RECURSIVELY ENUMERABLE *m*- AND *tt*-DEGREES III: REALIZING ALL FINITE DISTRIBUTIVE LATTICES

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1. *Introduction*

In [1], Degtev constructed a non-zero r.e. *tt*-degree containing a single r.e. *m*-degree, and it is certainly possible to construct an r.e. *tt*-degree with no greatest *m*-degree (Downey, [4]) and hence an r.e. *tt*-degree can also have infinitely many r.e. *m*-degrees (Fischer [8]). Odifreddi [12, Problems 10, 13] asked if each r.e. *tt*-degree had to contain either one or infinitely many r.e. *m*-degrees. The second author in [6] showed that it is possible to construct an r.e. *m*-degree with exactly 3 r.e. *m*-degrees. He also claimed that one could use the techniques of [6] to construct an r.e. *tt*-degree with exactly $2^n - 1$ r.e. *m*-degrees and hence arbitrarily large numbers of r.e. *m*-degrees.

Unfortunately (see Section 4 of the present paper) this claim is not quite correct. The methods there can be used to construct r.e. *tt*-degrees with arbitrarily large (but finite) numbers of r.e. *m*-degrees; they do not seem to give the precise number $2^n - 1$ (nor the structure of a boolean algebra with the least element removed) claimed in [6]. In the same paper, the second author also raised the question of whether $n = 2$ is possible.

In this paper we prove the following result that is definitive for finite lattices.

THEOREM. *The structure of the* r.e. *m-degrees within an* r.e. *tt-degree can be any given finite distributive lattice. Thus there exist* r.e. *tt-degrees with exactly n* r.e. *m-degrees* (*for any given n*).

The organization of the paper is as follows. As in [5, 6] to motivate the proof we indicate how to construct a *singular* r.e. *tt*-degree (that is, one with a single r.e. *m*-degree). In Section 3 we then construct an r.e. *tt*-degree with 2 r.e. *m*-degrees, and in Section 4 indicate how to extend the ideas to the *n* atom boolean algebra. In Section 5 we use an additional trick to get the main result. Section 6 is devoted to open questions.

Notation is standard and follows, for example, Soare [15]. The following exceptions are taken from [6]. We denote *tt*-functions by Γ and Δ and the uses are the corresponding lower case letters γ and δ. We let $\{\alpha_e : e \in \omega\}$ list all partial recursive unary functions. As usual all computations, etc., are bounded by s at stage s. We let $A[x] = \{z : z \in A \text{ and } z \leqslant x\}$. The tree argument of Section 3, implicit in Sections 4 and

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5, is similar in structure to that found in [6]. While we have given as many details as necessary to make the paper self-contained, we have sketched in a little in the verification of Section 3 rather than reproducing a lot of [6].

2. *One m-degree*

We begin our journey by briefly recalling the construction in [6] of a singular r.e. tt-degree. That is, one consisting of a single r.e. m-degree. We build $A = \bigcup_s A_s$ to meet the requirements

$$R_e : \Lambda_e(A) = V_e \quad \text{and} \quad \Gamma_e(V_e) = A \quad \text{implies} \quad V_e \equiv_m A.$$

We define the length of agreement functions as follows:

$$L(e,s) = \max\{x : \forall y < x(\Lambda_{e,s}(A_s; y) = V_{e,s}(y))\},$$
$$l(e,s) = \max\{x : \forall y < x(\Gamma_{e,s}(V_{e,s}; y) = A_s(y) \text{ and } L(e,s) \geqslant \gamma_e(y))\}.$$

As usual we do not allow sets in the range of a functional to change unless there is a corresponding change in the use. Thus $l(e,s)$ forms an 'A-controllable' length of agreement.

Let $\{a_{i,s} : i \in \omega\}$ list $\overline{A_s}$ in order of magnitude. At stage s define

$$A_s^i = A_s \cup \{a_{j,s} : i \leqslant j \leqslant i+s\}.$$

The basic idea behind meeting R_e is to monitor $l(e,s)$. When we see that $l(e,s) > a_{i,s}$ we can use the tt-reductions Λ_e and Γ_e to analyse the effect of possible future configurations of A. At this stage we agree that if $A_{s+1} \neq A_s$ then $A_{s+1} = A_s^i$ for some $i \leqslant s$. (Thus A is built by a 'dump' construction, it is, for example, retraceable and hyperimmune.)

Now the key idea is to see if we can diagonalize against e. This is possible if, for example, we can set $A_{s+1} = A_s^i$ but Γ_e says that there is no corresponding possible V_e-change. That is, $\Lambda_e(\Gamma_e(A_s^i))(z) \neq A_s^i(z)$ for some $z \leqslant a_{i,s}$. If this is so, we can immediately diagonalize against e by setting $A_{s+1} = A_s^i$ and restraining (with priority e) $A_{s+1}[a_{i,s}]$.

The other possible way of diagonalizing e is a *two step action*. If we cannot do it in a one step way, then for all i there must be some $y(i)$ such that if we set $A_{s+1} = A_s^i$ and wait till $l(e,t) > a_{i,s}$ again, then $y(i)$ must enter $V_e - V_{e,s}$. Now if, for some $j < i$, we have $y(i) \notin \Lambda_e(A_s^j) (= V_{e,s}^j$, say) then it follows that for some k we have $y(i) \in V_{e,s}^{k+1}$, but $y(i) \notin V_{e,s}^k$. We call k a *killing point* for e. The idea is straightforward. First set $A_{s+1} = A_s^{k+1}$ then wait till $y(i)$ enters $V_{e,t}$ by restraining $A_t[a_{k+1,s}]$ with priority e. When $y(i)$ enters,we then set $A_{t+1} = A_t^k$ and again restrain with priority e.

Now to see that the idea works. Suppose that we cannot win with a two-step action. Then for all i and s we have $V_{e,s}^{i+1} \subset V_{e,s}^i$. First $A \leqslant_m V_e$. To see if z enters A wait till $l(e,s) > z$. If $z \notin A$ then $z = a_{i,s}$ for some i. Use Γ_e and Λ_e to figure out the least $y(i)$ with $y(i) \in V_{e,s}^i - V_{e,s}$. Then $z \in A$ if and only if $y(i) \in V_e$. Similarly $V_e \leqslant_m A$. Wait till $l(e,s) > x$. If $x \notin V_e$ then we can analyse Λ_e to see if some legal A-configuration will cause x to enter. If it does, then for this configuration to be legal, it must be of the form $A_{s+1} = A_s^i$. For the largest such i, we see that $x \in V_e$ if and only if $a_{i,s} \in A$. Hence $V_e \equiv_m A$ via a standard application of the finite injury method. (Fuller intuitive discussion can be found in [6].)

