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RECURSIVELY ENUMERABLE m- AND tt-DEGREES. I: THE QUANTITY OF m-DEGREES

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Introduction. In [1], Degtëv constructed a nonzero r.e. tt-degree containing a single r.e. m-degree. It is not difficult to construct an r.e. tt-degree containing infinitely many r.e. m-degrees (Fischer [6]); indeed, in [3], the author constructed an r.e. tt-degree with no greatest r.e. m-degree. Odifreddi [12, Problem 10] asked if every r.e. tt-degree contains either one or infinitely many r.e. m-degrees. The goal of this paper is to solve Odifreddi's question by showing:

THEOREM. *There exists a nonzero r.e. tt-degree containing exactly* 3 *r.e. m-degrees.*

This theorem can be extended to show that there exist r.e. tt-degrees with arbitrarily large finite numbers of r.e. m-degrees.

We remark that save for the aforementioned results, very little is known about the structures that can be realized as the collection of r.e. m-degrees within an r.e. tt-degree. It seems conceivable that the methods of the present paper may be useful in, for example, embedding distributive (semi) lattices into such structures.

In part II of this paper [4], we continue our analysis of r.e. m- and tt-degrees. We define an r.e. tt-degree to be *singular* if it contains a single r.e. m-degree, and an r.e. T-degree **a** to be singular if **a** contains a singular r.e. tt-degree.

In [4] we study the distribution (in the r.e. T-degrees) of singular tt-degrees. We show that $\mathbf{0}'_T$ is singular (solving a question of Odifreddi [11]), and that the singular T-degrees are dense, but also we construct a nonsingular T-degree. The techniques used for the first results extend those of §2 of the present paper.

The organisation of this paper is as follows. As a warm-up, to (hopefully) help the reader with the proof of the main theorem, in §2 we shall give a new (direct) proof of Degtëv's result that there exists a nontrivial singular r.e. tt-degree. In §3 we prove the main result. This involves a substantial modification of the material of §2.

Notation and terminology is standard. A good reference is Soare [14]. The following are exceptions. We shall use the upper-case Greek letters Δ and Γ as tt-functionals. We let $\{\alpha_e\}_{e\in\omega}$ denote a list of all partial recursive functions. As usual, all computations, etc. are bounded by s at stage s. We warn the reader that §3 uses a tree of strategies argument and refer him to Soare [13], [14] if he is unfamiliar

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with this technique (for more on tree arguments). Finally, $A[x]$ denotes $\{z: z \in A \,\&\, z \leq x\}$.

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§2. One *m*-degree. To provide a basis for the main result, we give a direct proof of Degtëv's result on singular tt-degrees.

(2.1) THEOREM (DEGTËV [1]). *There exists an r.e. nonrecursive set A such that for all r.e. sets B if $B \equiv_{tt} A$ then $B \equiv_m A$.*

PROOF. We build $A = \bigcup_s A_s$ in stages. At each stage s we let $\{a_{i,s}: i \in \omega\}$ list in order $\bar{A}_s$. We must meet the requirements

$$P_e: \bar{A} \neq W_e,$$

$$N_e: \Delta_e(A) = V_e \,\&\, \Gamma_e(V_e) = A \text{ implies } V_e \equiv_m A.$$

Here the N_e is considered for all 3-tuples $(\Delta_e, \Gamma_e, V_e)_{e \in \omega}$ consisting of an r.e. set V_e and two partial tt-functionals. For notational convenience, let δ_e denote the (partial recursive) use of Δ_e and γ_e the use of Γ_e.

Associated with the N_e is a restraint $r(e, s)$. The argument is finite injury, and $\lim_s r(e, s) = r(e)$ exists and is finite. We meet the P_e by making A simple. Thus when we see some $x \in W_{e,s}$ such that $x > 2e$, $x > r(e, s)$ and $W_{e,s} \cap A_s = \varnothing$, we simply enumerate x into A, meeting P_e forever.

Thus we shall concentrate only on the N_e. We need the auxiliary functions

(2.2) $$L(e, s) = \max\{x: \forall y < x(\Delta_{e,s}(A_s; y) = V_{e,s}(y))\}$$

and

(2.3) $$l(e, s) = \max\{x: \forall y < x(\Gamma_{e,s}(V_{e,s}; y) = A_s(y) \,\&\, L(e, s) > \gamma_e(y))\}.$$

The reader should think of $l(e, s)$ as the "A-controllable" length of agreement function. Now in the construction to follow, it is convenient to *dump*. That is, at each stage s if we add some $a_{i,s}$ to $A_{s+1} - A_s$, then we also add $a_{j,s}$ to $A_{s+1} - A_s$ for $j > i$ and $j \leq s$. We call this the dump property of the construction. (*Remark*: this makes A semirecursive (cf. Jockusch [7]).) Jockusch has pointed out that any singular r.e. tt-degree contains a semirecursive set. This follows by [7, Theorem 3.6], which asserts that any r.e. tt-degree contains a semirecursive set and semirecursiveness is preserved downward under $\leq_m$: Jockusch asks if all r.e. sets in a singular tt-degree must be semirecursive and conjectures the answer is no.

The way in which we satisfy N_e is this. For a single requirement we monitor $l(e, s)$. When we see $l(e, s) > a_{i,s}$, we ask questions of the tt-reductions Δ_e and Γ_e concerning the effect of possible future configurations of A. If we assume that for all $j < e$, $r(j, s) = r(j)$ and the P_j for $j < e$ have ceased acting, and the N_j for $j < e$ do not act further, then N_e knows all possible A-configurations (below $a_{i,s}$) it can attain. That is, we can attain (by the dump property)

$$A_s^k = A_s \cup \{a_{j,s}: k \leq j \leq s\}$$

for any k with $a_{k,s} > R(e, s) = \max\{r(j, s): j \leq e\}$. Now we can ask (Δ_e, Γ_e) what the effect of changing A_s to A_s^k will have on $V_{e,s}$.

For example, if we set $A_{s+1} = A_s^i$ then the addition of $\{a_{j,s}: i \leq j \leq s\}$ must cause a change in $V_{e,s}$ below $\gamma_e(a_{i,s})$. Moreover since $l(e,s) > a_{i,s}$, we can use Δ_e to tell us exactly what that change will be (examine (2.3) and (2.2)). In particular, if $A_{s+1} = A_s^i = A[a_{s,s}]$ then we can use Γ_e and Δ_e to tell us (at least) one y which must enter $V_e - V_{e,s}$ if $\Delta_e(A) = V_e$ and $\Gamma_e(V_e) = A$. The idea is to tie $a_{i,s}$ and y together to get $A \leq_m V_e$.

The problem, of course, is that A_s^i predicts one $y(i)$ to enter $V_e - V_{e,s}$ and, for some $j < i$, A_s^j predicts $y(j) \neq y(i)$ to enter with $y(i) \notin V_e$. Then we will not know which y—$y(i)$ or $y(j)$—to blame for $a_{i,s}$'s entry into A. In general, let $V_{e,s}^i$ denote the result—predicted by Δ_e and Γ_e—on V_e of setting $A = A_s^i$. A moment's thought thus reveals that for the above strategy to work, it must be the case that, for all $j < i$ and s, $V_{e,s}^i \subset V_{e,s}^j$, for otherwise we cannot get the desired m-reduction.

If, for some s, $V_{e,s}^i \not\subset V_{e,s}^j$, then we use this fact to satisfy N_e. Define a number k to be a *critical number* if $V_{e,s}^{k+1} \not\subset V_{e,s}^k$. If we never see a critical number, then for all s we have $V_{e,s}^{k+1} \subset V_{e,s}^k$ for all k, and so $A \leq_m V_e$. If we see a critical number k, our idea is to first set $A = A_s^{k+1}$ and $r(e,s) = a_{k,s}$. Now we then wait till $l(e,t) > a_{k,s}$ at some stage $t > s$. Should this not occur, we win since $l(e,s) \not\to \infty$. On the other hand, if we see such a stage t, some number $y = y(k+1)$ must have entered $V_{e,t} - V_{e,s}$ with $y(k+1) \in V_{e,s}^{k+1} - V_{e,s}^k$. Such a $y(k+1)$ cannot be withdrawn. We can therefore ensure that $l(e,s) \not\to \infty$ by keeping $r(e,t) = r(e,s)$ but now setting $A_{t+1} = A_t^k$ (and note that $A_{t+1}[s] = A_t^k[s] = A_s^k[s]$), which predicts that $y(k+1) \notin V_e$. Such a disagreement kills N_e once and for all.

