

Completing pseudojump operators

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Abstract

We investigate operators which take a set X to a set relatively computably enumerable in and above X by studying which such sets X can be so mapped into the Turing degree of K . We introduce notions of nontriviality for such operators, and use these to study which additional properties can be required of sets which can be completed to the jump by given c.e. operators.

1 Introduction

The most natural example of a noncomputable set is the computably enumerable set K , the halting problem. The relativization of the construction of K produces, for any a set X of natural numbers, the jump of X , X' , which is the set $\{e : \Phi_e(X; e) \downarrow\}$. X' is computably enumerable in X , and $X <_T X'$. The operation, $X \mapsto X'$, can be generalized by considering, for any index e , the *eth pseudojump operator*, J_e , which maps X to $X \oplus W_e^X$. J_e is the *eth* way in which one can possibly increase the degree of X 's information in the simplest kind of uniform way, namely, by a Σ_1^X -definition. When the index of the operator does not explicitly need to be mentioned, we simply write capital letters such as V and W for pseudojump operators, as we do for the c.e. sets of which they provide the relativizations. While the jump operator itself has been the object of intense study throughout the history of computability theory, the explicit study of relative computable enumerability as produced by operators stems from the papers [3], [4] by Jockusch and Shore.

The fundamental theorem, Theorem 3.1 of [3], is a completion result for pseudojump operators, asserting that for any index e there is a noncomputable, computably enumerable set A such that $A \oplus W_e^A \equiv_T K$. Intuitively, this means that any construction e of a computably enumerable set can be relativized in such a way that, up to Turing degree, K itself has the properties of the result of that construction relative to the set to which the construction is relativized. For example, applying this theorem to the construction of a noncomputable c.e. low set yields an incomplete c.e. set A relative to which K is low, in other words, an incomplete high c.e. set. We consider here to what extent this basic result can be generalized by requiring various properties of the set A which completes

a pseudojump operator V . This is equivalent to considering the class of sets relative to which the degree of K is produced by the construction V .

Notice that if for all X , $V^X \leq_T X$, then only another Turing-complete set can complete V , so that no other special properties of the degree of the completing set can be demanded. On the other hand, natural constructions of nontrivial c.e. sets do incorporate some feature guaranteeing noncomputability. In this case, if V is such a construction, then, when relativized to any oracle X , we are guaranteed $X <_T V^X$. Thus, any such natural construction is strongly nontrivial for the purpose of relativization. Of course, it is in general impossible, given an index e , to determine what properties W_e^X has, even when X is the empty set, and it is in general quite likely that for some X , $W_e^X \leq_T X$, while for some Y , possibly of the same degree as X , $W_e^Y \not\leq_T Y$. Because of these considerations, we make the following definitions.

Definition 1. *Let \mathcal{C} be a class of subsets of ω . A pseudojump operator V is uniformly nontrivial with respect to \mathcal{C} if for all $X \in \mathcal{C}$, $X <_T V^X$. If \mathcal{C} is the set of all reals, then V is strongly nontrivial.*

In what follows, we first examine two finite injury priority arguments which give rise to computably bounded injury. We consider the existence of incomparable c.e. sets completing pseudojump operators, and the question of the existence of sets not of c.e. degree completing pseudojump operators. With strong enough hypotheses about uniform nontriviality, we show that each of these problems has a positive solution.

Theorem 2. *For any pseudojump operator V such that $X <_T V^X$ for all c.e. sets X , there exist Turing incomparable computably enumerable sets A and B such that $V^A \equiv_T V^B \equiv_T K$.*

Theorem 3. *For any pseudojump operator V such that $X <_T V^X$ for all d.c.e. sets X , there exists a set d.c.e. set A such that $V^A \equiv_T K$ and the degree of A is not c.e.*

Because constructions involving pseudojump operators applied to c.e. sets deal with sets which are explicitly given only by Σ_2^0 definitions, both of these constructions are more complex than those in the classical theorems. Essentially, the tightly-controllable computably bounded injury in the familiar constructions case gives way to essentially noncomputable activity for the corresponding requirements in the relative computable enumerability case because of the addition of a quantifier. These constructions give content to this somewhat vague idea, and therefore have an added technical interest. It is partly for this reason that we give a direct construction proving Theorem 2, although this fact can be derived from the stronger version of Jockusch-Shore's completion theorem, Theorem 4.1 of [3], together with the Sacks Splitting Theorem.

We still do not know whether the hypotheses of Theorem 3 can be weakened to mere uniform nontriviality on c.e. sets. In fact, the relationship between different nontriviality hypotheses and the classes of degrees into which completing sets must fall is still quite mysterious. We do illustrate how independent nontriviality with respect to one class of sets can be from nontriviality with respect

to even another closely related class. The classes used for this illustration are closely related classes from Ershov's difference hierarchy.

Theorem 5. *For every $n > 0$ there exists a pseudojump operator V and a co- n -c.e. set A such that $V^A \leq_T A$ but $V^X \not\leq_T X$ for all n -c.e. sets X .*

Our final result is the most difficult and surprising result of this paper. We show the impossibility of requiring cone-avoidance in a pseudojump completion theorem, given what seems to be a natural nontriviality hypothesis. Thus, the usual restraint functions for avoiding cones are not compatible with the pseudojump completion theorem.

Theorem 6. *There exist a non-computable, computably enumerable set C and a pseudojump operator V such that*

- (1) *for every $e \in \omega$, $W_e <_T V^{W_e}$, and*
- (2) *for every $e \in \omega$, if $V^{W_e} \equiv_T K$, then $C \leq_T W_e$.*

We leave open the question whether or not the condition in (1) can be strengthened to guarantee that V is strongly nontrivial, as any natural operator must be, rather than merely uniformly nontrivial for c.e. sets.

In what follows our notation is standard for priority constructions, as in [6]. One minor variation that we note explicitly to avoid confusion is that we often relativize functions or entire expressions to a particular stage by writing $[s]$ after them.

2 Two finite injury constructions and the completion construction

We first show how two standard finite injury constructions can be combined with the completion method of [3]. These are the Friedberg-Muchnik construction of two incomparable c.e. degrees and the Cooper-Lachlan construction of a properly d.c.e. degree. In the both of these case the number of times higher priority requirements inflict injury on lower ones can be bounded in advance by a computable function, thereby making it possible to avoid "over-shooting" $\mathbf{0}'$ with the degrees so constructed. This turns out to be easy in the case of the Friedberg-Muchnik construction, but in the second case, the necessity to preserve computations for a potentially successful diagonalization attempt later conflicts strongly with the coding of K into $A \oplus V^A$. Because of this, what is a relatively simple finite-injury construction has to be recast as an infinite-injury argument somewhat reminiscent of the proof of the Sacks Density Theorem. As we illustrate later in the case of the Sacks cone-avoidance method, it is not always possible to combine a finite injury method with a construction completing a c.e. operator to $\mathbf{0}'$, particularly in the case where one must set restraints that while eventually finite are not under the direct control of the construction.

The following result of Jockusch and Shore is proved in [3], Corollary 4.2.

Corollary 1. *For any low c.e. set L and any pseudojump operator V , there is a c.e. set $A \geq_T L$ such that $V^A \equiv_T K$.*

By the Sacks Splitting Theorem we can split K into two low sets, L_1 and L_2 say. If we take a pseudojump operator V which is uniformly nontrivial with respect to c.e. sets and apply Corollary 1 to V , L_1 and L_2 , then we get two c.e. sets $A_1 \geq_T L_1$ and $A_2 \geq_T L_2$ such that $V^{A_1} \equiv_T V^{A_2} \equiv_T K$. If A_1 and A_2 are Turing comparable, then A_1 or A_2 is Turing complete, and so, for some i , $V^{A_i} \leq_T K \leq_T A_i$, which contradicts the hypothesis that V is uniformly nontrivial with respect to c.e. sets.

Both to avoid direct dependence on [3], and as a warm-up for the more complex constructions necessary to prove our other results, we present a direct proof¹ of this fact which avoids using lowness.

Theorem 2. *For any pseudojump operator V with $X <_T V^X$ for all c.e. sets X , there exist Turing incomparable computably enumerable sets A and B such that $V^A \equiv_T V^B \equiv_T K$.*

Proof. The proof combines the method of proof of Theorem 3.1 of [3] with that of the Sacks splitting theorem. We construct computably enumerable sets A and B with $K \leq_T A \oplus B$ to meet the following requirements for every $n \in \omega$.

$$N_n^A : (\exists^\infty s)(n \in V^A[s]) \implies n \in V^A,$$

$$N_n^B : (\exists^\infty s)(n \in V^B[s]) \implies n \in V^B,$$

$$P_n^A : n \in K \text{ if and only if } \gamma^A(n) \in A,$$

$$P_n^B : n \in K \text{ if and only if } \gamma^B(n) \in B,$$

Above, γ^A will be a V^A -computable function, and γ^B will be a V^B -computable function. Hence the positive requirements P_n^A and P_n^B ensure that $K \leq_T V^A$ and $K \leq_T V^B$. The negative requirements N_n^A and N_n^B ensure that V^A and V^B are Δ_2^0 , and hence Turing reducible to K . Furthermore, if the negative requirements are satisfied and $K \leq_T A \oplus B$, it follows as above that A and B are Turing incomparable. (If not, either A or B is Turing complete, and so $V^A \leq_T 0' \leq_T A$ or $V^B \leq_T 0' \leq_T B$, in contradiction to the nontriviality hypothesis on V .)

To ensure that $K \leq_T A \oplus B$, we require that $(\forall k)[k \in K \iff 2k \in A \cup B]$, so that $K \leq_m A \cup B \leq_T A \oplus B$. In fact, when a number k enters K , the number $2k$ will enter exactly one of A , B , (where the lucky recipient is chosen to minimize injury to the negative requirements, just as in the proof of the Sacks splitting theorem). The positive requirements P_n^A and P_n^B will not interfere with this because they will cause only odd numbers to enter $A \cup B$.

The proof will be a finite injury argument with the priorities as follows:

$$P_0^A < P_0^B < N_0^A < N_0^B < P_1^A < P_1^B \dots$$

We briefly discuss the strategies for satisfying the requirements. We discuss the requirements for building A — the construction of B is a mirror image of that of A .

¹R. Molinari (unpublished) has independently given a similar argument.

The strategy for the requirement N_n^A is to impose a restraint $r^A(n, s)$ to preserve the computation (if any) witnessing $n \in V^A[s]$, where $V^A[s]$ abbreviates $V_s^{A_s}$. Specifically, $r^A(n, s)$ is the use of the computation showing that $n \in V^A[s]$, if $n \in V^A[s]$, and otherwise $r^A(n, s)$ is defined to be 0. This restraint will be injured only finitely often, so N_n^A will be satisfied.

For P_k^A we obtain $\gamma^A(k)$ as $\lim_s \gamma^A(k, s)$, where $\gamma^A(k, s)$ is computable. If $k \in K_s$ we ensure that $\gamma^A(k, s) \in A_{s+1}$. The values of $\gamma^A(k, s)$ will always be odd numbers exceeding $r^A(n, s)$ for all $n < k$. To avoid conflicts between P_j^A and P_k^A for $j \neq k$, we never allow these two requirements to use the same trace. Specifically, let R_0, R_1, \dots be a uniformly computable sequence of infinite, pairwise disjoint sets of odd numbers. We will always have $\gamma^A(k, s) \in R_k$. It is then easily seen that if $\gamma^A(k) = \lim_s \gamma^A(k, s)$ exists, then P_k^A is satisfied. The existence of $\gamma^A(k)$ will follow from the fact that $\lim_s r^A(n, s)$ exists for all $n < k$.

The only possibly subtle point in the argument is to check that $\gamma^A(k)$ is a V^A -computable function. To ensure this, we choose $\gamma^A(k, s)$ to be as small as possible, subject to the above restrictions. Thus, $\gamma^A(k, s)$ is defined to be the *least* number $z \in R_k$ such that $z > r^A(n, s)$ for all $n < k$. Now, since A is c.e., the limiting restraint $r^A(n) = \lim_s r^A(n, s)$ is V^A -computable, and from this it follows that $\gamma^A(\cdot)$ is V^A -computable. (Note: We do *not* require that $\gamma^A(k, s)$ be nondecreasing in s for fixed k , i.e. we allow the trace for P_k^A to drop back to a smaller value when higher priority restraints decrease.)

The requirement that $(\forall k)[k \in K \iff 2k \in A \cup B]$ also threatens to injure strategies for the negative requirements. To avoid infinite injury to these strategies, we use the method familiar from the proof of the Sacks splitting theorem: as each new number k enters K , we enumerate $2k$ into A if the highest priority requirement that would be injured by its entry is some N_n^B ; otherwise, we enumerate it into B .

Construction.

Let k_0, k_1, \dots be a computable enumeration of K without repetitions.

Stage 0. Let $A_0 = B_0 = \emptyset$, $r^A(n, 0) = r^B(n, 0) = 0$ for all n , and for each k let $\gamma^A(k, 0)$ and $\gamma^B(k, 0)$ be the least element of R_k .

Stage $s+1$. Let $r^A(n, s), r^B(n, s), \gamma^A(n, s)$ and $\gamma^B(n, s)$ be defined as above. Note that they depend only on A_s and B_s , the sets of numbers enumerated in A and B respectively by the end of stage s of the construction, so they are now defined. Let n_A be the least number $n \leq s$ such that $2k_s < r^A(n, s)$ if there is such an n , and otherwise let $n_A = s$. Define n_B analogously with B in place of A . If $n_A \leq n_B$, let

$$A_{s+1} := A_s \cup \{\gamma^A(k, s) : k \in K_s\}, \quad B_{s+1} := B_s \cup \{\gamma^B(k, s) : k \in K_s\} \cup \{2k_s\}$$

If $n_B < n_A$, the definition of A_{s+1} and B_{s+1} is the same except that $2k_s$ is enumerated into A_{s+1} rather than B_{s+1} .

Let $A = \cup_s A_s$ and $B = \cup_s B_s$.

This completes the construction.

Verification

The sets A and B are obviously c.e. Since all values of $\gamma^A(n, s)$ and $\gamma^B(n, s)$ are odd, we have that $(\forall k)[k \in K \iff 2k \in A \cup B]$, so $K \leq_T A \oplus B$.

We next prove the following facts by simultaneous induction on n :

- (a) $\lim_s \gamma^A(n, s)$ and $\lim_s \gamma^B(n, s)$ each exist
- (b) P_n^A and P_n^B are each satisfied
- (c) $\lim_s r^A(n, s)$ and $\lim_s r^B(n, s)$ each exist
- (d) N_n^A and N_n^B are each satisfied.

Assume that (a)-(d) hold for all $m < n$ in order to prove that they hold for n . Then (a) clearly holds for n from the definition of $\gamma^A(n, s)$ and $\gamma^B(n, s)$ and the assumption that (c) holds for all $m < n$. From this and the construction it follows that (b) also holds for n . For (c), we show that $\lim_s r^A(n, s)$ exists, and the proof for r^B is similar. Say that N_n^A is *injured* at stage $s+1$ if $A_{s+1} - A_s$ has an element less than $r(n, s)$. It is easily seen that if N_n^A is not injured at $s+1$ and $r^A(n, s) \neq 0$, then $r^A(n, s+1) = r^A(n, s)$. Thus it suffices to show that N_n^A is injured at only finitely many stages. But if N_n^A is injured at s , then $A_{s+1} - A_s$ has an element of the form $\gamma^A(m, s)$ for some $m \leq n$, or $2k_s < r^B(m, s)$ for some $m < n$. Since by inductive hypothesis and the fact that (a) holds for n there are only finitely many values of $\gamma^A(m, s)$ over all $m \leq n$ and all s and only finitely many values of $r^B(m, s)$ over all $m < n$ and all s , it follows that N_n^A is injured only finitely often, so (c) holds. Essentially the same argument also proves (d), which completes the induction.

It remains to show that $\gamma^A(\cdot)$ is V^A -computable, and the analogous fact for γ^B . Let $r^A(n) = \lim_s r^A(n, s)$, and define $r^B(n)$ analogously. Note that $r^A(\cdot)$ is V^A -computable. (If $n \notin V^A$, then $r(n) = 0$ since N_n^A is satisfied. If $n \in V^A$, then $r(n)$ is the use of the computation showing that $n \in V^A$, and this can be computed from A , and $A \leq_T V^A$.) Then $\gamma^A(n)$ is the least element of R_n which exceeds $r(m)$ for all $m < n$, so $\gamma^A(\cdot)$ is A -computable. □

It is reasonable to ask why we did not use the standard Friedberg-Muchnik strategy to ensure the Turing incomparability of A and B in the above theorem. The reason is that there is no apparent way for V^A or V^B to compute the limiting value of the restraints imposed by the usual Friedberg-Muchnik requirements. Thus, the incomparability requirements are replaced by the purely positive requirement $K \leq_T A \oplus B$.

On the other hand, the direct use of a splitting strategy for K to ensure $A \upharpoonright_T B$ is not necessary — one can use the hypothesis of nontriviality directly to achieve permission to set restraints and diagonalize as in the proof of Theorem 3 below. However, the resulting A and B still have the property that $A \oplus B \equiv_T K$. This leaves open the following question:

Question 1. *Does there exist a nontrivial pseudojump operator V such that for all c.e. A and B , if both $A \upharpoonright_T B$, and $A \oplus V^A \equiv_T B \oplus V^B \equiv_T K$, then $A \oplus B \equiv_T K$?*

There are many other natural questions concerning the existence of various kinds of c.e. sets completing a given operator. We list a few of them in section 7 below.

We next solve the problem of completing a given pseudojump operator by a set of non-c.e. degree. In fact, the simplest such kind of degree will always do — a d.c.e. degree. We use as hypothesis nontriviality of the operator relative to all d.c.e. sets.

Theorem 3. *For every pseudojump operator V such that $V^X \not\leq_T X$ for all d.c.e. sets X , there exists a d.c.e. set A such that $V^A \equiv_T K$ and the degree of A is not c.e.*

Proof. We construct a d.c.e. set A and a function $\gamma \leq_T V^A$ to satisfy the following requirements for all computable functionals Φ and Ψ and all $n \in \omega$

$$N_n : (\exists^\infty s)[n \in V^A[s]] \implies n \in V^A,$$

$$P_n : n \in K \text{ if and only if } (\exists y < \gamma(n))[y \in A^{[2^n]}], \text{ and}$$

$$R_{\Phi, \Psi, e} : \Phi(W_e) \neq A \text{ or } \Psi(A) \neq W_e.$$

As in Theorem 2, the function γ is the limit as $s \rightarrow \infty$ of a computable $\gamma(n)[s]$.

