey [10]) *Every* $\mathbf{a} \neq \mathbf{0}$ *contains a non-USP r.e. set.*
(ii) (Downey [12]) *Indeed, if A is hypersimple, then A is non-USP.*

We would like to point out several interesting applications of (3.1), (3.11) and (3.12):

(3.14) **Theorem.** (i) *Let* $A_0 \leqslant_T A_1 \leqslant_T \cdots \leqslant_T B$ *be any uniformly r.e. list of r.e. sets such that* $\mathbf{a}_0$ *contains a UWP r.e. set (e.g.* $\mathbf{a}_0$ *(semi-)contiguous). Then there exists* $V \in L(V_\infty)$ *such that V has UBP,* $D(V) \equiv_T B$ *and* $D_i(V) \equiv_T A_{i-1}$. *(Here as usual,* $D_i(V)$ *denotes the collection of i-tuples dependent over V.)*
(ii) *Every nonrecursive* $V \in L(V_\infty)$ *has an r.e. basis of contiguous degree.*
(iii) *Hence, in particular, if* $V \in L(V_\infty)$ *and* $V \not\equiv_T \emptyset$, *then there exists* $\mathbf{a} \leqslant \mathbf{v}$ *such that* $\mathbf{a} \neq \mathbf{0}$ *and if* $\mathbf{b} \leqslant \mathbf{a}$, *then V has an r.e. basis* $B \in \mathbf{b}$.
(iv) *Moreover, if V is any r.e. subspace such that* $\mathbf{v}$ *is promptely simple, then V does not have the antibasis property.*

Remark. We remind the reader that $\mathbf{a}$ is promptly simple iff (by [3]) $\mathbf{a}$ is not cappable. That is for all $\mathbf{b} \neq \mathbf{0}$, $\mathbf{b} \cap \mathbf{a} \neq \mathbf{0}$. Also V has the antibasis property if there exists $\mathbf{0} < \mathbf{d} \leqslant \mathbf{v}$ such that B an r.e. basis of V and $\mathbf{b} \leqslant \mathbf{d}$ implies $B \equiv_T \emptyset$. Here see [16].

Proof. (i) This follows by applying Theorem (3.10) to any r.e. subspace V constructed by Shore's [39] technique. We remark that Shore's [39] actually gives W-reductions.
(ii) This again follows by Ladner and Sasso [26]. If $E \not\equiv_T \emptyset$, then there exists A with $\emptyset <_T A \leqslant_W E$ and A contiguous. Now apply (3.10) to any r.e. nonrecursive basis of V.
(iii) This follows by (ii) and (3.10) applied to an r.e. basis of contiguous degree.
(iv) If V is promptly simple, then by [16] (actually [4] also), $\mathbf{V}$ is not *W-cappable* (rather than just T-cappable). Hence given any $E \not\equiv_T \emptyset$, there exists B with $\emptyset <_T B \leqslant_W E, V$. Now, take a contiguous predecessor of $\mathbf{B}$ and apply (3.10). $\square$

Again, we have contrasts with the USP for r.e. sets. For example, we may constrast (iii) with the fact that in [16], Downey and Stob constructed a complete r.e. set with the antisplitting property, where A has the *antisplitting property* if there exists $\mathbf{b}$ with $\mathbf{0} < \mathbf{b} < \mathbf{a}$ such that $A_1 \sqcup A_2 = A$ and $a_1 \leqslant \mathbf{b}$ implies $\mathbf{a}_1 = \mathbf{0}$.

The degrees containing r.e. subspaces with the anti-basis property would seem difficult to classify. For example, in [16], it is shown that there exists an r.e.

degree $\mathbf{a} \neq \mathbf{0}$ such that for all $V \in L(V_\infty)$ with $V \in \mathbf{a}$, V has the antibasis property, yet it is a corollary to Lachlan's nonbounding theorem [23] that there are nonzero r.e. degrees bounding no nonrecursive r.e. subspace with the antibasis property.

As a final remark for this section, we wish to point out that we do not know if there is any r.e. subspace with the *strong antibasis property*, namely $V \in L(V)$ such that there exists an r.e. set B with $\emptyset <_T B <_T V$ such that if Q is an r.e. basis of V, then $Q \geqslant_T B$ implies $Q \equiv_T V$ and $Q \leqslant_T B$ implies $Q \equiv_T \emptyset$. The analogue for r.e. sets was treated in [11] and [12].

4. Splittings, complements and singular degrees

In this section we analyse the degrees of complements and splittings (by direct sum $\oplus$) of r.e. subspaces. We remind the reader that the symbol $\oplus$ is reserved purely for *direct sum* rather than *sum*. Thus $V \oplus W = Q$ means that $V + W = Q$ *and* $V \cap W = \{\vec{0}\}$. The crucial fact here is:

(4.1) **Theorem** (Downey, Remmel and Welch [14]). (i) *Let* $\mathbf{A} \neq \mathbf{0}$ *and* $V \in L(V_\infty)$. *Then* $\mathbf{A}$ *is the W-degree of a basis of an r.e. basis of V iff there exist* $V_1, V_2 \in L(V_\infty)$ *with* $V_1 \oplus V_2 = V$ *and* $A \equiv_W V_1 \equiv_W D(V_1)$.

