AUTOMORPHISMS OF THE LATTICE OF RECURSIVELY ENUMERABLE SETS: ORBITS

R. G. Downey
Michael Stob

Introduction. An important program in the study of the structure of $\mathcal{E}$, the lattice of r.e. sets, is the classification of the orbits under $\mathbf{Aut}(\mathcal{E})$, the automorphism group of $\mathcal{E}$. Obviously, the class of finite sets of a given cardinality and the class of infinite-coinfinite recursive sets form orbits. But other than these, relatively few orbits are known. Harrington has shown that the creative sets form an orbit. The most important result in the area is still Soare's proof [So1]that maximal sets (sets which are coatoms in $\mathcal{E}^* = \mathcal{E}$ modulo the ideal of finite sets) form an orbit. The proof of this result contains an intricate and powerful construction which has since been used to construct automorphisms by Maass [M], Maass and Stob [MSt], and Stob [St]. It remains the most powerful method of constructing automorphisms.

The principal result of this paper is the existence of a new orbit, the hemimaximal sets. An r.e. set H is *hemimaximal* if there is a maximal set M and disjoint r.e sets M_0, M_1 such that $M_0 \cup M_1 = M$, each M_i is nonrecursive, and $M_0 = H$. In section 1, we detail the properties of splittings of maximal sets which led us to discover this result. We also prove that the hemimaximal sets form an orbit by extending Soare's proof for maximal sets. In section 2, we study the degrees of hemimaximal sets. (All degrees mentioned in this paper are r.e.) We show that hemimaximal sets exist in every high degree, and that for every degree $\mathbf{a} > \mathbf{0}$ there is a hemimaximal set H such that $\mathbf{deg}(H) \leq \mathbf{a}$. However, there are some r.e degrees which contain no hemimaximal sets. Thus the class of hemimaximal sets, $\mathcal{H}$, provides an example of a definable (in $\mathcal{E}$) class of r.e. sets which forms an orbit but contains r.e. sets of low r.e. degree. This is the first example of such an orbit. Furthermore, the class of degrees of sets in $\mathcal{H}$ is not closed upward; again this is the first example of a class definable over $\mathcal{E}$ with this property.

In section 3, we turn to the question of which orbits contain complete r.e. sets. Soare has conjectured that the orbit of every nonrecursive r.e. set contains a complete set. Exploiting further our idea of finding automorphism types from splittings, we introduce the halfhemimaximal sets and show that every halfhemimaximal set is automorphic to a complete set. An r.e. set A is *halfhemimaximal* if A has a splitting into disjoint, nonrecursive, r.e. sets A_0, A_1 such that A_0 is hemimaximal . We show that, although not every set is halfhemimaximal , there are large natural classes of halfhemimaximal r.e. sets (such as all low simple r.e. sets).

In section 4 we discuss the prospects of extending the idea of section 1 to splittings of other sets which form orbits. Our main positive result is the classification

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of all orbits which arise from splitting quasimaximal sets (sets which are the intersection of finitely many maximal sets). We conclude in section 5 with further remarks and a list of open questions.

Our notation is standard; a reference is Soare [So2]. In particular, a * affixed to a binary relation means up to a finite set; e.g., $A \subseteq^* B$ means $A - B$ is finite. Also, we will use heavily the notion of e-states. If $\{V_e\}_{e\in\omega}$ is a a recursive array of r.e. sets with simultaneous enumeration given by $\{V_{e,s}\}_{e,s\in\omega}$ then $\sigma_{e,s}(x)$, the e-state of x at s, is $\{i \le e \mid x \in V_{i,s}\}$. The e-state of x, $\sigma_e(x)$, is $\{i \le e \mid x \in V_e\}$. e-states are ordered by the relation $\sigma > \tau$ if and only if $(\exists i)(\forall j < i)[(j \in \sigma \Leftrightarrow j \in \tau)$ and $i \in \sigma - \tau]$. Thus the empty set is the least e-state and $\{0, 1, 2, \ldots, e\}$ is the greatest. If we have a simultaneous enumeration of a recursive array of finite sets including A and B, $A\backslash B = \{x \mid (\exists s)[x \in A_s - B_s]\}$ and $A \searrow B$ denotes $(A\backslash B) \cap B$.