3. *Two m-degrees*

In this section, we shall describe how to construct an r.e. *tt*-degree consisting of two r.e. *m*-degrees. This was a question left open in [5, 6]. The principal technique is a new flexible *tt*-reduction (simpler than those employed in [6]) together with the tree machinery of [6].

We shall build A_0 and A_1 in stages with $A_0 \leqslant_m A_1$, $A_0 \equiv_{tt} A_1$ and such that we meet the requirements below:

$$P_e: \neg(A_1 \leqslant_m A_0) \quad \text{via} \quad \alpha_e,$$

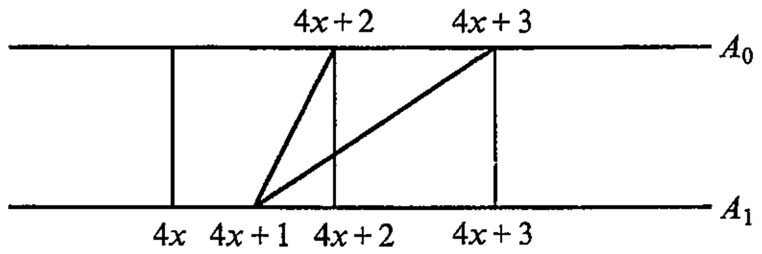
$$R_e: \Lambda_e(A_0) = V_e \quad \text{and} \quad \Gamma_e(V_e) = A_0 \quad \text{implies} \quad V_e \equiv_m A_0 \quad \text{or} \quad V_e \equiv A_1.$$

We shall ensure that there are four infinite disjoint recursive sets Q_0, Q_1, Q_2, and Q_3 so that $A_0 \subseteq \bigcup_{i \neq 1} Q_i$, $A_0 \cap \bigcup_{i \neq 1} Q_i = A_1 \cap \bigcup_{i \neq 1} Q_i$, and $A_1 \subseteq \bigcup_i Q_i$. We can consider $\bigcup_i Q_i$ as partitioning ω and, for example, $Q_i = \{4x+i : i \in \omega\}$. Then the reductions are

(3.1) $\quad z \in A_0$ if and only if $z = 4x, 4x+2$ or $4x+3$ and $z \in A_1$,

(3.1)′ $\quad 4x+1 \in A_1$ if and only if $4x+2 \in A_0$ or $4x+3 \in A_0$.

For those familiar with [6], the diagram of this reduction is the following.



DIAG. 1

We first describe the manner by which we satisfy P_e. For P_e alone, we pick some follower $q = 4x+1 \notin A_1$ targeted for A_1. We wait till $\alpha_{e,s}(q)\downarrow$. If such a stage s does not occur then α_e is partial and surely cannot witness an *m*-reduction reducing A_1 to A_0.

Let $F_x = \{4x, \ldots, 4x+3\}$. If $\alpha_e(q)\downarrow$ and $\alpha_e(q) \in A_{0,s}$ we win by simply keeping q out of A_1. If $\alpha_e(q) > 4x+3$, similarly we can win by dumping $4x+4, \ldots, 4s+3$ into $A_{0,s+1}$, putting $\alpha_e(q)$ into A_0 since $\alpha_e(q) < s$ by convention. Note that, to make our life simpler here, we ask that if $z \in F_x$ enters $A_{1,s}$ then for all $y > x$ with $y < s$, all the elements of F_y enter $A_{1,s}$ too. Thus we have a dump construction on the blocks. Also we agree that $4x$ is the 'garbage element'. So if $4x$ enters then all of F_x enters. Finally, if $\alpha_e(q) \notin A_{0,s}$ and $\alpha_e(q) \leqslant 4x+3$, then we win as follows. For some $j \in \{2, 3\}$ we have $4x+j \neq \alpha_e(q)$. We shall put q into A_1 and $4x+j$ into both A_0 and A_1 and protect the result with priority e. Note that this means that $q \in A_1$ yet $\alpha_e(q) \notin A_0$.

We now turn to the satisfaction of the R_e. The fundamental idea is to pursue the strategy of Section 2 – namely, to analyse the effect of potential A-configurations – but now in the context of the reductions (3.1) and (3.1)′.

Let $\{a_{i,s} : i \in \omega\}$ list in order $\overline{A_{0,s}} - \{4x+1 : x \in \omega\}$ and let $l(e, s)$ be the analogous length of agreement function of Section 2. (Here we would use A_0 in place of A.) Note that for any unused F_x, we have $F_x = \{a_{i,s}, 4x+1, a_{i+1,s}, a_{i+2,s}\}$ for some i. Now by our agreed construction, at any stage s we can make $A_{0,s+1} = A_{0,s}$, $A_{0,s+1} = A^i_{0,s}$ for $a_{i,s} \in \{4x, 4x+3\}$ or $A_{0,s+1} = A^{i+1}_{0,s} \cup \{4x+2\}$, where $a_{i,s} = 4x+4$. In the last case we write $A_{0,s+1} = A^{i+}_{0,s}$.

Now, as in Section 2, we shall argue that if V_e does not *locally emulate* one of A_0 or A_1, we can diagonalize α_e forever. First we ask if there are legal future configurations of A_0 that allow us to diagonalize for R_e. That is, is there a one or two stage action (as in Section 2) for such a diagonalization. If so we take the appropriate action. Note that, as in [6], actions are *constrained*. If we set $A_{0,s+1} = A^i_s$ for $a_{i,s} = 4x+3$ then we must put $4x$ into A_0 if we put $4x+2$ into A_0. Similarly, if we set $A_{0,s+1} = A^{i+}_{0,s}$ then if we put either $4x+3$ or $4x$ into A_0 we must put both in. Suppose we cannot so diagonalize. First note that V_e can m-compute A_0. To see if $z \notin \{4x+1 : x \in \omega\}$ enters A_0 wait till $l(e,s) > a_{j,s}$ for $j = 4z+4$ or z enters A_0. Now assuming that $z \notin A_{0,s}$, we have $z = a_{i,s}$ for some i. First suppose that F_x contains z and F_x is unused. Then we have a number of cases. If $z = 4x+3$ or $z = 4x$ then $A_{0,s+1} = A^i_s$ is possible. Otherwise $A_{0,s+1} = A^{i+}_{0,s}$ is possible. To the appropriate configuration will correspond the entry into V_e (as in Section 2) of a unique least $y(i)$. One can readily see that $z \in A_0$ if and only if $y(i) \in V_e$. If F_x is used then the argument is essentially the same, but easier, for z will enter if and only if all of F_x enters A_0 which happens if and only if $a_{k,s} = 4x$ enters A_0, where $z = 4x+k$. So find the $y(k)$ which enters if we set $A_{0,s+1} = A^k_{0,s}$ and then $z \in A_0$ if and only if $y(k) \in V_e$.