(ii) *Moreover, if* $V_1 \oplus V_2 = V$ *and* $\mathbf{A}$ *is the W-degree of an r.e. basis of* V_1, *then if* $\dim(V_2) = \infty$, *there exists* $Q \in L(V_\infty)$ *with* $A \equiv_W Q \equiv_W D(Q)$ *and* $Q \oplus V_2 = V$.

Now, we may immediately apply all the results of Section 3 concerning bases to give results about splittings. In particular, we have:

(4.2) **Theorem.** $\mathbf{a}$ *is completely UDP iff* $\mathbf{a}$ *is completely UBP iff* $\mathbf{a}$ *is completely UWP.*

(4.3) **Theorem.** $\mathbf{A}$ *is the degree of an r.e. summand of V iff* $\mathbf{A} \leqslant_W \mathbf{V}$.

We feel that these results really provide a striking contrast to the non-USP results of [11] and [16]. The reader unfamiliar with [14] might wish to review Section 3 in terms of splittings.

Thus there is almost total freedom for (W-) degrees of halves of splittings of r.e. subspaces. This leads to the suggestion that perhaps there is virtually no restriction on the degrees of *pairs* of splittings. That is, perhaps we can find $V_1 \oplus V_2 = V$ with $\mathbf{V}_i = \mathbf{A}_i$ for any $\mathbf{A}_i \leqslant \mathbf{V}$. This natural enough suggestion, and several others were destroyed in [14]. For example, in [14] it is shown that there exists $V_1, V_2, V \in L(V_\infty)$ such that $V_1 \oplus V_2 = V$ and for all $W_1, W_2 \in L(V_\infty)$ with $W_1 \oplus W_2 = V$ if $V_1 \equiv_T W_1$ then $\mathbf{w}_1 \vee \mathbf{w}_2 \neq \mathbf{v}$.

We shall now use the results of Section 3, plus some new constructions to significantly improve these results. First we shall construct a specific type of r.e. subspace. Recall that $V \in L(V_\infty)$ is called *hyperatomic* if given any $V_1 \oplus V_2 = V$, we have $d(D(V_1)) \cap d(D(V_2)) = \mathbf{0}$. These spaces are called hyperatomic since they have stronger properties than the *strongly atomic* spaces of [14].

(4.4) Theorem. *There exists an r.e. nonrecursive hyperatomic r.e. subspace.*

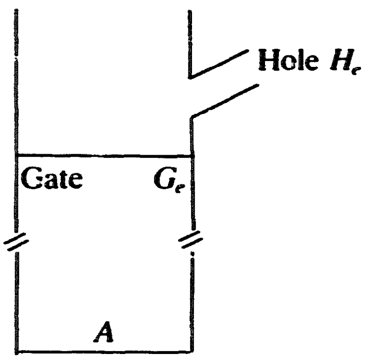
Proof. First we give a basic construction of a hyperatomic space. We construct an r.e. subset $A = \bigcup_s A_s$ of a recursive basis B of V_∞ in stages. Let Q_e denote the e-th r.e. subset of B and let $\langle W_e, V_e, \Phi_e, \Gamma_e \rangle$ denote a standard enumeration of all 4-tuples consisting of two r.e. sets (W_e, V_e) and two reduction (Φ_e, Γ_e). We shall satisfy

P: $B - A \neq Q_e$.

N_e: If $W_e \oplus V_e = (A)^*$, then $\Phi_e(D(W_e)) = \Gamma_e(D(V_e)) = f$ and f total implies f recursive.

At each stage s we implicitly consider (for example) $D(W_{e,s})$ as $D(W_{e,s}) = \{x: (D_x)^* \cap (W_{e,s})^* \neq \{\vec{0}\}$ and $x \leq s\}$ where D_x denotes the x-th canonical finite set. Now let $l(e,s) = \max\{z: \forall y < z\ ((W_{e,s} \oplus V_{e,s})^*(y) = (A_s)^*(y)$ and $\Phi_{e,s}(D(W_{e,s}); y) = \Gamma_{e,s}(D(V_{e,s}); y))\}$, (where $l(e,s) = 0$ if $(W_{e,s})^* \cap (V_{e,s})^* \neq \{\vec{0}\}$).

To make some of our later comments more perspicuous, we shall use a pinball construction following Lerman [27]. Holes H_e are associated with P_e and gate G_e with N_e as in the diagram.



We shall say P_e *requires attention* at stage $s+1$ if $A_s \cap Q_{e,s} = \emptyset$ and one of the following options holds:

(4.5) P_e has a follower x at gate G_j for $j \leq e$ and $l(j,s) > x_t$ for some $t < s$, $t > \gamma(j,s)$ and $x_t > \hat{u}$ where $\hat{u} = \max\{l(e,s), y: y \in D_j$ for $j \leq u$ or $y \leq u$ where $u = \max\{u(\Phi_{e,s}(D(W_{e,s}); \gamma(j,s))), u(\Gamma_{e,s}(D(V_{e,s}); \gamma(j,s)))\}\}$ ($\gamma(e,s)$ is defined later).

(4.6) P_e has a follower x at hole H_e and $x \in Q_{e,s}$.

(4.7) P_e has no follower at hole H_e.

Followers have the priority of the hole they are emitted from. If x and y both follow P_e, then $x<y$ means x is given higher priority than y.