1. Hemimaximal sets form an orbit.

Definition 1. If A is any nonrecursive r.e. set, a *nontrivial splitting* of A is a pair of disjoint nonrecursive r.e. sets A_0, A_1 such that $A = A_0 \cup A_1$.

Friedberg was the first to show that every nonrecursive r.e. set A has a nontrivial splitting. In fact, the splittings that Friedberg produced had the slightly stronger property of Definition 2.

Definition 2. A nontrivial splitting A_0, A_1 of A is a *Friedberg splitting* if for every r.e. W, if $W - A$ is not r.e., then each of $W - A_0$, $W - A_1$ is not r.e.

Sacks showed that nontrivial splittings A_0, A_1 of A always exist with the added property that $A_0 <_{\mathbf{T}} A$ and $A_1 <_{\mathbf{T}} A$ (and in fact it can be guaranteed that each of $\mathbf{deg}(A_0)$ and $\mathbf{deg}(A_1)$ is low).

Definition 3. A set A is *hemimaximal* if there is a maximal set M and a nontrivial splitting M_0, M_1 of M such that $A = M_0$. (More generally, if P is any property of r.e. sets, A is *hemiP* is there is a set M with property P and a nontrivial splitting M_0, M_1 of M such that $A = M_0$.)

Our investigation of hemimaximal sets was initially motivated by the following two theorems.

Theorem 1. *Every nontrivial splitting of a maximal set is a Friedberg splitting.*

Proof. Suppose that M_0, M_1 form a nontrivial splitting of the maximal set M and suppose that $W - M$ is not r.e. Then $W - M$ is infinite so $\overline{M} \subseteq^* W$. Now $W - M_0 \supseteq W - M =^* \overline{M}$ so that if $W - M_0$ is r.e., $\overline{M}_0 =^* M_1 \cup (W - M_0)$ and so $\overline{M}_0$ is r.e. This contradicts the nonrecursiveness of M_0. Similarly, $W - M_1$ is not r.e. □

Definition 4. (Downey, [D]) A pair of disjoint r.e. sets A_0, A_1 forms a *maximal pair* if whenever B_0, B_1 are such that $B_i \supseteq A_i$, $i = 0, 1$, and $B_0 \cap B_1 = \emptyset$ then $B_i - A_i$ is finite for $i = 0, 1$.

Theorem 2. *Every Friedberg splitting of a simple set forms a maximal pair.*

Proof. Suppose that A_0, A_1 is a Friedberg splitting of A, $B_0 \supseteq A_0$, $B_1 \supseteq A_1$, and $B_0 \cap B_1 = \emptyset$. Now $B_0 - A_1 = B_0$ is r.e., thus $B_0 - A$ is r.e. (since A_0, A_1 form a Friedberg splitting of A). Since A is simple, this implies that $B_0 - A$ is finite. Similarly, $B_1 - A$ is finite. □

Theorems 1 and 2 led us to conjecture the following.

Theorem 3. *If C, D are hemimaximal sets, then there is $\Phi \in \mathbf{Aut}(\mathcal{E})$ such that $\Phi(C) = D$.*

Proof. Let A and B maximal be given such that A_0, A_1 and B_0, B_1 form nontrivial splittings of A and B respectively. Soare[So1] shows us how to construct $\Phi \in \mathbf{Aut}(\mathcal{E})$such that $\Phi(A) = B$. We will modify his proof so that in addition $\Phi(A_i) = B_i$ for each i.