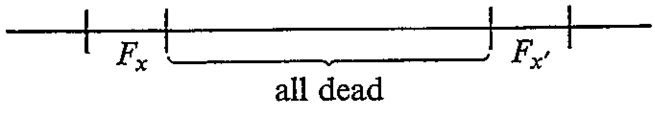
Now we need to do some work to make $V_e \equiv_m A_0$ or $V_e \equiv_m A_1$. Assume we are unable to diagonalize R_e. The key idea from [6] is to attach a *state* to F_x, according to which of A_0 or A_1, V_e locally emulates. First note that $V_e \leqslant_m A_1$. By the modified dump construction we use to build A_1, we see that if z can enter V_e there is some unique largest block F_x that will cause its entry. This can be seen when $l(e,s) > F_x$. Similarly by the possible transitions of $A_{0,s}$ to $A_{0,s+1}$, we know that z can enter in only one of precisely two ways; either (1) through a unique largest i, $A_{0,s+1} = A^i_{0,s}$ or $A_{0,s+1} = A^{i+}_{0,s}$, or (2) through either of the two possible configurations, $A^i_{0,s}$ or $A^{i+}_{0,s}$. In the former case $z \in V_e$ if and only if $a_{i,s} \in A_1$. In the latter case $z \in V_e$ if and only if $4x+1$ $(= a_{i,s} - 1)$ is an element of A_1. In any case $V_e \leqslant_m A_1$ always. The key is to make $\overline{A_0}$ 'pure' for V_e.

Reversing the above reasoning, in this context, V_e locally emulates A_1 only if there is some $z(i)$ such that

$$z(i) \in V_e \text{ if and only if } (A_0 \text{ contains either } A^i_{0,s} \text{ or } A^{i+}_{0,s}).$$

We assign to this the high e-state 2 (of course σ with $|\sigma| = e$ and $\sigma(e) = 2$). This F_x, if unused, will first have state 0 (waiting). When $l(e,s) > F_x$ if F_x is unused it is given e-state 1 unless such a $z(i)$ exists for F_x. If, at some $t > s$ such a $z(i)$ appears F_x gets state 2. Note that if F_x is used but $F_x - \{4x+1\} \not\subseteq A_{0,s}$ then F_x is essentially like a single element. That is, if any more of F_x enters A_0, then all of it so enters. So the used F_x do not raise their states. Note also that if currently F_x is devoted to meeting P_k and its state rises to σ, it may be reassigned to the least unsatisfied P_f with no follower block of e-state σ.

This is the machinery of [6]. At any stage s we have a largest live F_x with e-state 2. When we see an $F_{x'}$ for $x' > x$ with state 2 we declare those currently live F_z with $x < z < x'$ as 'e-dead'. Thereafter, these F_z will enter A_0 if and only if $F_z \cap A_0 \neq \varnothing$. Diagram 2 may help here to visualise the situation.



DIAG. 2

The remaining details for the case of 2 r.e. m-degrees are to implement the above via a tree of strategies argument akin to a full approximation argument. (Actually, as Jockusch pointed out (in a personal communication), the whole argument is rather similar to an η-maximal set argument except that at any stage R_e has the potential Σ_2^0 outcome where its effect is finite (as it is diagonalized).)

For the construction to follow, it is convenient to let $\{F_{x,s}: x \in \omega\}$ list in order the set of F_x with $F_x \cap \overline{A_s} \neq \varnothing$.

We shall now give some formal details but these are essentially similar to those of [6]. If the reader is familiar with them, she or he should skip to the next section.

Let $\Lambda = \{0, 1, 2, w, k\}$ be ordered by $<_\Lambda$ in the manner given, so that $0 > 1 > 2 > w > k$. We refer to $\sigma, \tau \in \Lambda^{<\omega}$ as *guesses*. We write $\sigma \subset \tau$ if σ is an initial segment of τ. We use λ as the empty guess. Let $\leqslant_L$ denote the lexicographic order:

$$\sigma \leqslant_L \tau \leftrightarrow (\sigma \subseteq \tau \vee \exists \gamma (\gamma * i \subseteq \sigma \wedge \gamma * j \subseteq \tau \wedge i <_\Lambda j)).$$

We let $|\sigma|$ denote the length of σ. If $|\sigma| = e$ then σ is assigned to R_e.

The intention of Λ is: k for 'kill' meaning we believe we are preserving a disagreement; w for 'wait' meaning that we are in the middle of a 2-step attack on R_e waiting for computations to recover; 1 and 2 correspond to $l(e,s) \to \infty$ and the cases discussed above, and 0 is meant to indicate $l(e,s) \not\to \infty$.

(3.2) DEFINITION. Define the notions *σ-stage*, *e-state* σ and *σ-dead* by simultaneous induction on $|\sigma|$.

(i) Every stage s is a λ-stage. Every block has -1-state λ.
(ii) We are given $(e-1)$-states and $\tau \in \Lambda^{<\omega}$ with $|\tau| = e$ and s a τ-stage. In order of $x \leqslant s$, generate the e-state of $F_{x,s}$ with $F_{x,s} \cap \overline{A_{0,s}} \neq \varnothing$ as follows.

Case 1. First, if for some $\gamma \leqslant_L \tau$ with $|\gamma| = e$ we have $r(\gamma * k, s) \neq 0$, we do nothing save to declare s a $\tau * k$-stage. (This indicates that we are currently believing that we are preserving a R_e-disagreement.)

Case 2. Case 1 does not pertain and R_e requires attention in a τ-legal way. That is, one of the following options holds.

Case 2(a). There exists $a_{i,s} > \max\{a_{e,s}, q(\gamma, s), r(\gamma, s): \gamma \leqslant_L \tau\}$ such that
(i) $a_{i,s}$ is not τ-dead and is not constrained,
(ii) $l(e,s) > a_{i,s}$ and for all sets U with $V_{e,s} \subseteq U \subseteq \{0, \ldots, s\}$, we have either

$$(\Gamma_e(U) \neq A_{0,s}^i)[l(e,s)] \quad \text{or} \quad (\Delta_e(A_{0,s}^i) U)[u(\Gamma_e(U; l(e,s)))].$$

Case 2(b). There exists $a_{i,s} > \max\{a_{e,s}, q(\gamma,s), r(\gamma,s): \gamma \leqslant_L \tau\}$ such that $l(e,s) > a_{i,s}$ and we can τ-legally win a two step action at $a_{i,s}$: there exists $j > i$ such that $l(e,s) > a_{j,s}$ and

(i) $a_{j,s}$ is not τ-dead and is not constrained,
(ii) if we set $A_{0,s+1} = A^j_{0,s}$ then $a_{i,s}$ is not constrained at stage $s+2$,
(iii) for all sets U with $V^j_{e,s} \subseteq U \subset \{0, \ldots, s\}$ we have either

$$(\Gamma_{e,s}(U) \neq A^i_{0,s})[l(e,s)] \quad \text{or} \quad (\Delta_e(A^i_{0,s}) \neq U)[u(\Gamma(A^i_{0,s}(l(e,s))))].$$

If Case 2 pertains, we declare s to be a $\tau * w$-stage.