Construction stage $s+1$

Step 1. Define x_{s+1} to be the least number x such that

$$x \notin \Big(A_s \cup \bigcup_{y \leq x_s+1} D_y \cup \{z: z \leq \max\{x_s+1, s\}\Big)^* \quad (\text{where } x_0 = 1).$$

Step 2. Find e least such that P_e requires attention. If (4.5) or (4.6) hold, find the corresponding highest priority x. Cancel all followers of lower priority than e (and x). Adopt the appropriate case below:

Case 1: (4.5) *holds*. Allow x to drop to the first unoccupied gate G_k for $k<j$. If no such gate exists, set $A_{s+1}=A_s \cup \{x\}$ and cancel all followers of P_e. If k exists, set $\gamma(k, s+1)=x_{s+1}$.

Case 2: (4.6) *holds*. As in case 1, with $j=e+1$.

Case 3: (4.7) *holds*. Appoint x_{s+1} as a follower of P_e and place it above hole H_e. □End of construction

Verification. The reader may easily establish by induction that each gate G_e has at most one permanent resident, and hence all the P_e receive attention at most finitely often and are met. Thus, it remains to show that all the N_e are met. Suppose therefore that $W_e \oplus V_e=(A)^*$ and $\Phi_e(D(W_e))=\Gamma_e(D(V_e))=f$ total. Let z be given. Find a stage s_0 such that

(i) $\forall s>s_0 \forall j<e$ (P_j does not receive attention at stage s).

(ii) $\forall k \leq e$ (if G_k has a permanent resident, then it does so at stage s_0).

(iii) $l(e, s_0)>x_s$, for some $s>z$, $s_0>s$ and $x_s>\hat{u}$ where $\hat{u}=\max\{y: y \in D_j$; for $j \leq u$ or $y<u$ where $u=\max\{u(\Phi_{e,s_0}(D(W_{e,s_0}); z)), u(\Gamma_{e,s_0}(D(V_{e,s_0}); z))\}\}$.

(iv) $\forall k<e$ (if G_k has any resident, then it is a permanent one).

We claim that

$$\text{(4.8)}\ \forall s>s_0\ (\text{one of } \Phi_{e,s}(D(W_{e,s}); z)=\Phi_{e,s_0}(D(W_{e,s_0}); z) \text{ or } \Gamma_{e,s}(D(V_{e,s}); z)=\Gamma_{e,s_0}(D(V_{e,s_0}); z) \text{ holds}).$$

To prove (4.8) we need some algebraic lemmas. Let $R=\{x_0, x_1, \ldots\}$.

(4.9) **Lemma.** *Suppose $r \leq x_n$ and $r \in (R)^*$. Then* $\mathrm{supp}_R(r) \subseteq \{x_0, \ldots, x_n\}$. *In particular, if $r<x_n$ and $r \in (A)^*$, then* $\mathrm{supp}_R(r) \subseteq \{x_0, \ldots, x_n\} \cap A$.

Proof. Let j be greatest with $x_j \in \mathrm{supp}_R(r)$ and suppose $j>n$. Then $r \in (\{x_i: i \leq j\})^*-(\{x_i: i<j\})^*$. By exchange $x_j \in (\{x_i: i<j\} \cup \{r\})^*$. As $r<x_n$ and $n<j$, we see $r \leq x_{j-1}$. It follows that $x_j \in (\{z: z \leq x_{j-1}\})^*$ and this contradicts the definition of x_j in step 1 of the construction. Hence $\mathrm{supp}_R \subseteq \{x_0, \ldots, x_n\}$. The second statement is a consequence of this fact and the fact that $A \subseteq R$. □

(4.10) **Lemma.** *Let $s_1 \geq s$ be such that $A_{s_1}[x_s]=A[x_s]$. Then $(A_{s_1})^*[x_s]=(A)^*[x_s]$.*

Proof. If $r<x_s$ enters $(A_t)^*-(A_{s_1})^*$, since $A_{s_1}[x_s]=A[x_s]$, this can only be as a consequence of x_j—for some $j>s$—entering $A_t-A_{s_1}$. Then it follows that $x_j\in \mathrm{supp}_R(r)$ contradicting (4.9). $\square$

The decisive lemma is

(4.11) **Lemma.** *Suppose $s<t$ are stages such that*

(i) $(W_{e,s})^*\oplus(V_{e,s})^*[x_n]=(A_s)^*[x_n]$,

(ii) $(W_{e,t})^*\oplus(V_{e,t})^*[x_n]=(A_t)^*[x_n]$, *and*

(iii) *either* $A_t[x_n]=A_s[x_n]$ *or* $A_t[x_n]=A_s[x_n]\cup\{x_i\}$ *for some* $i\leq n$.

Then either $(W_{e,t})^*[x_n]=(W_{e,s})^*[x_n]$ *or* $(V_{e,s})^*[x_n]=(V_{e,t})^*[x_n]$.

Proof. It clearly suffices to establish the lemma for the case $x_i\in A_t-A_s$ since the other one will follow *a fortiori*.