Soare constructs four recursive arrays,$\{U_e\}_{e\in\omega}$, $\{V_e\}_{e\in\omega}$, $\{\hat{U}_e\}_{e\in\omega}$, $\{\hat{V}_e\}_{e\in\omega}$, such that the map Φ defined by $\Phi(U_e) = \hat{U}_e$, and $\Phi^{-1}(V_e) = \hat{V}_e$ induces an automorphism of $\mathcal{E}^*$. (It is enough to construct an automorphism of $\mathcal{E}^*$ since Soare has also shown that if A and B are r.e. sets which are infinite and coinfinite, and there is $\Phi \in \mathbf{Aut}(\mathcal{E}^*)$ such that Φ maps the equivalence class of A to B, then there is an automorphism of $\mathcal{E}$ which maps A to B.) To insure that Φ defined in this way is defined on all of $\mathcal{E}^*$ and is onto $\mathcal{E}^*$, Soare guarantees that

$$(1) \qquad (\forall e)(\exists n)[W_e =^* U_n] \quad \text{and} \quad (\forall e)(\exists n)[W_e =^* V_n]$$

To guarantee that Φ preserves inclusions, the only other requirement on Φ, Soare divides the problem into two subproblems, the socalled $\overline{A}$ to $\overline{B}$ part and the A to B part. To state exactly what each part requires, we need the following definition.

DEFINITION 6. Let $\{X_e\}_{e\in\omega}$ and $\{Y_e\}_{e\in\omega}$ be recursive arrays of r.e. sets. The *full e-state, ν of x with respect to* $\{X_e\}_{e\in\omega}$, $\{Y_e\}_{e\in\omega}$ is the triple $\langle e, \sigma, \tau\rangle$ where σ is the e-state of x with respect to $\{X_e\}_{e\in\omega}$ and τ is the e-state of x with respect to $\{Y_e\}_{e\in\omega}$. (Given x and s, $\nu_{e,s}(x)$ is the approximation to the full e-state of x at stage s in some fixed simultaneous enumeration of all the sets in the arrays $\{X_e\}_{e\in\omega}$, $\{Y_e\}_{e\in\omega}$.)

Now the $\overline{A}$ to $\overline{B}$ part of the requirement amounts to

(2)

for each full e-state ν,

infinitely many elements of $\overline{A}$ have e-state ν with respect to $\{U_e\}_{e\in\omega}, \{\hat{V}_e\}_{e\in\omega}$

iff

infinitely many elements of $\overline{B}$ have e-state ν with respect to $\{\hat{U}_e\}_{e\in\omega}, \{V_e\}_{e\in\omega}$.

Similarly, the A to B requirement is

(3)

for each full e-state ν,

infinitely many elements of A have e-state ν with respect to $\{U_e\}_{e\in\omega}, \{\hat{V}_e\}_{e\in\omega}$

iff

infinitely many elements of B have e-state ν with respect to $\{\hat{U}_e\}_{e\in\omega}, \{V_e\}_{e\in\omega}$.

It is clear that (1),(2), and (3) guarantee that Φ as defined above is an automorphism such that $\Phi(A) =^* B$. Now for our theorem, it is enough to replace (3) by

(4)

for each i and for each full e-state ν,

infinitely many elements of A_i have e-state ν with respect to $\{U_e\}_{e\in\omega}, \{\hat{V}_e\}_{e\in\omega}$

iff

infinitely many elements of B_i have e-state ν with respect to $\{\hat{U}_e\}_{e\in\omega}, \{V_e\}_{e\in\omega}$.

The most difficult of the three conditions (1), (2) and (3) is (3) (and our (4)). The primary reason for this difficulty is the conflict between (2) and (3). To see why this is so, suppose that U_0 is given. (Suppose for instance, because of (1), that U_0 is enumerated to satisfy $U_0 = W_0$.) Then, as we observe elements in $U_0 \cap \overline{A}$, we must enumerate certain elements in $\hat{U}_0$ while they remain in $\overline{B}$. However, these elements may later enter B thereby threatening (3) with respect to U_0. For if $\hat{U}_0 \cap B$ is infinite, we must have that $U_0 \cap A$ is infinite but we have no control over U_0. Thus a necessary condition for meeting (3) seems to be that if infinitely many elements enter B while in $\hat{U}_0$, infinitely many elements of A must be in U_0. Soare extends this analysis to all e-states to get a sufficient condition on the enumeration on all the sets in the four arrays above for (3) to be met. A preliminary definition is needed.