For the reduction $V_e \leq_m A$ we reason similarly. We attempt to build the m-reduction as follows. For $z \in \omega$ we wait till $l(e,s) > z$. Now if $z \in V_{e,s}$ we need not worry about z. If $z \notin V_{e,s}$, we need only look at numbers $\leq \delta_e(z)$ to decide if it is possible for z to enter V_e. Also we know that future configurations of A are of the form A_s^i. If there is no $i < s$ with $z \in V_{e,s}^i$ then $z \notin V_e$. (The reader should note that we need only look at $i < s$ for the least s whose $l(e,s) > z$, since any other i will be beyond the use function of the reduction.) Now if there is some i which predicts $z \in V_e$, *it must be the case that, for all* $j < i$, $z \in V_{e,s}^j$ *too, for otherwise we could use* i *as a killing point for* N_e *as above.* That is, if such an i exists, then $i - 1$ is a critical number and so we would win by first setting $A_{s+1} = A_s^i$ and $r(e,s) = s$, waiting for $l(e,t)$ to recover so that $V_{e,t}[z] = V_{e,s}^i[z]$, and then setting $A_{t+1} = A_t^{i-1}$, creating a disagreement at z.

Thus, in summary we only need attack N_e when we see a way (in a one or two step action) of making a disagreement predicted by Δ_e and Γ_e. Such an attack, if not injured by N_j for $j < e$, will succeed in meeting N_e forever. The result now follows by a standard application of the finite injury method. □

In part II we analyse the distribution in the r.e. T-degrees of singular r.e. tt-degrees. In particular, we show that singular r.e. tt-degrees also live in $\mathbf{0}'_T$. The interesting point about that result is that it requires an infinite injury argument.

§3. The main result. In this section we prove the main result:

(3.1) THEOREM. *There is a nonzero r.e. tt-degree containing exactly three r.e. m-degrees.*

We then indicate how to extend this to arbitrarily large finite numbers. The requirements are the following:

P_e: $\neg(A \leq_m B)$ via α_e.

$\hat{P}_e$: $\neg(B \leq_m A)$ via α_e.

N_e: $\Delta_e(A) = V_e$ and $\Gamma_e(V_e) = A$ implies $V_e \equiv_m A$ or $V_e \equiv_m B$ or $V_e \equiv_m A \oplus B$.

We remind the reader that α_e denotes the eth unary partial recursive function. To ensure that $A \equiv_{tt} B$ we fix (in advance) the following (bounded) tt-reductions.

(3.2) $A \leq_{tt} B$: $3x \in A$ iff $3x \in B$,
$3x + 2 \in A$ iff $3x + 2 \in B$,
$3x + 1 \in A$ iff $(3x + 2 \in B \;\&\; 3x + 1 \notin B) \vee (3x \in B)$.

(3.3) $B \leq_{tt} A$: $3x \in B$ iff $3x \in A$,
$3x + 1 \in B$ iff $(3x + 2 \in A \;\&\; 3x + 1 \notin A) \vee (3x \in A)$,
$3x + 2 \in B$ iff $3x + 2 \in A$.

We also build A and B using the dump of §2, in the sense that if $y \in A_{s+1} - A_s$ then $\forall z(y \leq z \leq s \rightarrow z \in A_{s+1} - A_s)$; and similarly for B.

A good way to picture the situation at stage s is to view ω as broken up into triples of the form $\{3y, 3y + 1, 3y + 2\}$. This is the basic unit of the construction. At stage s let $F_{x,s}$ denote the xth triple with $\bar{A}_s \cap F_{x,s} \neq \varnothing$ (equally, $\bar{B}_s \cap F_{x,s} \neq \varnothing$). Hence $D = \{3, 4, 5\}$ is deleted from the F_x-list only when $D \subset A$ (& $D \subset B$). We say $F_{x,s}$ is *unused* at stage s if $F_{x,s} \cap A_s = \varnothing$ (and hence $F_{x,s} \cap B_s = \varnothing$).

The reader should note that to be consistent with (3.2), (3.3) and the dump property, only certain membership patterns for A and B are possible. For a triple $\{3y, 3y + 1, 3y + 2\}$ unused at stage s, let $a_{i,s} = 3y$, with $\{a_{i,s}: i \in \omega\}$ listing $\bar{A}_s$. Now, for example, if we set $A_{s+1} = A_s^{i+2}$ (in the notation of §2) then we must set $B_{s+1} = B_s^{i+1}$. Having done this, since both $3y + 1$ and $3y + 2$ are in B_{s+1} we can only enumerate $3y + 1$ into A if we enumerate $3y$ into B (by (3.2)) and hence $3y$ into A by (3.3). In general, the possible membership patterns for such a triple as above and stages $s < t$ are summarized below:

$$(3.4) \quad \begin{cases} \text{(i) } A_s^{i+2}, B_s^{i+1} \text{ then later } A_t^i, B_t^i, \text{ or} \\ \text{(ii) } A_s^{i+1}, B_s^{i+2} \text{ then later } A_t^i, B_t^i, \text{ or} \\ \text{(iii) } A_s^i, B_s^i. \end{cases}$$

We meet the requirement P_e by operating on an *unused* triple $F = \{3x, 3x + 1, 3x + 2\}$ currently devoted to satisfying P_e. We first wait for a stage s to occur with $\alpha_{e,s}(3x + 1)\downarrow$. Note that as F is unused, $F \cap A_s = F \cap B_s = \varnothing$. We can now meet P_e according to one of the two cases below:

Case 1. $\alpha_{e,s}(3x + 1) \notin \hat{B}_s$, where $\hat{B}_s = B_s \cup \{z: 3x + 2 \leq z \leq F_{s,s}\}$. Note that here and henceforth we identify $F_{s,s}$ with its maximum member where required for inequalities such as $z \leq F_{s,s}$ to make sense.

Actions (*to meet* P_e). (i) Set $A_{s+1} = A_s \cup \{z: 3x + 1 \leq z \leq F_{s,s}\}$.

(ii) Set $B_{s+1} = B_s \cup \{z: 3x + 2 \leq z \leq F_{s,s}\}$.

(iii) Set $q(e, s) = 3x + 2$ (restraint).

Analysis of outcome. Note that we have remained consistent with $\equiv_{tt}$ of (3.2) and (3.3). With priority e we are now preserving "$3x + 1 \in A$ and $\alpha_e(3x + 1) \notin B$". This follows by putting together the facts that $\alpha_e(3x + 1) \notin \hat{B}_s$ and $\alpha_e(3x + 1) < s$ (convention, since $\alpha_{e,s}(3x + 1)\downarrow$) to imply that $\alpha_e(3x + 1) < 3x + 2$.

Case 2. $\alpha_e(3x + 1) \in \hat{B}_s = B_s \cup \{z: 3x + 2 \leq z \leq F_{s,s}\}$.

Actions. (i) Set $A_{s+1} = A_s \cup \{z: 3x + 2 \le z \le F_{s,s}\}$.
(ii) Set $B_{s+1} = B_s \cup \{z: 3x + 1 \le z \le F_{s,s}\}$.
(iii) Set $q(e, s) = 3x + 2$.

Analysis of outcome. Again we remain consistent with $A \equiv_{tt} B$ of (3.2) and (3.3). Note that as $\alpha_e(3x + 1) \in \hat{B}_s$ and $\hat{B}_s \subset B_{s+1}$ in this case, we must have $\alpha_e(3x + 1) \in B_{s+1}$. Hence in this case we $q(e, s)$-preserve "$\alpha_e(3x + 1) \in B$ but $3x + 1 \notin A$," with priority e.