Let $A_s[x_n]=\{x_{j_1},\ldots,x_{j_m}\}$, so that $A_t[x_n]=\{x_i,x_{j_1},\ldots,x_{j_m}\}$. Let $r_1,r_2<x_n$ with $r_1\in(W_{e,t})^*-(W_{e,s})^*$ and $r_2\in(W_{e,t})^*-(V_{e,s})^*$. By Lemma (4.9) and the fact that $x_i\in\mathrm{supp}_R(r_j)$ (by exchange) we know for some $\lambda_{k,g}$ and γ_k that

$$(4.12)\qquad r_k=\sum_{j\leq m}\lambda_{k,g}x_{j_g}+\gamma_k x_i\quad \text{for } k=1,2.$$

Let $\{v_1,\ldots,v_d\}$, $\{w_1,\ldots,w_p\}$ be bases of $(V_{e,s})^*$ and $(W_{e,s})^*$ respectively. Since $A_s[x_n]\subseteq(V_{e,s})^*\oplus(W_{e,s})^*$ it follows, in particular, that for all $g\leq m$,

$$x_{j_g}\in(\{v_1,\ldots,v_d,w_1,\ldots,w_p\})^*.$$

Hence, by (4.12) we see

$$r_1\in(\{v_1,\ldots,v_d,w_1,\ldots,w_p,x_i\})^*-(\{v_1,\ldots,v_d,w_1,\ldots,w_p\})^*.$$

By exchange,

$$x_i\in(\{r_1,v_1,\ldots,v_d,w_1,\ldots,w_p\})^*.$$

However this implies that

$$r_2\in(\{r_1,v_1,\ldots,v_d,w_1,\ldots,w_p\})^*.$$

It follows that $\{r_1,v_1,\ldots,v_d,w_1,\ldots,w_p,r_2\}$ is dependent. Therefore $(V_{e,t})^*\cap(W_{e,t})^*\neq\{\bar{0}\}$, a contradiction. $\square$

Remark. The argument above simply sets out combinatorial ideas that allow us to adapt splitting type arguments to our setting. Similar ideas are implicit in several other papers such as [14] and [5]. We thank the referee for suggesting that we separate out the key algebraic point.

Verification of (4.8). Suppose that (4.8) fails. The only way both sides of the computations fail to hold is if numbers enter both sides between steps where $l(e,s)>ml(e,s)$. We shall argue that this is impossible. By conditions (i)–(iv) above, the way we appoint followers, and Lemma (4.10) the only way numbers can enter A below $u(\Phi_{e,s}(D(W_{e,s}); z))$ or $u(\Gamma_{e,s}(D(V_{e,s}); t))$ is if they are already on the surface at stage s_0 and above gate G_e. (All others will exceed x_s, and there are no movable numbers alive below gate G_e.)

Let $x_{i_1} \leqslant x_s$ be the first follower below either uses to enter A and set $y_1 = x_s$. Since we only pass over a gate if it is already occupied by a higher priority ball, x_{i_1} must stop at gate G_e. Otherwise the higher priority ball would have cancelled x_{i_1}. Indeed, all balls $< x_{i_1}$ must remain above G_e until x_{i_1} actually enters A. When at stage $s_1 \geqslant s_0$, x_{i_1} enters G_e we set $\gamma(e, s_1) \equiv x_{s_1} > x_s$. At stage $s_2 + 1$ when x_{i_1} leaves G_e both sides of the computation hold and indeed by choice of x_{s_1} nothing has changed since stage s_0.

Let $x_{i_1} \in A_{s_3+1} - A_{s_3}$ when x_{i_1} so moves; it cancels x_j for $i_1 < j \leqslant s_3$ as followers. By Lemma (4.11) it follows that until a stage $s_4 > s_3$ occurs with $x_k \in A_{s_4} - A_{s_3}$ and $k < s$, (4.8) must hold. To see this we need note that (by 4.11) until such a number x_k enters A, one of $(W_{e,t})^*[x_s] = (W_{e,s_0})^*[x_s]$ or $(V_{e,t})^*[x_s] = (V_{e,s_0})^*[x_s]$ must hold. Recall that $x_s > \hat{u}$ where

$$(*) \qquad \hat{u} = \max\{y : y \in D_j \text{ for } j \leqslant u \text{ or } y \leqslant u \text{ where } u = \max\{u(\Phi_{e,s_0}(D(W_{e,s_0}); t)), u(\Gamma_{e,s_0}(D(V_{e,s_0}); z))\}\}$$

by condition (iii). Suppose that $(W_{e,s_0})^*[x_s] = (W_{e,t})^*[x_s]$. It follows that $D(W_{e,s_0})[u] = D(W_{e,t})[u]$ since, by (*) and exchange,

$$\{z : (D_z)^* \cap (W_{e,s_0})^* \neq \{0\}\}[u] = \{z : (D_z)^* \cap (W_{e,s_0})^*[x_s] \neq \{\vec{0}\}\}[u]$$
$$= \{z : (D_z)^* \cap (W_{e,t})^*[x_s] \neq \{0\}\}[u] = D(W_{e,t})[u].$$

In other words, by holding $(W_{e,t})^*[x_s] = (W_{e,s_0})^*[x_s]$ we preserve $D(W_{e,t})[u] = D(W_{e,s_0})[u]$ and so (4.8) holds until the stage where x_k enters $A_{s_4} - A_{s_3}$. Now when x_k stops at gate G_e, it remains there until $l(e, s_5) > x_j$ for some $y_2 = x_j$ fulfilling the same conditions as (*) above but with s_5 in place of s_0 (see (4.5)). When it leaves, both sides of the computation hold and all followers $x_{\hat{i}}$ with $j < \hat{i} \leqslant s_5$ are cancelled. Thus exactly the same argument as above shows that one side of the computation holds until the next stage where $l(e, s) > ml(e, s)$. In this way we see that N_e is met since such injuries are finite in number. $\square$

Remark. Save for the algebraic complications, the above argument is an essentially standard pinball one that easily admist Cooper's high permitting as for example [7, 14]. Indeed, with more effort one can similarly modify Miller's [30] construction to show all high r.e. degrees bound high hyper-atomic r.e. subspaces. As these arguments really require no further *algebraic* difficulties than that above, we leave then to the reader.