Case 3. None of the above pertains and there exists a least x such that
(i) $F_{x,s}$ is unused,
(ii) $F_{x,s} > \max\{F_{e,s}, q(\gamma, s), r(\gamma, s) : \gamma \leqslant_L \tau\} =_{\text{def}} m(\tau, s)$,
(iii) $l(e,s) > F_{x,s}$,
(iv) $F_{x,s}$ has $(e-1)$-state τ,
(v) one of the subcases below pertains.

Let $F_{x,s} = \{a_{i,s}, a_{i+1,s}, a_{i+2,s}\}$. We keep the $V^j_{e,s}$ notation of Case 2 and the intuitive discussion preceding the construction.

Case 3(a) $F_{x,s}$ has e-state $\tau * m$ for $2 <_\Lambda m$, and some $y(x)$ occurs so that

$$y(x) \in (V^{i+3}_{e,s} - V^{(i+1)+}_{e,s}) \cap (V^{i+3}_{e,s} - V^{i+2}_{e,s}).$$

In this case, give block $F_{x,s}$ the e-state $\tau * 2$. Find the greatest y with $y < x$ such that $F_{y,s} \leqslant m(\gamma, s)$ for some $\gamma \leqslant_L \tau$ or $F_{y,s}$ has e-state $\tau * 2$. Declare all $a_{n,s}$ for $F_{y,s} < a_{n,s} < \min\{z : z \in F_{x,s}\}$ to be $\tau * 2$-dead. Finally declare s to be a $\tau * 2$-stage.

Case 3(b) $F_{x,s}$ has $(e-1)$-state $\tau * 0$ and $l(e,s) > F_{x,s}$. In this case declare $F_{x,s}$ to have e-state $\tau * 1$. Find the greatest y with $y < x$ such that $F_{y,s} \leqslant_m (\gamma, s)$ for some $\gamma \leqslant_L \tau$ or $F_{y,s}$ has e-state $\tau * 1$ or $\tau * 2$. Declare all $a_{n,s}$ for $F_{y,s} < a_{n,s} < \min\{z : z \in F_{x,s}\}$ to be $\tau * 1$-dead. Declare s to be a $\tau * 1$-stage.

Case 4. None of the above pertains. Declare s to be a $\tau * 0$-stage, otherwise change nothing.

Notation. Let σ_s denote the unique string of length s with s a σ_s-stage.

(3.3) Definition. We say that P_e *requires attention* at stage $s+1$ if P_e is not currently declared satisfied and one of the following options holds:
(3.4) P_e has a follower block $F_{x,s}$ with e-state $\subset \sigma_s$ such that $\alpha_{e,s}(4x+1)\downarrow$ where $4x+1 \in F_{x,s}$;
(3.5) P_e has no follower block with e-state $\sigma \subset \sigma_s$, where $|\sigma| = e+1$, and there is a currently unassigned (unused) block $F_{x,s}$ with e-state σ, with

$$\max\{q(\tau, s), r(\tau, s) : \tau \leqslant_L \sigma\} < \min\{z : z \in F_{x,s}\},$$

and such that $F_{x,s}$ is not γ-dead for any $\gamma \leqslant_L \sigma$.

Finally we shall use, at stage s, the phrase *initialize* (for example $\gamma \geqslant_L \sigma$). As with standard practice this means γ-assignments, states, satisfaction etc. become reset to the initial values. Note that $\sigma * 0$ is the initial e-state if we so initialize.

Construction. *Stage* 0. Initialize all $\sigma \in \Lambda^{<\infty}$, and define $q(\sigma, 1) = r(\sigma, 1) = 0$ for all σ.

Stage $s+1$. *Step* 1. Compute σ_{s+1}. Initialize all γ with $\gamma \nleqslant_L \sigma_{s+1}$.

Step 2. Find the least e, if any, such that R_e or P_e requires attention. If more than one does, sort them out by the given priority order. Adopt the first case below to pertain. Initialize all those γ with $\sigma \leqslant_L \gamma$ and $\sigma \neq \gamma$, where $\sigma \subset \sigma_s$ and $|\sigma| = e+1$. Let $\sigma = \tau * n$.

Case 1 (*R_e receives attention at τ*). Case 2 of (3.2) pertains. Initialize P_e at guess $\tau * j$ for all j. Set $r(\tau * w, s+1) = s+1$. Adopt the first subcase below to pertain.

Subcase 1. Case 2(a) of (3.2) holds. Set $A_{s+1} = A_s^i$. Set $r(\tau * k, s+1) = s+1$.

Subcase 2. Case 2(b) of (3.2) holds. Set $A_{s+1} = A_s^j$.

Case 2 (*P_e receives attention at τ*). Set $q(\sigma, s+1) = s+1$. Adopt the first subcase below to pertain.

Subcase 1 (3.4) holds. Let $4y+1 = a_{i,s}-1$.

Option (a) $\alpha_e(4y+1) \in A_{0,s} \cup \{z : 4y+3 < z\}$. *Action*. Set $A_{0,s+1} = A_{0,s}^{i+3}$.

Option (b) $\alpha_e(4y+1) \notin A_{0,s}^{i+2}$. *Action*. Set $A_{0,s+1} = A_{0,s}^{i+2}$.

Option (c) otherwise (so that $\alpha_e(4y+1) = 4y+3$). *Action*. Set $A_{0,s+1} = A_{0,s}^{(i+1)+}$.

In all cases above declare P_e as satisfied at guess σ.

Subcase 2 (3.5) holds. Assign $F_{x,s}$ to be a follower block of P_e at guess σ.

This completes the construction.

Verification. Let β denote the leftmost path. Thus β is defined by induction: $\lambda \subset \beta$ and for all σ if $\sigma \subset \beta$ then one of $\sigma * i \subset \beta$ for some i. This is according to the rule that i is $<_\Lambda$-least with infinitely many $\sigma * i$-stages.

The exact analysis of the construction is almost [6] verbatim. We repeat enough of it here to enable the paper to be self contained.

The following technical lemma is easily established by induction and is left to the reader. (We shall use it implicitly.)