Coherence of the P_e & $\equiv_{tt}$. Note that the above strategies cohere in the following way. First we make sure we use different (with priority e) triples for each e with assignments in order of priority; and we initialize P_e when P_j for $j < e$ acts. Thus, in particular, if $i < e$ and $F_{x,s}$ is assigned to P_i and $F_{y,s}$ to P_e, then $x < y$ (and $e < s$). However the key to coherence is the "$3x$" term in the definition of $\equiv_{tt}$. Suppose that we are currently satisfying P_e at $F_{y,s}$ (as above). Now P_i acts and the dump property dumps all of $F_{y,s}$ into B_s and A_s. The cruical point that must be realized is that this is compatible with $A \equiv_{tt} B$ of (3.2) and (3.3). (As we shall see below, not all actions are compatible with $\equiv_{tt}$.) Note that by a finite injury argument the activity above is fine from P_e's point of view since it can be so injured only finitely often.

Meeting $\hat{P}_e$. Obviously we meet $\hat{P}_e$ in exactly the same way as P_e only with the roles of A and B reversed; and we use the notation $\hat{q}$ for $\hat{P}_e$'s restraint.

Constrained actions. Before we discuss the satisfaction of the N_e, we wish to point out the important constraints imposed by (3.4) on our enumeration of numbers into A. At stage s, let $\{a_{i,s}: i \in \omega\}$ list in order $\bar{A}_s$. Now, by the dump, $A_{s+1} = A_s$ or $A_{s+1} = A_s \cup \{a_{k,s}: i \le k \le i(s)\}$ (where $i(s) > s$). The crucial point is that *not all i's are available for such enumeration.* The sequence of events which can so constrain $a_{i,s}$ is the following. It must be that $a_{i,s} = 3x + 1$ for some x, but $a_{i+1,s} \ne 3x + 2$. The only way this is possible (to remain consistent with the dump construction and (3.4)) is that at some stage $\hat{s} < s$ we enumerated $3x + 2$ into $A_{\hat{s}+1} - A_{\hat{s}}$. At this stage we must also have enumerated $3x + 1$ into $B_{\hat{s}+1} - B_{\hat{s}}$, since we did not enumerate either $3x$ or $3x + 1$ into $A_{\hat{s}+1} - A_{\hat{s}}$ (they are still alive at $s > \hat{s}$). The point is that *the only way we can remain consistent with* (3.4) *is to allow* $3x + 1 = a_{i,s}$ *to enter* $A_{s+1} - A_s$ *iff* $3x = a_{i-1,s}$ *also enters* $A - A_s$. (We actually ask that $3x \in A_{s+1} - A_s$.) We therefore refer to $a_{i,s}$ as a *constrained point,* and for all future stages *we can essentially regard* $3x$ *and* $3x + 1$ *as one unit.* It is the existence of constrained points that gives rise to the three options in N_e as we see below.

Meeting N_e. To meet a single N_e we attempt to implement the strategy of §2 in the new environment forced by (3.2) and (3.3).

First, the simplest situation occurs when we see a *legal way* of killing N_e. In §2 there were two ways of killing N_e. The first way was that we saw an i such that if we set $A_{s+1} = A_s^i$ and $r(e, s) = s$ then $V_{e,s}$ could not respond and remain consistent with Δ_e and Γ_e. In our setting, this first case is the same with the twist that we must be able to set $A_{s+1} = A_s^i$. The point here is that perhaps $a_{i,s}$ is a constrained point and indeed for all *unconstrained* $a_{j,s}$ there is a $\hat{V}_e \supset V_{e,s}$ such that $\hat{V}_e \equiv_{tt} A_s^j$ via Δ_e and Γ_e. We point out that the fact that perhaps some constrained i might kill (Δ_e, Γ_e) is fine from the point of view of building, say, $V_e \equiv_m A$, since $A_{s+1} = A_s^j$ only for legal j.

The other way we would kill (Δ_e, Γ_e) in §2 was to use a two step action. That is, we saw an $i < j$ such that if we first set $A_{s+1} = A_s^j$ and $r(e, s) = s$ and waited for the

(Δ_e, Γ_e)-computations to recover, and then later (at $\hat{s} > s$) set $A_{\hat{s}+1} = A^i_{\hat{s}}$, then $V_{e,\hat{s}}$ could not respond. In our case again i and j must be legal. However, the crucial observation we need also make is that we must also ensure that we do not *make* $a_{i,s}$ constrained when we first set $A_{s+1} = A^j_s$. The one problem case dictated by (3.4) is if $a_{j,s} = 3y + 2$, and $a_{i,s} = 3y + 1$. Now if we first set $A_{s+1} = A^j_s$ then we automatically make $A_{t+1} = A^i_t$ illegal at all $t > s$ (as $a_{i,s+1} = a_{i,s}$ is now constrained). The problem of course is that although setting $A_{\hat{s}+1} = A^i_{\hat{s}}$ might kill N_e were it legal, $A_{\hat{s}+1} = A^{i-1}_{\hat{s}}$ might be completely consistent with (Δ_e, Γ_e).

Summarizing, there are again two legal ways to kill N_e. First we might see a legal i such that setting $A_{s+1} = A^i_s$ and $r(e, s) = s$ might be inconsistent with $(\Delta_e, \Gamma_e, \hat{V}_e)$ for any $\hat{V}_e \supset V_{e,s}$. This simply asks that $a_{i,s}$ is unconstrained. The second way is to see unconstrained $i < j$ such that we can win with a two step action involving i and j and furthermore if $a_{j,s} = 3y + 2$ then $a_{i,s} \neq 3y + 1$.

The important cases are the ones where we never see a legal method of killing N_e, and we shall assume we are in this situation henceforth. We must prove that $V_e \equiv_m A$, $V_e \equiv_m B$ or $V_e \equiv_m A \oplus B$. To do this, we analyse the effect on V_e of various incarnations of A_{s+1} depending on an unused $F_{x,s} = \{3y, 3y + 1, 3y + 2\}$ with $l(e, s) > F_{x,s}$. Let $a_{i,s} = 3y$.

Roughly speaking we examine the effect—predicted by Δ_e and Γ_e—on V_e of setting $A = A^{i+2}_s$, again denoted by $V^{i+2}_{e,s}$; and compare the result of setting $A = A^{i+2}_s$, again denoted by $V^{i+1}_{e,s}$. Of course, we are only interested in A-controlled versions of V_e, that is those below the current length of agreement. We suppress this to save on terminology.

Certainly it must be the case that $V^{i+1}_{e,s} \neq V^{i+2}_{e,s}$ (by definition of a use function for Δ_e). The critical points come from analysing the manner in which $V^{i+1}_{e,s}$ and $V^{i+2}_{e,s}$ differ. At this point we will speak quite broadly: as the reader might guess, there are three cases.

Case 1. $V^{i+1}_{e,s} \supsetneqq V^{i+2}_{e,s}$. In this case, roughly, V_e is "looking like A" at least on $F_{x,s}$ (up to $\equiv_m$, anyhow). To see this, we argue as follows. Suppose that for (almost) all x, at the first stage s when $l(e, s) > F_{x,s}$ either $F_{x,s}$ is not unused or $V^{i+1}_{e,s} \supset V^{i+2}_{e,s}$, where $F_{x,s} = \{3y, 3y + 1, 3y + 2\}$ and $3y = a_{i,s}$ (and we cannot see a legal way of killing N_e). Then $A \leq_m V_e$ as follows. To decide if $3y + j \in A$, compute the least stage s where $l(e, s) > 3y + 3$. Now, assuming $a_{i,s} = 3y + j$, if $3y + 2 \in A_s$ then $a_{i,s} = 3y$ or $a_{i,s} = 3y + 1$. In either case $3y + j$ may enter A after stage s precisely if $3y$ enters A too (if $a_{i,s} = 3y + 1$ then $a_{i,s}$ is constrained). As we argued in §2, we simply tie $a_{i,s}$ to a number it causes to enter $V^{\bar{i}}_{e,s} - V_{e,s}$, where $a_{\bar{i},s} = 3y$. The crucial case is therefore if $3y + 2 \notin A_s$, so that the block $\{3y, 3y + 1, 3y + 2\}$ is unused. In the case that $a_{i,s} = 3y$ there is no problem, since we can reason exactly as we did above. Finally, in the case $a_{i,s} = 3y + 1$ or $3y + 2$, the fact that $V^{\bar{i}+1}_{e,s} \supset V^{\bar{i}+2}_{e,s}$, where $3y + 1 = a_{\bar{i},s}$, ensures that we can tie $3y + 2$ to some element in $V^{i+2}_{e,s} - V_{e,s}$ and $3y + 1$ to some element in $V^{\bar{i}+1}_{e,s} - V^{\bar{i}+2}_{e,s}$. The point is that since there is no legal way to kill N_e, all legal future configurations of A must agree with this. The reduction $V_e \leq_m A$ is similar.