We shall call an r.e. degree hyperatomic if it contains a hyperatomic subspace.

(4.13) **Theorem.** (i) *Let* $\mathbf{A}$ *be hyperatomic and if* $\mathbf{0} < \mathbf{B} < \mathbf{A}$, *then* $\mathbf{B}$ *is hyperatomic.*

(ii) *There exist contiguous hyperatomic r.e. subspaces.*

Proof. (i) Let $\mathbf{A}$ be hyperatomic, and without loss let $A \in L(V_\infty)$ with A hyperatomic and $A \in \mathbf{A}$. Now let $\mathbf{0} < \mathbf{B} < \mathbf{A}$. By (4.3), let $V, W \in L(V_\infty)$ with $\mathbf{V} = \mathbf{B}$ and $V \oplus W = A$. Now as A is hyperatomic V must be hyperatomic. Suppose not. Then there must exist $V_1, V_2 \in L(V_\infty)$ with $V_1 \oplus V_2 = V$ and $d(D(V_1)) \cap d(D(V_2)) \neq \mathbf{0}$. Now, $d(D(V_2 \oplus W)) \geqslant d(D(V_2))$. Consequently, $d(D(V_1)) \cap d(D(V_2 \oplus W)) \neq \mathbf{0}$, contradicting the hyperatomicity of A. Hence V is hyperatomic, clinching (i). Evidently, by Ladner and Sasso [26], (i)$\rightarrow$(ii). $\square$

The existence of hyperatomic spaces has very interesting consequences in $L(V_\infty)$. Perhaps the most striking is the existence of *singular degrees*. We recall that a nonzero r.e. degree $\mathbf{a}$ is *singular* if there exists $V \in L(V_\infty)$ such that $\mathbf{a} < \mathbf{v}$, for which there exists $W, A \in L(V_\infty)$ with $A \in \boldsymbol{a}$ and $A \oplus W = V$ such that if $V_1, V_2 \in L(V_\infty)$ with $V_1 \equiv_T A$ and $V_1 \oplus V_2 = V$ then $D(V_2) \equiv_T \emptyset$. That is, every co-summand of any space of the same degree as A is decidable. We shall say an r.e. subspace V *has singularities* if $\mathbf{v}$ bounds a singular degree. We have the following very strong extension of the [14] result.

(4.14) **Theorem.** (i) *Suppose V is hyperatomic. Then V has singularities.*

(ii) *Hence every high degree bounds an r.e. degree containing a space with a singularity.*

Proof. From [17] we have the following result.

(4.15) $\quad \forall \mathbf{a} \neq \mathbf{0}\, \exists \mathbf{b}\, (\mathbf{0} < \mathbf{b} < \mathbf{a}\ \&\ \forall \mathbf{c} \leqslant \mathbf{a}\, (\mathbf{c} \cap \mathbf{b} = \mathbf{0} \rightarrow \mathbf{c} = \mathbf{0}))$.

Now take $\mathbf{a}$ to contain a hyperatomic space V. Then $\mathbf{b}$ is a singularity for V. $\square$

Actually for (ii), we need only a much easier result; namely that the analogue of (4.15) holds for the W-degrees, which is an unpublished result of Downey, Ingrassia, Stob and Welch (applied to a contiguous hyperatomic space). For an extension of this, we refer the reader to [11] where it is proved that

$$\forall \mathbf{A} \neq \mathbf{0}\, \exists \mathbf{b}\, (\mathbf{0} < \mathbf{B} < \mathbf{A}\ \&\ \forall \mathbf{C} \leqslant \mathbf{A}$$
$$((\mathbf{C} \cap \mathbf{B} = \mathbf{0} \rightarrow \mathbf{C} = \mathbf{0})\ \&\ (\mathbf{C} \cup \mathbf{B} = \mathbf{A} \rightarrow \mathbf{C} = \mathbf{A})).$$

These results and similar results make the possible pairs of degrees of V, W with $V \oplus W = Q$ very much more difficult to classify. We feel our next results provide some of the most striking contrasts between USP and UDP known. Recall that an r.e. set A has the *strong universal splitting property* (SUSP) if given any r.e. sets B_1, B_2 with $B_1 \oplus B_2 \equiv_T A$, there exists an r.e. splitting $A_1 \sqcup A_2 = A$ of A with $A_i \equiv_T B_i$. Analagously, we say $V \in L(V_\infty)$ has the *strong universal decomposition property* (SUDP) if given any r.e. sets $B_1, B_2 \leqslant_T V$, there exists $V_1, V_2 \in L(V_\infty)$ such that $V_i \equiv_T B_i$ and $V_1 \oplus V_2 = V$. In [AF] it is shown that $\mathbf{0}'$ is not SUSP although it is USP since it contains a creative set. We have the following result:

(4.16) **Theorem.** (i) *There exists* $\mathbf{a} \neq \mathbf{0}$ *such that* $\mathbf{a}$ *is completely SUDP.*

(ii) *There exists* $\mathbf{b} \neq \mathbf{0}$ *such that* $\mathbf{b}$ *bounds no completely SUDP r.e. degree.*

(iii) $\{b \mid b$ *is not completely SUDP*$\}$ *is dense in the r.e. degrees.*

Proof. (i) Ladner [25] constructed a contiguous completely mitotic r.e. degree $\mathbf{a}$, where A is mitotic if there exists A_1, A_2 with $A_1 \sqcup A_2 = A$ and $A_1 \equiv_T A_2 \equiv_T A$. Now, let $B_1, B_2 \leqslant_T V$ for some $V \in L(V_\infty)$ and $V \in \mathbf{a}$. Take an r.e. basis $R \equiv_W V$. By mitoticity, we can split R as $R_1 \sqcup R_2$ with $R_i \equiv_W R$. Now by contiguity $B_i \leqslant_W R_i$. by (3.1) we thus have a basis N_i of $(R_i)^*$ with $N_i \equiv_W B_i$. By (4.1)(ii) we have $V_1 \in L(V_\infty)$ with $V_1 \equiv_W D(V_1) \equiv_W B_1$ and $V_1 \oplus (N_2)^* = V$. By (4.1)(ii) again we have V_2 with $V_2 \equiv_T D(V_2) \equiv_T B_2$ and $V_1 \oplus V_2 = V$.

(ii) Let $\mathbf{b}$ be hyperatomic and contiguous. Then $\mathbf{b}$ bounds no completely SUDP r.e. degree by (4.13) (ii).

(iii) Finally, (iii) will follow from (4.17) below. □

(4.17) **Theorem.** *The degrees containing nonmitotic r.e. subspaces are dense in* $\mathbf{R}$.

In an earlier draft of this paper we proved (4.17) by a direct construction based on [5] and [18]. Subsequently, we discovered the following very useful coding argument which allows us to derive (4.17) from [15], or Ingrassia's results on 'P-generic' sets [19].

(4.18) **Theorem.** *Let B be a recursive basis of V_∞ and R an r.e. subset of B. Suppose for $V, W \in L(V_\infty)$, $V \oplus W = (R)^*$. Then there exist r.e. disjoint sets C, D with $C \cup D = R$ and with $V \leqslant_W C$ and $W \leqslant_W D$.*

Proof. First, without loss of generality we can suppose $B = \{b_0 < b_1 < \cdots\}$ ordered so that $\#(b_i) \leqslant \#(\sum_{j \in J} \lambda_j b_j)$ if $i \in J$ and $\lambda_i \neq 0$. (This does no harm since we can map B to such a basis if necessary.)

Let $\{b_{f(s)}: s \in \omega\}$ a 1–1 enumeration R with f recursive. Let $B(V) = \{v_0, v_1, \ldots\}$ and $B(W) = \{w_0, w_1, \ldots\}$ list r.e. bases of V and W respectively with $B(V) \equiv_W V$ and $B(W) \equiv_W W$. Let $\{d_i: i \in \omega\}$ be an r.e. basis of $(R)^*$ defined via $d_{2i} = v_i$ and $d_{2i+1} = w_i$. We first define a partial recursive bijection $g: B(V) \cup B(W) \to R$ in stages.

Construction

Stage 0. Compute the least d_j such that $b_{f(0)} \in \operatorname{supp}_B(d_j)$ and $d_j \in \operatorname{supp}_R(b_{f(0)})$. Define $g(d_j) = b_{f(0)}$. Set $R_0 = (R - \{d_j\}) \cup \{b_{f(0)}\}$.

Stage $s+1$. Assume we have defined $g^{-1}(b_{f(i)})$ for $i \leqslant s$ and

$$R_s = (R - \{g^{-1}(b_{f(0)}), \ldots, g^{-1}(b_{f(s)})\}) \cup \{b_{f(0)}, \ldots, b_{f(s)}\}.$$

Compute the least $d_j \in R_s$ such that $b_{f(s+1)} \in \mathrm{supp}_B(d_j)$ and $d_j \in \mathrm{supp}_{R_s}(b_{f(s)})$. Define $g(d_j) = b_{f(s+1)}$. □End of construction

We note that the construction does not break down: By induction and the exchange lemma, at each stage s, $(R_s)^* = (R)^*$. Thus, for each t, $b_{f(t)} \in (R_s)^*$. Now if there is no $d_j \in R_s \cap \mathrm{supp}_{R_s}(b_{f(s+1)})$, then it follows that $b_{f(s+1)} \in (\{b_{f(j)}: j \leqslant s\})^*$ this contradicting the fact that B is a basis. Furthermore for some $d_j \in \mathrm{supp}_{R_s}(b_{f(s+1)})$, $b_{f(s+1)} \in \mathrm{supp}_B(d_j)$: If, for all $d_j \in \mathrm{supp}_R(b_{f(s+1)})$, $b_{f(s+1)} \notin \mathrm{supp}_B(d_j)$ then we get a contradiction as follows.