Definition 6. Given full e-states $\nu = \langle e, \sigma, \tau \rangle$ and $\nu' = \langle e, \sigma', \tau' \rangle$, $\nu \leq \nu'$ if $\sigma \subseteq \sigma'$ and $\tau \supseteq \tau'$. (The relation $\leq$ is pronounced "is covered by.")

Lemma 1 (Soare's Extension Lemma). *Assume that A and B are infinite r.e sets and $\{U_n\}_{n\in\omega}$, $\{\tilde{V}_n\}_{n\in\omega}$, $\{\tilde{U}_n\}_{n\in\omega}$, $\{V_n\}_{n\in\omega}$ are recursive arrays of r.e. sets. Suppose that there is a simultaneous enumeration of a recursive array including all the above such that $A \searrow \tilde{V}_n = \emptyset = B \searrow \tilde{U}_n$, for all n. For each full e-state ν with respect to $\{U_n\}_{n\in\omega}$, $\{\tilde{V}_n\}_{n\in\omega}$, define the r.e. set $D^A_\nu = \{x \mid x \in A_{s+1} - A_s$ for some s such that $\nu_{e,s}(x) = \nu\}$. Similarly, define D^B_ν with B, $\{\tilde{U}_n\}_{n\in\omega}$, and $\{V_n\}_{n\in\omega}$ in place of A, $\{U_n\}_{n\in\omega}$, and $\{\tilde{V}_n\}_{n\in\omega}$. Furthermore suppose that*

$$(\forall \nu)\left[D^B_\nu \text{ infinite } \Rightarrow (\exists \nu')\left[\nu \leq \nu' \text{ and } D^A_\nu \text{ infinite}\right]\right]$$

and

$$(\forall \nu)\left[D^A_\nu \text{ infinite } \Rightarrow (\exists \nu')\left[\nu' \leq \nu \text{ and } D^B_\nu \text{ infinite}\right]\right]$$

Then there are r.e. sets $\hat{U}_n$ extending $\tilde{U}_n$ and $\hat{V}_n$ extending $\tilde{V}_n$ such that (3) above is satisfied.

The first step then in our proof is to extend the Extension Lemma to pairs of r.e. sets to guarantee that (4) above is met. To this end, define $D^{A_i}_\nu$ as D^A_ν above but with A_i in place of A for each $i = 0, 1$. Similarly, define $D^{B_i}_\nu$. Then we have

Lemma 2. *Let A and B be infinite r.e. sets and A_0, A_1 and B_0, B_1 form splittings of A and B respectively. Suppose that $\{U_n\}_{n\in\omega}$, $\{\tilde{V}_n\}_{n\in\omega}$, $\{\tilde{U}_n\}_{n\in\omega}$, $\{V_n\}_{n\in\omega}$ are recursive arrays of r.e. sets and that there is a simultaneous enumeration of a recursive array including all the above such that $A_i \searrow \tilde{V}_n = \emptyset = B_i \searrow \tilde{U}_n$, for all n and i. Furthermore suppose that for each i, $i = 0, 1$,*

$$(\forall \nu)\left[D^{B_i}_\nu \text{ infinite } \Rightarrow (\exists \nu')\left[\nu \leq \nu' \text{ and } D^{A_i}_\nu \text{ infinite}\right]\right] \tag{5}$$

and

$$(\forall \nu)\left[D^{A_i}_\nu \text{ infinite } \Rightarrow (\exists \nu')\left[\nu' \leq \nu \text{ and } D^{B_i}_\nu \text{ infinite}\right]\right] \tag{6}$$

Then there are r.e. sets $\hat{U}_n$ extending $\tilde{U}_n$ and $\hat{V}_n$ extending $\tilde{V}_n$ such that (4) above is satisfied.