The negative requirements N_i are met by imposing restraints. If i enters V^A at s , then N_i imposes the restraint $r(i)[s] = s$, and this restraint remains in effect until (if ever) i leaves V^A . We make the usual convention that every computation existing at stage s has use less than s , so this restraint suffices to preserve $i \in V^A$ if A does not change below its value. Furthermore, it will be technically useful to choose the restraint to be s rather than the use of the computation establishing that $i \in V^A$. The values of the $\gamma(i)[s]$ are chosen to be greater than $r(j)[s]$ for all $j \leq i$, as in Theorem 2. However, a new feature of the current theorem (as will be seen below) is that a value of $\gamma(i)[s]$ can be put into A and then removed from A . Of course, such a value $\gamma(i)[s]$ is not suitable at the value of $\gamma(i)[t]$ for any $t > s$ since it cannot again be put into A . Thus, we will define $\gamma(i)[s]$ to be the least number in $\omega^{[2^i]}$ which exceeds $r(j)[s]$ for all $j \leq i$ and is not in $\cup_{t \leq s} A[t]$.

The usual strategy for satisfying a requirement $R_{\Phi, \Psi, e}$ ensuring that A is not of c.e. degree is based on diagonalization. One chooses a witness x which has never been in A , and then waits for a stage s at which the length of agreement between $\Phi(W_e)[s]$ and $A[s]$ has increased beyond x , and that between $\Psi(A)[s]$ and $W_e[s]$ has increased beyond $\phi(x)[s]$. At stage $s + 1$, one adds x to A , thereby creating a disagreement between $\Phi(W_e; x)[s] = 0$ and $A(x)[s + 1] = 1$, meanwhile restraining A from changing otherwise on $\psi(\phi(x))[s]$. If at some stage $t > s$ a change in W_e below $\phi(x)[s]$ causes $\Phi(W_e; x)[t] = 1$, then one removes x from A , an action that restores the values of $\Psi(A) \upharpoonright \phi(x)[s] = W_e \upharpoonright \phi(x)[s] \neq W_e \upharpoonright \phi(x)[t]$. Since W_e is c.e., the change between s and t on the use $\phi(x)$ is irreversible, and hence $\Psi(A) \neq W_e$. This win is preserved by the restraint $\psi(\phi(x))[s]$.

The success of the strategy described for R_n requires restraining A on $\psi(\phi(x))[s]$ so that the removal of x from A at stage t will restore the original computation and so that the win is preserved if x is put back into A . As in Theorem 2, the diagonalization requirements R_n are unable to impose restraints directly because the imposition of such restraints could increase the values of γ markers

in a way that could not be calculated by $A \oplus V^A$. Instead, as in Theorem 2, the requirement R_n is aided by the restraints imposed by negative requirements. Specifically, we attempt to meet R_n as follows (ignoring for the moment the other diagonalization requirements). Initially, R_n is *waiting for a witness*. Suppose R_n is waiting for a witness at the beginning of stage $s + 1$ and for some sufficiently large i and some $x \in \omega^{[2^{(e,i)+1}]}$, $x \notin A[t]$ for all $t \leq s$, $x > r(j)[s]$ for all $j < i$, $l(n)[s] > x$, and $r(i) > \psi(\phi(x))[s]$. (The meaning of “sufficiently large” will be clarified later.) Put the least such x into A_{s+1} , and say that R_n is *waiting for agreement (via x, i and s)*. (The enumeration of x into A may cause i to leave V^A and hence the restraint $r(i)$ to drop below $\psi(\phi(x))$, but we must live with this possibility.) This action is based on the assumption that, for all $j < i$, $r(j)$ and $\gamma(j)$ will not change in the future. (If this assumption should be seen to be incorrect at some later stage, then R_n returns at that stage to the state of waiting for a witness.) Now suppose that R_n is waiting for agreement via x, i and s at stage t and that $\Phi(W_e; x)[t] = 1$. This can happen only if W_e has changed below $\phi(x)[s]$ since stage s , since $\Phi(W_e; x)[s] = 0$. In this situation, we remove x from A at $t + 1$ and we also remove from A all $z \leq \psi(\phi(x))[s]$ such that $z \in A[t] \setminus A[s]$. These z 's may include numbers of the form $\gamma(j)[t + 1]$ for $j \geq i$. This causes $r(i)[t + 1]$ to be greater than $\psi(\phi(x))[s] = \psi(\phi(x)[t + 1])$, for the same reason that the corresponding fact held at s . Hence, at the end of stage $t + 1$ all γ markers below $r(i)$ have the form $\gamma(k)[t + 1]$ for some $k < i$. It follows that R_n is met unless for some $k < i$, some $\gamma(k)$ or $r(k)$ changes after stage $t + 1$. Note that R_n puts only finitely many numbers into A via any fixed i . Thus, the negative requirements N_i should be satisfied, and the restraints $r(i)[s]$ should have finite limits as $s \rightarrow \infty$.

Suppose that R_n is not met. One can then reach a contradiction by inductively computing $V^A(i)$ from an A -oracle. Suppose that $V^A(j)$ has been computed for all $j < i$. From this it is easy to compute the limiting value of $r(j)[s]$ for all $j < i$ using an A -oracle. Let b_i exceed all these limiting values. Using an A -oracle, search simultaneously for the following:

(a) A stage z such that $i \in V^A[z]$ via an A -correct computation

(b) A stage r and a number $x > b_i$ with $x \in \omega^{[2^{(e,i)+1}]}$ such that $x \notin A_t$ for all $t \leq r$, $x \notin A$, $i \notin V^A[r]$, A_r, A agree below $\psi(\phi(x))[r]$, and $l(n)[r] > x$. (Here $l(n)[r]$ is the least y such that it is not the case that both $\Phi(W_e)[r]$ and $A[r]$ agree below y and $\Psi(A)[r]$ and $W_e[r]$ agree below $\phi(y)[r]$.)

Of course, if z is found as in (a), then $i \in V^A$. Suppose now that r and x are found as in (b). Then we claim that $i \notin V^A$. To prove this, assume that $i \in V^A$. Suppose for the moment that there is a stage s with $x \in A[s + 1]$ and consider the least such s . Then $s \geq r$, since $x \notin \cup_{t \leq r} A_t$. Since R_n is not met, W_e must change below $\phi(x)[r]$ after s and hence after r . But this implies that $\Psi(A) \neq W_e$, since $\Psi(A)$ was A -correctly defined and agreed with $W_e[r]$ on all arguments less than $\phi(x)[r]$, and the change in W_e is irreversible. This implies that R_n is met, which is a contradiction. Thus, it suffices to show that there is a stage s with $x \in A[s + 1]$. If not, observe that x meets the criteria for being added to A at all sufficiently large stages at which R_n has no witness in A . If R_n has a witness which is permanently in A , it is clearly met. Otherwise, consider

a stage s after which no $x' < x$ is added to A and at which R_n has no witness currently in A . Then x is added to A at $s + 1$ as needed to complete the proof that $i \notin V^A$ if r is found as in (b).

Finally, we observe that one of the searches in (a) and (b) above must be successful. If $i \in V^A$, clearly (a) is successful. If $i \notin V^A$, then any sufficiently large r, x satisfy (b). (Here we are using the previous remark that R_n puts only finitely many numbers into A for each fixed i .) Thus, $i \in V^A$ if search (a) succeeds first, and $i \notin V^A$ if search (b) succeeds first. It follows that $V^A \leq_T A$, which contradicts the nontriviality of V on d.c.e. sets. This contradiction shows that R_n is met.

It follows by a small extension of the above argument that if $\Phi(W_e)$ and $\Psi(A)$ are total, then R_n changes its state only finitely often as it permanently succeeds on some fixed witness. However, without these totality assumptions, there is no reason to think that R_n changes its state only finitely often. However, infinitary action of R_n can cause difficulty for the negative requirements N_i , the positive requirements P_i and the other diagonalization requirements R_m .

The difficulties just alluded to are all resolved by the standard device of using a tree T of strategies. We assume that the reader is familiar with such arguments. The nodes of the tree are just the finite binary strings so $T = 2^{<\omega}$. Every node $\alpha \in T$ of length n is associated with the diagonalization requirement R_n . Let R_α be the version of R_n associated with the node α . If $\alpha, \beta \in T$ and $\beta \hat{\ } 0 \subseteq \alpha$, then R_α assumes that R_β changes state infinitely often. If $\beta \hat{\ } 1 \subseteq \alpha$, then R_α assumes that R_β changes state only finitely often. The state transitions are arranged so that if R_α changes state infinitely often, then it is infinitely often waiting for a witness. It will be an important feature of the construction that if $\beta \supseteq \alpha \hat{\ } 0$, then β acts only when α is waiting for a witness. We write $\alpha <_L \gamma$ if $\alpha(i) < \gamma(i)$ for the least i (if any) with $\alpha(i) \neq \gamma(i)$. In this case, we say that α is to the left of γ , or γ is to the left of α . If α is to the left of γ and R_α acts, then γ is “initialized”, which means that $x(\gamma)$, $i(\gamma)$, and $s(\gamma)$ all become undefined.

All witnesses for R_β will be elements of $\omega^{[2^{(\beta, i)+1}]}$ for some i , where $\beta \in T$ is identified with its numerical code.

We now consider the interaction of two diagonalization requirements R_β and R_α . Suppose that R_α is associated with the triple (Ψ, Φ, W_e) . A potentially very bad sort of interaction is the following. Suppose that at stage $s + 1$ R_α places a witness x into A . Then R_β later removes a witness y from A , with $y < \psi(\phi(x)[s])$ and $y \in A[s + 1]$. Later still, R_α wishes to remove x from A and to restore A so that it agrees with A_s below $\psi(\phi(x)[s])$. This is of course impossible because the removal of y from A is irreversible. The handling of this depends on the relationship between α and β . If $\beta \hat{\ } 0 \subseteq \alpha$, then R_α acts only when R_β is waiting for a witness. Since R_β is waiting for a witness at $s + 1$, no number in $A[s + 1]$ is removed from A by R_β after stage $s + 1$, so the situation described above does not arise. If $\beta \hat{\ } 1 \subseteq \alpha$, then α is assuming that R_β acts only finitely often. Hence it is safe to initialize α whenever R_β acts, and in particular when R_β removes y from A . (In this case α abandons the witness x .) Similarly, if $\beta <_L \alpha$, α is initialized when β removes y from A . If $\alpha <_L \beta$, then β is

initialized when x is added to A by α . If $\alpha \hat{\ } 0 \subseteq \beta$, then β is assuming that α will remove x and waits for it to do so before removing y . If $\alpha \hat{\ } 1 \subseteq \beta$, then β is safely initialized when α puts x into A . Finally, if $\alpha = \beta$, the situation does not arise because R_α does not put a witness into A if it currently has a witness in A which entered since the last time it was initialized. Thus, it is possible to restore A on given intervals, as far as witnesses are concerned. A similar argument applies to γ -traces, since they are removed from A only when associated witnesses are removed.

Another potentially serious difficulty is that a fixed R_α can act infinitely often and thus prevent the satisfaction of a negative requirement N_j . This is prevented by requiring that R_α can add a number x to A via i only if i is greater than $|\alpha|$ and also greater than the the last stage at which R_α was initialized by any other node. If $i \in V^A[s]$, then the only numbers below the use of this computation allowed to enter A at stage $s + 1$ are those of the form $\gamma(j)[s]$, where $j < i$ and $\gamma(j)[s] \neq \gamma(j)[s + 1]$, and witnesses x for R_α with $|\alpha| \leq i$. Once γ has settled down below i , each such R_α can be initialized only by other nodes and puts at most one number into A between consecutive stages at which it is initialized. Once it is initialized i times (if ever) by other nodes, it can never injure N_i . This makes it straightforward to show that N_i is injured only finitely often and in fact gives a computable bound on the number of times which it is injured.

A final difficulty involves the positive requirements. Here the bad scenario is as follows. Suppose a number j enters K and the corresponding trace $\gamma(j)[s]$ enters A . Then, as described above, it is possible that $\gamma(j)[s]$, will be removed from A for the sake of some diagonalization. This in itself is no problem, as $\gamma(j)$ takes a new large value which can be added to A . The problem arises if $\gamma(j)$ later decreases in value (which can happen when restraints drop), but its former value $\gamma(j)[s]$ is no longer available because it has been previously added to and removed from A . In this case, the new value for $\gamma(j)$ may be larger than $\gamma(j)[s]$, and it is not immediately apparent how $A \oplus V^A$ can compute the limiting value of $\gamma(j)$. (The problem is that the increase in size of $\gamma(j)$ is caused in part by the insertion and removal of $\gamma(j)[s]$ from A , and these events leave no trace in A or V^A .) This is overcome by a counting argument closely related to the fact, mentioned above, that there is a computable bound on the number of times that a given negative requirement N_i is injured. This will give a computable bound $f(i)$ on the number of elements of $\omega^{[2^i]}$ which ever enter A , and this in turn will make it possible to show that an upper bound on $\lim_s \gamma(j)[s]$ is $A \oplus V^A$ -computable. (Specifically, we can take this upper bound to be the least z such that there are more than $f(i)$ numbers which are in $\omega^{[2^i]}$ which are less than z and greater than the limiting restraints $r(j)$ for each $j \leq i$.)

Construction

As mentioned above, we use the tree of strategies $T = 2^{<\omega}$ for our construction. Our notation is standard, as found in [6], XIV, except as noted below. We order the diagonalization requirements by means of some fixed computable indexing of computable partial functionals and a standard function for coding

the ordered triples, and assign requirement R_n to each node $\alpha \in T$ of length n . For each $n \in \omega$, if $n = \langle \Phi, \Psi, e \rangle$, we define the length of agreement function for R_n , $l(n)[s] = \max\{x : (\forall y < x)[\Phi(W_e; y) = A(y) \text{ and } \Psi(A) \upharpoonright \phi(W_e; y) = W_e \upharpoonright \phi(W_e; y)][s]\}$.

At any point in the construction, each diagonalization requirement R_α may have associated with it the parameters $x(\alpha)$, $i(\alpha)$ and $s(\alpha)$. Whenever any of these parameters is defined, all three will be defined. We write $x(\alpha)[s]$ for the value of $x(\alpha)$ at the *end* of stage s , and similarly for the other parameters. The parameter $x(\alpha)$ is the *current witness* for α . This witness was chosen at stage $s(\alpha) + 1$, and $i(\alpha) \in V^A[s_\alpha]$. We say that R_α is *waiting for a witness* when $x(\alpha)$ is undefined. Suppose that α is working on the requirement $R_{\Phi, \Psi, e}$. We say that R_α is *waiting for a W -change* when $x(\alpha) \downarrow \in W_e$, and is *finished* when $x(\alpha) \downarrow \notin W_e$.

We have restraint functions $r(j)[s]$ for the negative requirements that ensure $V^A \leq_T K$. We also have a sequence of trace functions $\gamma(k)[s]$ for the positive requirements. For the purpose of setting restraints it is convenient to increase the use of each convergent computation $V^A(i)$ by setting it equal to a stage at which the computation converges without any possibility of interference. To do this, we use the ideas of the Soare-Lachlan hat-trick. We let, for every stage t , a_t be the least y such that $A[t] \upharpoonright y \neq A[t+1] \upharpoonright y$ or t if A does not change at stage $t+1$. We then define $\hat{V}^A(i) \downarrow [s]$ if and only if there exists a $t \leq s(V^A(i) \downarrow [t], A[s] \upharpoonright a_t = A[t] \upharpoonright a_t)$, and $v^A(i) < a_t$.

Since A will be a Δ_2^0 set, $\hat{V}^A = V^A$ in the limit. We define the use $\hat{v}^A(i)$ to be the least t such that $(V^A(i) \downarrow [t] \wedge A \upharpoonright t = A[t] \upharpoonright t)$. Note that if $i \in V^A$, this use is computable from A , since we merely have to wait for A to achieve its correct values on every number less than the first stage at which $V^A(i)$ converges A -correctly. The construction at each stage s is divided into substages at which all calculations take place with the current value for A — in other words, elements enumerated or removed at one substage are taken into account at all later ones.

Stage 0: For all i , we let $r(i)[0] = 0$, and $\gamma(i)[0]$ be the least element of $\omega^{[2^i]}$. Initialize all $\alpha \in T$. (These are considered to be initialized by the empty node.)

Stage $s+1$:

Case 1. s is even. In this case, we perform all enumeration necessary to correct our intended reduction of K to $A \oplus V^A$. If $n \in K[s]$ and $\neg(\exists x < \gamma(n)[s])[x \in A[s] \cap \omega^{[2^n]}]$, then enumerate $\gamma(n)[s]$ into $A[s+1]$, and, for all $\alpha \in T$ if $i(\alpha) \downarrow \geq n$ or $|\alpha| \geq n$, initialize α . Go to the final substage (substage s) to set restraints and update values of γ .

Case 2. s is odd. In this case, we work on the diagonalization requirements. We define an *approximation to the true path* $g[s]$ at stage s of length at most s by recursion. For any node $\alpha \in T$, s is an α -stage if and only if $\alpha \subset g[s]$. Let $\alpha = g[s] \upharpoonright n$. At substage n we take action for α and define $g[s] \upharpoonright n+1$.

Substage n ($n < s$). Let R_n be the requirement $R_{\Phi, \Psi, e}$, and let $\alpha = g[s] \upharpoonright n$. Apply the first applicable subcase below. In Case 2B, let t be the greatest stage

such that $t < s$ and α changed states at t , or $t = 0$.