Case 2. $V^{i+1}_{e,s} \subsetneqq V^{i+2}_{e,s}$. In this case V_e is "looking like B", as this is how B behaves on $F_{x,s}$ in response to changes in A. The reader should note that as there is no legal way to kill N_e, we must here have $V^{i}_{e,s} \supsetneqq V^{i+2}_{e,s}$ lest we be able to use $A_{i+2,s}$ and $a_{i,s}$ to kill N_e in a two step action. Concentrating on $F_{x,s}$, the reader should note that V_e can, for example, compute $F_{x,s} \cap B$ using essentially the same argument as in Case 1, since if we denote by $\hat{V}^j_{e,s}$ the result of setting $B_{s+1} = B^j_s$ (with the obvious meaning), we have, as our Case 2 hypothesis restated, that $\hat{V}^{i+1}_{e,s} \supsetneqq \hat{V}^{i+2}_{e,s}$. Thus if almost all unused blocks fall into Case 2, then $V_e \equiv_m B$.

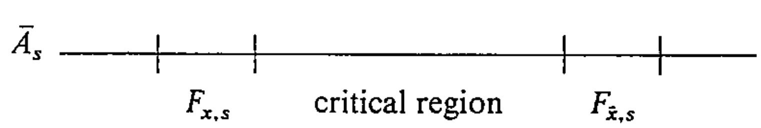
Case 3. $V^{i+2}_{e,s} \not\subset V^{i+1}_{e,s}$ *and Case* 2 *does not hold.* This means that some numbers are caused by both A^{i+2}_s and A^{i+1}_s to enter V_e, but some numbers which A^{i+2}_s causes to enter are not caused to enter by A^{i+1}_s, and vice versa. In this case V_e is "looking like" $A \oplus B$, and this turns out to be an amalgam of the two previous arguments.

REMARK. We remark that the reason we only look at unused blocks is that used blocks all "look the same" to both A and B as far as $\equiv_m$ is concerned. It turns out to be impossible, in general, to ensure that almost all unused blocks fall into one of the above cases, but we try to do this as much as we can.

Our idea is to be very selective as to which $F_{x,s}$ we shall use to build A, and force V_e to look like exactly one of the above cases for almost all "live" $F_{x,s}$. We implement this as follows.

First the reader should note that exactly one of the above cases must pertain for an unused block at any stage s. We should also note that a Case 1 can later become a Case 3 but *not*, of course, a Case 2. A Case 1 cannot become a Case 2 since to have Case 1 at some stage s we must have had $V^{i+1}_{e,s} \supsetneqq V^{i+2}_{e,s}$. Thus there is some number $z < l(e,s)$ caused by $a_{i+1,s}$ to enter $V^{i+1}_{e,s} - V^{i+2}_{e,s}$. If we are to remain consistent with Δ_e and Γ_e this number can never enter V_e unless we set $A_{t+1} = A^j_t$ for some $j \leq i+1$ at some $t \geq s$. In particular, $z \in V^{i+1}_{e,t} - V^{i+2}_{e,t}$ for all such t. Our idea is to assign a sort of e-state to each unused block. The e-states roughly correspond to the cases above. Thus an e-state of $\sigma^\wedge i$ for $F_{x,s}$ (with σ its $(e-1)$-state) corresponds to case i. We order the cases as they are given.

Now, a typical situation will be that we have a block $F_{x,s}$ with e-state $\sigma^\wedge 2$ and discover a new block $F_{\hat{x},s}$ with $x < \hat{x}$, and also with e-state $\sigma^\wedge 2$. We suppose, without loss, that all blocks between $\hat{x}$ and x have e-state $\sigma^\wedge 1$ or were used before e-states could be assigned. The situation is diagrammed below:



As the name suggests, the *critical region* is the region we must be wary of. The point is that at $F_{x,s}$ and $F_{\hat{x},s}$, V_e is looking purely like B, and in the critical region V_e is looking purely like A. To get around this difficulty we simply declare the region as $\sigma^\wedge 2$-dead and enumerate the region into $A_{\hat{s}+1} - A_{\hat{s}}$ at some stage $\hat{s} \geq s$ iff also $3y + 2 \in A_{\hat{s}+1} - A_{\hat{s}}$ where $F_{x,s} = \{3y + i : i = 0, 1, 2\}$. The crucial point is that in the critical region, any block $F_{z,s} = \{3\hat{y} + i : i = 0, 1, 2\}$ must behave exactly like $3y + 2$, and so be *known by both A and B*. Once a region is $\sigma^\wedge 2$-dead, we must reassign any P_e using a block in the region to another block. The situation for $\sigma^\wedge 3$ is similar.

Of course, we do not know which outcome is the correct one, but a fairly familiar π_2-priority argument (along the lines of a "full approximation" argument) deals with the coherence of the strategies. We assume that the reader is at least familiar with e-state type constructions. We remark that the P_e and $\hat{P}_e$ will be met via their *σ-correct* *versions*; that is, the version we have of P_e for the $\mathbf{0}''$-construction along the true path. For more on tree of strategies arguments, we refer the reader to Soare [13], [14].

The reader should think of the construction to follow as attempting to assign to P_e ($\hat{P}_e$) a follower block F_x of the highest priority e-state, and thus in some ways our construction might be thought to resemble a maximal set construction (except that when we move "markers", we do not enumerate). However there is an additional complication caused by the fact that if we see a possible legal way of killing N_e, then we pursue this strategy instead of the e-state one. For example, suppose we simply assigned to P_e ($\hat{P}_e$) a moving marker $\wedge_{2e}$ ($\wedge_{2e+1}$) which moved to the unassigned unused block of highest e-state. Suppose this e-state is σ, and we are at stage s where P_e has block $F_{x,s}$ with e-state σ. Now let us suppose that we see a chance to kill N_e using $F_{y,s}$ for some $y > x$ with, say, a two step action beginning at stage s. As N_e has higher priority than P_e, it seems reasonable to move $\wedge_{2e}$, as we cannot allow P_e to later upset N_e. Thus we might move $\wedge_{2e}$ to some $F_{z,s+1}$ for some $z > y$. Now what might happen is that at some stage $\hat{s} > s$ before we get our chance to finish killing N_e some N_j for $j < e$ requests us to declare $F_{y,s}$ as dead since it is not in the high j-state. Thus we never get our chance to kill N_e at $F_{y,s}$, but we have moved $\wedge_{2e}$ off $F_{x,s}$. This process might reoccur infinitely often.

There are several ways to overcome this difficulty. One is to use various markers $\wedge^{\sigma}_{2e}$ equipped with guesses. One version of P_e will wait for the restraints to all drop before it can be attacked and another will begin anew. Such coordination can be formalized using the "hat trick" of Lachlan-Soare (see e.g. [14]); however, it is our belief that a much clearer presentation of the details is obtained by explicitly putting all the outcomes on a tree.

We now give the formal details, but expect that the reader familiar with tree of strategies arguments might wish to supply them for himself. Let $\Lambda = \{0, 1, 2, 3, \mathrm{w}, \mathrm{k}\}$ be ordered by $>_\Lambda$ in the manner given, so that $0 > 1 > \cdots > k$. We refer to $\sigma, \tau \in \Lambda^{<\omega}$ as *guesses*. We write $\sigma \subset \tau$ if σ is an initial segment of τ. We use λ as the empty guess. Let $\leq_{\mathrm{L}}$ denote the lexicographic order:

$$\sigma \leq_{\mathrm{L}} \tau \leftrightarrow (\sigma \subset \tau \vee \exists\gamma(\gamma\hat{\ }i \subset \sigma \;\&\; \gamma\hat{\ }j \subset \tau \;\&\; i <_\Lambda j)).$$

We let $\mathrm{lh}(\sigma)$ denote the length of σ. If $\mathrm{lh}(\sigma) = e$ then σ is assigned to N_e.