Let $\mathrm{supp}_{R_s}(b_{f(s+1)}) - \{b_{f(i)}: i \leqslant s\} = \{d_{i_1}, \ldots, d_{i_k}\}$. Then assume $b_{f(s+1)} \notin \mathrm{supp}_B(d_{i_j})$. We have $b_{f(s+1)} \in (\{b_{f(i)}: i \leqslant s\} \cup \bigcup_{j \leqslant k} \mathrm{supp}_B(d_{i_j}))^*$ and so $b_{f(s+1)} \in (B - \{b_{f(s+1)}\})^*$, a contradiction.

Finally, to see that g is onto we need only observe that—by exchange—at each stage s, R_s is independent. Now if some $d_i \notin \mathrm{ran}(g)$ find the least stage s where $\mathrm{supp}_B(d_i) \subset R_s$. Then if $d_i \in R_s$ we would have R_s dependent.

Thus g is a partial recursive bijection with domain $B(V) \cup B(V)$, range R such that for all $z \in \mathrm{dom}(g)$,

(4.19) $\quad g(z) \in \mathrm{supp}_B(z)$.

We remark that (4.19) is the crucial property. Now define $C = \bigcup_s C_s$ and $D = \bigcup_s D_s$ via

(4.20) $\quad C_{s+1} = C_s \cup \{g(v_s)\}, \qquad D_{s+1} = D_s \cup \{g(w_s)\}$.

By construction $C \cup D = R$ and $C \cap D = \emptyset$ as g is a bijection. Finally by hypotheses on B—as $g(z) \in \mathrm{supp}_B(z)$—it follows that for all $z = v_s$ or w_s, $\#(g(z)) \leqslant \#(z)$. Therefore, by permitting we see that $V \leqslant_{\mathrm{W}} C$ and $W \leqslant_{\mathrm{W}} C$. (For example, to decide if $y \in V$ find the least stage s where $C_s[y] = C[y]$. Then $B(V)_s[y] = B(V)[y]$ and so [illegible] $\leqslant_{\mathrm{W}} C$ and hence $V \leqslant_{\mathrm{W}} C$ since $V \equiv_{\mathrm{W}} B(V)$.) □

Theorem (4.18) has several interesting consequences. For example, we can solve problems from [illegible] and [5].

(4.21) **Theorem.** *Let R be an r.e. subset of a recursive basis. Then*

(i) *If R is nonmitotic so is $(R)^*$.*

(ii) *If R is (strongly) atomic, so is $(R)^*$.*

Proof. Immediate. □

Remarks. (1) the results of Lachlan [21], Ingrassia [19] Ladner [24], and Downey–Slaman [15] now give numerous results on the distribution of nonmitotic r.e. degrees. For example, they are dense and contain $\mathbf{0}'$. Similar remarks apply to strongly atomic degrees using the results of Ambos-Spies [1] and Downey–Welch [18].

(2) We do not know if a strongly atomic r.s. subset of a recursive basis of V_∞ generated a hyperatomic r.e. space. In particular, we do not know if we can improve (4.18) to say $C \geq_{\mathrm{W}} D(V)$, $D \geq_{\mathrm{W}} D(W)$.

Finally, we address some comments to complements of fully co-r.e. subspaces. Recall that a subspace V of V_∞ is *fully co-r.e.* if it is generated by a co-r.e. subset of a recursive basis of V_∞. By Downey [9], we know every $V \in L(V_\infty)$ has a fully co-r.e. complement R with $R \leq_{\mathrm{W}} D(V)$. Now, as we observed in [14] if $\mathbf{R}$ is fully co-r.e. and $V \oplus R = V_\infty$ then V, $D(V) \leq_{\mathrm{W}} R$. Thus limiting results on the structure of W below R transfer to results about complements. For example by [16], there exists a nonzero r.e. degree $\mathbf{a}$ such that if R is fully co-r.e. of degree $\mathbf{a} \neq \mathbf{0}$, then R has the *anticomplementation property*, namely, there exists an r.e. $\mathbf{0} < \mathbf{b} \leq \mathbf{r}$ such that $V \oplus R = V_\infty$ and $\mathbf{v} \leq \mathbf{b}$ implies $\mathbf{v} = \mathbf{0}$. However, what we lack is the analogue of Theorem (3.10) to obtain the positive results like the existence of completely UCP r.e. degrees. As our final result, we shall provide such an analogue.

(4.22) **Theorem.** (i) *Let R be fully co-r.e. and suppose Q is an r.e. independent set with $(Q)^* \oplus R = V_\infty$. Then there exists an r.e. $V \in L(V_\infty)$ such that $D(V) \equiv_{\mathrm{W}} V \equiv_{\mathrm{W}} Q$ and $V \oplus R = V_\infty$.*

(ii) *Consequently $\mathbf{A}$ is the (dependence) degree of a complement of R iff $\mathbf{A} \leq \mathbf{R}$.*

Proof. (i) Let Q, R be as given, and let A be an r.e. subset of B a recursive basis of V_∞ with $(B - A)^* = R$. The construction is again in two parts, the second part being very similar to Lemma (3.1), Construction 2. For the first part, we shall construct $E = \bigcup_s E_s$ a decidable space in stages, satisfying certain additional hypotheses. Let $B = \{b_0, b_1, \ldots\}$ and, at each stage s let $\mathrm{card}(A_{s+1}) - \mathrm{card}(A_s) = 1$ with $\{z(s)\} = A_{s+1} - A_s$ (with $A_0 = \emptyset$).