- Subcase 2A: Let m be the least integer $\leq s$ such that $m = s$ or $r(m)[s] \neq r(m)[s-1]$ or $\gamma(m)[s] \neq \gamma(m)[s-1]$. If $m \leq |\alpha|$ or $m < i(\alpha) \downarrow$, then initialize α and go to substage $n+1$.
- Subcase 2B: The node α is waiting for a witness, and there is no stage u such that $t \leq u < s$ and $\alpha \hat{\ } 0$ was accessible at u . Then define $g[s] \upharpoonright n+1 = \alpha \hat{\ } 0$ and go to the next substage (without taking any action for α). (This gives nodes $\beta \supseteq \alpha \upharpoonright n \hat{\ } 0$ the opportunity to act.)
- Subcase 2C: α is waiting for a witness, and there exist $x < s$ and $i < s$ such that $x \in \omega^{[2(\alpha, i)+1]}$, $x \notin \cup_{t \leq s} A_t$, $x < l(n)[s]$, $x > r(j)[s]$ for all $j < i$, i is at least as large as the last stage at which R_α was initialized by some other node, and $r(i) > \phi(\psi(x))[s]$. Then choose the least x such that this holds for some i , and choose the least possible i for this x . Set $x(\alpha)[s+1] = x$, enumerate x into $A[s+1]$, and let R_α be in the state of waiting for a W -change. Also, set $i(\alpha)[s+1] = i$, and $s(\alpha)[s+1] = s$. Let $g[s] = \alpha$ and let α initialize all $\beta \in T$ such that $\alpha <_L \beta$. Go to the final substage, substage s .
- Subcase 2D: α is waiting for a witness. Define $g[s] \upharpoonright n+1 = \alpha \hat{\ } 1$ and go to the next substage (without taking any action for α).
- Subcase 2E: The node α is waiting for a W -change, and $x(\alpha) \geq l(n)[s]$. Define $g[s] \upharpoonright n+1 = \alpha \hat{\ } 1$ and go to the next substage (without taking any action for α).
- Subcase 2F: The node α is waiting for a W -change, and $x(\alpha) < l(n)[s]$. Then remove from A all $z < r(i(\alpha))[s(\alpha)-1]$ such that $z \in A[s] - A[s(\alpha)]$. Let $g(s) = \alpha$. Declare α to be *finished*. Let α initialize all $\beta \in T$ such that $\alpha <_L \beta$. Go to the final substage.
- Subcase 2G: The node α is finished. Define $\alpha \upharpoonright n+1 = \alpha \hat{\ } 1$ and go to the next substage (without taking any action for α).

Substage s . This is the final substage. $A[s+1]$ has already been determined by the previous substages. If $g[s]$ has not already been completely defined, let it be $g[s] \upharpoonright s$. For each i , if $i \in \hat{V}^A[s+1]$, let $r(i)[s+1]$ be $\hat{v}(i)[s+1]$, and otherwise let $r(i)[s+1] = 0$. For each m , let $\gamma(m)[s+1]$ be the least $z \in \omega^{[2m]}$ such that $z > r(i)[s+1]$ for all $i \leq m$ and $z \notin \cup_{t \leq s+1} A[t]$.

We initialize all β such that $g[s] <_L \beta$ at s . Unless stated otherwise, all parameters, functionals and states remain the same at $s+1$ as at s . Go to stage $s+1$.

This completes the construction. First, note that A is d.c.e. To see this, it suffices to show that no number y can be removed from A and subsequently re-enter A . Each y has the form $x(\alpha)[s]$ or $\gamma(n)[s]$ for at most one α or n , and it has at most one of these forms. Suppose that $y = x(\alpha)[s]$ and y is removed from

A at stage $s+1$. Then Subcase 2G will apply to α at all α -stages $t > s+1$ until, if ever, α is initialized. When Subcase 2G applies, α is finished so $x(\alpha)$ does not enter A . If α is initialized, y is never subsequently the value of $x(\alpha)$ and so does not enter A again. If $y = \gamma(n)$ is removed from A at stage $s+1$, then it is because some $x(\alpha)[s]$ is removed from A at the same substage of $s+1$. Then y never subsequently re-enters A by an argument analogous to the preceding case where $y = x(\alpha)$.

Lemma 2.0.1. (i) For each n , $\lim_s \gamma(n)[s]$ exists.

(ii) Each negative requirement N_n is met.

Proof. We prove the above two statements by simultaneous induction on n . First, prove (i) for n , assuming that (ii) holds for all $m < n$. If $\gamma(n)[s] \neq \gamma(n)[s+1]$, then either $n \in K[s+1] \setminus K[s]$ or there exists $m < n$ such that $r(m)[s] \neq r(m)[s+1]$. But since we are assuming that N_m is met for each $m < n$, it follows that there are only finitely many s such that $r(m)[s] \neq r(m)[s+1]$ for some $m < n$.

Next, we prove (ii) for n , assuming that (i) holds for all $m \leq n$. We say that N_n is *injured* at stage $s+1$ if some number less than or equal to $r(n)[s]$ enters or leaves A at stage $s+1$. It suffices to show that N_n is injured only finitely often. If N_n is injured at stage $s+1$ where s is even, then $\gamma(m)[s]$ enters A at $s+1$ for some $m \leq n$. Since (i) holds for all $m \leq n$ there are only finitely many such stages. Say that N_n is injured at stage $s+1$ by β if some number less than or equal to $r(n)[s]$ enters or leaves A at stage $s+1$ because of the action of R_β .

Note that if $i(\beta)[s] > n$, then β does not injure N_n at stage $s+1$. Hence no β of length greater than n ever injures N_n . Also, if β is initialized by other nodes more than n times, then $i(\beta)[s] > n$ for all sufficiently large s , so β injures N_n at most finitely many times. Suppose that β is initialized at most n times by other nodes, and choose s_1 so large that β is not initialized by other nodes at any stage $s > s_1$, and also no $\gamma(m)[s]$, $m \leq n$, enters A at any stage $s > s_1$. Finally suppose that $r(m)[s] = r(m)[s_1]$ for all $s > s_1$ and all $m < n$. Such an s_1 exists because (i) holds for $m \leq n$ and (ii) holds for all $m < n$. If β injures N_n after s_1 , then choose $s_2 > s_1$ such that β injures N_n at s_2+1 . Then $x(\beta)[s_2] \downarrow$ and $i(\beta)[s_2] \leq n$. It follows that β is never initialized after s_2 (by other nodes, or enumeration of γ -values or changes in r -values). Hence β acts at most once after s_2 (as it keeps the same witness after s_2 and acts at most twice on any given witness). Thus β injures N_n at most finitely many times. In summary, there are most finitely many β which ever injure N_n , and each of these β injures N_n only finitely many times. Thus N_n is injured only finitely often at stages $s+1$ with s odd, and hence it is injured only finitely often altogether. \square

It follows from Lemma 2.0.1 that $V^A \leq_T K$. To show that $K \leq_T V^A$, we need the following lemma, which is a quantitative version of Lemma 2.0.1.

Lemma 2.0.2. There are computable functions f and g with the following properties.

(i) There are at most $f(n)$ stages s such that $\gamma(n)[s] \neq \gamma(n)[s+1]$.

(ii) There are at most $g(n)$ stages s such that N_n is injured at $s+1$.

Proof. The proof is parallel to that of Lemma 2.0.1. The functions f and g are defined by simultaneous induction. In defining $f(n)$, we assume that $g(m)$ has been defined for all $m < n$. In defining $g(n)$, we assume that $f(m)$ has been defined for all $m \leq n$ and that $g(m)$ has been defined for all $m < n$.

To define $f(n)$ recall from the proof of Lemma 2.0.1 that if $\gamma(n)[s] \neq \gamma(n)[s+1]$, then either $n \in K[s+1] \setminus K[s]$ or there exists $m < n$ such that $r(m)[s] \neq r(m)[s+1]$. If $r(m)[s] \neq r(m)[s+1]$, then either N_m is injured at $s+1$, or m enters V^A at $s+1$ and remains there until, if ever, N_m is injured. It follows that we may take $f(n) = 1 + 2 \sum_{m < n} g(m)$.

To define $g(n)$, first note that if N_n is injured at stage $s+1$ with s odd, then some $\gamma(m)[s]$ enters A at stage $s+1$. There are at most $\sum_{m \leq n} (f(m) + 1)$ such stages since $\gamma(m)[s]$ takes on at most $f(m) + 1$ values as s varies. For each β let n_β be the number of stages s such that $i(\beta)[s] \leq n$ and β is initialized at $s+1$. Note that $n_\beta = 0$ if $|\beta| > n$. Suppose $|\beta| \leq n$. Then $n_\beta \leq n + 1 + \sum_{m \leq n} f(m) + \sum_{m < n} g(m)$. To see this, note that there are at most $n + 1$ stages s such that β is initialized by some other node at $s+1$ and $i(\beta)[s] \leq n$, and the other terms correspond to initialization of β by γ and r changes, respectively. If s_0 and s_1 are stages such that there is no s with $s_0 \leq s < s_1$ and, for some β of length at most n , $i(\beta)[s] \leq n$ and β is initialized at $s+1$, then there are at most 2 stages s such that $s_0 \leq s < s_1$ and N_n is injured at $s+1$ (corresponding to adding and removing a fixed witness). As there $2^{n+1} - 1$ binary strings of length at most n , we may take

$$g(n) = \sum_{m \leq n} (f(m) + 1) + 2^{n+2}(n + 1) + \sum_{m \leq n} f(m) + \sum_{m < n} g(m)$$

□

We can now show easily that $K \leq_T V^A$. Let f be as in Lemma 2.0.2. Let $r(n) = \lim_s r(n)[s]$, where this limit exists by Lemma 2.0.1. Then $r \leq_T V^A$. Let $h(n)$ be the least number z such that in $\omega^{[2n]}$ there are at least $f(n) + 1$ numbers $y < z$. Then $h \leq_T V^A$ and there is a number $y \in \omega^{[2n]}$ such that $r(n) < y < h(n)$ and $y \neq \gamma(n)[s]$ for all s , and thus, for all s , $y \notin A[s]$. Clearly, $n \in K$ if and only if there exists $y < h(n)$ such that $y \in \omega^{[2n]} \cap A$. It follows that $K \leq_T A$, and so $K \equiv_T V^A$.

We turn now to the verification that the diagonalization requirements R_n are met. We say that α is on the *true path* if it is the leftmost node of length $|\alpha|$ which is accessible infinitely often.

Lemma 2.0.3. *Suppose that α is on the true path. If α changes state infinitely often, then $\alpha \frown 0$ is on the true path, and otherwise $\alpha \frown 1$ is on the true path.*

Proof. Suppose first that α changes state infinitely often. Then α is waiting for a witness at infinitely many α -stages. (Otherwise, it could change only from

“waiting for W ” to “finished” at all sufficiently large α -stages.) Whenever α arrives in the state “waiting for a witness”, it makes $\alpha \frown 0$ accessible at the next α -stage. Hence $\alpha \frown 0$ is accessible infinitely often and is on the true path. Now suppose that α changes state only finitely often. Then, by construction, $\alpha \frown 1$ is accessible at every sufficiently large α -stage. It follows that $\alpha \frown 1$ is on the true path. \square

Lemma 2.0.4. *For each n , the requirement R_n is met.*

Proof. Let n be given, and let α be the node of length n on the true path. Assume for a contradiction that $R_n = R_{\Phi, \Psi, e}$ is not met.

Consider first the case where the witness $x(\alpha)[s]$ is defined with a fixed value $x(\alpha)$ for all sufficiently large s . Let s be minimal with $x(\alpha)[s+1] = x(\alpha)$, so that α acts via Subcase 2C at stage $s+1$ and $x(\alpha) \in A[s+1] \setminus A[s]$. Note that $x(\alpha)[t] = x(\alpha)$ for all $t > s$. Since R_n is not met, $l(s) > x(\alpha)$ for all sufficiently large s . Since there are infinitely many α -stages, there is an α -stage $s_1 > s$ with $l(s) > x(\alpha)$. For the least such s_1 , $x(\alpha)$ is removed from A at stage s_1+1 , and some number enters W_e below $\phi(x(\alpha))[s]$ after stage s . We claim that $A[s]$ and A agree below $r(i(\alpha))[s]$ and hence below $\psi(\phi(x(\alpha)))[s]$. The claim implies that R_n is met because $\psi(A)$ and W_e disagree below $\phi(x(\alpha))[s]$, as in the basic module.

As a first step to proving the above claim, we show that $A[s]$ and $A[s_1+1]$ agree below s and hence below $r(i(\alpha))[s]$. At stage s_1+1 , all elements of $A[s_1] \setminus A[s]$ are removed from A , so to obtain the claimed agreement between $A[s]$ and $A[s_1+1]$ it suffices to show that no numbers $y \in A[s]$ less than or equal to s are removed from A at any stage $t+1$ such that $s \leq t < s_1$. Suppose that such a number y were removed from A at stage $t+1$, where $s+1 \leq t < s_1$. Then t is odd and some unique β causes y to be removed from A at stage $t+1$. It cannot be that $\beta <_L \alpha$ or $\beta \frown 1 \subseteq \alpha$, since in either case α would be initialized at $t+1$, contrary to the fact that $x(\alpha)[t] = x(\alpha)$ for all $t > s$. Suppose now that $\alpha <_L \beta$. Then β is initialized at stage $s+1$, so every number added to or removed from A by β after stage $s+1$ is bigger than $s+1$. Suppose now that $\beta \frown 0 \subseteq \alpha$. Then α cannot act at stage $t+1$ because β is not waiting for a witness at stage $t+1$. The only remaining case is where $\alpha = \beta$, but this can be ruled out because α does not act at any stage $t+1$ such that $s+1 < t+1 < s_1+1$. This completes the proof that $A[s]$ and $A[s_1+1]$ agree below s .

It remains to be shown that $A[s_1+1]$ and A agree below $r(i(\alpha))[s]$. This is done by showing by induction on $t > s_1$ that $A[t]$ and $A[s_1+1]$ agree below $r(i(\alpha))[s]$. The base step is immediate, and the inductive step for $t+1$ odd is similar to the proof in the previous paragraph and so is omitted. Suppose now that $t+1$ is even and $A[t]$ and $A[s_1+1]$ agree below $r(i(\alpha))[s]$. Hence $A[t]$ and $A[s]$ agree below $r(i(\alpha))[s]$. It follows that $r(i(\alpha))[s] \leq r(i(\alpha))[t]$. Hence if $\gamma(m)[t] \in A[t+1] - A[t]$, then $m < i(\alpha)[t]$ in which case α is initialized at $t+1$, contrary to our hypothesis. This completes the proof that $A[s_1+1]$ and A agree below $r(i(\alpha))[s]$, and hence the proof the R_n is met if $x(\alpha)$ is defined with a fixed witness at all sufficiently large stages.

We now show that R_n is met if α is waiting for a witness at infinitely many stages. This part of the proof uses the nontriviality of V on the d.c.e. sets.

We claim first that α is initialized only finitely often by other nodes. α is initialized only finitely often by nodes $\beta <_L \alpha$ since there are only finitely many such nodes which are ever accessible, and each such β is accessible only finitely often. The only other nodes which can initialize α are nodes β such that $\beta \frown 1 \subseteq \alpha$. There are only finitely many such β and $\beta \frown 1$ is on the true path for each such β . Hence by Lemma 2.0.3, each such β changes state only finitely often. But each such β initializes α only when it has just changed state. Hence each such β initializes α only finitely often.

We next observe that for all i , $\omega^{[2(\alpha, i)+1]} \cap \cup_s A[s]$ is finite. Given i , let s_0 be a stage such that α is not initialized by any other node after s_0 , no number $z \leq i$ enters K after s_0 , and $\gamma(m)[s] = \gamma(m)[s_0]$ for all $s \geq s_0$ and $m \leq i$. Suppose that $s \geq s_0$ and $x(\alpha)[s] \in A[s+1] \setminus A[s]$. (If there is no such s , the desired conclusion is immediate.) Then α is never initialized after s , and so acts at most once after s . The desired conclusion follows.

We now complete the proof that R_n is met. Recall that we have assumed that it is not met, and we may also assume that α is waiting for a witness at infinitely many α -stages, since otherwise α has a fixed witness at all sufficiently large stages and hence R_n is met, as shown above. We reach a contradiction by showing that $V^A \leq_T A$. Let i_0 exceed the final stage at which α is initialized by other nodes. We calculate $V^A(i)$ using an A oracle for $i \geq i_0$ by induction on i . Note that there are only finitely many stages s with $x(\alpha) \downarrow$ and $i(\alpha) \leq i$. We give ourselves $V^A(i)$ for $i < i_0$ and assume inductively that we have calculated $V^A(j)$ for $i_0 \leq j < i$. From these we can calculate $r(j) := \lim_s r(j)[s]$ for each $j < i$.

We must calculate $V^A(i)$ from an A -oracle. Search simultaneously for a stage s such that $V^A(i)[s] \downarrow$ via an A -correct computation and for a stage $t \geq i_0$ and a number $y \in \omega^{2(\alpha, i)+1}$ such that $y > r(m)[t]$ for all $m < i$, for all $u \leq t$, $y \notin A_u$, $i \notin V^A[t]$, and $A[s] \upharpoonright \psi(\phi(y)) = A \upharpoonright \psi(\phi(y))$, and $x(\alpha)[t] \uparrow$. If the search for s succeeds first, then clearly $i \in V^A$. We claim that if the search for t and y succeeds first, then $i \notin V^A$. We first note that there is no stage $s > t$ with $x(\alpha)[s] = y$. (Otherwise consider the first such stage s and note that R_n is met, as y enters A at $s+1$, and then W_e must subsequently change below $\phi(y)[s]$, so that $\Psi(A)$ and W_e must disagree below $\phi(y)[s]$.) Thus, for all s , $y \notin A_s$. It follows that y meets the criteria (except possibly for minimality of i) for being chosen as $x(\alpha)[s+1]$ at every sufficiently large stage s at which α is eligible to choose a witness, and there are infinitely many such s because α is initialized infinitely often. Hence there are infinitely many stages s with $i(\alpha)[s] \leq i$. This is a contradiction because for each j there are only finitely many stages s with $i(\alpha)[s] = j$. Finally, if $i \in V^A$ the first search must succeed, and if $i \notin V^A$ the second search must succeed. \square

\square

3 Pseudajump operators on co- n -c.e. sets

Although the hypothesis that the operator V be nontrivial on all d.c.e. sets in Theorem 3 seems to be the most natural one, it may be possible to weaken it:

Question 2. *Given a pseudajump operator V such that $V^X \not\leq_T X$ for all c.e. sets X , need there exist a set A of properly d.c.e. degree such that $A \oplus V^A \equiv_T K$?*

We can show that, in general, mere nontriviality on the class of c.e. sets does not ensure nontriviality on other classes, as we show with our next result.