Proof (of Lemma 2). Apply Soare's Extension Lemma (Lemma 1) to the pair A_0, B_0, in place of A, B. The extension guaranteed there meets (4) with respect to A_0 and B_0. Further, the proof of Soare's Extension Lemma guarantees that $\hat{U}_n - \tilde{U}_n \subseteq B_0$ and $\hat{V}_n - \tilde{V}_n \subseteq A_0$ for all n. Now, renaming the sets $\hat{U}_n$ and $\hat{V}_n$ which result from this application of Lemma 1 to $\tilde{U}_n$ and $\tilde{V}_n$, we see that the hypotheses of the Lemma 1 are now satisfied with A_1 and B_1 in place of A and B. Thus, applying the Extension Lemma again, we get that (4) is satisfied with respect to A_1, B_1.

It remains to meet (1) and (2) and satisfy the hypotheses of Lemma 2. Obviously, (1) can be satisfied by taking $U_e = W_e$ and $V_e = W_e$ for all e. However, to facilitate meeting (2) and the hypotheses of Lemma 1, Soare chooses the sets U_n and V_n to have certain further properties. He proves

Lemma 3. *Let M be maximal. Then there is a simultaneous enumeration of a recursive array $\{Z_n\}_{n\in\omega}$ satisfying (1), with $Z_0 = M$, and such that*

$$(\forall n)\left[Z_n\backslash M \textit{ infinite } \Leftrightarrow Z_n - M \textit{ infinite } \Leftrightarrow Z_n \supseteq \overline{M}\right] \tag{7}$$

and

$$(\forall n)(\forall m)_{>n}(a.a.x)\,[x \in (Z_n\backslash M) \cap (Z_m\backslash M) \Rightarrow x \in (Z_n\backslash Z_m)] \tag{8}$$

where a.a.x denotes for almost all x.

We shall choose $\{U_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$ as the arrays resulting from the application of Lemma 3 to A and B respectively. Thus, (1) is satisfied. It remains to meet (2) and the hypotheses of the Extension Lemma. Now for maximal sets, (2) amounts to the following:

$$(\forall n)\left[\overline{A} \subseteq U_n \Leftrightarrow \overline{B} \subseteq^* \hat{U}_n\right]$$

and similarly with B and V_n in place of A and U_n. Thus, to complete the proof, we need to prove the following.

Lemma 4. *Given maximal sets A and B let $\{U_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$ be the recursive arrays given by Lemma 3 for A and B respectively. Then there exist recursive arrays $\{\tilde{U}_n\}_{n\in\omega}$, $\{\tilde{V}_n\}_{n\in\omega}$ together with a simultaneous enumeration of a recursive array including all of $\{U_n\}_{n\in\omega}$, $\{V_n\}_{n\in\omega}$, $\{\tilde{U}_n\}_{n\in\omega}$, and $\{\tilde{V}_n\}_{n\in\omega}$ which satisfy the hypotheses of Lemma 2 and also (2) which here asserts that*

$$\left[\overline{A} \subseteq U_n \Leftrightarrow \overline{B} \subseteq^* \tilde{U}_n\right] \textit{ and } \left[\overline{B} \subseteq V_n \Leftrightarrow \overline{A} \subseteq^* \tilde{V}_n\right]. \tag{9}$$