Construction 1, stage $s+1$. Find $b \in B - A_{s+1}$ sufficiently large such that for all $j \leq b_s$ we have

$$j + z(s) + b \notin (\{i + d_k : i \leq \max\{b_s, d_s\}, k \leq s\} \cup \{D_k : k \leq s\})^*,$$

and for all $j \leq b_s$ we have

$$\#(j) \leq \#(j + z(s) + b).$$

Set $E_{s+1} = E_s \cup \{d_{s+1}\}$ where $d_{s+1} = z(s) + b$. Set $E = (\bigcup_s E_s)^*$. □End of construction

Let $g(\omega)$ list the basis $\{d_i : i \in \omega\}$ in order. Again assume we have a

W-reduction $\hat{\Gamma}(A) = Q$ with increasing use γ, where now $x < \gamma$ implies $\gamma(x) + 3 < \gamma(y)$. We now construct the desired complement V in stages.

Construction 2, stage $s+1$

Step 1. Say s is an expansion stage if, putting $l(s) = \max\{y: \hat{\Gamma}_s(A_s; x) = Q_s(x)$ for each $x < y\}$, then $l(s) > l(t)$ for all $t < s$. If s is not an expansion stage, this step is vacuous.

If s is an expansion stage, let $t > s$ be the least stage such that $l(t) > l(s)$. Let $y(s)$ be be the least number in $Q_t - Q_s$. If none exists, this step is vacuous. If $y(s)$ be the least number in $Q_t - Q_s$. If none exists, this step is vacuous. If $y(s)$ exists, find the least $x(s)$ with $x(s) \in A_t - A_s$. Now one of $g(\gamma(y(s)))$, $g(\gamma(y(s)) + 1)$ or $g(\gamma((y(x) + 2))$ is *unused* and has $x(s)$ not in its support relative to B. Let $y(s) + i$ denote the least such choice, and now define

$$r(s) = g(\gamma(y(s)) + i) + x(s).$$

Declare $g(\gamma(y(s) + i)$ and $x(s)$ as *used*. Now if $z(s)$ is unused find $g(n) \in E$ with $g(n) > 3(k)$ where $k > \max\{\mathrm{supp}_B(m): m \in V_s \cup E_z \cup Q_t \cup D_y; y \leq t\}^*$ and set $V_{s+1} = (V_s \cup \{r(s), z(s) + g(n)\})$. Otherwise, if $z(s)$ is used, set $V_{s+1} = (V_s \cup \{r(s)\})$.

Step 2. If Step 1 didn't pertain, see if $z(s)$ is used. If it is not find $g(n)$ similarly as above and set $V_{s+1} = (V_s \cup \{z(s) + g(n)\})$. If $z(s)$ is used, set $V_{s+1} = V_s$. □End of construction

Using a very similar argument to that used for Theorem (3.10), we can show $Q \equiv_W \{r(i): i \in \omega\}$. Notice that $V - Q$ is recursive. Hence $Q \leq_W (V)^*$. It remains to show that $(V)^* \leq_W Q$. Define a co-r.e. co-basis C of $(V)^*$ as follows:

Stage 0. Set $C_0 = B$.

Stage $s+1$. *Case* 1: $V_{s+1} = V_s$. Do nothing.

Case 2: $V_{s+1} = V_s \cup \{x\}$. Find $y_1 = \max\{y: y \in \mathrm{supp}_B\{x\} \cap C_s\}$ and set $C_{s+1} = C_s - \{y_1\}$.

Case 3: $V_{s+1} = V_s \cup \{x_1, x_2\}$. Find $y_i = \max\{y: y \in \mathrm{supp}_B(x_i)\}$. Set $C_{s+1} = C_s - \{y_1, y_2\}$.

To complete the construction, set $C = \bigcup_s C_s$. It is really quite easy to show that for all s, if s is expansionary, then $Q_s[x] = Q[x]$ implies $C_s[x] = C[x]$. Therefore $C \leq_W Q$. However, as C is a co-r.e. co-basis of V, we have $D((V)^*) \leq_W C$. That is, $Q \leq_W (V)^* \leq_W D((V)^*) \leq_W C \leq_W Q$ as required. □

5. Generalizations

For the reader familiar with the generalizations of vector space results to abstract dependence systems, we remark that they hold in any recursive Steinitz system with the appropriate regularity condition (cf. [31, 32]) and in particular $L(F_\infty)$ the lattice of algebrically closed fields. However, once we lose a global exchange, the situation is not nearly as clear. For boolean subalgebras of $\bar{Q}$, the free boolean algebra, Remmel has shown that there are no restrictions on the degrees of summands. However, in the lattice of ideals of $\bar{Q}$, we don't even know if every nonrecursive ideal has a nonrecursive independent basis. Thus, a classification for this lattice along the lines we have described here would seem very difficult.

Another interesting lattice here is the lattice of r.e. subgroups of the free group $G = \langle X, Y | \rangle$. C.F. Miller and Downey have some unpublished results for this lattice which suggest that is is very different. For example, although any recursive subgroup of G has a recursive free basis, there are *r.e.* (free) subgroups of G with no *r.e.* free basis. We remark that the result for recursive bases requires some care since the classical proof do not give recursive bases. Actually, these results came as corollaries to some 'reverse mathematics' investigations. Miller and Downey show that RCA_0 is enough to prove the Neilson–Schreier theorem.

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