Theorem 4. *There exists a pseudajump operator V and a co-c.e. A such that $V^A \leq_T A$ but $V^W \not\leq_T W$ for all c.e. sets W .*

Proof. We construct a co-c.e. set A and an A -computable function Γ^A to satisfy the following requirement for each $i, j \in \omega$:

$$N_{\langle i, j \rangle} : V^{W_i} \neq \overline{W_j^{W_i}}$$

Each requirement $N_{\langle i, j \rangle}$ is assigned one witness $x = \langle i, j \rangle$. The basic strategy for satisfying such a requirement is to wait for a stage s such that $x \in W_j^{W_i}[s]$ with use $\phi_j(W_i; x)[s]$, then let $x \in V^{W_i}[s+1]$ with use $v(x)[s] = \phi_j(W_i; x)[s]$. If we do this at every such stage s , there are two possible outcomes for the strategy. If $x \in W_j^{W_i}$, then $x \in V^{W_i} - \overline{W_j^{W_i}}$ permanently after some stage s . If $x \notin W_j^{W_i}$, then $x \in \overline{W_j^{W_i}} - V^{W_i}$, since W_i must have changed permanently on each use $v(x)[s+1]$, as these are all values of $\phi_j(W_i; x)[s]$. Either way, $N_{\langle i, j \rangle}$ is satisfied. Of course, if there exist infinitely many s such that $x \in W_j^{W_i}[s]$, but $x \notin W_j^{W_i}$, then $v(x)$ will increase at infinitely many stages.

To achieve, $V^A \leq_T A$, we also must build an A -computable function Γ^A such that $V^A = \Gamma^A$. Initially we have $A[0] = \omega$ and traces $\gamma(x) \uparrow [0]$ for all $x \in \omega$. At each stage s we set, for each $x \leq s$ such that $\gamma^A(x) \uparrow [s], \Gamma^A(x) \downarrow [s] = 0$, with use $\gamma^A(x)$ some large number. While $x \notin V^A[s]$ we maintain $\Gamma^A(x) \downarrow [s] = 0$. Suppose at some later stage we find that, $A \upharpoonright_{\phi_j(x)} [s]$ agrees with $W_i \upharpoonright_{\phi_j(x)} [s]$, where $x = \langle i, j \rangle$. We then must correct Γ^A since $\Gamma^A(x)[s] = 0$ yet $V^A(x)[s] = 1$. To do this we extract $\gamma(x)[s]$ from A and define $\Gamma^A(x)[s+1] = V^A(x)[s+1]$. Of course, if there exist infinitely many s such that $x \in W_j^{W_i}[s]$, but $x \notin W_j^{W_i}$, then $v(x)$ will increase at infinitely many stages, and there are in this case infinitely many stages at which new uses for $x \in V^{W_i}$ are set from V . However, provided we ensure that the V -use at s , $v(W_i; x)[s]$, has length greater than $\gamma(x)[s]$, we will have a permanent disagreement between A and W_i in the use of any subsequent axioms for V since $W_i[s+1] \upharpoonright_{\phi_j(W_i; x)[s]} \not\subseteq A[s+1] \upharpoonright_{\phi_j(W_i; x)[s]}$. Thus none of these subsequent uses can effect $x \in V^A$, hence $\Gamma^A(x)$ converges permanently after s with the correct value.

This strategy clearly succeeds in the presence of a single $N_{\langle i, j \rangle}$ requirement; however, we must modify this strategy when dealing with more than one negative requirement. The problem is that after extracting $\gamma(x)[s]$ from A some higher priority requirement may extract $\gamma(y)[t]$ from A for some $y < x$, causing

x to enter $V^A[t+1]$ because of some axiom enumerated for x in V at an earlier stage having a use agreeing with an initial segment of $A[t+1]$. We would then be powerless to correct Γ^A . We prevent this in a natural way by choosing new traces $\gamma(z)[t]$ for all $z > y$ whenever we extract $\gamma(y)[t]$ from A , setting the new values $\gamma(z)[t+1]$ greater than the longest string $v(W_i; y)[t]$ currently used in a V -axiom for y . We can now give the formal details.

Construction.

For convenience, we assume all use functions from c.e. oracles are nondecreasing in the stage and increasing in the argument. We write $v(W; x)[s]$ for the use of $x \in V^W$ if $x \in V^W[s]$. Of course, in this case, we are the ones defining $v(W; x)$, in contrast to the case of v^A and v^B in the previous theorems. Recall that witness $x = \langle i, j \rangle$ is assigned to requirement $N_{\langle i, j \rangle}$.

Stage $s = 0$: Let $A[0] = \omega$ and $\gamma^A(0) \downarrow [0] = 0$.

Stage $s + 1$: At this stage, we have already defined $A[s]$, and, for all $x \leq s$, $\gamma^A(x)[s] \downarrow$,

For each $x = \langle i, j \rangle \leq s + 1$ in turn, we perform the following actions.

If $x \in W_j^{W_i}[s+1] - V^{W_i}[s]$ then let $x \in V^{W_i}[s+1]$ with use $v(W_i; x)[s+1] = \max\{\gamma^A(x)[s] + 1, \phi_j(W_i; x)[s+1]\}$.

If $\gamma^A(x)[s] \in A[s]$ and there exists some $t \leq s$ such that $A[s] \upharpoonright_{v(W_i; x)[t]} = W_i[t] \upharpoonright_{v(W_i; x)[t]}$, then remove $\gamma^A(x)[s]$ from A and reset $\Gamma^A(x)[s+1] = V^A(x)[s+1]$ with use $\gamma^A(x)[s+1] = \gamma^A(x)[s]$. For all $z > x$, reset $\Gamma^A(z)[s+1] = V^A(z)[s+1]$ with use $\gamma(z) \downarrow [s+1]$ equal to the least number never yet mentioned in the construction greater than both every $\gamma(y)[s+1]$ for $y < z$ and every $v(W_{i'}; y)[s+1]$ for $y = \langle i', j' \rangle \leq z$. Go immediately to stage $s + 2$.

If $x = s + 1$ then let $\gamma(x) \downarrow [s+1]$ equal to the least number never yet mentioned in the construction greater than both every $\gamma(y)[s+1]$ for $y < z$ and every $v(W_{i'}; y)[s+1]$ for $y = \langle i', j' \rangle \leq x$. Go to stage $s + 2$.

This completes the construction.

Verification.

Note that for every x , $\gamma^A(x)[s]$ is nondecreasing in the stage. For all $x \in \omega$ we let $\gamma(x) = \lim_s \gamma(x)[s]$. By construction, $\gamma^A(x)[s+1] \neq \gamma^A(x)[s]$ if and only if there is some $z < x$ such that $\gamma^A(z)[s] \in A[s] - A[s+1]$. By a straightforward induction on x , then, $\gamma(x)[s]$ can only change value a finite number of times. Hence $\gamma^A(x) \downarrow$.

Let $x = \langle i, j \rangle$. Choose a stage s_0 such that $\gamma(y) = \lim_s \gamma(y)[s] = \gamma(y)[s_0]$ for all $y \leq x$ and $s > s_0$, and $W_i \upharpoonright_{\gamma(x)[s_0]} = W_i[s_0] \upharpoonright_{\gamma(x)[s_0]}$. Without loss of generality, we may assume that $\phi_j(W_i; x)[s] > \gamma(x)[s_0]$ for every $s > s_0$. If $x \in W_j^{W_i}$, then there is a stage $s_1 > s_0$ such that $x \in W_j^{W_i}[t]$ for all $t \geq s_1$. By construction, $x \in V^{W_i}[s_1 + 1]$ and, hence, since $v(W_i; x)[s_1 + 1] = \phi(W_i; x)[s_1]$, we must have $x \in V^{W_i} - \overline{W_j^{W_i}}$. If $x \notin W_j^{W_i}$, then since each value of $v(W_i; x)[s] > \phi_j(W_i)[s]$, $x \notin V^{W_i}$, hence $x \in \overline{W_j^{W_i}} - V^{W_i}$. Hence N_x is satisfied.

We now show $\Gamma^A = V^A$. Assume as inductive hypothesis that for all $y < x$ there is a stage t_y such that for all $t > t_y$, $V^A(y)[t] = V^A(y) = \Gamma^A(y) \downarrow [t] =$

$\Gamma^A(y)$, $A[s] \upharpoonright_{\gamma(y)[t]+1} = A[t_y] \upharpoonright_{\gamma(y[t_y]+1}$. Note that $\gamma(y)[t] = \lim_s \gamma(y)[s]$.

Choose the least stage $t_x \geq t_{x-1}$ such that $A[t_x] \upharpoonright_{\gamma(x-1)[t_x]+1} = A \upharpoonright_{\gamma(x-1)[t_x]+1}$ and $\gamma(x)[t_x] \downarrow \in A[t_x]$. Such a stage exists by the inductive hypothesis and the definition of trace values. By choice of t_x we must have for all $t \geq t_x$, $\gamma(x)[t] = \gamma(x)[t_x]$. There are two possibilities.

First, suppose there is a stage $s+1 > t_x$ such that $A[s] \upharpoonright_{v(W_i;x)[t]} = W_i[t] \upharpoonright_{v(W_i;x)[t]}$ for some $t \leq s$. Then $\gamma(x)[s]$ is removed from A at stage $s+1$ and $\Gamma^A(x) \downarrow [s+1] = V^A(x)[s+1]$. By construction, and inductive hypothesis, $\gamma^A(x)$ must attain its final value at stage $s+1$, and $\Gamma^A(x)[s+1] = \Gamma^A(x)$. Because any axiom enumerated in V for x after stage $s+1$ has length at least $\gamma(x)[s+1]$, we have for all $s' > s$, that $V^A(x)[s'] = V^A(x)[s+1]$. Because any axiom enumerated in V for x after stage $s+1$ has length at least $\gamma(x)[s+1]$, and $\gamma(x)[s+1] \in W_i$, A disagrees with W_i on all later axioms in V for x . Also for every $y > x$, $\gamma(y)[s+1]$ is reset larger than any $v^{W_i}(x)[t]$ for $t \leq s+1$, so that no change on such a trace can cause A to agree with W_i on one of these previous uses. Hence, if $x \notin V^A[s+1]$, then $x \notin V^A[t]$ for all stages $t > s$. If $x \in V^A[s+1]$, then since all $\gamma(y)[s+1]$ are reset larger than $v^A(x)[s+1]$ if this value is defined, we must have $V^A(x)[s+1] = V^A(x)$. Therefore $V^A(x) = \Gamma^A(x) \downarrow$ as required, and this value can never change after stage s .

Otherwise, suppose there are no stages $s \geq t_x$ and $t \leq s$ such that $A[s] \upharpoonright_{v(W_i;x)[s+1]} = W_i[s+1] \upharpoonright_{v(W_i;x)[s+1]}$. Then $\gamma(x)[t_x] \in A$ and $A[s]$ never agrees with any axiom in V for x at any $s > t_x$. Then $\Gamma^A(x) = 0 = V^A(x)$ as required. This establishes the result. \square

It is not hard to extend this result to higher levels in the difference hierarchy.

Theorem 5. *For every $n > 0$, there exists a pseudojump operator V and a co- n -c.e. set A such that $V^A \leq_T A$ but $V^X \not\leq_T X$ for all n -c.e. sets X .*

Sketch of Proof. We sketch the result for $n = 2$, and then make brief comments indicating the proof of the general case.

Fix an $n > 0$. Given an enumeration of the n -c.e. sets, X_i , $i \in \omega$, we must construct a co- n -c.e. set A to meet the following requirement for all $i, j \in \omega$:

$$N_{\langle i, j \rangle} : V^{X_i} \neq \overline{W_j^{X_i}}.$$

The strategy for requirement $N_{\langle i, j \rangle}$ is exactly the same as that in the proof of Theorem 4 above. The previous strategy for building the reduction $\Gamma^A = V^A$, however, faces the following problem: Let $x = \langle i, j \rangle$. Once x enters A , $\gamma^A(x)[s]$ is removed from A , and we move all traces for $y > x$ to avoid any further interference. If this causes $x \notin V^A[s+1]$, we reset $\Gamma^A(x)[s+1] = 0$. Since A must actually be co-2-c.e., however, x may return to V^A at some $t > s$ because X_i returns to a previous state and $x \in W_j^{X_i}[t]$. If this happens, however, we cannot correct $\Gamma^A(x)$ by merely restoring $\gamma^A(x) \in A[t+1]$, since this in general will restore $x \in V^A[t+1]$ and $\gamma^A(x)[t+1] = \Gamma^A(x)[s] = 0$. Notice that increasing the use $\gamma(x)$ at $s+1$ would not solve this problem, since we might then have to go through the same sequence over and over again at later stages, inasmuch as x can enter and leave V^{W_i} infinitely often. This will result in $\gamma^A(x) \uparrow$ in the

limit. Hence, we must ensure that A disagrees with W_i permanently after some finite sequence of changes on $V^{W_i}(x)$, and we can only achieve this by keeping our traces bounded. We can do this most simply by using two traces for x , $\gamma^A(x, 1) < \gamma^A(x, 2)$, using larger trace to deal with the first change in $V^{W_i}(x)$ and the smaller one to deal with the second change. More precisely, we begin at stage 0 with $\gamma^A(1, x)[0] \notin A$ and $\gamma^A(2, x)[0] \in A$. If, at some $s > 0$, A agrees with X_i on the use of some computation $x \in V^{W_i}[t]$, then we extract $\gamma^A(2, x)[s]$ from A to correctly define $\Gamma^A(x) \downarrow [s+1] = V^A(x)[s+1]$. As before, we want the use $v^A(x)[s]$ to be greater than $\gamma^A(2, x)[s]$ in the hope of winning a permanent disagreement between A and all future states of W_i at which $x \in V^{X_i}$. Now X_i can return to a previous configuration on the initial segment $X_i \upharpoonright_{\gamma^A(2, x)[s]}$ via $\gamma^A(2, x)[s]$ leaving X_i . As discussed above, if x later enters $W_j^{X_i}[t]$ at some stage $t > s$ with use $\phi_j(X_i; x)[t] > \phi_j(X_i; x)[s]$, it is not enough to simply enumerate $\gamma^A(2, x)[s]$ back into A because some other axiom for x in V may apply to A causing $V^A(x)[t+1] = 1$ yet $\Gamma^A(x) \downarrow [t+1] = 0$ whether $\gamma^A(2, x)[t]$ is in or out of A . In this case we can use the second trace, $\gamma^A(1, x)[s]$, to correct the definition of $\Gamma^A(x)$. Therefore we enumerate both $\gamma^A(1, x)[s]$ and $\gamma^A(2, x)[s]$ into A , and since $\gamma^A(1, x)[s]$ is smaller than $\gamma^A(2, x)[s]$ we can rectify $\Gamma^A(x)$. More importantly, we will now have a disagreement between A and all future axioms for x in V via $\gamma^A(2, x)[s] \in A - X_i$ since X_i is a 2-c.e. set. If some higher priority requirement performs some trace activity after we have begun some activity for x , then this will result in permanent A -changes below $\gamma^A(x, 1)$, so that we can simply throw away any work done for defining $\Gamma^A(x)$ and begin again with a new sequence of traces for x . Because we will only need to change the value of traces finitely often, we can succeed in building Γ^A in the presence of the requirements $N_{(i,j)}$.

Notice that A is of the form $W \cup A^*$, where W is a c.e. set consisting of the traces $\gamma^A(1, x)[s]$ used to correct $\Gamma^A(x)$ for the last time, and A^* is a co-2-c.e. set consisting of traces $\gamma^A(2, x)[s]$ used to create a permanent disagreement between A and X_i on all axioms in V for x after a certain stage.

The only difference between the proof of the full result and that of the case $n = 2$ is the need for n traces for each x . We use $\gamma^A(x, n)[s]$ to guarantee a disagreement between A and X_i on all axioms for x in V that are enumerated after X_i changes for the last time on $\gamma^A(x, n)$, and $\gamma^A(x, 1), \dots, \gamma^A(x, n-1)$ to ensure that we always have the freedom to correct $\Gamma^A(x)$. In this case A will be some $W \cup A^*$ where W is a c.e. set consisting of traces $\gamma^A(1, x)[s]$ used to correct $\Gamma^A(x)$ for the last time, and A^* is a co- n -c.e. set consisting of traces $\gamma^A(2, x)[s], \dots, \gamma^A(n, x)[s]$. The details are straightforward, so we leave them to the interested reader. □

4 Cone avoidance and pseudojump completion

Theorem 6. *There exist a non-computable, computably enumerable set C and a pseudojump operator V such that*

- (1) for every $e \in \omega$, $W_e <_T W_e \oplus V^{W_e}$, and
(2) for every $e \in \omega$, if $W_e \oplus V^{W_e} \equiv_T K$, then $C \leq_T W_e$.

The proof consists of a $\mathbf{0}'''$ -priority argument using a tree of strategies. We construct an auxiliary computably enumerable set B to approximate for each computably enumerable W whether or not $K \leq_T W \oplus V^W$. Since B will be computably enumerable, $B \leq_T K$. (In fact, $B \equiv_T K$.)

We have to satisfy the following three types of requirements

$$N_{\Phi, \Psi, k}: (\Phi(W_k \oplus V^{W_k}) = B \ \& \ \Psi(K) = V^{W_k}) \implies C \leq_T W_k,$$

$$P_j: C \neq \overline{W_j}, \text{ and}$$

$$R_{i,l}: V^{W_i} \neq \overline{W_l^{W_i}}.$$

A construction satisfying all these requirements is given in Section 5. The central technique of the proof is the ensuring of cooperation between a strategy σ to which some requirement P_j is assigned and a higher priority strategy τ to which some requirement $N_{\Phi, \Psi, k}$ is assigned. Because of the complexity of the full construction, we discuss first a basic module which ensures cooperation between just one pair of strategies. The actual construction, however, incorporates several technical devices to overcome the various obstacles which arise in this simplest case, as well as in the coordination of strategies for many requirements. Because of this, after our first informal sketch of the key idea, we proceed to describe in some detail the problems which arise in implementing it. In this way the technicalities involved in the full construction can be motivated before they arise.

4.1 Basic strategies for the requirements

The basic strategy for satisfying a requirement $P = P_j$ is the familiar diagonalization strategy: a witness c is assigned to P at some stage s_0 , large enough so that $c \notin C[s_0]$. If at some $s > s_0$, c enters $W_j[s]$, then we add c to $C[s+1]$, thereby ensuring that either $W_j \cap C \neq \emptyset$, or $\overline{W_j} \cap \overline{C} \neq \emptyset$ (if s never appears).

The strategy for satisfying a requirement $R = R_{i,l}$ is a relativized version of this basic diagonalization strategy: a witness x is assigned to R at some stage s_0 , large enough so that $x \notin V^{W_i}[s_0]$. If at some $s > s_0$, x enters $W_l^{W_i}[s]$, then we add x to $V^{W_i}[s+1]$, by enumerating the axiom $\langle x, W_i \upharpoonright_{\phi_l(W_i; x)[s]} \rangle$ into $V[s+1]$. It is straightforward to check that this strategy satisfies the requirement R essentially as in the unrelativized case for P , although if $x \notin W_l^{W_i}$, R may require attention infinitely often. While this introduces problems, we refrain from discussing in more detail the interaction with these kinds of requirements for a while in the sequel. For the purposes of intuition, it suffices for the moment merely to remember that these requirements force us to add axioms of the form $\langle x, \mathcal{W} \rangle$ to V at various stages in the construction.