The intention of $\wedge$ is: k for "kill" meaning we believe we are preserving a disagreement; w for "wait" meaning that we are in the middle of a 2-step attack on N_e waiting for computations to recover; 1, 2, 3 correspond to $l(e,s) \to \infty$ and the cases discussed above, and 0 is meant to indicate $l(e,s) \not\to \infty$.

(3.5) Definition. Define the notions *σ-stage*, *e-state σ* and *σ-dead* by simultaneous induction on $\mathrm{lh}(\sigma)$.

(i) Every stage s is a λ-stage. Every block has -1-state λ.

(ii) We are given $(e-1)$-states and $\tau \in \Lambda^{<\omega}$ with $\mathrm{lh}(\tau) = e$ and s a τ-stage. In order of $x \leq s$, generate the e-state of $F_{x,s}$ with $F_{x,s} \cap \bar{A}_s \neq \varnothing$ as follows:

Case 1. First, if for some $\gamma \leq_{\mathrm{L}} \tau$ with $\mathrm{lh}(\gamma) = e$ we have $r(\gamma\hat{\ }\mathrm{k}, s) \neq 0$, do nothing save to declare s a $\tau\hat{\ }\mathrm{k}$-stage. (This indicates we are currently believing that we are preserving a N_e-disagreement.)

Case 2. Case 1 does not pertain and N_e requires attention *in a τ-legal way*. That is, one of the following options holds:

Case 2(a). There exists $a_{i,s} > \max\{a_{e,s}, \hat{q}(\gamma, s), q(\gamma, s), r(\gamma, s): \gamma \leq_L \tau\}$ such that

(i) $a_{i,s}$ is not τ-dead and is not constrained, and

(ii) $l(e,s) > a_{i,s}$ and for all sets $\hat{V} \supset V_{e,s}$ and $\hat{V} \subset \{0, \ldots, s\}$, we have either $(\Gamma_e(\hat{V}) \neq A^i)[l(e,s)]$ or $\Delta_e(A_s^i)[u] \neq \hat{V}[u]$, where $u = u(\Gamma_e(V; l(e,s)))$.

Case 2(b). There exists $a_{i,s} > \max\{a_{e,s}, \hat{q}(\gamma, s), q(\gamma, s), r(\gamma, s): \gamma \leq_L \tau\}$ such that $l(e,s) > a_{i,s}$ and we can τ-legally win a two step action at $a_{i,s}$: There exists $j > i$ such that $l(e,s) > a_{j,s}$ and

(i) $a_{j,s}$ is not τ-dead and is not constrained,

(ii) if we set $A_{s+1} = A_s^j$ then $a_{i,s}$ is not constrained at stage $s + 2$, and

(iii) For all sets $\hat{V} \subset \{0, \ldots, s\}$ and $V_{e,s}^j \subset \hat{V}$ either $(\Gamma_{e,s}(\hat{V}) \neq A_s^i)[l(e,s)]$ or $\Delta_e(A_s^i)[u] \neq \hat{V}[u]$, where $u = u(\Gamma_e(A_s^i; l(e,s)))$.

If this case pertains, we declare s to be a $\tau\hat{}w$-stage.

Case 3. None of the above pertain and there exists a least x such that

(i) $F_{x,s}$ is unused,

(ii) $F_{x,s} > \max\{F_{e,s}; \hat{q}(\gamma, s), q(\gamma, s), r(\gamma, s): \gamma \leq_L \tau\} = m(\tau, s)$,

(iii) $l(e,s) > F_{x,s}$,

(iv) $F_{x,s}$ has $(e-1)$-state τ, and

(v) one of the subcases below pertains.

Let $F_{x,s} = \{a_{i,s}, a_{i+1,s}, a_{i+2,s}\}$. We keep the $V_{e,s}^j$ notation of Case 2 and the intuitive discussion preceding the construction.

Case 3(a). $F_{x,s}$ has e-state $\tau\hat{}m$ for $3 <_\Lambda m$, $V_{e,s}^{i+2} \not\subset V_{e,s}^{i+1}$ and $V_{e,s}^{i+1} \not\subset V_{e,s}^{i+2}$.

In this case, give block $F_{x,s}$ the e-state $\tau\hat{}3$. Find the greatest y with $y < x$ such that $F_{y,s} \leq m(\gamma, s)$ for some $\gamma \leq_L \tau$ or $F_{y,s}$ has e-state $\tau\hat{}3$. Declare all $a_{n,s}$ for $F_{y,s} < a_{n,s} < \min\{z: z \in F_{x,s}\}$ to be $\tau\hat{}3$-dead. Finally declare s to be a $\tau\hat{}3$-stage.

Case 3(b). $F_{x,s}$ has $(e-1)$-state $\tau\hat{}0$ and $V_{e,s}^{i+1} \subset V_{e,s}^{i+2}$. Declare $F_{x,s}$ to have e-state $\tau\hat{}2$. Find the greatest y with $y < x$ such that $F_{y,s} \leq m(\gamma, s)$ for some $\gamma \leq_L \tau$ or $F_{y,s}$ has e-state $\tau\hat{}2$ or $\tau\hat{}3$. Declare all $a_{n,s}$ for $F_{y,s} < a_{n,s} < \min\{z: z \in F_{y,s}\}$ to be $\tau\hat{}2$-dead. Declare s to be a $\tau\hat{}2$-stage.

Case 3(c). $F_{x,s}$ has $(e-1)$-state $\tau\hat{}0$ and $V_{e,s}^{i+1} \supset V_{e,s}^{i+2}$. Declare $F_{x,s}$ to have e-state $\tau\hat{}1$. Find the greatest y with $y < x$ such that $F_{y,s} \leq m(\gamma, s)$ for some $\gamma \leq_L \tau$ or $F_{y,s}$ has e-state $\tau\hat{}1$, $\tau\hat{}2$, or $\tau\hat{}3$. Declare all $a_{n,s}$ for $F_{y,s} < a_{n,s} < \min\{z: z \in F_{x,s}\}$ to be $\tau\hat{}1$-dead. Declare s to be a $\tau\hat{}1$-stage.

Case 4. None of the above pertains. Declare s to be a $\tau\hat{}0$-stage, and otherwise change nothing.

DEFINITION. Let σ_s denote the unique string of length s with s a σ_s-stage.

(3.6) DEFINITION. We say that P_e *requires attention* at stage $s + 1$ if P_e is not currently declared satisfied and one of the following options holds.

(3.7) P_e has a follower block $F_{x,s}$ with e-state $\subset \sigma_s$ such that $\alpha_{e,s}(3y + 1)\downarrow$, where $3y + 1 \in F_{x,s}$.

(3.8) P_e has no follower block with e-state $\sigma \subset \sigma_s$ where $\mathrm{lh}(\sigma) = e + 1$ and there is a currently unassigned (unused) block $F_{x,s}$ with e-state σ, $\max\{\hat{q}(\tau, s), r(\tau, s), q(\tau, s): \tau \leq_L \sigma\} < \min\{z: z \in F_{x,s}\}$, and such that $F_{x,s}$ is not γ-dead for any $\gamma \leq_L \sigma$.

(3.9) REMARK. Obviously $\hat{P}_e$ is like (3.6) above with $\hat{P}_e$ in place of P_e. We refer to these as $(3.\hat{6})$, $(3.\hat{7})$ etc.

Finally we shall use, at stage s, the phrase initialise. (e.g. $\gamma \geq \sigma$.) As with standard practice this means γ-assignments, states, satisfaction etc. become reset to the initial values. Note that $\sigma \hat{} 0$ is the initial e-state if we so initialise.

CONSTRUCTION. *Stage* 0. Initialize all $\sigma \in T$, and define $q(\sigma, 1) = \hat{q}(\sigma, 1) = r(\sigma, 1) = 0$ for all $\sigma \in T$.

Stage $s + 1$. *Step* 1. Compute σ_{s+1}. Initialise all γ with $\gamma \not\leq_L \sigma_{s+1}$.