Proof. This proof follows the proof of the analogous lemma of Soare [So, Theorem 3.2] very closely. At the end of stage s we will have defined A_s and B_s. Given A_s and B_s, define $\psi_s(x)$ to be the function which maps $\overline{A}_s$ onto $\overline{B}_s$ in increasing order. Let ψ_{-1} be the identity permutation. Let $\psi = \lim_s \psi_s$. We will use ψ to govern our enumeration of the sets $\tilde{U}_n$ and $\tilde{V}_n$. We must enumerate $\tilde{U}_n$ and $\tilde{V}_n$ to satisfy (9). We will thus attempt to guarantee that $x \in U_n \Leftrightarrow \psi(x) \in \tilde{U}_n$ and similarly for $\tilde{V}_n$. The two problems with this strategy are that we only have an approximation $\psi_s(x)$ to $\psi(x)$ at any stage s, and that we must not enumerate so much in $\tilde{U}_n$ so as to violate the hypotheses of the Extension Lemma. Actually, the first problem is minor; the second is handled by not enumerating an element y in $\tilde{U}_n$ until the state to which it will be raised is successfully covered by elements entering both A_0 and A_1.

CONSTRUCTION.

Stage $s = 4t$. Let x be the unique element enumerated in some U_e in the given simultaneous enumeration of $\{U_n\}_{n\in\omega}$. Enumerate x in $U_{e,s}$. Let $A_s = U_{0,s}$.

Stage $s = 4t+1$. As in stage $4t$ with V_n for U_n and B for A.

Stage $s = 4t+2$. Let (x', e') be the first pair (x, e) with the following properties P1–P4. If there exist such a pair, enumerate $\psi_{s-1}(x') \in \tilde{U}_{e',s}$. Otherwise do nothing at this stage.

$$\text{(P1)} \qquad (x \in U_{e,s-1} - A_{s-1}) \text{ and } (\psi_{s-1}(x) \notin \tilde{U}_{e,s-1})$$

$$\text{(P2)} \qquad (\forall i)_{<e}\left[x \in U_{i,s-1} \Leftrightarrow \psi_{s-1}(x) \in \tilde{U}_{i,s-1}\right]$$

$$\text{(P3)} \qquad (\forall i)_{<e}\left[x \in \tilde{V}_{i,s-1} \Leftrightarrow \psi_{s-1}(x) \in V_{i,s-1}\right].$$

Define

$$\sigma_0 = \left\{i \mid i \le e \text{ and } \psi(x) \in \tilde{U}_{i,s-1}\right\}$$
$$\tau_0 = \{i \mid i \le x \text{ and } \psi_{s-1}(x) \in V_{i,s-1}\}$$
$$u_x = (\text{the unique } u)_{<s-1}\,[x \in U_{e,u+1} - U_{e,u}]$$

$$\text{(P4)} \quad (\forall i)_{i=0,1}(\exists v)(\exists y)(\exists\sigma_1)(\exists\tau_1)[u_x < v < s-1 \text{ and } y \in A_{i,v+1} - A_{i,v} \text{ and}$$
$$\sigma_1 \supseteq \{e\} \cup \sigma_0 \text{ and } \tau_1 \subseteq \tau_0 \text{ and}$$
$$\sigma_1 = \{i \mid i < e \text{ and } y \in U_{i,v}\} \text{ and } \tau_1 = \{i \mid i \le x \text{ and } y \in \tilde{V}_{i,v}\}].$$

Stage $s = 4t+3$. Similarly, attempt to enumerate some element $\psi^{-1}_{s-1}(x')$ in $\tilde{V}_{e',s}$ where A_i, U_n, $\tilde{U}_n$, V_n, $\tilde{V}_n$ and $\psi_{s-1}(x)$ in the preceding stage are replaced by B_i, V_n, $\tilde{V}_n$, U_n, $\tilde{U}_n$ and $\psi^{-1}_{s-1}(x)$. Also, in P3, replace $(\forall i)_{i<e}$ by $(\forall i)_{i\le e}$ to reflect the priority of $\tilde{U}_e$ over $\tilde{V}_e$.

Claim 1. The hypotheses of the Extension Lemma, Lemma 2, are met by the above arrays and simultaneous enumeration.