In its crudest form, the strategy for the requirement $N = N_{\Phi, \Psi, k}$ is relatively straightforward. Suppose we have some way to approximate whether or not

$\Phi(W_k \oplus V^{W_k}) = B \& \Psi(K) = V^{W_k}$, so that if this condition holds, it will appear more and more likely at infinitely many stages s , as measured by the increase in some length-of-agreement function $l^N[s]$. We satisfy the requirement by constructing a functional $\Gamma = \Gamma^N$, extending our construction at each such stage s . For every $c < s$ such that $\Gamma(W_k; c) \uparrow [s]$, we set $\Gamma(W_k; c)[s] = C(c)[s]$ with use $\gamma(W_k; c)$ equal to the amount of W_k used in checking that the l^N has increased at s . We then restrain C from ever changing on any $c < s$ until W_k changes on $\gamma(W_k; c)[s]$. As long as our approximation has the property that there are infinitely many stages s at which W_k is stable on these $\gamma(W_k; c)[s]$, this procedure will succeed in satisfying N .

These two strategies clash very badly in these crude forms. After all, there is in general nothing to keep N , when it has higher priority, from imposing infinite restraint on requirement P , keeping us from ever enumerating any c into $W_j \cap C$. The key allowing escape from these restraints is the fact that we are in control of B as well as C , giving us the potential of forcing W_k to change on $\gamma(W_k; c)[s]$ when c needs to enter C by enumerating a relatively small number into B and hence changing the approximation to B given by $\Phi(W_k \oplus V^{W_k})$ at s .

More precisely, we link the two strategies together as follows: when we choose some c at stage s for the purpose of satisfying P_j , we simultaneously choose an element $b \notin B[s]$ which is greater than the current length of agreement for N . If a stage s' arrives such that $l^N[s'] > b$, we set $\Gamma(W_k; c)[s'] = 0$ with use $\gamma(W_k; c)[s'] = s'$, and we restrain V below the use $\phi[s'] = \phi(W_k \oplus V^{W_k}; b)[s']$. If at some later stage $s'' > s'$, c enters W_j , we then *attack* with b by enumerating $b \in B[s'']$, while continuing to restrain V below the old use $\phi[s']$. Notice that this means that no axiom $\langle x, \mathcal{W} \rangle$, with $x < \phi[s']$ can be enumerated into V after s' . Therefore, by the usual convention that the stage number s' bounds all the computations existing at s' , any axiom $\langle x, \mathcal{W} \rangle$ has $|\mathcal{W}| < s'$. Hence, if $x < \phi[s']$, x can neither enter nor leave V^{W_k} after s' , without W_k changing below $\gamma(W_k; b)$, and, of course, $\phi[s'] < s'$ in any case. Therefore, at any stage $t > s'$, either

- (a) $(W_k \oplus V^{W_k})[t] \upharpoonright_{\phi[s']} = (W_k \oplus V^{W_k})[s'] \upharpoonright_{\phi[s']}$, or
- (b) $W_k[t] \upharpoonright_{\gamma(W_k; b)[s']} \neq W_k[s'] \upharpoonright_{\gamma(W_k; b)[s']}$.

Because $\Phi(W_k \oplus V^{W_k}; b)[s'] = 0 \neq 1 = B(b)$, as long as (a) remains true $\Phi(W_k \oplus V^{W_k}) \neq B$, so that N is satisfied finitarily through diagonalization. In this case we can play another strategy for P , which merely has to respect the finite restraint involved, and so is guaranteed to win. On the other hand, once (a) fails to hold at some t , (b) becomes true, so that $\gamma(W_k; c) \uparrow [t]$, and c can then be freely added to $C[t]$ and $\Gamma(W_k; c)$ can be corrected permanently.

Of course, W_k may change below the original use s' of γ at some $t > s'$ while $c \notin W_j[t]$. However, as long as such a change does not disturb the computation $\Phi(W_k \oplus V^{W_k}; b)[s']$, we can continue to reset γ with the same value, and win via the same linked strategy. Hence, as long as our method of approximation is good enough to eventually become stable, we can define $\Gamma(W_k; c)$ permanently.

4.2 A technical obstacle

When sketched in such a broad fashion, the basic strategy seems relatively simple. Its implementation, however, faces a series of technical obstacles, the first of which arises in defining the approximation to the truth of the condition $\Phi(W_k \oplus V^{W_k}) = B \ \& \ \Psi(K) = V^{W_k}$. For while this approximation can only be true if V^{W_k} is a Δ_2^0 set, the representation of V which we have available to us when approximating the condition is essentially a Σ_2^0 one. Most immediately, this seems to leave open the disturbing possibility that for every $b \in \omega$, $\Phi(W_k \oplus V^{W_k}; b) = B(b)$, but at infinitely many stages s , $\Phi(W_k \oplus V^{W_k}; b)$ does not converge, since some $y < \phi(W_k \oplus V^{W_k}; b)$ is an element of $V^{W_k}[s]$ at every sufficiently large stage s , but fails to be in V^{W_k} . A natural solution to this problem, is to use the Lachlan-Soare “hat trick” method of true stages. This replaces the ordinary approximation $V^{W_k}[s]$ with a modified approximation \widehat{V}^{W_k} consisting of only those elements of V^{W_k} with axioms of length less than $w_k[s]$, the least element recently enumerated into W_k . This ensures that infinitely often, at so called W_k -true stages, longer and longer substrings of our approximation to V^{W_k} actually agree with V^{W_k} . This will ensure in turn that any true computation $\Phi(W_k \oplus V^{W_k}; b)$ will appear at every sufficiently large W_k -true stage.

Unfortunately, this use of the hat trick complicates our basic strategy. Suppose we believe that $\Phi(W_k \oplus \widehat{V}^{W_k}; b)[s] \downarrow = 0$. It may be that some element $x < \phi(W_k \oplus \widehat{V}^{W_k}; b)[s]$ is actually in $V^{W_k}[s]$ by some axiom $\langle x, \mathcal{W} \rangle \in V[s]$ with $W_k \upharpoonright_{|\mathcal{W}|} = \mathcal{W}$. If $|\mathcal{W}| > w_k[s]$, then $x \notin \widehat{V}^{W_k}[s]$, so that the computation $\Phi(W_k \oplus \widehat{V}^{W_k}; b)[s]$ will change without any later change occurring in W_k below s . This will defeat our purpose in setting $\gamma(W_k; c) = s$. The natural solution is to restrain V below the use $\phi(W_k \oplus \widehat{V}^{W_k}; b)[s]$ and wait for the next N-expansionary stage s' . Since we restrain any elements below this use from entering V after stage s , we only need worry about elements that entered at stage s itself, or before stage s . If $w_k[s'] < |\mathcal{W}|$, for some such $\langle x, \mathcal{W} \rangle \in V[s]$, then $x \notin V^{W_k}$ if $x \notin \widehat{V}^{W_k}$, since $w_k[s']$ injures its axiom. If, on the other hand, some new $x \in \widehat{V}^{W_k}$, $x \in V^{W_k}$, and we restrain again and wait for the next expansionary stage s'' , since the situation at s no longer looks good. Once we get stability at successive stages s and s' , we can set $\gamma(W_k; c)[s'] = s'$ and proceed with our strategy as before.

Because the available approximation to V^{W_k} is not Δ_2^0 , we are clearly in danger of introducing infinite restraint again at this point in the construction, simply because we may always have a change in V^{W_k} below the use at s before the stage s' appears. If we merely drop all restraint at such a stage, we will face almost the same difficulty as before, since $\Phi(W_k \oplus V^{W_k}; b)$ might actually converge, while appearing not to at infinitely many stages. It is for this reason that we require the second condition, $\Psi(K) = V^{W_k}$, in the condition for requirement N, since this gives a Δ_2^0 representation of V^{W_k} . Thus, if we need to satisfy requirement N, we will eventually be working with $\Phi(W_k \oplus \Psi(K); b)$, which will be well-behaved in just the way that $\Phi(W_k \oplus V^{W_k}; b)$ need not be.

We can now give a more detailed description of our basic module. Suppose there are infinitely many N-expansionary stages. (Otherwise, we eventually stop acting for requirement N.) We define $\Gamma(W_k; c)$ in steps as follows:

Step 1. Choose $b \notin B$ and $c \notin C$ at stage s_{-1} .

Step 2. Wait for a stage s_0 such that $b < l^N[s_0]$. Impose restraint at $s_0 + 1$ on V below $\phi(W_k \oplus \widehat{V}^{W_k}; b)[s_0]$.

Step 3. At the next expansionary stage $s_1 > s_0$, if both

$$\begin{aligned} \Psi(K)[s_1] \upharpoonright_{\phi(W_k \oplus \widehat{V}^{W_k}; b)[s_0]} &= \widehat{V}^{W_k}[s_0] \upharpoonright_{\phi(W_k \oplus \widehat{V}^{W_k}; b)[s_0]}, \text{ and} \\ W_k[s_1] \upharpoonright_{\phi(W_k \oplus \widehat{V}^{W_k}; b)[s_0]} &= W_k[s_0] \upharpoonright_{\phi(W_k \oplus \widehat{V}^{W_k}; b)[s_0]}, \end{aligned}$$

then set $\gamma(W_k; c)[s_1 + 1] = s_1 + 1$. Otherwise, return to step 2.

Step 4. At each N-expansionary stage $s_2 > s_1$, if $\gamma(W_k; c) \uparrow[s_2]$ and either

- (a) K has changed below $\max\{\psi(K; y)[s_1] : y < \phi(W_k \oplus \widehat{V}^{W_k}; b)[s_1]\}$,
or
- (b) W_k has changed below $\phi(W_k \oplus \widehat{V}^{W_k}; b)[s_1]$,

then drop all restraint and return to step 2. Otherwise, s_1 still looks good at s_2 . If $\gamma(W_k; c) \uparrow[s_2]$, set $\gamma(W_k; c)[s_2 + 1] = s_1 + 1$, if $c \notin C[s_2 + 1]$; and set $\gamma(W_k; c)[s_2 + 1] = 0$, if $c \in C[s_2 + 1]$.

As pointed out above, if $\Phi(W_k \oplus V^{W_k}) = B$ and $\Psi(K) = V^{W_k}$, this process must eventually terminate, since we never again return to Step 2 after some stage at which W_k and K have stabilized on the total use involved in the computation $\Phi(W_k \oplus \Psi(K); b)$. It is straightforward to verify that this defines γ in such a way that the basic strategy for linking requirements can still work: an attack with b at some stage $s_2^- > s_1$ will ensure $\gamma(W_k; c) \uparrow[s_2]$ at the next N-expansionary stage s_2 , permitting c to be added to $C[s_2]$. We have essentially, then, three outcomes for the basic strategy: If we eventually define some use $\gamma(c)$, then either $c \in C$ because of a permanent win on requirement P_j , or $c \notin C$ because $c \notin W_j$. Both of these outcomes impose some finite restraint on lower priority requirements. If, on the other hand, the use tied to b is unstable, this causes infinite restraint, which drops back to 0 infinitely often, because b witnesses that $N_{\Phi, \Psi, k}$ is satisfied. This is the typical situation in an $\mathbf{0}'''$ -priority construction, with the higher priority requirement won by infinitary action at the lower.

4.3 The priority arrangement

We next describe how we intend to organize the action of strategies. We use the familiar tree-of-strategies technique for organizing our construction. In the discussion that follows, we assume familiarity with the $\mathbf{0}'''$ -priority method using this technique. We use the notation of Soare, [6]. We face two main problems

here. The first problem involves the mechanism used to impose restraint for the many type-P strategies below one type-N strategy; the second problem involves coordinating the activity of many type-N strategies above a given type-P strategy.

A strategy τ for some higher priority requirement $N_{\Phi, \Psi, k}$ has in general a great number of elements b assigned to lower priority requirements below it which are waiting for an appropriate time to initiate an attack. Whenever some such b causes a return to Step 2 in the procedure for defining some $\gamma(W_k; c)$, this action should immediately introduce a new restraint on all requirements below the strategy σ for P_j to which b and c are assigned. However, we have no reason to think that the particular lower priority strategy σ to which b and c are assigned will act at this stage, since its activity depends on the state of many intermediate strategies. If we allow σ to act whenever τ would like it to, we will injure all these intermediate strategies. Because there are in general infinitely many such b and c , we cannot afford to do this without infinitely injuring all strategies below τ .² We solve this problem by using a *proxy* for σ at any such stage s . Notice that if the approximation to the true path $f_s <_L \sigma$, σ will be initialized at s , so we need not consider this case. On the other hand, if $\sigma <_L f_s$, then some ξ with $b^\sigma < b^\xi$ acts at s , and, since the use tied to b^σ appears bad, the use tied to b^ξ appears bad as well. Thus we can let the least such ξ stand in for σ , giving it a τ -infinitary outcome at this stage and tying the use of both c^σ and c^ξ to b^ξ , since the τ - ξ strategy has been protected at this stage.

The immediate problem with this procedure is that it threatens to make our functional Γ undefined in the long run, since as the approximation to the true path moves right, we tie $\gamma(c^\sigma)$ to greater and greater b^ξ s. Clearly, when the path branches back to the left, we must give up the current σ -proxy and choose a new one. In this way, we will eventually tie $\gamma(c^\sigma)$ to some fixed b , namely b^ξ , where ξ is the least type-P strategy which must respect τ such that $\sigma \leq \xi \subseteq f$. The obstacle to merely redefining the σ -proxy whenever the path moves left, is that $\gamma(c^\sigma)$ may look good at the τ -expansionary stage where this happens, so that we have no justification for changing the use. In other words, we may have the following situation: some original use for $\gamma(c^\sigma)$ is tied to b^σ at a σ -stage s_0 . At a later stage, s_1 , this use looks bad, so we tie $\gamma(c^\sigma)$ to some b^{ξ_0} and make ξ_0 the σ -proxy. Now at stage $s_2 > s_1$, $f_{s_2} <_L \xi_0$, causing b^{ξ_0} to become undefined. There will be some appropriate $\xi \subseteq f_{s_2}$, but, if $\gamma(c^\sigma)$ is not undefined at s_2 , then we cannot reassign its value to ξ , and, even if we did, we have no reason to think that ξ itself has permission from τ to set new restraints at this stage. Notice, however, that because $\sigma \leq \xi <_L \xi_0$, $\gamma(c^\xi)$ must also have looked bad at stage s_1 , hence, and ξ_0 must have become the ξ -proxy then as well. Because ξ_0 is initialized at s_2 , we now have b^{ξ_0} available to us to use in any manner we choose. We keep ξ from acting immediately at stage s_2 , and instead enumerate b^{ξ_0} into B and set a link from ξ to τ , performing an attack for the sake of correcting our use on ξ and σ . At the next τ -expansionary stage,

²It is possible to approach the proof in this way, using a complex technique involving “*toplinking*” and “*scouting reports*”, as in the density theorem of Downey-Lempp, [2], but this results in an even more difficult construction.

$\gamma(c^\sigma)$ and $\gamma(c^\xi)$ must diverge, and we can reset the σ -proxy to be ξ and allow a τ -infinitary outcome at ξ , setting restraints to protect both strategies.

Our second problem arises from the fact that we are attempting to diagonalize against every computably enumerable set. Because of this, a given type-P strategy can have in general many different infinitary outcomes, each of which depends on a different use associated to a different type-N strategy being eventually unstable. The fact that we have no control over the order in which these instabilities may occur is what causes a problem here. In fact, it is this that is the most significant obstacle to the construction. Suppose σ is a type-P strategy and τ_0 and τ_1 are two type-N strategies which σ must respect. In other words, σ believes it must define both γ^{τ_0} and γ^{τ_1} . There are four possible outcomes for the σ strategy: the two that impose finite restraint, and an infinitary outcome for each of the type-N strategies. Suppose τ_0 has higher priority than τ_1 , and let b_{τ_0} and b_{τ_1} be the attackers to which the uses for $\gamma^{\tau_0}(c^\sigma)$ and $\gamma^{\tau_1}(c^\sigma)$ are tied. Since τ_0 has higher priority, we must initialize τ_1 to set a higher restraint whenever we get a change in the use tied to b_{τ_0} . This involves picking a new b_{τ_1} and a new c^σ , injuring the σ -strategy. Below this τ_0 -infinitary outcome, we no longer have to respect τ_0 's requirement, so that we have freedom to try again using a new σ' that only respects τ_1 's requirement. But σ' cannot attempt to coordinate its strategy with τ_1 itself, since σ' can only be allowed to act when the τ_0 use tied to σ looks bad. Coordination with τ_1 involves making an attack on τ_1 and waiting for success at the next τ_1 -expansionary stage. Since we have no means to ensure that the τ_0 -use at σ will look bad at such a stage, we would be forced to wait for the next such stage in order to protect τ_0 . (σ' cannot attack with σ 's τ_0 -attacker without introducing infinite injury from below.) But by this stage, the permission from τ_1 will in general have gone away.

We solve this problem by forcing both τ_1 and τ_0 to automatically give permission whenever σ' acts. As in the case of the need to reset the σ -proxy, we do this by introducing an auxiliary attack in order to correct our uses. We associate to the τ_0 -attacker of σ , b_{τ_0} , a pair of τ_0 -correctors, $b(\tau_0, \tau_1)$ and $b(\tau_0, \tau_0)$, with $b_{\tau_0} < b(\tau_0, \tau_0) < l^{\tau_0}$. We set the uses γ^{τ_1} and γ^{τ_0} for σ' using these correctors. When the use tied to τ_0 looks bad, and we wish to allow σ' to act, we first attack τ_1 by enumerating $b(\tau_0, \tau_1)$ into B . At the next τ_1 -expansionary stage, we attack τ_0 by enumerating $b(\tau_0, \tau_0)$ into B , and linking over τ_1 . At the next τ_0 -expansionary stage, σ' is free to act as if neither τ_0 nor τ_1 existed. This involves using up the two correctors, so that new ones have to be chosen at the next σ -stage. This procedure only happens when the use tied to the τ_0 -attacker at σ changes, and this attacker is itself not given up in the process of correcting for σ' . Thus, the infinitary outcome of σ is correct in the sense that τ_0 's requirement need never be reassigned below σ , because instability in the use tied to the attacker b_{τ_0} witnesses its satisfaction. Of course, we must reassign τ_1 's requirement to some τ_1' below this τ_0 -infinitary outcome of σ , since this procedure does injure τ_1 by "artificially" increasing all of τ_1 's uses to protect it from τ_0 and σ' .