3.6 Lemma. (i) *Let $F_{x,s}$ be a live block (that is, with $F_{x,s} \cap \overline{A_s} \neq \varnothing$). Suppose that s is a $\sigma * i$-stage with $|\sigma| = e$ for $i = 1, 2$, or 3. Then one of the following options holds:*

(a) *$F_{x,s}$ is $\sigma * i$-dead,*

(b) *$F_{x,s} < r(\gamma, s)$ or $q(\gamma, s)$ for some $\gamma \leqslant_L \sigma$,*

(c) *$F_{x,s}$ has e-state $\sigma * i$ at stage $s+1$,*

(d) *for all $y \geqslant x$ neither* (a), (b) *nor* (c) *hold for $F_{y,s}$.*

(ii) *If $F_{x,s}$ and $F_{y,s}$ are live blocks at stage s, $x < y$ and $F_{x,s} > \max\{r(\gamma, s), q(\gamma, s)\}$ for all $\gamma \leqslant_L \sigma$, then if $F_{y,s}$ has e-state $\sigma * i$ and $F_{x,s}$ has e-state $\sigma * j$ then $j \geqslant_\Lambda i$.*

(iii) *If $F_{x,s}$ has e-state $\sigma * i$ for $i = 1, 2$, then $F_{x,s}$ was assigned e-state $\sigma * i$ at a $\sigma * i$-stage $t \leqslant s$ when $F_{x,s} = F_{x,t}$ was unused at stage t.*

3.7. LEMMA. *Let $\sigma \subset \beta$ with $|\sigma| = e+1$. Then*

(a) *P_e and R_e receive attention finitely often at γ-stages for $\gamma \leqslant_L \sigma$,*

(b) *for all $\gamma \leqslant_L \sigma$, $\lim_s q(\gamma, s) = q(\gamma)$ and $\lim_s r(\gamma, s) = r(\gamma)$ exist,*

(c) *P_e is met,*

(d) *if $\sigma = \tau * 0$ or $\sigma = \tau * k$ then R_e is met,*

(e) *if $r(\gamma * k) \neq 0$ for some γ with $|\gamma| = e$ and $\gamma * k \leqslant_L \sigma$, then R_e is met and $\sigma = \tau * k$ or $\tau * 0$ for some τ.*

Proof. Let s_0 be a stage such that for $s > s_0$ we have

(i) $\sigma \leqslant_L \sigma_s$,

(ii) $\forall j < e$ (P_j and R_j do not receive attention at σ-stages),

(iii) if we let $\sigma = \sigma^+ * i$, then, for all $\rho \leqslant \sigma^+$, $r(\rho, s) = r(\rho, s_0)$ and $q(\rho, s) = q(\rho, s_0)$.

First note that if $r(\gamma * k, s) \neq 0$ for some $\gamma \leqslant_L \sigma$ with $|\gamma| = e$ and $\gamma \neq \sigma$, then $r(\gamma * k, s) = r(\gamma * k)$ and, as in the intuitive discussion and Section 2, this is preserving an e-disagreement which will not be violated. Thus we shall suppose that, for all such γ, $r(\gamma * k, s) = 0$. We claim that R_e can receive attention at most twice more at σ-stages. Again this is like Section 2. If, for example, Case 1, Subcase 1 pertains to some $a_{i,s}$ at a σ-stage $s_1 > s_0$, then we set $A_{s_1+1} = A^i_{s_1}$, and all η with $\sigma <_L \eta$ are initialized. We also set $r(\sigma^+ * k, s_1) = s_1$, and by construction this restraint cannot be violated. Note that in this case (by definition of σ-stage), since $\sigma^+ \subset \beta$, it must be that $\sigma = \sigma^+ * k$.

The other case (Subcase 2, then Subcase 1) is entirely similar to this and Section 2 and is left to the reader.

Finally we consider P_e. Once R_e ceases receiving attention we are free to attack P_e at will. Thus once we have a σ-stage t such that $t > s_0$ and for all σ-stages R_e will not receive attention at stage s, it must be that P_e gets a follower block with e-stage σ (by induction). By the technical lemma and the choice of t this assignment cannot be cancelled. As in the intuitive description, we must win P_e on this block. Thus P_e is met and $\lim_s q(\sigma, s)$.

Now we must check the key lemma (that all of our machinery was set up to establish).

3.8 LEMMA (Truth of outcome for R_e). *Suppose that $l(e,s) \to \infty$. Then $\sigma = \sigma^+ * n \subset \beta$, where $|\sigma^+| = e$ and $n = 1$ or 2. Furthermore,*

(i) *if $n = 1$ then $V_e \equiv_m A_0$,*

(ii) *if $n = 2$ then $V_e \equiv_m A_1$.*

Proof. Let s_0 be as in (3.7). If $l(e,s) \to \infty$, then by the definition of σ-stage (and the fact that there will be infinitely many P_e with $\operatorname{dom} \alpha_e = \emptyset$) it must be that either $\sigma^+ * 1 \subset \beta$ or $\sigma^+ * 2 \subset \beta$. Fix $\sigma = \sigma^+ * n$. We must verify (i) and (ii) above.

(i) $n = 1$. We must show that $V_e \equiv_m A_0$. Let x be such that at some σ-stage $s_1 > s_0$ we have

(a) $F_{x,s_1} = F_x > \max\{r(\gamma), q(\gamma) : \gamma \leqslant_L \sigma\}$,

(b) $\forall y > x \forall s > s_1$ ($F_{y,s}$ has e-state γ at stage s implies $\sigma \leqslant_L \gamma$).

The existence of x and s_1 is justified by choice of s_0 and the technical lemma. The intuition here is that, beyond F_x, if F_y gets an e-state it is at best σ.

We need to prove that $V_e \leqslant_m A_0$. Let $z > F_x$ be given. Find the least σ-stage $s = s(z)$ such that the e-state σ is assigned to some block $F_{y,s}$ with $F_{y,s} > F_x$ and $z < F_{y,s}$. Note that, as in Section 2, $l(e,s) > z$. We might as well also suppose that