Proof of Claim 1. It is obvious that $B_i \searrow \tilde{U}_n = \emptyset = A_i \searrow \tilde{V}_n$ for all n and i since we enumerate only elements $\psi(x)$ in $\tilde{U}_n$. By "speeding up" the enumeration of A if necessary, we may assume that infinitely many x appear in each A_i before appearing in any U_j, $j > 0$, or V_j. Thus we need only verify (5) and (6) for states $\sigma \ne \emptyset$. We verify (5); the verification of (6) is dual.

Fix i, e and full e-state $\nu_0 = \langle e, \sigma_0, \tau_0\rangle$ with $\sigma_0 \ne \emptyset$. Assume that $D^{A_i}_{\nu_1}$ is finite for all $\nu_1 \ge \nu_0$. We must show that for each $\tau_1 \subseteq \tau_0$, only finitely many x whose e-state with respect to $\{V_n\}_{n\in\omega}$ is τ_1 are allowed to enter $\tilde{U}_{\sigma_0} = \bigcap\left\{\tilde{U}_n \mid n \in \sigma_0\right\}$. Since e-states increase with time it follows that $D^{B_i}_{\nu_0}$ must be finite also.

Let $e_0 = \max\sigma_0$. If follows by P2 of the construction and (8) of Lemma 3 , that $\tilde{U}_{e_0} \searrow \tilde{U}_n$ is finite for all $n < e_0$. Hence, if there are infinitely many x in $\tilde{U}_{\sigma_0}$, almost all such x enter $\tilde{U}_{e_0}$ only after $x \in \tilde{U}_n$, for all $n \in \sigma_0 - \{e_0\}$. But by P4 of the construction and our assumption that $D^{A_i}_{\nu_1}$ is finite for all $\nu_1 \geq \nu_0$, at most finitely many x already in $\bigcap \left\{ \tilde{U}_n \mid n \in \sigma_0 - \{e_0\} \right\}$ will be allowed to enter $\tilde{U}_{e_0}$ while the e-state of x with respect to $\{V_n\}_{n\in\omega}$ is some $\tau_1 \subseteq \tau_0$.

Claim 2 (9) is satisfied.

Proof of Claim 2. It is only here that we use that A_0, A_1 and B_0, B_1 are nontrivial splittings. This implies (by way of Theorem 2) that if U is any r.e. set such that $U \supseteq \overline{A}$, then $U \searrow A_i$ is infinite for each i.

We prove (9) by induction on e. Assume then that (9) is true for $i < e$. If $U_e \not\supseteq \overline{A}$ then $U_e \backslash A$ is finite by (7) and hence $\tilde{U}_e \backslash B$ and $\tilde{U}_e - B$ are finite. Assume then that $U_e \supseteq \overline{A}$. Choose y_0 such that for all $x \in \overline{A}$, $x \geq y_0$ implies

$$(\forall i)_{i<e} \left[\left[x \in U_i \Leftrightarrow \psi(x) \in \tilde{U}_i \right] \text{ and } \left[x \in \tilde{V}_i \Leftrightarrow \psi(x) \in V_i \right] \right].$$

We claim that for any $x_0 \in \overline{A}$, $x_0 \geq y_0$ implies $\psi(x_0) \in \tilde{U}_e$. Let $m = \max\{e, x_0\}$. Choose s_0 such that for all $s \geq s_0$, $\psi_s(x_0) = \psi_{s_0}(x_0)$. To see that $\psi(x_0)$ must be in $\tilde{U}_e$, define

$$U_{\sigma_0} = \bigcap \left\{ U_i \mid i \leq e \text{ and } U_i \supseteq \overline{A} \right\}.$$