Step 2. Find the least e, if any, such that N_e, P_e or $\hat{P}_e$ requires attention. If more than one does, sort them out by the given priority order. Adopt the first case below to pertain. Initialise all those γ with $\sigma \leq_L \gamma$ and $\gamma \neq \sigma$, where $\sigma \subset \sigma_s$ and $\mathrm{lh}(\sigma) = e + 1$. Let $\sigma = \tau \hat{} n$.

Case 1 (*N_e receives attention at τ*). Case 2 of (3.5) pertains. Initialise P_e and $\hat{P}_e$ at guess $\tau \hat{} j$ for all j. Set $r(\tau \hat{} w, s + 1) = s + 1$. Adopt the first subcase below to pertain.

Subcase 1. Case 2(a) of (3.5) holds. Set $A_{s+1} = A_s^i$. Set $r(\tau \hat{} k, s + 1) = s + 1$.

Subcase 2. Case 2(b) of (3.5) holds. Set $A_{s+1} = A_s^j$.

Case 2 (*P_e receives attention at τ*). Initialise $\hat{P}_e$ at guess $\tau \hat{} n$. Set $q(\sigma, s + 1) = s + 1$. Adopt the first subcase below to pertain.

Subcase 1. (3.7) holds. Let $a_{i,s} = 3y + 1$.

Option (a). $\alpha_e(3y + 1) \notin B_s \cup \{z: 3y + 2 \leq z \leq s\}$.

Action. Set $A_{s+1} = A_s^i$.

Option (b). Otherwise.

Action. Set $A_{s+1} = A_s^{i+1}$.

In either case declare P_e as satisfied at guess σ.

Subcase 2. (3.8) holds. Assign $F_{x,s}$ to be a follower block of P_e at guess σ.

Case 3. $\hat{P}_e$ receives attention as in Case 2 except P_e is not initialized.

End of construction.

VERIFICATION. Let β denote the leftmost path. Thus β is defined by induction: $\lambda \subset \beta$ & for all σ if $\sigma \subset \beta$ then one of $\sigma \hat{} i \subset \beta$ for some $i \in \Lambda$. This is according to the rule that i is $<_\Lambda$-least with infinitely many $\sigma \hat{} i$-stages.

The following technical lemma is easily established by induction and is left to the reader. (We shall use it implicitly.)

(3.10) LEMMA. (i) *Let $F_{x,s}$ be a live block (i.e. with $F_{x,s} \cap \bar{A}_s \neq \emptyset$). Suppose that s is a $\sigma \hat{} i$-stage with* $\mathrm{lh}(\sigma) = e$ *for $i = 1, 2$ or 3. Then one of the following options holds.*

a) *$F_{x,s}$ is $\sigma \hat{} i$-dead,*

b) *$F_{x,s} < r(\gamma, s)$, $q(\gamma, s)$ or $\hat{q}(\gamma, s)$ for some $\gamma \leq_L \sigma$,*

c) *$F_{x,s}$ has e-state $\sigma \hat{} i$ at stage $s + 1$, or*

d) *for all $y \geq x$ neither* a), b) *nor* c) *hold for $F_{y,s}$.*

(ii) *If $F_{x,s}$ and $F_{y,s}$ are live blocks at stage s, $x < y$ and $F_{x,s} > r(\gamma, s), q(\gamma, s)$ and $\hat{q}(\gamma, s)$ for all $\gamma \leq_L \sigma$, then if $F_{y,s}$ has e-state $\sigma \hat{} i$ and $F_{x,s}$ has e-state $\sigma \hat{} j$ then $j \leq_\Lambda i$.*

(iii) *If $F_{x,s}$ has e-state $\sigma \hat{} i$ for $i = 1, 2$ or 3 then $F_{x,s}$ was assigned e-state $\sigma \hat{} i$ at a $\sigma \hat{} i$-stage $t \leq s$ when $F_{x,s} = F_{x,t}$ was unused at stage t.*

(3.11) LEMMA. *Let $\sigma \subset \beta$ with* $\mathrm{lh}(\sigma) = e + 1$. *Then*

a) *P_e, $\hat{P}_e$ and N_e receive attention finitely often at γ-stages for $\gamma \leq_L \sigma$,*

b) *for all $\gamma \leq_L \sigma$, $\lim_s q(\gamma, s) = q(\gamma)$, $\lim_s \hat{q}(\gamma, s) = \hat{q}(\gamma)$ and $\lim_s r(\gamma, s) = r(\gamma)$ exist,*

c) *P_e and $\hat{P}_e$ are met,*
d) *if $\sigma = \tau^\wedge 0$ or $\sigma = \tau^\wedge k$ then N_e is met, and*
e) *if $r(\gamma^\wedge k) \neq 0$ for some γ with $\mathrm{lh}(\gamma) = e$ and $\gamma^\wedge k \leq_L \sigma$, then N_e is met and $\sigma = \tau^\wedge k$ or $\sigma = \tau^\wedge 0$ for some τ.*

PROOF. Let s_0 be a stage such that for $s > s_0$
(i) $\sigma \leq_L \sigma_s$,
(ii) $\forall j < e$ (P_j, $\hat{P}_j$ and N_j do not receive attention at σ-stages), and
(iii) if we let $\sigma = \sigma^{+\wedge} i$, then, for all $\rho \leq_L \sigma^+$,

$$r(\rho, s) = r(\rho, s_0), \qquad q(\rho, s) = q(\rho, s_0), \qquad \hat{q}(\rho, s_0) = \hat{q}(\rho, s).$$

First note that if $r(\gamma^\wedge k, s) \neq 0$ for some $\gamma \leq_L \sigma$ with $\mathrm{lh}(\gamma) = e$ and $\gamma \neq \sigma$, then $r(\gamma^\wedge k, s) = r(\gamma^\wedge k)$ and, as in the intuitive discussion and §2, this is preserving an e-disagreement which will not be violated. Thus we shall suppose that, for all such γ, $r(\gamma^\wedge k, s) = 0$. We claim that N_e can receive attention at most twice more at σ-stages. Again this is like §2. If, for example, Case 1, Subcase 1 pertains to some $a_{i,s}$ at a σ-stage $s_1 > s_0$, then we set $A_{s_1+1} = A^i_{s_1}$ and all $\hat{\sigma}$ with $\sigma \leq \hat{\sigma}$ are initialised. We also set $r(\sigma^{+\wedge} k, s_1) = s_1$, and by construction this restraint cannot be violated. Note that in this case (by definition of σ-stage) since $\sigma^+ \subset \beta$ it must be that $\sigma = \sigma^\wedge k$.

The other case (Subcase 2, then Subcase 1) is entirely similar to this and §2, and is left to the reader.

Finally for the P_e ($\hat{P}_e$). Once N_e ceases receiving attention we are free to attack P_e at will. Thus once we have a σ-stage $\hat{s}_0$ such that $\hat{s}_0 > s_0$ and for all σ-stages $s > \hat{s}_0$, N_e will not receive attention at stage s, it must be that P_e gets a follower block with e-state σ. (By induction.) By the technical lemma and the choice of $\hat{s}_0$ this assignment cannot be cancelled. As in the intuitive description, we must win P_e on this block. This P_e (and $\hat{P}_e$) is met and $\lim_s q(\sigma, s) = q(\sigma)$ exists ($\lim_s \hat{q}(\sigma, s) = \hat{q}(\sigma)$ exists). □

Now we must check the key lemma (that all of our machinery was set up to establish).

(3.12) LEMMA (Truth of outcome for N_e). *Suppose that $l(e, s) \to \infty$. Then $\sigma^{+\wedge} n \subset \beta$, where $\mathrm{lh}(\sigma)^+ = e$ and $n = 1, 2$ or 3. Furthermore*
(i) *if $n = 1$ then $V_e \equiv_m A$,*
(ii) *if $n = 2$ then $V_e \equiv_m B$, and*
(iii) *if $n = 3$ then $V_e \equiv_m A \oplus B$.*

PROOF. Let s_0 be as in (3.11). If $l(e, s) \to \infty$, then by the definition of σ-stage (and the fact that there will be infinitely many P_e with $\mathrm{dom}\, \alpha_e = \emptyset$) it must be that one of $\sigma^{+\wedge} 1$, $\sigma^{+\wedge} 2$, or $\sigma^{+\wedge} 3 \subset \beta$. Fix $\sigma = \sigma^{+\wedge} n$. We must verify (i), (ii) and (iii) above.