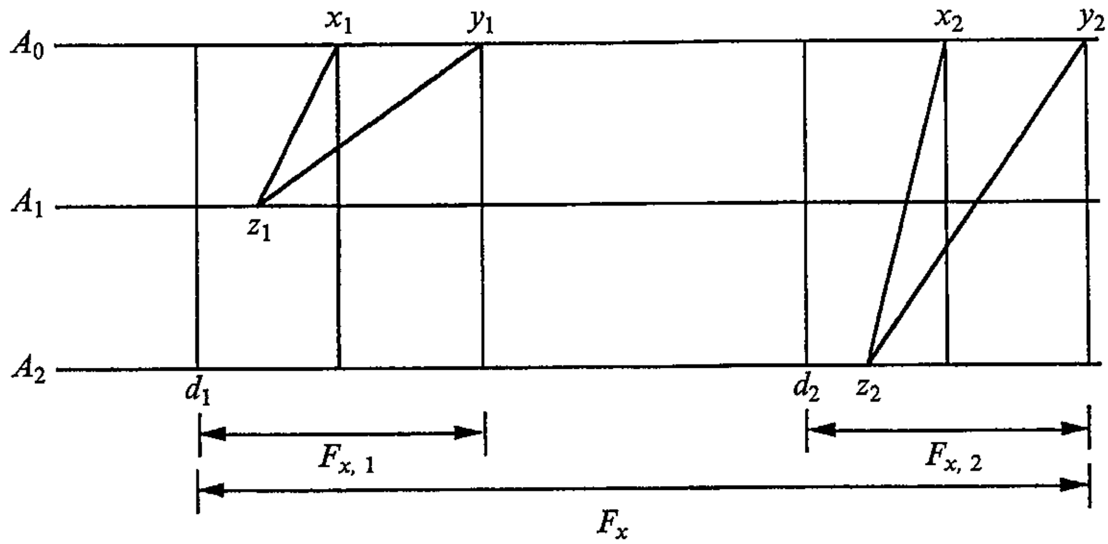
$z \notin V_{e,s}$. First, we ask if any σ-legal configuration of A_0 can cause z to enter V_e. Thus we see if there exists a σ-live unconstrained number $a_{i,s}$ such that $\Delta_{e,s}(A^i_{0,s}) \models z \in V_e$. Note that if i is the largest such and $a_{i,s} < F_x$, then $z \notin V_e$ since $A_{0,s}[F_x] = A_0[F_x]$. We define an i to be *attainable* if $a_{i,s} > F_x$ and $a_{i,s}$ is unconstrained and $a_{i,s}$ is not σ^+-dead. If no attainable i exists then $z \notin V_e$. If an attainable i exists it must also be that, for all σ^+-legal attainable configurations $A^j_{0,s}$ with $j < i$, we have $\Delta_{e,s}(A^j_{0,s}) \models z \in V_e$ except for $j = i-1$ if $a_{i,s} = 4y+3$. (If $j = i-1$ and $a_{i,s} = 4y+3$, then it must be that $\Delta_{e,s}(A^{i-1}_{0,s}) \models z \notin V_e$ since the state has $n = 1$.) It is not difficult to see that if this condition does not hold for all $j < i$ then we can win by a diagonalization. So we see that $z \in V_e$ if and only if $a_{i,s} \in A_0$. As with the basic module we can m-compute A_0 from V_e and hence $A_0 \equiv_m V_e$.

(ii) The proof in the case $n = 2$ is essentially similar. For further details see [6].

4. *Finite Boolean algebras*

The next step in the proof is to extend the ideas of Section 3 to construct an r.e. tt-degree whose r.e. m-degree structure realises a given finite boolean algebra. We remarked in the Introduction that in [6] it is claimed that the ideas used for 3 r.e. m-degrees could be easily extended to construct one with $2^n - 1$ r.e. m-degrees whose m-degree structure was a boolean algebra missing a zero, and hence that there are r.e. tt-degrees with arbitrarily large, but finite, numbers of r.e. m-degrees. This claim is not quite correct in the sense that, although the argument of [6] can be used to show that there are r.e. tt-degrees with arbitrarily large (but finite) numbers of r.e. m-degrees, the exact number does not seem to follow. We shall make some further comments on that paper at the end of this section.

In this section we first indicate how to modify Section 3 to get all finite boolean algebras. Let B_n denote the n-atom boolean algebra. The tt-reductions for B_2 are shown in the Diagram 3.



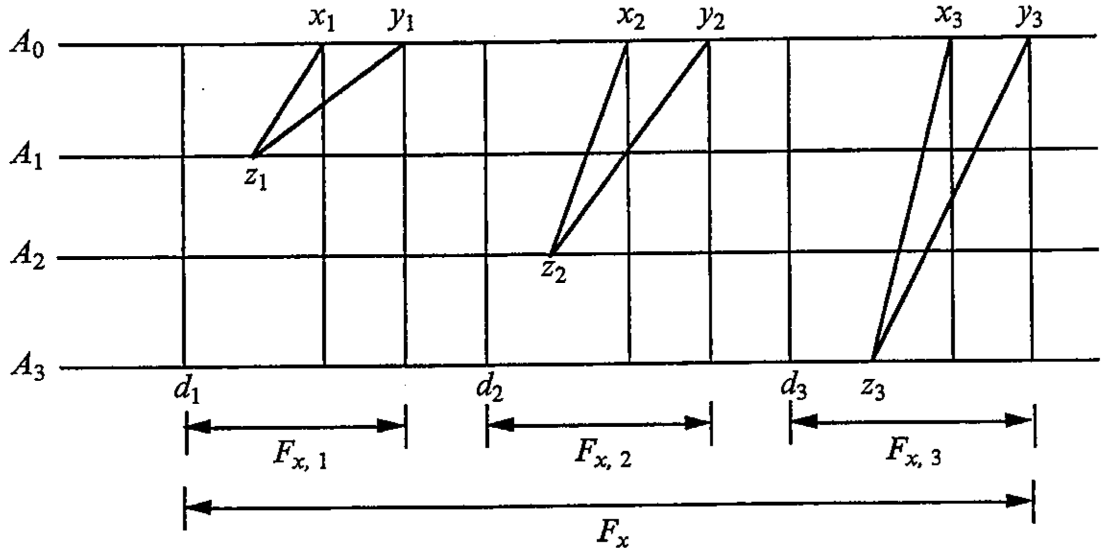
DIAG. 3

Here F_x is a block of 8 (not 4 as in Section 3) so that $z_1 = 8x+1$ and $z_2 = 8x+5$. We promise that $z_i \in A_i$ if and only if x_i or $y_i \in A_0$. Then for $q = d_i$, x_i, or y_i, $q \in A_k$ if and only if $q \in A_i$. Finally we have a modified dump. That is, we use a dump construction on the blocks and on the subblocks $F_{x,1} = \{d_1, z_1, x_1, y_1\}$ and $F_{x,2} = \{d_2, z_2, x_2, y_2\}$. In other words, if any of $F_{x,1}$ enters A_0 then all of $F_{x,2} - \{z_2\}$ enters A_0.

It is only within the subblocks on the x_i and y_i again that we can set $A_{0,s+1} = A_{0,s}^{i+}$ exactly as in Section 3. Now also we agree that a block can be used only once. So as soon as it is used, for example, we put z_2 into A_2 and x_2 into A_0 (to diagonalize $A_2 \not\leq_m A_1$), the rest of the block can only enter A_0 all together. Hence only unused blocks can change states.

States are now generated by the subblocks. A state is now an ordered pair $\{p, q\}$ with $p, q \in \{0, 1\}$. The idea is that on the $F_{x,i}$, subblock V_e can locally emulate only either A_i or A_0. We shall let the state of $F_{x,i}$ be 1 if and only if $F_{x,i}$ locally emulates A_i, and 0 otherwise. Note that this gives 4 states: $\{0,0\}$, $\{0,1\}$, $\{1,0\}$, and $\{1,1\}$. These correspond to V_e emulating respectively A_0, A_2, A_1, and $A_1 \oplus A_2$. These are of course the 4 vertices of the boolean algebra, and are ordered lexicographically. With these modifications the argument clearly goes through precisely as before.