Notice that τ_1 does not need a pair of correctors at σ , since an attack on τ_0 by any node turns τ_1 off for the duration of the attack. However, because

we have to introduce correctors to perform the auxiliary attacks anyway, we also use these correctors in the construction below when we need to reset the σ -proxy, rather than keeping track of what attacker was used to set the use at a lower priority proxy. This means that even when τ is the lowest priority requirement which σ must respect, we introduce an auxiliary corrector $b(\tau, \tau)$ solely for this purpose. Since $b(\tau, \tau)$ will be less than any attacker or corrector for lower priority strategies, this will work in a natural way.

There is one slight technical difficulty with our correcting strategy. It is not clear what action we should take when the use tied to b_{τ_0} looks good, but the use tied to the associated corrector $b(\tau_0, \tau_0)$ looks bad. The \mathbf{O}''' -method is based on the fact that σ' below the τ_0 -infinitary outcome at σ *does not* explicitly respect τ_0 , and hence does not have a τ_0 -infinitary outcome. Since we can only let σ' act when the use tied to b_{τ_0} looks bad, we would be prevented from using such an outcome to set restraints for protecting the use tied to $b(\tau_0, \tau_0)$ in any case. What we are forced to do to get around this problem is to use the next greatest node which does have a τ_0 -infinitary outcome as another kind of proxy for σ' . In the case we have described, this will be the next type-P strategy below the τ_1 -infinitary outcome at σ . This strategy has a τ_0 -attacker which is greater than $b(\tau_0, \tau_0)$, and hence has a τ_0 -use which looks bad whenever the use tied to $b(\tau_0, \tau_0)$ looks bad. Its τ_0 -infinitary action will therefore set restraints which are sufficient until the τ_0 -infinitary action at σ occurs again, setting an even better restraint. Of course, in order to allow this strategy to act, the coordinated σ - τ_1 strategy must be injured, and we must attack with $b(\tau_1, \tau_1)$, giving a false τ_1 -infinitary outcome at σ . But when this occurs, we get a true τ_0 -infinitary outcome just below this, allowing us to reassign τ_1 's requirement. If this happens infinitely often, this new version of τ_1 will succeed in satisfying the requirement. Intuitively this procedure makes sense because the coordinated σ - τ_0 strategy, and hence σ' which depends on it, has higher priority than the coordinated σ - τ_1 strategy which is injured each time we perform this procedure. If this procedure takes place infinitely often at σ , then this actually injures τ_1 itself, but it gives a τ_0 -infinitary outcome on the true path and therefore enables τ_1 's requirement to be satisfied.

4.4 Interference from the nontriviality requirements

We have so far avoided discussing in detail an important aspect of our construction, namely the effect which a strategy for some requirement $R_{i,l}$ (that is, $V^{W_i} \neq \overline{W_l^{W_i}}$) has on the coordinated strategy for requirements $N_{\Phi, \Psi, k}$ and P_j . Recall that the strategy for $R_{i,l}$ involves enumerating some number $x^{R_{i,l}}$ into V^{W_i} , possibly at infinitely many stages. Whenever some new axiom is enumerated into V for the sake of enumerating $x^{R_{i,l}}$ into V^{W_i} , we run the risk of unintentionally enumerating $x^{R_{i,l}}$ into many other relatively computably enumerable sets V^W , without any change in the oracle W . Enumeration of this kind into V^{W_k} directly injures the $N_{\Phi, \Psi, k}$ strategy which seeks to define $\Gamma(W_k)$ by means of $\Phi(W_k \oplus V^{W_k})$. When the strategy for $R_{i,l}$ affects V^{W_k} in this way

only finitely often, there is essentially no problem; it is dealing with the infinite injury that can occur when W_k and W_i turn out to be the same set which causes problems in the construction.

There are three possibilities, depending on the relative priorities involved. If $R_{i,l}$ has higher priority than $N_{\Phi,\Psi,k}$, then the strategy for $N_{\Phi,\Psi,k}$ can merely approximate the eventual status of $x^{R_{i,l}}$ in the usual way for $\mathbf{0}''$ -priority arguments, initializing all lower-priority strategies guessing that $x^{R_{i,l}} \in V^{W_k}$. If $R_{i,l}$ has lower priority than P_j , then the coordinated strategy for P_j and $N_{\Phi,\Psi,k}$ is explicitly designed to force the strategies for $R_{i,l}$ to respect the use of $\Phi(W_k \oplus V^{W_k}; b)$, when b is the current $N_{\Phi,\Psi,k}$ -attacker for P_j . There is a slight technical problem here arising from our procedure of using proxies. The $R_{i,l}$ -strategy can have lower priority than the coordinated strategy for P_j and $N_{\Phi,\Psi,k}$, yet fail to be initialized after a change in the relevant use because it does not lie below any infinitary outcome for a strategy which can serve as proxy for this coordinated strategy. In this case, we must restrain $R_{i,l}$ by preventing it from acting with its current witness if that witness is below the relevant use. When this happens, we force the $R_{i,l}$ strategy to choose a new witness, although we do not otherwise initialize it. This may injure this strategy infinitely often, but only through infinitary activity arising from a higher priority coordinated strategy. Thus, if this strategy for $R_{i,l}$ lies on the true path, we are assured that some infinitary outcome for a higher priority strategy lies below it on the true path, and hence its requirement can be satisfied. This situation is very much like the situation of the false τ_1 -infinitary outcome caused by the corrector $b(\tau_0, \tau_0)$ described at the end of the last section.

When the $R_{i,l}$ -strategy lies between the the $N_{\Phi,\Psi,k}$ -strategy and some P_j -strategy which must respect $N_{\Phi,\Psi,k}$, however, a kind of injury can occur which is more difficult to deal with. Suppose τ is some $N_{\Phi,\Psi,k}$ -strategy, σ some P_j -strategy, and α some $R_{i,l}$ -strategy such that $\tau \subseteq \alpha \subseteq \sigma$. Recall that the problem occurs when $x^\alpha < \phi(b_\tau^\sigma)$ at stage s . This means that a σ -attack changing B 's value on b_τ^σ can be affected when x^α enters V^{W_k} . If this entry occurs because of the appearance in W_k of a number greater than $\gamma_\tau(c^\sigma)$, the attack will fail, and this is exactly what may happen if α has acted at any stage after $\gamma(c^\sigma)$ was last set.

We avoid this problem in the natural way by introducing a pair of outcomes 0 and 1 at α , with 0 indicating that $x^\alpha \notin V^{W_i}$ and 1 indicating that $x^\alpha \in V^{W_i}$. This by itself, however, is not enough. The problem is that there is in general no relationship between $V^{W_i}(x^\alpha)$, which determines α 's ultimate outcome, and $V^{W_k}(x^\alpha)$, on which the success of the linked σ - τ strategy may depend. Thus, even if σ only acts when $x^\alpha \notin V^{W_i}$, W_k can come to resemble an *old* version of W_i on some initial segment much longer than the even-older use $\gamma(c^\sigma)$ and thereby allow x^α to enter V^{W_k} without this entry being detectable at σ .

The key to solving this problem is the recognition of the fact that it can only occur infinitely often when both of the sets W_i and W_k are the same, although with different enumerations. This is because α only ever enumerates axioms that agree with W_i on longer and longer apparent initial segments. We can therefore avoid this problem by embedding a further action at α to check whether the

sets W_k and W_i are tending to agree with each other. In fact, however, we can achieve the same result by the device of replacing the ordinary enumeration $\{W_k\}$ indexing the computably enumerable sets by an enumeration $\{W_k^*\}$ of these sets without repetitions. The existence of such an enumeration is an old result due to Friedberg. We can now know in advance whether W_i and W_k are the same set. Since the effect of α on any strategy for a requirement involving a different set W' is guaranteed to be finite, we can therefore initialize every strategy below α whenever such an injury to a coordinated strategy involving a different set occurs. This also makes it possible to more conveniently treat the case $\alpha \subseteq \tau$, since we can initialize τ finitely often for each such α when injury occurs in this way.

5 The full construction

5.1 Preliminary definitions and the priority tree

We use a priority tree T which is isomorphic to a subtree of ${}^{<\omega}\omega$. Using standard coding functions for triples and pairs, as well as standard indexing for computable functionals and a listing without repetitions of the computably enumerable sets, we order the requirements in a priority listing. We assign requirements recursively along each path in T and we simultaneously define T . To achieve this we define a listing function, $L(\rho, k)$, listing, for each $\rho \in T$ the requirements that still need to be satisfied at ρ . The requirement $L(\rho) = L(\rho, 0)$ is assigned to ρ . A natural notational abbreviation is the writing of L^ρ for the functional $\lambda x L(\rho, x)$. We also define $L(\rho) < L(\rho')$ whenever $k < k'$ such that $L(\rho) = L(\emptyset, k)$ and $L(\rho') = L(\emptyset, k')$. We define L by recursion on $\rho \in T$ and $m \in \omega$, after first making some preliminary definitions.

A node is said to be of type N if it has some requirement $N_{\Phi, \Psi, k}$ assigned to it. A node is said to be of type P if it has some requirement P_j assigned to it. A node is said to be of type R if it has some requirement $R_{i,l}$ assigned to it.

Let $\rho \in T$. Suppose τ is a node of type N such that $\tau \frown \langle \infty \rangle \subseteq \rho$. If $\sigma \subseteq \rho$, σ has type P, and $\sigma \frown \langle \tau \rangle \subseteq \rho$, then σ has a τ -infinitary outcome at ρ . A node ρ respects $\tau \frown \langle \infty \rangle \subseteq \rho$ if there do not exist any $\tau_0 \subseteq \tau$ and σ such that σ has a τ_0 -infinitary outcome at ρ .

Nodes τ of type N have outcomes of the form ∞ and 1, where $\infty < 1$.

Nodes α of type R have outcomes of the form $0 < 1$.

Nodes σ of type P have outcomes $\langle \text{win} \rangle$, $\langle \tau \rangle$, and $\langle \text{fin} \rangle$, where τ is a node (of type N) included in σ such that σ respects $\tau \frown \langle \infty \rangle$. We order the outcomes using the inclusion ordering on the nodes τ and the additional rule that $\langle \text{win} \rangle < \langle \tau \rangle < \langle \text{fin} \rangle$ for any τ .

We can now define the function L . Let λ be the empty string.

- For every $m \in \omega$, $L(\lambda, 3m) = N_m$, $L(\lambda, 3m+1) = P_m$, and $L(\lambda, 3m+2) = R_m$.

- If $\beta \neq \lambda$, $\beta = \beta_0 \widehat{\mathcal{O}}$ for some outcome \mathcal{O} , and β_0 has type N, or R, then for every $m \in \omega$, $L(\beta, m) = L(\beta_0, m + 1)$.
- Suppose $\beta \neq \lambda$, $\beta = \beta_0 \widehat{\mathcal{O}}$ for some outcome \mathcal{O} , and β_0 has type P. There are two possibilities:
 - Case 1.** If $\mathcal{O} = \langle \tau \rangle$ for some $\tau \subseteq \beta_0$, then for every $m \in \omega$, $L(\beta, m) = L(\tau, m + 1)$.
 - Case 2.** Otherwise, for every $m \in \omega$, $L(\beta, m) = L(\beta_0, m + 1)$.

As usual, we have an *approximation to the true path* f_s defined at each $s > 0$. For any node $\beta \in T$, s is a β -stage if $\beta \subseteq f_s$; s is an *active* β -stage if β was allowed to act at stage s . If s is an active β -stage, then we use s_{β}^- to denote the last previous β -stage. When β is clear from the context, we merely write s^- for s_{β}^- . Whenever $f_s <_L \beta$, we *initialize* β at s , meaning that we undefine all of β 's parameters and functionals, and start over completely with a new version of β . At stage 0 we initialize all nodes in T . We then take action as follows at each stage $s + 1$.

5.2 A node τ of type N

Suppose τ has requirement $N_{\Phi, \Psi, k}$ assigned to it. Our first task is to make explicit how we intend to approximate the truth of the condition $\Phi(W_k \oplus V^{W_k}) = B \ \& \ \Psi(K) = V^{W_k}$.

For each τ -stage t let

$$w_t^{\tau} = \begin{cases} \mu w (w \in W_k[t] - W_k[t^-]), & \text{if } W_k[t] - W_k[t^-] \neq \emptyset, \text{ and} \\ t, & \text{otherwise.} \end{cases}$$

Let $\widehat{V}_{\tau}^{W_k}[t] = \{x : \exists \langle x, W \rangle \in V[t] (|W| < w_t^{\tau} \wedge W_k[t] \upharpoonright_{|W|=W})\}$. In other words, $\widehat{V}_{\tau}^{W_k}[t]$ consists of just those elements of $V^{W_k}[t]$ with axioms smaller than w_t^{τ} . A stage t is said to be a τ -true stage, if t is a τ -stage and $W_k \upharpoonright_{w_t^{\tau}} = W_k[t] \upharpoonright_{w_t^{\tau}}$. This means that no element $w < w_t^{\tau}$ is ever enumerated into W_k at any stage after t .

Let s be a τ -stage. We define the set $S^{\tau}[s]$ of *apparent τ -true stages at s* to be the set of τ stages $t < s$ such that for all $t' \leq s$, if $t < t'$ and t' is an active τ -stage, then $w_t^{\tau} < w_{t'}^{\tau}$. When a fixed τ is under consideration, we usually write $w_{k,t}$ for w_t^{τ} and \widehat{V}^{W_k} for $\widehat{V}_{\tau}^{W_k}$, and we call τ -true stages W_k -true stages.

At each τ -stage t , we define the τ -length-of-agreement at t , $l^{\tau}[t]$, to be the least x such that for every $y < x$, $\Phi(W_k \oplus \widehat{V}^{W_k}; y)[t] = B(y)[t]$ and for every $z < \phi(W_k \oplus \widehat{V}^{W_k}; y)[t]$, $\widehat{V}^{W_k}(z)[t] = \Psi(K; z)[t]$. We define the *maximum previous τ -length-of-agreement at t* by $m^{\tau}[t] = \max\{l^{\tau}[s] : s < t\}$. A τ -stage t is τ -expansionary whenever $l^{\tau}[t] > m^{\tau}[t]$.

The strategy for satisfying N also depends on keeping track of the stage t at which an eventual computation reaches its final state. Suppose $l^{\tau}[t] > b$. Let

$$\begin{aligned}
\phi(t) &= \phi(W_k \oplus \widehat{V}^{W_k}; b)[t], \\
\phi^+(t) &= \max(\{\phi(t)\} \cup \{v : \exists x < \phi(t) \langle x, W_k[t] \upharpoonright_v \rangle \in V[t]\}), \text{ and} \\
\psi(t) &= \max\{\psi(K; y)[t] : y < \phi(t)\}.
\end{aligned}$$

Then t looks good for b with respect to τ at s if and only if there exists some $t_0 \leq t$ such that

1. $t_0 \in S^\tau[s]$,
2. $(W_k \oplus \widehat{V}^{W_k})[t_0] \upharpoonright_{\phi^+(t_0)} = (W_k \oplus \widehat{V}^{W_k})[t] \upharpoonright_{\phi^+(t_0)} = (W_k \oplus \widehat{V}^{W_k})[s] \upharpoonright_{\phi^+(t_0)}$,
3. $l^\tau[t_0] > b$,
4. for every t' such that $t_0 \leq t' \leq s$, $K[t_0] \upharpoonright_{\psi(t_0)} = K[t] \upharpoonright_{\psi(t_0)} = K[s] \upharpoonright_{\psi(t_0)}$,
5. for every $x < \phi(t)$ and t' such that $t_0 \leq t' \leq s$, if t' is τ -expansionary, then $x \in \widehat{V}^{W_k}[t_0]$ if and only if $x \in V^{W_k}[t']$.

As discussed in 4.2, the reason for condition 5 is to ensure that no $x < \phi(t_0)$ can enter \widehat{V}^{W_k} later when it was actually already in V^{W_k} . This could produce a change in \widehat{V}^{W_k} that would be undetectable by a later W_k -change. If $\Psi(K) = V^{W_k}$, every such x will eventually be counted as in at a true stage t_0 with a use below $w_{t_0}^\tau$. Note that 4 implies $\Psi(K)[t_0] \upharpoonright_{\phi(t_0)} = \Psi(K)[t] \upharpoonright_{\phi(t_0)} = \Psi(K)[s] \upharpoonright_{\phi(t_0)}$. When τ and s are clear from the context, as they often will be, we merely say t looks good for b .

Let s^- be the greatest stage τ -expansionary stage before s . If σ extends $\tau \widehat{\langle \infty \rangle}$, and some witness c^σ is eventually chosen permanently by σ , then τ has the task of eventually defining some $\gamma^\tau(\sigma)$ (to be used to define $\Gamma^\tau(W_k) = C$.) It is because σ 's witness c^σ changes over time to protect lower priority type-N requirements that we define $\gamma^\tau(\sigma)$, rather than $\gamma^\tau(c^\sigma)$. (Since the enumeration of potential witnesses is increasing, this procedure succeeds in ensuring the totality of Γ .) For each $\sigma \in T$, if there exists a greatest stage t such that $s^- \leq t < s$ and $\gamma(\sigma) \downarrow [t]$, then let $\gamma^-(\sigma)[s] = \gamma(\sigma)[t]$ and $t^-(\sigma)[s] = t$.

A node $\sigma \supseteq \tau \widehat{\langle \infty \rangle}$ may have an incorrect use because of the unpredictable activity of some α of type R such that $\alpha < \tau$. Suppose there exists some $\alpha < \tau$ such that $L(\alpha) = R_{k', l'}$, $k \neq k'$, and $V^{W_k}(x^\alpha)[s^-] \neq V^{W_k}(x^\alpha)[s]$. Then we say τ discovers an error at s .

There are four cases to consider. We take the first one that applies.

Case 1. If τ discovers an error at s , then initialize all $\beta \geq \tau$ and proceed immediately to stage $s + 2$.

Case 2. If s is not τ -expansionary then let $\tau \widehat{\langle \text{fin} \rangle}$ act at stage $s + 1$.

Case 3. If s is τ -expansionary, and there is some $\sigma \supseteq \tau \widehat{\langle \infty \rangle}$ with a link in place from σ to τ then we let σ act at stage $s + 1$.

Case 4. Otherwise, let $\tau \widehat{\langle \infty \rangle}$ act at stage $s + 1$.

We also have to define the functional γ^τ .