Case 1. $n = 1$. We must show that $V_e \equiv_m A$. Let x be such that at some σ-stage $s_1 > s_0$ we have
(a) $F_{x,s_1} = F_x > \max\{r(\gamma), q(\gamma), \hat{q}(\gamma) : \gamma \leq_L \sigma\}$, and
(b) $\forall y > x\, \forall s > s_1$ ($F_{y,s}$ has e-state γ at stage s implies $\sigma \leq_L \gamma$).

The existence of x and s_1 is justified by choice of s_0 and the technical lemma. The intuition here is that, beyond F_x, if F_y gets an e-state it is at best σ.

$V_e \leq_m A$. Let $z > F_x$ be given. Find the least σ-stage $s = s(z)$ such that the e-state σ is assigned to some block $F_{y,s} > F_x$ with $z < F_{y,s}$. Note that, as in §2, $l(e, s) > z$. We might as well also suppose that $z \notin V_{e,s}$. First we ask if any σ-legal configuration of A can cause z to enter V_e. Thus, see if there exists a σ^+-live unconstrained number $a_{i,s}$ such that $\Delta_{e,s}(A_s^i) \models z \in V_e$. Note that if i is the largest such and $a_{i,s} < F_x$, then $z \notin V_e$ since $A_s[F_x] = A[F_x]$. We define an i to be *attainable* if $a_{i,s} > F_x$ and $a_{i,s}$ is unconstrained and $a_{i,s}$ is not σ^+-dead. If no attainable i exists then $z \notin V_e$. If an attainable i exists it must also be that for all σ^+-legal attainable configurations A_s^j with $j < i$, $\Delta_{e,s}(A_s^j) \models z \in V_e$.

This follows, for suppose i and j existed with i and j attainable, $j < i$, and $\Delta_{e,s}(A_s^i) \models z \in V_e$ and $\Delta_{e,s}(A_s^j) \models z \notin V_e$. There are then two cases.

Case (a). $a_{j,s}$ is unconstrained if $A_{s+1} = A_s^i$. In this case, exactly as in §2 we would use i and j to kill N_e by a two step action at σ-stages (and so $\sigma = \sigma^+ \hat{}\, k$, contradiction).

Case (b). Otherwise. In this case it can only be that $a_{i,s} = 3y + 1$ and $a_{i+1,s} = 3y + 2$ for some y. But now, by definition, the block $F_{n,s}$ containing $a_{i,s}$ would have been given e-state $\sigma^+ \hat{}\, 2$ or $\sigma^+ \hat{}\, 3$, contradicting the choice of s_1 and x. (See (3.5), Cases 3(a) and 3(b).)

But now we see that $z \in V_e$ iff $a_i \in A$, and so $V_e \leq_m A$.

$A \leq_m V_e$. Let $z > F_x$ be given. Compute a σ-stage $s = s(z)$ as above, and now suppose that $z \notin A_s$. Now if there is no block $F_{\hat{x},s}$ with $z \in F_{\hat{x},s}$ or $F_{\hat{x},s} \leq z$, and $F_x \leq F_{\hat{x},s}$ with $F_{\hat{x},s}$ having e-state σ, then z must be σ-dead at all stages $\hat{s} > s$. Hence in this case $z \notin A$. If $z \in F_{y,s}$ for some block $F_{y,s}$ with e-state σ, then—assuming some (largest) $\hat{x}$ above exists—z will enter A precisely if the largest member of $F_{\hat{x},s}$ does. Let $a_{i,s}$ be the largest member of $F_{\hat{x},s}$ if $z \notin F_{y,s}$, and $a_{i,s} = z$ otherwise. It obviously suffices to argue that we can decide $a_{i,s}$'s entry into A from V_e.

But now, as in §2 and the intuitive discussion, there must be a unique least $\hat{z}$ such that $a_{i,s} \in A$ iff $\hat{z} \in V_e$. Otherwise, as above, either we could use $a_{i,s}$ to kill N_e, or we could improve the e-state of $F_{\hat{x},s}$ $(F_{y,s})$ to either $\sigma^+ \hat{}\, 1$ or $\sigma^+ \hat{}\, 2$.

Specifically, there are three possibilities. Namely, $a_{i,s} = 3q + j$ for some $j = 0, 1, 2$ and some q. Now if $j = 0$ then clearly for all attainable σ^+-live numbers $a_{\hat{\imath},s}$ for $\hat{\imath} < i$ it must be that if $\hat{z}$ is any number caused by A_s^i to enter V_e then $A_s^{\hat{\imath}}$ must also cause $\hat{z}$ to enter V_e. (Otherwise we could use the largest such $\hat{\imath}$ to kill N_e.) If $j = 1$ then by the fact that $a_{i,s}$ has e-state $\sigma^+ \hat{}\, 1$ it must be that $A_s^{\hat{\imath}}$ causes a fixed number $\hat{z}$ to enter V_e for all attainable σ^+-live numbers $\hat{\imath} < i$, and for all unconstrained σ^+-live $k > i$, A_s^k does not cause $\hat{z}$ to enter. (Otherwise, again we get a killing point.) Finally for $j = 2$ the e-state $\sigma^+ \hat{}\, 1$ guarantees that $\hat{z}$ exists. Hence $A \leq_m V_e$.

Case 2. $i = 2$. First find the σ-stage s_1 and x as in Case 1. $V_e \leq_m B$. Let $z > F_x$. As in Case 1, find a σ-stage $s = s(z)$ such that some block $F_{y,s} > z$ attains e-state $\sigma = \sigma^+ \hat{}\, 2$. Without loss $z \notin V_{e,s}$. Now we must decide from B if z can enter V_e. First we see if there exists a (largest) σ^+-live unconstrained attainable number $a_{i,s}$ with $\Delta_e(A_s^i) \models z \in V_e$. Since N_e does not receive attention, it must be that for all σ^+-live unconstrained attainable numbers $j < i$ *not in the same block as* $a_{i,s}$ that $\Delta_e(A_s^j) \models z \in V_e$ (otherwise as in §2 and Case 1 we could win N_e with i and j). Thus if there exists a σ^+-live unconstrained attainable number $j < i$ with $\Delta_e(A_s^j) \models z \notin V_e$ it can only be that $a_{j,s}$ and $a_{i,s}$ are in the same block and also $3\hat{y} + 1 = a_{j,s}$ and $3\hat{y} + 2 = a_{i,s}$ for some $\hat{y}$, and note that $\Delta_e(A_s^{j-1}) \models z \in V_e$ (since a_{j-1} is still attainable and unconstrained whether or not we ever add $a_{i,s}$ to A). It follows then that in this case $z \in V_e$ iff $3\hat{y} \in A$ or ($3\hat{y} + 2 \in A$ and $3\hat{y} + 1 \notin A$). But this means that in this case, $z \in V_e$ iff $3\hat{y} + 1 \in B$.

On the other hand, if for all unconstrained σ^+-live attainable numbers $j < i$ we have that $\Delta_e(A_s^i) \models z \in V_e$, then choose the largest such i (assuming $z \notin V_{e,s}$). Now suppose $a_{i,s} \in F_{y,s}$.

If $F_{y,s}$ is σ-live then $F_{y,s}$ has e-state σ (by construction). Now if $a_{i,s} = 3\hat{y}$ then $z \in V_e$ iff $3\hat{y} \in B$. If $a_{i,s} = 3y + 2$ then $z \in V_e$ iff $3\hat{y} + 2 \in B$. Finally we claim $a_{i,s} = 3\hat{y} + 1$ is impossible. To see this we need only note that otherwise $z \in V_{e,s}^{i+1} - V_{e,s}^{i+2}$ and hence $V_{e,s}^{i+2} \not\supset V_{e,s}^{i+1}$, forcing $F_{y,s}$ to have state $\sigma^+ \hat{\ } 3$ or $\sigma^+ \hat{\ } 1$ (see Case 3(a) or 3(c) of (3.5)).