The situation for B_3 is given in Diagram 4.

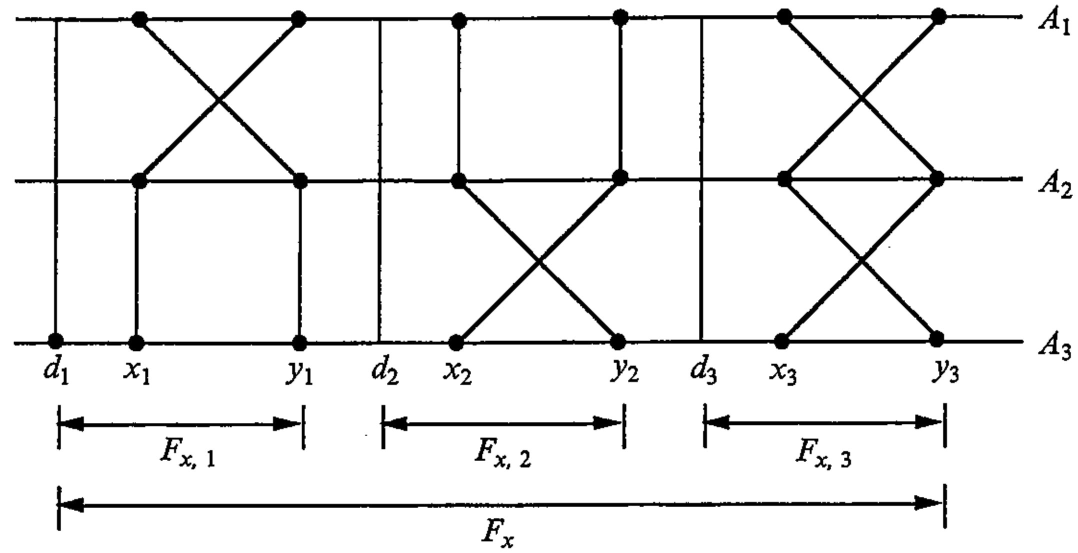


DIAG. 4

Here note that $F_{x,1}$ is used for $A_1 \not\leq_m A_2 \oplus A_3$, $F_{x,2}$ for $A_2 \not\leq_m A_1 \oplus A_3$ and $F_{x,3}$ for $A_3 \not\leq_m A_1 \oplus A_2$.

Now note that the modified dump of the construction ensures that if y enters V_e due to the change of an A-configuration, again it blames a unique $F_{x,i}$ subblock (again the largest one that can cause the entry of y). On this subblock the configurations coincide exactly with the 2 element case. Hence the analysis goes through as before.

To conclude the section, we demonstrate that the reduction used in the last section of [6] is flawed. There it is claimed that one can construct $2^n - 1$ r.e. m-degrees via the following reduction (for $n = 3$) shown in Diagram 5.



DIAG. 5

Now, we use $F_{x,i}$ to make $A_i \not\leqslant_m A_j \oplus A_k$ for $i \neq j \neq k$. The reductions are the following (illustrated for $F_{x,2}$). First $z \in A_1$ if and only if $z \in A_3$. Then $y_2 \in A_2$ if and only if $y_2 \in A_1$; $d_2 \in A_2$ if and only if $d_2 \in A_1$; and $x_2 \in A_2$ if and only if ($y_2 \in A_1$ and $x_2 \notin A_1$) or $d_2 \in A_2$. The construction is a dump one. Diagonalization is achieved via the x_i. We compute $\alpha_e(x_i)$, then if $\alpha_e(x_i) > y_i$ we can win by dumping all $k > y_2$. If $\alpha_e(x_i) = y_i$, we put x_i and y_i into A_q for $q \neq i$. This puts y_i into A_i but x_i not into A_i. So $\alpha_e(x_i) \in A_q$ for $q \neq i$ yet $x_i \notin A_i$. On the other hand, if $\alpha_e(x_i) \leqslant x_i$ we put z for $z \geqslant y_i$ into A_q for $q \neq i$. Then $x_i \in A_i$ yet $\alpha_e(x_i) \notin A_q$.

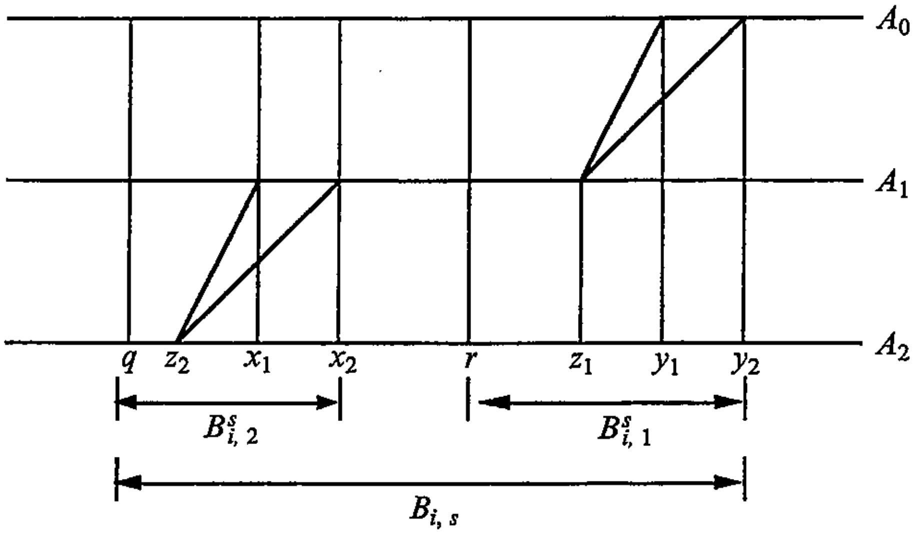
The problem in the above is instructive. It comes from a phenomenon we call *mixed states*. Suppose for instance that F_x emulates $A_1 \oplus A_3$ on $F_{x,1}$ and $F_{x,2}$ yet it emulates A_3 on $F_{x,3}$. Surely this cannot correspond to $A_1 \oplus A_3$ alone nor to A_3 alone. The problem arises because, for example, A_3 occurs too often in the $F_{x,i}$. Note that the above can be used to construct arbitrarily large numbers of r.e. m-degrees. This is done by assigning one of the mixed states to F_x instead of one from the boolean algebra. Thus we assign a state to each $F_{x,i}$ (rather than to F_x) and concatenate these 3 possible states for each $F_{x,i}$ to give 27 possible states to F_x. This gives an upper bound of 27 r.e. m-degrees and a lower bound of $2^3 - 1 = 7$ r.e. m-degrees.

We do not at present know how to realise the structure with $2^n - 1$ r.e. m-degrees and n zero elements. Of course we do get the relevant number in the next section.