Setting γ^τ : If s is τ -expansionary (Cases 3 and 4), at the end of stage $s + 1$, if any type P node $\sigma <_L f_s$, σ respects $\tau \widehat{\langle \infty \rangle}$, and $\gamma^\tau(\sigma) \uparrow[s]$, then there will be some σ -proxy $\xi(\sigma) \downarrow[s]$. If $f_s <_L \xi(\sigma)[s]$, or if s^- looks bad at s for $b_\tau^{\xi(\sigma)}[s]$, then redefine $\xi(\sigma)[s + 1]$ to be the least $\xi \subseteq f_s$ such that $b_\tau^\sigma \leq b_\tau^\xi < l^\tau[s]$.

- If there exists some $\tau_0 \subseteq \tau$ such that $\xi(\sigma)[s + 1] \widehat{\langle \tau_0 \rangle} \subseteq f_s$ or if $\xi(\sigma)[s^- + 1] \neq \xi(\sigma)[s + 1]$, then $\gamma^\tau(\sigma) \uparrow[s + 1]$.
- Otherwise, if s^- looks good at s for $b_\tau^{\xi(\sigma)}[s]$ and $\gamma^\tau(\xi(\sigma)) \downarrow[s + 1]$, let $\gamma^\tau(\sigma) \downarrow[s + 1] = \gamma^\tau(\xi(\sigma)) \downarrow[s + 1]$.
- Otherwise, if s^- looks good at s for $b_\tau^{\xi(\sigma)}[s]$ and $\gamma^\tau(\xi(\sigma)) \uparrow[s + 1]$, let $\gamma^\tau(\sigma) \downarrow[s + 1] = \gamma^\tau(\xi(\sigma)) \downarrow[s + 1] = s + 1$.

5.3 A node σ of type P

Suppose σ has some $P = P_j$ assigned to it. For each $\tau \widehat{\langle \infty \rangle}$ which σ must respect, σ has a τ -attacker, b_τ^σ , and for each $\tau_0 \widehat{\langle \infty \rangle} \subseteq \tau \widehat{\langle \infty \rangle}$ which σ must respect with $\tau_0 \subseteq \tau$, σ has a (τ_0, τ) -corrector, $b^\sigma(\tau_0, \tau)$.

Let s^- be the greatest active σ -stage since σ was last initialized, or the stage at which σ was last initialized, if no such stage exists.

σ discovers an α -error at $s + 1$ if $\alpha < \sigma$, $L(\alpha) = R_{i,l}$, and there exists either some $\tau < \alpha$ such that $L(\tau) = N_{\Phi, \Psi, k}$ or some $\beta < \alpha$ such that $L(\beta) = R_{k, l'}$, $k \neq i$, and $V^{W_k}(x^\alpha)[s^-] \neq V^{W_k}(x^\alpha)[s]$ or $V^{W_i}(x^\beta)[s^-] \neq V^{W_i}(x^\beta)[s]$.

We act according to the first case that applies below.

Case 1. There exists some $\alpha < \sigma$ such that σ discovers an α -error at stage $s + 1$. Then initialize all $\beta \geq \alpha$, and go to stage $s + 2$.

Case 2. There exists some least $\tau \widehat{\langle \infty \rangle}$ which σ must respect such that s^- does not look good for $b^\sigma(\tau, \tau)[s]$ at s , and either σ is not currently attacking, or σ is currently performing a τ_0 -correction for some τ_0 which $\tau \subseteq \tau_0$. (In other words, at most a lower priority correction is taking place.) There are three subcases.

Subcase 2a. s^- does not look good for $b_\tau^\sigma[s]$ at s . Then σ initiates a τ -correction. Let τ_1 be greatest such that σ must respect $\tau_1 \widehat{\langle \infty \rangle}$, $\tau \subseteq \tau_1$, and $\gamma^{\tau_1}(\sigma) \downarrow[s]$. Enumerate $b^\sigma(\tau, \tau_1)[s] \in B[s + 1]$, let $b^\sigma(\tau, \tau_1) \uparrow[s + 1]$, and set a link from σ to τ_1 . For all τ' such that σ must respect $\tau' \widehat{\langle \infty \rangle}$ and $\tau \subseteq \tau'$, let $b_{\tau'}^\sigma \uparrow[s + 1]$ and let, for all ρ , $b^\sigma(\tau', \rho) \uparrow[s + 1]$. Let $c^\sigma \uparrow[s + 1]$. End stage $s + 1$ and proceed to stage $s + 2$.

Subcase 2b. s^- looks good for $b_\tau^\sigma[s]$ at s and there exists some τ_0 such that σ must respect $\tau_0 \widehat{\langle \infty \rangle}$ and $\tau \subseteq \tau_0$. Then σ *initiates a τ_0 -correction*. Let τ_1 be greatest such that σ must respect $\tau_1 \widehat{\langle \infty \rangle}$, $\tau_0 \subseteq \tau_1$, and $\gamma^{\tau_1}(\sigma) \downarrow [s]$. Enumerate $b^\sigma(\tau_0, \tau_1)[s] \in B[s+1]$, let $b^\sigma(\tau_0, \tau_1) \uparrow [s+1]$, and set a link from σ to τ_1 . For all τ' such that σ must respect $\tau' \widehat{\langle \infty \rangle}$ and $\tau_0 \subseteq \tau'$, let $b_{\tau'}^\sigma \uparrow [s+1]$ and let, for all ρ , $b^\sigma(\tau', \rho) \uparrow [s+1]$. Let $c^\sigma \uparrow [s+1]$. End stage $s+1$ and proceed to stage $s+2$.

Subcase 2c. s^- looks good for $b_\tau^\sigma[s]$ at s and τ is greatest such that σ must respect $\tau \widehat{\langle \infty \rangle}$. Let $c^\sigma \uparrow [s+1]$ and let $\sigma \widehat{\langle \text{fin} \rangle}$ act at stage $s+1$.

Case 3. σ is currently performing a τ -correction, and there does not exist any $\tau_0 \widehat{\langle \infty \rangle}$ which σ must respect such that $\tau_0 \subseteq \tau$ and s^- does not look good for $b^\sigma(\tau_0, \tau_0)[s]$ at s . There are two subcases.

Subcase 3a. There is some greatest τ_1 such that σ must respect $\tau_1 \widehat{\langle \infty \rangle}$, $\tau \subseteq \tau_1$, and $\gamma^{\tau_1}(\sigma) \downarrow [s]$. Then we continue the τ -correction. Enumerate $b^\sigma(\tau, \tau_1)[s] \in B[s+1]$, let $b^\sigma(\tau, \tau_1) \uparrow [s+1]$, and set a link from σ to τ_1 . (For all τ' such that σ must respect $\tau' \widehat{\langle \infty \rangle}$ and $\tau \subseteq \tau'$, let $b_{\tau'}^\sigma \uparrow [s+1]$ and let, for all ρ , $b^\sigma(\tau', \rho) \uparrow [s+1]$. Let $c^\sigma \uparrow [s+1]$.) End stage $s+1$ and proceed to stage $s+2$.

Subcase 3b. There is no τ_1 such that σ must respect $\tau_1 \widehat{\langle \infty \rangle}$, $\tau \subseteq \tau_1$, and $\gamma^{\tau_1}(\sigma) \downarrow [s]$. Then we end the τ -correction. If the τ -correction was begun because some $\tau_0 \subseteq \tau$ experienced a stage s which looked good for $b_{\tau_0}^\sigma$, but bad for $b^\sigma(\tau_0, \tau_0)$ (as in subcase 1b. above), then we let $b_\tau^\sigma \uparrow [s+1]$. We let $\sigma \widehat{\langle \tau \rangle}$ act at stage $s+1$.

Case 4. σ is not currently attacking or correcting, and there is some least τ such that $b^\sigma(\tau, \tau) \uparrow [s]$. Let $c^\sigma[s+1] \uparrow$. Then σ sets a link from σ to τ . End stage $s+1$ and proceed to stage $s+2$.

Case 5. σ was visited by a link from τ which was set because $b^\sigma(\tau, \tau) \uparrow$. s is τ -expansionary, and we say that any b such that $m^\tau[s] \leq b < l^\tau[s]$ is an *available τ -attacker*. For each $\tau_0 \subseteq \tau$, such that $b^\sigma(\tau_0, \tau) \uparrow [s]$, let $b^\sigma(\tau_0, \tau)[s+1]$ be the next available τ -attacker. Once all of these are assigned, let $b_\tau^\sigma[s+1]$ be the next available attacker, and, finally, let $b^\sigma(\tau, \tau)[s+1]$ be the next available τ -attacker. If $b^\sigma(\tau, \tau) \uparrow [s+1]$ (*i.e.*, there are not enough available τ -attackers), then σ sets a link from σ to τ . If $b^\sigma(\tau, \tau) \downarrow [s+1]$, let $\xi(\sigma)[s+1]$, the σ -*proxy*, be σ . End stage $s+1$ and proceed to stage $s+2$.

Case 6. σ is not currently attacking or correcting, and for every τ such that σ respects $\tau \widehat{\langle \infty \rangle}$, $b^\sigma(\tau, \tau) \downarrow [s]$ and s^- looks good for $b^\sigma(\tau, \tau)[s]$. If $c^\sigma \uparrow [s]$, then let $c^\sigma[s]$ be some number greater than any yet mentioned in the construction. If $c^\sigma \downarrow [s]$ and $(c^\sigma \notin W_j)[s]$, then do nothing. In either case, let $\sigma \widehat{\langle \text{fin} \rangle}$ act at $s+1$.

Case 7. $c^\sigma \downarrow [s]$ and $(c^\sigma \in W_j - C)[s]$. There are two subcases.

Subcase 7a. If there is no node τ such that $\tau \frown \langle \infty \rangle \subseteq \sigma$ and $\gamma^\tau(\sigma) \downarrow [s]$, then let $c^\sigma \in C[s+1]$. Let $\sigma \frown \langle \text{win} \rangle$ act at stage $s+1$.

Subcase 7b. Otherwise, suppose τ is greatest such that $\tau \frown \langle \infty \rangle \subseteq \sigma$ and $\gamma^\tau(\sigma) \downarrow [s]$. Let $b_\tau^\sigma[s] \in B[s+1]$, $b_\tau^\sigma[s+1] \uparrow$, and set a link from σ to τ . End stage $s+1$ and proceed to stage $s+2$.

If σ was not already currently attacking, then we say σ *begins an attack at s* .

Case 8. $C[s] \cap W_j[s] \neq \emptyset$. Then $\sigma \frown \langle \text{win} \rangle$ acts at stage $s+1$.

5.4 A node α of type R

Suppose α has some $R = R_{i,l}$ assigned to it.

Let s^- be the last previous active α -stage since α was last initialized, or the stage at which α was last initialized if no such active α -stage exists.

Suppose $\sigma < \alpha$, σ has type P, and both α and σ respect $\tau \frown \langle \infty \rangle$, where $L(\tau) = N_{\Phi, \Psi, k}$. Let s_σ be $\gamma^\tau(\sigma)[t]$ for the greatest $t \leq s$ such that $\gamma^\tau(\sigma)[t] \downarrow$.

σ *restricts α at $s+1$ because of τ* if there is some b such that $b = b_\tau^\sigma[s+1]$ or there is some $\tau_0 \subseteq \tau$ such that $b = b^\sigma(\tau_0, \tau)[s+1]$, and $x^\alpha(s) < \phi(W_k \oplus V^{W_k}; b)[s_\sigma]$.

α *discovers an error at $s+1$* if there exists either some $\tau < \alpha$ such that $L(\tau) = N_{\Phi, \Psi, k}$ or some $\beta < \alpha$ such that $L(\beta) = R_{k,l}$, $k \neq i$, and $V^{W_k}(x^\alpha)[s^-] \neq V^{W_k}(x^\alpha)[s]$ or $V^{W_i}(x^\beta)[s^-] \neq V^{W_i}(x^\beta)[s]$.

We act according to the first case below that applies.

Case 1. $x^\alpha[s] \uparrow$. Choose $x^\alpha[s+1]$ greater than any number yet mentioned in the construction, immediately initialize all $\beta > \alpha$, and let $\alpha \frown \langle 0 \rangle$ act at stage $s+1$.

Case 2. α has discovered an error at $s+1$. Then initialize all $\beta \geq \alpha$. Go immediately to stage $s+2$.

Case 3. $x^\alpha \notin V^{W_i}[s]$, and $x^\alpha \notin \widehat{W}_l^{W_i}[s^-]$ or $x^\alpha \notin \widehat{W}_l^{W_i}[s]$. Let $\alpha \frown \langle 0 \rangle$ act at stage $s+1$.

Case 4. $x^\alpha \notin V^{W_i}[s]$ (and $x^\alpha \in \widehat{W}_l^{W_i}[s^-]$ and $x^\alpha \in \widehat{W}_l^{W_i}[s]$). There are two possibilities.

Subcase 4a. If there exists a σ which restricts α from acting at s , then let $x^\alpha \uparrow [s+1]$ and let $\alpha \frown \langle 0 \rangle$ act at stage $s+1$.

Subcase 4b. Otherwise, let $\langle x^\alpha[s], W_i \upharpoonright \phi_l(W_i; x^\alpha)[s] \rangle \in V[s]$, and let $\alpha \frown \langle 1 \rangle$ act at stage $s+1$.

Case 5. $x^\alpha \in V^{W_i}[s]$. Let $\alpha \frown \langle 1 \rangle$ act at stage $s+1$.

6 Verification

The verification of the construction splits naturally into three main parts. First, we must show that when σ must respect $\tau \frown \langle \infty \rangle$, the coordinated σ - τ strategy is protected from injury by lower priority strategies. From this it will follow that Γ^τ is correct and that σ witnesses satisfaction of its requirement if neither is initialized infinitely often. We will also be able to show that the true path is infinite. Second, we must show that if $\Phi(W_k \oplus V^{W_k}) = B$ and $\Psi(K) = V^{W_k}$, then Γ^τ is a total function, where τ is greatest such the $\tau \subseteq f$ and $L(\tau) = N_{\Phi, \Psi, k}$. This makes it possible to show that the type-N requirements are satisfied. Finally, we must show that for each requirement $R_{i,l}$, the greatest α on the true path with $L(\alpha) = R_{i,l}$ is only prevented from acting finitely often. From this it follows that α witness satisfaction of $R_{i,l}$.

6.1 The coordinated σ - τ strategy

To avoid needless repetition in the statement of the next three lemmas, we stipulate that we are always considering some σ which must respect $\tau \frown \langle \infty \rangle$, with $L(\tau) = N_{\Phi, \Psi, k}$. We let s be a σ -stage and b be either $b_\tau^\sigma[s]$ or $b^\sigma(\tau_0, \tau)[s]$ for some $\tau_0 \subseteq \tau$. As in section 5.2, for any t , we write $\phi(t)$ for $\phi(W_k \oplus V^{W_k}; b)[t]$, and $\phi^+(t)$ for the maximum of $\phi(t)$ and all $|\mathcal{W}|$ such that $\langle x, \mathcal{W} \rangle \in V[t]$, $x < \phi(t)$ and there is some v with $W_k \upharpoonright_v = \mathcal{W}$.

Lemma 6.1.1. *Suppose $\alpha < \sigma$, $\gamma^\tau(\sigma) \downarrow [s] = s_0 + 1$, and there is a $t \leq s$ such that $x^\alpha[t] < \phi(s_0)$. Then, $x^\alpha(t) \in V^{W_k}[s_0]$ if and only if $x^\alpha(t) \in V^{W_k}[s]$.*

Proof. Otherwise, let s be least such that there exist σ, τ, t , and s_0 such that this fails. Let $x = x^\alpha(t)$. If $L(\alpha) = R_{i,l}$ and $i \neq k$, then either α or τ is initialized at some s' with $s_0 < s' < s$, so σ is initialized then too, a contradiction. From this it follows that α initializes σ whenever V^{W_k} changes value, unless $\alpha \frown \langle 0 \rangle \subseteq \sigma$. In this case, $x \notin V^{W_k}[s]$, since α only picks a new witness without being initialized when the old one is out, by 5.4, subcase 4a. If $x \in V^{W_k}[s_0]$, then $s_0 > |\mathcal{W}|$ for all $\langle x, \mathcal{W} \rangle \in V[s_0]$. But then $W_k[s_0] \upharpoonright_{s_0} \neq W_k[s] \upharpoonright_{s_0}$ and s_0 can no longer ever look good for b by (5) in the definition of looking good, since $x < \phi(s_0)$. \square

Lemma 6.1.2. *If $\gamma^\tau(\sigma) \downarrow [s] = s_0 + 1$, then*

$$(W_k \oplus V^{W_k})[s] \upharpoonright_{\phi^+(s)} = (W_k \oplus V^{W_k})[s_0] \upharpoonright_{\phi^+(s_0)}.$$