In the case that $F_{y,s} \cap A_s \neq \emptyset$, by construction it must be that $3\hat{y} + 2 \in A$ (where $F_{y,s} = \{3\hat{y}, 3\hat{y} + 1, 3\hat{y} + 2\}$). Consequently $a_{i,s} = 3\hat{y} + 1$ or $a_{i,s} = 3\hat{y}$. In any case we easily see that $z \in V_e$ iff $3\hat{y} \in B$.

Finally, if $a_{i,s}$ is a member of a σ-dead block $F_{y,s}$ as above (but now with $F_{y,s} \cap A_s = \emptyset$), then $a_{i,s}$ can henceforth only enter A when the whole of $F_{y,s}$ enters A. (Remember $F_{y,s}$ will remain σ-dead forever by choice of x and s_1.) Thus again $z \in V_e$ iff $3\hat{y} \in B$.

Therefore $V_e \leq_m B$.

$B \leq_m V_e$. This is very similar to Case 1, and so we only sketch the details. Let $z > F_x$ be given. Compute a stage $s = s(z) > s_1$ (as above) and again suppose $z \notin B_s$. Again we can suppose that there is some σ-live block $F_{\hat{x},s} \leq z$ with $F_{x,s} \leq F_{\hat{x},s}$. In the case when the z is a member of a σ-live block it must be that this block has e-state σ. Now if $z = 3\hat{y}$ then we proceed exactly as in Case 1. If $z = 3\hat{y} + 1$ and then $z \in B$ iff $(3\hat{y} + 2 \in A \ \& \ 3\hat{y} + 1 \notin A)$ or $3\hat{y} \in A$, let $z = a_{i,s}$. Since $F_{\hat{x},s}$ has final e-state σ, as above there must exist a least $\hat{z} \notin V_{e,s}$ such that A_s^i and A_s^{i+2} both predict that $\hat{z} \in V_e$ and A_s^{i+1} predicts that $\hat{z} \notin V_e$ (after all, this is what e-state $\sigma^+ \hat{\ } 2$ means). Consequently $z \in B$ iff $\hat{z} \in V_e$. $3\hat{y} + 2$ is similar.

In the case that $F_{\hat{x},s}$ is σ-dead, then the same reasoning as for Case 1 (with the above modifications) works, and this is left to the reader.

Case 3. $i = 3$. This is an amalgam of the preceding arguments. Let x and s_1 be as usual. First, to see that $V_e \leq_m A \oplus B$, find the stage $s(z)$ as in Cases 1 and 2. Now either for all attainable i we have that $\Delta_e(A_s^i) \models z \in V_e$, or for some (unique) i we have $\Delta_e(A_s^i) \models z \in V_e$, $\Delta_e(A_s^{i-1}) \models z \notin V_e$ and $\Delta_e(A_s^{i-2}) \models z \in V_e$ as in Case 2. In the former case z's entry can be ascertained by A, and in the latter by B. Hence $V_e \leq_m A \oplus B$.

To see that $A \oplus B \leq_m V_e$ it suffices to decide this for σ-live $F_{\hat{x},s}$ with e-state $\sigma = \sigma^+ \hat{\ } 3$. Let $F_{\hat{x},s} = \{a_{i,s}, a_{i+1,s}, a_{i+2,s}\}$. By the definition of e-state $\sigma \hat{\ } 3$ it is clear that our previous arguments give the existence of elements z_1, z_2 such that

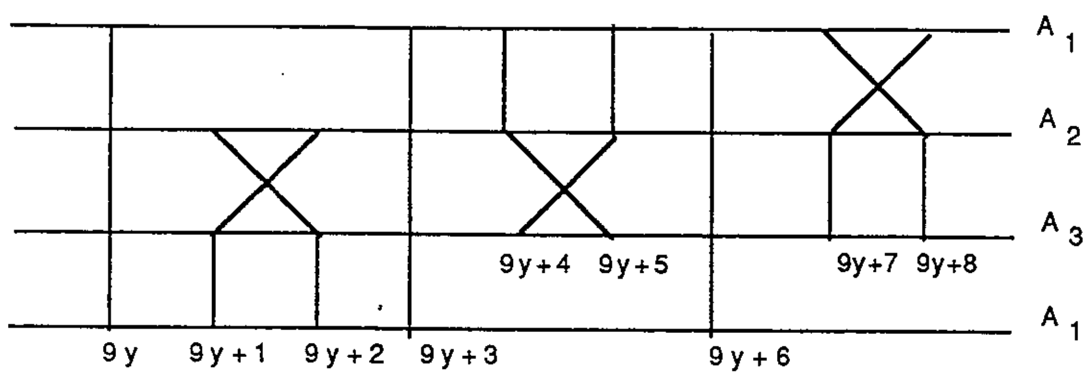
$$\left.\begin{array}{lll} a_{i+2,s} \in A & \text{iff} & z_1 \in V_e \\ a_{i+1,s} \in A & \text{iff} & z_2 \in V_e \end{array}\right\} \quad \text{using the Case 1 argument.}$$

Similarly, using the Case 2 argument, $a_{i+2,s} \in B$ iff $a_{i+1,s} \in A$ (and $a_{i+2,s} \in A$) iff $z_2 \in V_e$. Also $a_{i+1,s} \in B$ iff $(a_{i+1,s} \notin A$ and $a_{i+2,s} \in A)$ or $(a_{i,s} \in A)$. These events must be able to blame a single z_3's entry into V_e (since $V_{e,s}^{i+2} \not\subset V_{e,s}^{i+1}$). Hence $A \oplus B \leq_m V_e$. □

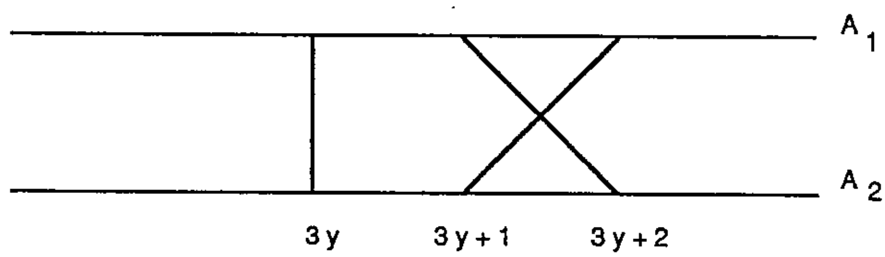
The above ideas can be extended to build:

(3.13) COROLLARY *Let* $n \in \omega - \{0\}$. *There exists an r.e. tt-degree containing exactly* $2^n - 1$ *r.e. m-degrees.*

PROOF (sketch). The idea extends quite easily—but with great detail—and so we will simply do the next case (i.e. 7 m-degrees) and leave the general formulation to the reader. The key of course is to get the correct sort of tt-reduction and then apply the above machinery. We shall need $9 = 3 \times 3$ elements ($3n$ for $n \geq 3$) for the "basic blocks". The reader may find the diagram below helpful for the tt-reduction.



The relevant tt-reductions are found by following the relevant lines. For example, $9y + 1 \in A_2$ iff ($9y + 2 \in A_3$ and $9y + 1 \notin A_3$) or $9y \in A_3$ iff ($9y + 2 \in A_1$ and $9y + 1 \notin A_1$) or $9y \in A_1$. For the case of Theorem (3.1) the diagram would be



Now using this procedure, it is clear that we can meet the relevant requirements that $A_i \not\leq_m A_{i+1} \oplus A_{i+2}$ giving A_1, A_2, A_3, $A_1 \oplus A_2$, $A_2 \oplus A_3$, $A_1 \oplus A_3$ and $A_1 \oplus A_2 \oplus A_3$ as representatives of the relevant m-degrees. The remainder of the argument is essentially the same and left to the reader. □

§4. Open questions. A natural question suggested by the above is whether or not all finite n can be realized. That is, whether or not there exist r.e. tt-degrees containing exactly n r.e. m-degrees for any given finite $n > 1$.

An apparently more difficult question is whether or not all finite distributive upper semilattices can be realized. For example, our construction realizes any "Boolean" upper semilattice (that is, any Boolean algebra with 0 removed). The most interesting test case seems to be linear orders. The fundamental point is, of course, to find the relevant tt-reduction; the e-state machinery ought to take care of the rest.

It would also seem interesting to extend the above ideas to infinite lattices. Exactly what sorts of infinite lattices can be embedded seems completely open.

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