5. *Finite distributive lattices*

In this section, we shall achieve our overall goal by proving that if L is a finite distributive lattice then there is an r.e. tt-degree **a** such that the structure of the m-degrees inside **a** is isomorphic to L. This involves only one new idea, together with some technical devices which are fairly standard. Again we remark that **a** will consist of a single tt-degree of norm 2 and all reductions will be positive.

To illustrate this new idea we begin with the 3-element chain. This has bottom A_0, middle A_1 and top A_2. The requirements are, as before, an R_e which says that if $A_0 \equiv_m V_e$ then V_e is one of the A_i in m-degree. As well we have diagonalization requirements saying that $A_1 \not\leqslant_m A_0$ and $A_2 \not\leqslant_m A_1$. Again our strategy is based on the simple one of Section 3. A block B_i^s devoted to meeting such requirements consists of a pair of subblocks, and is of the form described below in Diagram 6.



DIAG. 6

Here $B_{i,j}^s$ is devoted to $A_j \not\leqslant_m A_{j-1}$. Note that it is immediate that $A_j \leqslant_m A_{j+1}$. The problem we must avert is, again, the one of mixed states. Note that any subblock $B_{i,j}^s$ can have 'state' A_j or A_0 by abuse of terminology. The crucial problem is that if the final σ-live blocks have state A_2 *then they must also have state* A_1, *since* $A_2 \geqslant_m A_1$. One can consider states as pairs (p_1, p_2) where $p_j \in \{0, 1\}$ and $p_j = v$ if and only if $B_{i,j}^s$ has state A_j. The possible states are then again $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$. If these states were the well resided states we would argue that $(0,0)$ means $A_0 \equiv_m V_e$, $(0,1)$ means $A_1 \equiv_m V_e$, $(1,1)$ means $A_2 \equiv_m V_e$ but $(1,0)$ is a mixed state. In particular as V_e has state 1 on $B_{i,2}^s$, there is some $g(2)$ that enters V_e if and only if either x_1 or x_2 enters A_0 so that V_e can figure out if z_2 enters A_2, but since $B_{i,1}^s$ is only in state 0, there is no such $g(1)$ for V_e corresponding to z_1.

The solution to this dilemma is to *convert* B_i^s into a pair where V_e can so figure out if z_1 enters.

First the reader should note that it seems reasonable to believe that infinitely many $B_{i,2}^s$ have state A_2 only if infinitely often at least another k of them have this state (where k is fixed). In particular, instead of using $k = 1$, here we ask that at least 2 new blocks B_i^s and B_j^s with $i < j$ occur with both B_i^s and B_j^s having state A_2. The idea is then to avoid the situation of a $(1,0)$-state, that is, the A_1-state being uncovered by the A_2 state, by *converting an* A_1 vs A_2 *subblock into an* A_1 vs A_0 *subblock*, in the situation where we would have B_i^s followed by B_j^s with states $(1,0)$. From an infinite recursive set set aside for this purpose, we would pick a large fresh number z_3, and ask that $z_3 \in A_1$ if and only if x_1^j or $x_2^j \in A_0$ if and only if $z_2^j \in A_2$, where $B_{j,2}^s$ is the subblock $\{q^j, z_2^j, x_1^j, x_2^j\}$. We now redefine the blocks so that B_i^{s+1} consists of $B_{2i,2}^s$ and $B_{j,2}^s$. All the other elements in the relevant subblocks between $B_{i,2}^s$ and $B_{i,1}^s$ are declared to be σ-dead as in Section 3. Note that since z_2^j enters A_2 if and only if z_3 enters A_1, since $B_{j,2}^s$ had state A_2, V_e can figure out if z_3 enters A_1. Of course now we use z_3 for the diagonalizations. The main point is that now we shall get at most one block in state $(1,0)$ since pairs of such blocks get converted as above into blocks with state $(1,1)$. Now the analysis goes through as before.

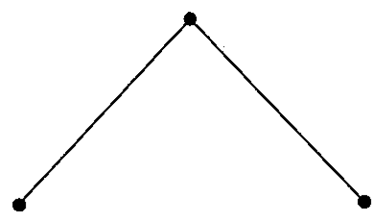
The key dynamical feature in the above is that, if in the lattice L node d_1 is above d_2, then if a subblock has state A_{d_1}, at stage s we can convert it into one with state A_{d_2}, since on the subblock the set A_{d_2} will currently be equal A_0 in the current reductions.

To complete this section, we now discuss the manner by which we use the above for an arbitrary finite distributive lattice. The point the reader should get from the discussion of the 3-chain is that a state will correspond to a set of subblocks, and we have a strategy that allows us to ensure that if $d_1 > d_2$ in the poset, then a d_1-strategy can be converted into a d_2-strategy.

Now let L be a finite distributive lattice. Let $|L| = n$. We can enumerate L so that if $\sigma > \tau$ in L then the number $n(\sigma)$ corresponding to σ exceeds $n(\tau)$ using the so-called topological ordering. Let $d_1, \ldots, d_k$ list the labels corresponding to join irreducibles of L. As L is a finite distributive lattice, if $\sigma \in L$ then σ is expressible as a unique join of join irreducibles. We then need k subblocks, one for each join irreducible, where the ith subblock is devoted to making $A_{d_i} \not\leq_m \oplus_{j \in S_i} A_{d_j}$, where the sum S_i is taken over those d_j in L not above d_i in L. Note that this suffices, as L is finite and distributive and the d_i are join irreducible. Note that for those $j \in S_i$, we set A_{d_j} equal to A_0 and on the remaining join irreducibles, (those above d_i) we set A_{d_v} equal to A_{d_i}. We can convert this into an $A_{d_q} \not\leq_m \oplus_{j \in S_q} A_{d_j}$ subblock for any d_q below d_i in L as above. A state is an ordered binary k-tuple with 1 in the ith position if the subblock of $B_{e,s}$ is in state A_{d_i}. We shall then change the block exactly as in the linear case, if we see enough (for example, at least n) blocks of the same uncovered state. This allows us to convert these into a single block of a covered state. By the enumeration of the nodes, ordering the states lexicographically, we see that the argument goes through exactly as before. This concludes our proof for the finite distributive case.

6. *Open questions*

By Downey [4], we know that the structure of r.e. m-degrees inside an r.e. tt-degree need not be a lattice, merely a distributive upper semilattice (USL). Downey [4] showed, for instance, that the USL below can be realised.



DIAG. 7

A natural conjecture is that if U is a finite distributive USL then U can be realised. A good test case is B_3 without the zero element. Note that our reductions are positive and need a least element. The best result for finite USL would be that every finite filter of a countable distributive USL can be realized.

Turning to infinite (upper semi-) lattices, it is clear that such lattices need to be Δ^0_3. Now such lattices are recursive direct limits of finite ones. The problem is that states do not seem to settle down in such instances. A new idea is needed here. We feel that a good test case is the linear ordering of type ω.

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