Proof. Otherwise, choose s_0 least for which this fails (for some σ, τ , etc) and let s be least for s_0 . Note that $W_k[s] \upharpoonright_{\phi^+(s)} = W_k[s_0] \upharpoonright_{\phi^+(s_0)}$, since otherwise there exists some τ -expansionary stage s_1 such that $s_0 < s_1 < s$, s_0 does not look good at s_1 , and $W_k[s_1] \upharpoonright_{s_0} \neq W_k[s_0] \upharpoonright_{s_0}$. In this case, we would have $\gamma^\tau(s) \geq s_1 + 1 > s_0 + 1$, a contradiction. Hence, there must be some x such that $x \in V^{W_k}[s] \upharpoonright_{\phi^+(s_0)} - V^{W_k}[s_0] \upharpoonright_{\phi^+(s_0)}$. Let $s_1 > s_0$ be the least stage such that x entered V^{W_k} at $s_1 + 1$ and remained in V^{W_k} at the next τ -expansionary stage. By Lemma 6.1.1, $x = x^\alpha[s_0]$ for some $\alpha > \sigma$. Since s_0 was a τ -expansionary

stage, σ extends $\tau \hat{\ } \langle \infty \rangle$. σ prevents α from acting at stage $s_1 + 1$ unless α does not respect $\tau \hat{\ } \langle \infty \rangle$. Hence there must be some $\sigma_1 \hat{\ } \langle \tau_1 \rangle \subseteq \alpha$ such that $\tau_1 \subseteq \tau$. Suppose $\langle x, \mathcal{W} \rangle \in V[s_1 + 1] - V[s_1]$. Since $x < \phi(s_0)$, s_1 , and *a fortiori* s_0 , cannot look good for b_τ^σ at the least τ -expansionary stage $s_2 \geq s_1 + 1$. Hence, if $\gamma^\tau(\sigma) \uparrow [s_1 + 1]$, there is nothing more to prove. Note that $\xi(\sigma)[s_1 + 1]$ was visited between s_0 and s_1 . Since α was not initialized between s_0 and s_1 , $\sigma_1 \leq \xi(\sigma)[s_1 + 1]$ or $\xi(\sigma)[s_1 + 1] \subseteq \sigma_1$. If $\xi(\sigma)[s_1 + 1] \subseteq \sigma_1$, then $\xi(\sigma)[s_1 + 1]$ was visited at stage $s_1 + 1$. Now, $\xi(\sigma)[s_1 + 1] \hat{\ } \langle \tau_0 \rangle$ cannot have been visited at $s_1 + 1$ for any $\tau_0 \subseteq \tau$, since otherwise σ_1 cannot respect τ . However, since $x < \phi(s_0)$, and $\sigma \leq \xi(\sigma)[s_1 + 1]$, ϕ must have increased since the last stage t at which any such $\xi(\sigma)[s_1 + 1] \hat{\ } \langle \tau_0 \rangle \subseteq f_t$. But then, at the next $\xi(\sigma)[s_1 + 1]$ -stage, $\xi(\sigma)[s_1 + 1]$ must initiate a τ_0 -correction for some $\tau_0 \subseteq \tau$. But then no node extending $\xi(\sigma)[s_1 + 1]$ can act until some $\xi(\sigma)[s_1 + 1] \hat{\ } \langle \tau_1 \rangle$ acts for some $\tau_1 \subseteq \tau$. (Considering the case $\tau = \tau_0 = \tau_1$ to get the intuition.) This contradicts the choice of σ_1 . So $\sigma_1 \leq \xi(\sigma)[s_1 + 1]$. $\gamma^\tau(\sigma_1) \leq \gamma^\tau(\xi(\sigma)[s_1 + 1])$, and, by 5.3, Subcase 3b, $\gamma^\tau(\sigma_1) \uparrow [s_1]$. Hence $\gamma^\tau(\sigma) \uparrow [s_1]$, so that $\xi(\sigma)[s_1 + 1] = \sigma_1$. Since $\sigma_1 \hat{\ } \langle \tau_0 \rangle \subseteq f_{s_1}$, $\gamma^\tau(\sigma) \uparrow [s_1 + 1]$, as required. \square

Lemma 6.1.3. *Suppose σ puts up a link to τ at stage $s + 1$ as part of an attack or a correction. If s^+ is the next τ -expansionary stage, $\gamma^\tau(\sigma) \uparrow [s^+]$.*

Proof. σ enumerated an attacker b into B at stage $s + 1$. Since $b < l^\tau[s^+]$, and $B(b)[\gamma^\tau(\sigma)[s]] \neq B(b)[s^+]$, this follows immediately from Lemma 6.1.2. \square

We can now show that the true path is infinite.

Lemma 6.1.4. *f is infinite, and each $\rho \subseteq f$ is only initialized finitely often.*

Proof. By induction, let $\rho \subseteq f$ and choose s_0 such that ρ is never initialized after stage s_0 . If $\rho \subseteq f_s$, then some $\rho \hat{\ } \mathcal{O} \subseteq f_s$, unless ρ is a type P node which is attacking, correcting, or waiting to be assigned available attackers. Each attack or correction must eventually end by Lemma 6.1.3. Eventually all available attackers must also be assigned, since otherwise some $\tau \hat{\ } \langle \infty \rangle$ acts infinitely often, but $l^\tau[s]$ has a finite limit. So there is some $\rho \hat{\ } \mathcal{O} \subseteq f$. $\rho \hat{\ } \mathcal{O}$ can only be initialized after s_0 if some $\alpha \leq \rho$ causes an error to be discovered. There are only finitely many such α , and each can only produce finitely many errors, since our enumeration of computably enumerable sets is without repetitions. This is sufficient for the Lemma. \square

Because f is infinite, it is routine to check using the definition of L that for every requirement Q of the construction, there is some greatest $\rho \subseteq f$ such that $L(\rho) = Q$. We can now show that each requirement of type P is satisfied.

Let σ be the greatest node included in f such that $L(\sigma) = P_j$. Note that $\sigma \hat{\ } \langle \text{fin} \rangle \subseteq f$, or $\sigma \hat{\ } \langle \text{win} \rangle \subseteq f$, since otherwise σ cannot be greatest on the true path with $L(\sigma) = P_j$. If $\sigma \hat{\ } \langle \text{win} \rangle \subseteq f$, then $C \cap W_j \neq \emptyset$, so P_j is satisfied. So suppose $\sigma \hat{\ } \langle \text{fin} \rangle \subseteq f$. First, note that 5.3, Subcase 2b cannot apply infinitely often, for, if ξ is the next type-P node included in f , $b_\tau^\xi > b^\sigma(\tau, \tau)$. Whenever a stage looks bad for $b^\sigma(\tau, \tau)$, it must therefore look bad for b_τ^ξ as well, so

that $\xi \smallfrown \langle \tau_0 \rangle \subseteq f$ for some $\tau_0 \subseteq \tau$. But then, since $\tau \smallfrown \langle \infty \rangle \subseteq \sigma$, σ cannot be greatest such that $L(\sigma) = P_j$. Hence, eventually c^σ is defined permanently. By Lemma 6.1.3, an attack once started would eventually come to an end, with $\sigma \smallfrown \langle \text{win} \rangle \subseteq f$. From all this, it follows that σ must act under 5.3, Case 6 at almost every σ -stage. But then $c^\sigma \notin W_j$ and $c^\sigma \notin C$, so that P_j is satisfied.

6.2 Hat-trick lemmas and type N requirements

Let τ be a node such that $L(\tau) = N_{\Phi, \Psi, k}$. We first list a few technical facts about the hat trick. The significance of true stages lies in the following fact.

Lemma 6.2.1. *If there exist infinitely many τ -stages and u is any natural number, then there exists a least τ -true stage $t(u)$ such that for all $t \geq t(u)$, if t is a τ -true stage, then $\widehat{V}^{W_k}[t] \upharpoonright_u = V^{W_k} \upharpoonright_u$.*

Proof. The lemma follows straightforwardly from the definitions. \square

Lemma 6.2.2. *If there are infinitely many τ -stages, $\Phi(W_k \oplus V^{W_k}) = B$, and $\Psi(K) = V^{W_k}$, then there exist infinitely many τ -expansionary stages.*

Proof. There are infinitely many τ -true stages, and, by Lemma 6.2.1, every relevant computation eventually appears cofinitely often at such stages. Hence the limit of l^τ tends to infinity on the sequence of τ -true stages. This is sufficient to verify the claim. \square

Lemma 6.2.3. *Suppose $\Phi(W_k \oplus V^{W_k}) = B$, $\Psi(K) = V^{W_k}$, and $b \in \omega$. Then there exists a stage t_0 such that if t and s are any τ -expansionary stages with $t_0 \leq t \leq s$, then t looks good for b with respect to τ at s .*

Proof. Choose $\sigma \subseteq f$ such that $b_\tau^\sigma \geq b$, and choose t_{-1} such that for every $t \geq t_{-1}$, $\sigma \leq f_t$. Once $\Psi(K)$ and W_k become stable on the total use involved in $\Phi(W_k \oplus V^{W_k}; b)$ and the computation converges, $\widehat{V}^{W_k} \upharpoonright_{\phi(t)}$ must always be the same set at any τ -expansionary t . ϕ^+ must eventually get the same value on every τ -expansionary stage, since those elements that are in V^{W_k} below the use are eventually in this set permanently with some fixed use from W_k . But then any τ -true stage $t_0 > t_{-1}$ after this point will have the properties required, since the initialization that takes place for the sake of b_τ^σ ensures that $\widehat{V}^{W_k} \upharpoonright_{\phi(t_0)} = V^{W_k} \upharpoonright_{\phi(t_0)}$ at every subsequent τ -expansionary stage. \square

Lemma 6.2.4. *Suppose $\tau \subseteq f$ is greatest such that $L(\tau) = N_{\Phi, \Psi, k}$, $\Phi(W_k \oplus V^{W_k}) = B$, $\Psi(K) = V^{W_k}$, and σ must respect $\tau \smallfrown \langle \infty \rangle$. If $\sigma < f$, then there exists some s_0 such that for all $s > s_0$, $\gamma^\tau(\sigma) \downarrow [s] = \gamma^\tau(\sigma)[s_0]$.*

Proof. Note that $\tau \smallfrown \langle \infty \rangle \subseteq f$, and there exist infinitely many available τ -attackers. Let $\xi(\sigma)$ be the least node on the true path such that $\sigma \leq \xi(\sigma)$. Since there are only finitely many $\sigma' < \xi(\sigma)$, eventually all such nodes which require τ -attackers or τ -correctors receive them. This follows because no $\tau_0 \subseteq \tau$ can cause these attackers and correctors to become undefined infinitely often

without some τ_0 -infinitary outcome appearing on the true path below τ , which contradicts the choice of τ . (Recall, $\sigma < f$.) Eventually, then, b_τ^σ and $b_\tau^{\xi(\sigma)}$ are chosen permanently. Once $b_\tau^{\xi(\sigma)}$ is chosen permanently, there must exist some stage t_0 as in Lemma 6.2.3, after which every τ -expansionary stage looks good for $b_\tau^{\xi(\sigma)}$. By definition, eventually $\xi(\sigma)[t] = \xi(\sigma)$ at every τ -expansionary stage, and hence eventually $\gamma^\tau(\sigma) = \gamma^\tau(\xi(\sigma))$, and this value remains fixed forever after the stage t_0 which always looks good for $b_\tau^{\xi(\sigma)}$. \square

If $\Phi(W_k \oplus V^{W_k}) \neq B$, or $\Psi(K) \neq V^{W_k}$, then $N_{\Phi, \Psi, k}$ is trivially satisfied. Suppose $\tau \subseteq f$ is greatest such that $L(\tau) = N_{\Phi, \Psi, k}$, and suppose $\Phi(W_k \oplus V^{W_k}) = B$, and $\Psi(K) = V^{W_k}$. By Lemma 6.2.2, $\rho \hat{\ } \langle \infty \rangle \subseteq f$. By Lemma 6.2.4, any $\sigma < f$ such that σ must respect τ must eventually have a stable use $\gamma^\tau(\sigma)$ defined. After this point, σ can no longer receive a τ_0 -infinitary outcome for any $\tau_0 \subseteq \tau$. Thus every $\sigma \subseteq f$ must respect $\tau \hat{\ } \langle \infty \rangle$. C is a computably enumerable set, the set of witnesses chosen by σ with $f <_L \sigma$ which never enter C is a computably enumerable set, and the set of numbers never chosen as witnesses is a computable set. No σ that respects τ can ever enumerate a number into C without $\gamma^\tau(\sigma)$ being undefined, by 5.3, Subcase 7a. If σ does not respect τ , yet $\tau \hat{\ } \langle \infty \rangle \subseteq \sigma$, then there must exist some $\sigma_0 \subseteq \sigma$ such that σ_0 does respect τ and $\sigma_0 \hat{\ } \langle \tau_0 \rangle \subseteq \sigma$ for some $\tau_0 \subseteq \tau$. By Lemma 6.1.3, σ cannot act at s unless $\gamma^\tau(\sigma_0) \uparrow[s]$. Hence, letting $\gamma^\tau(\sigma) = \gamma^\tau(\sigma_0)$ ensures that we can correctly define a functional $\Gamma^\tau(W_k) = C$.

6.3 Satisfaction of type R requirements

Let α be the greatest node on the true path f such that $L(\alpha) = R_{i,l}$.

Lemma 6.3.1. *There exists some stage s_0 such that for all $s > s_0$, $x^\alpha[s] = x^\alpha[s_0]$.*

Proof. If $x^\alpha[s]$ changes value infinitely often, there must exist some $\tau \hat{\ } \langle \infty \rangle$ which α must respect and some $\sigma < \alpha$ that prevents α from acting because of τ infinitely often. This follows since there are only finitely many $\sigma < \alpha$ that ever act. Since α respects τ , $L(\tau) < L(\alpha)$, and since α is the greatest node with $L(\alpha)$ assigned to it on f , there can exist no $\tau_0 \subseteq \tau$ and σ_0 such that $\tau_0 \subseteq \tau \subseteq \sigma_0 \hat{\ } \langle \tau_0 \rangle \subseteq f$. Hence, every such σ respects τ , and no such σ can have $\gamma^\tau(\sigma)$ undefined infinitely often. There are $2^{|\alpha|}$ computably enumerable sets consisting solely of numbers less than $|\alpha|$, yet α has only $|\alpha|$ nodes included in it. Therefore, there is some σ_0 such that $\alpha \subseteq \sigma_0 \subseteq f$, and $L(\sigma_0) = P_j$ for some $W_j \subseteq \{0, 1, \dots, |\alpha|\}$ with a witness c^{σ_0} which is eventually chosen permanently. $c^{\sigma_0} \notin W_j$, since $c^{\sigma_0} > |\alpha|$. On the other hand, $\sigma \hat{\ } \langle \tau \rangle <_L \sigma_0$, since σ_0 respects τ . This implies that $b_\tau^{\sigma_0}$ is greater than any of σ 's correctors associated with τ and greater than b_τ^σ . σ_0 can therefore serve as the σ -proxy. But then $\sigma_0 \hat{\ } \langle \tau_0 \rangle \subseteq f_s$ at infinitely many stages s for some $\tau_0 \subseteq \tau$, since some corrector or attacker associated with σ has a use which is unbounded on τ -expansionary stages, otherwise it would stop interfering with α . Since $\sigma_0 \hat{\ } \langle \text{win} \rangle \not\subseteq f$, this is a contradiction. \square

Using Lemma 6.3.1, choose s_0 so that for every $s \geq s_0$ for every $s \geq s_0$, $x^\alpha[s] = x^\alpha[s_0]$. Let $x = x^\alpha[s_0]$. Note that $\alpha \leq f_s$ for all $s \geq s_0$. No node other than α ever puts any axiom of the form $\langle x, \mathcal{W} \rangle$ into V , and $x \in V^{W_i}$ only if some $\langle x, \mathcal{W} \rangle \in V$ and \mathcal{W} is W_i restricted to the use of some computation $\phi_l(W_i; x)[s]$. Hence, if $x \notin W_l^{W_i}$, $x \notin V^{W_i}$.

Suppose on the other hand that $x \in W_l^{W_i}$. Let $s_1 \geq s_0$ be the least stage such that for every stage s after s_1 , $x \in W_l^{W_i}[s]$. Since $x^\alpha[s]$ never changes after s_0 , no σ can prevent α from acting after s_1 . Hence 5.4, Subcase 4b must eventually apply at some stage after s_1 . In this case $x \in V^{W_i}$, since it is enumerated with a correct W_i -axiom. Hence if $x \in W_l^{W_i}$, $x \in V^{W_i}$. This shows $R_{i,l}$ is satisfied and finishes the proof of Theorem 6.

7 Further questions

Here we list just a few of the natural questions that are suggested by the general problem of completing a pseudojump operator. From a technical standpoint, probably the most interesting involve removing restrictions to computable enumerability or n -computable enumerability in our results. For instance the best possible strengthening of Theorem 6 would be one constructing an operator that was nontrivial on all sets and forced all of its completions into some upper cone.

Question 3. *Does there exist a noncomputable $C \subseteq \omega$ and a pseudojump operator V such that*

- (1) *for every $A \subseteq \omega$, $A <_T A \oplus V^A$, and*
- (2) *for every $A \subseteq \omega$, if $A \oplus V^A \equiv_T K$, then $C \leq_T A$?*

Any such result faces the immediate problem of constructing V by enumeration, yet forcing it to be nontrivial on every possible oracle. Our method for proving Theorem 6 is not easily strengthened to demand nontriviality of V on Δ_0^2 sets or even merely d.c.e. sets, since we face the problem of coordinating the nontriviality requirements that forced us to use an enumeration of the c.e. sets without repetition. Some other questions raised by Theorem 6 involve what properties we can demand of C . For instance, can we even ensure that C itself completes the operator V ? It is not hard to see that the C constructed there has low degree. Must this always be the case? More generally, we can ask about the relationship of such cones to the jump operator:

Question 4. *Given a Σ_2^0 A with $\mathbf{0}' \leq_T A$, does there exist a non-computable, computably enumerable set C with $C' \leq_T A$ and a pseudojump operator V such that*

- (1) *for every $e \in \omega$, $W_e <_T W_e \oplus V^{W_e}$, and*
- (2) *for every $e \in \omega$, if $W_e \oplus V^{W_e} \equiv_T K$, then $C \leq_T W_e$?*

There are also various questions involving the relationship of the completions of a pseudojump operator to the usl structure of the c.e. degrees. For example:

Question 5. *Given a pseudojump operator V nontrivial on the c.e. sets, must there exist a noncomputable, c.e., cappable C such that $C \oplus V^C \equiv_T K$?*

We remark that such a construction would have to be nonuniform by an argument similar to M. Simpson's proof of the Sacks Jump Theorem (see [6], p. 115), using Shore's noninversion theorem for the jump operator, [5]. On the other hand, it is easy to see that there must always be a noncappable C completing a given operator, by applying Corollary 1 to a low promptly simple degree.

The fact that we can find an operator that cannot avoid a particular upper cone suggests various other questions about completion and cones of degrees. An extreme case is the following:

Question 6. *Does there exist a c.e. $C <_T K$ such that for every V nontrivial on c.e. sets there exists a c.e. $A \leq_T C$ such that $A \oplus V^A \leq_T K$?*

Even if the answer to question 1 should turn out to be negative, there remains the problem of producing even the simplest nontrivial linear order with sets that both complete the same operator.

Question 7. *Given a nontrivial pseudojump operator, V , must there always exist a pair of c.e. sets $A <_T B$ such that $A \oplus V^A \equiv_T B \oplus V^B \equiv_T K$?*

Investigating the relationship of minimal degrees to pseudojump operators is another natural area of investigation. By Cooper, [1], there can be no high minimal degree below $\mathbf{0}'$. Hence we cannot expect to complete every operator with a set of minimal degree. We might hope for something weaker, however:

Question 8. *For all nontrivial V must there exist a C such that for all D with $C \leq_T D$ there exists a set of minimal degree A with $A \oplus V^A \equiv_T D$.*

As pointed out, such a C would have to be strictly above $\mathbf{0}'$. This also raises the possibility of investigating the range of pseudojump operators above $\mathbf{0}'$ in general. This seems to require some new idea for coding into the completing set, however.

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