Now $U_{\sigma_0} \supseteq \overline{A}$ and hence $U_{\sigma_0} \searrow A_i$ must be infinite for $i = 0, 1$ by the property of nontrivial splittings of A mentioned at the beginning of the proof of this claim. Define

$$\tau_0 = \{ i \mid i \leq x_0 \text{ and } \psi(x_0) \in V_i \}.$$

Now if $i \leq x_0$ and $i \notin \tau_0$, then $V_i - B$ is finite and hence $V_i \backslash B$ is finite by (7). But for any i, $V_i \backslash B$ finite implies $\tilde{V}_i \backslash A$ finite. Hence there exist $\sigma_1 \supseteq \sigma_0$ and $\tau_1 \subseteq \tau_0$, such that

$$\begin{aligned}(\forall i)_{i=0,1} (\forall t) (\exists v)_{>t} (\exists y) [y \in A_{i,v+1} - A_{i,v} & \text{ and} \\ \sigma_1 = \{ i \mid i \leq e \text{ and } y \in U_{i,v} \} & \text{ and } \tau_1 = \{ i \mid i \leq x_0 \text{ and } y \in \tilde{V}_{i,v} \}].\end{aligned}$$

But then by P4, $\psi(x_0)$ is eventually enumerated in $\tilde{U}_e$. The case of $\psi^{-1}(V_e) =^* \tilde{V}_e$ is handled similarly.

The combination of Lemmas (2) and (3) establish Theorem 1. □

2. Degrees of hemimaximal sets. The degrees of maximal sets are precisely the high degrees. One goal of this work was to find an orbit in which a much larger class of degrees is represented. It is not known, for instance, whether there is an orbit in which every nonzero r.e. degree is represented.

DEFINITION 7. $\mathbf{H} = \{\mathbf{deg}(A) \mid A \text{ is hemimaximal}\}$.

We summarize the results of this section:

(a) For every nonzero $\mathbf{c}$ r.e., there is $\mathbf{a} < \mathbf{c}$ such that $\mathbf{a} \in \mathbf{H}$. (Theorem 4)
(b) $\mathbf{H} \subseteq \mathbf{H}_1$. (Theorem 5)
(c) There is a nonzero degree $\mathbf{c}$, such that $\mathbf{c} \notin \mathbf{H}$. (Theorem 6)
(d) For all $\mathbf{a} < \mathbf{b}$, if $\mathbf{b} \in \mathbf{L}_1$, there is a degree $\mathbf{c} \notin \mathbf{H}$ such that $\mathbf{a} \leq \mathbf{c} \leq \mathbf{b}$. (Theorem 7).

By (a) and (b), $\mathbf{H}$ is an orbit class which contains more than only high degrees. (An orbit class is a class of r.e. degrees determined by an orbit of $\mathbf{Aut}(\mathcal{E})$.) This is the first example of such an orbit class. All previously known orbits consisted of only recursive sets or only high sets. By (a) and (d), $\mathbf{H}$ is not closed upwards in the r.e. degrees; this is the first example of any definable (in $\mathcal{E}$) class of r.e. sets with this property. We turn now to the proofs of the theorems mentioned above.

Theorem 4. *Suppose that $\emptyset <_{\mathbf{T}} C$. Then there is a hemimaximal set A, such that $\emptyset <_{\mathbf{T}} A \leq_{\mathbf{T}} C$.*

Proof. We will construct M maximal and A, B a nontrivial splitting of M such that A is nonrecursive and $A \leq_{\mathbf{T}} C$. $A \leq_{\mathbf{T}} C$ is guaranteed by the usual technique of permitting. M will be constructed by the usual e-state construction of a maximal set. The requirements, including those to make A nonrecursive, are thus as follows:

$$\mathbf{N}_e : \quad |\, \overline{M}\, | \geq e;$$

$$\mathbf{P}_e : \quad W_e \neq \overline{A};$$

$$\mathbf{Q}_e : \quad \overline{M} \text{ has almost constant } e\text{-state.}$$

The priority ranking is $\mathbf{Q}_0 < \mathbf{N}_0 < \mathbf{P}_0 < \mathbf{Q}_1 \ldots$ At the end of stage s, the elements of $\overline{M}$ are denoted by $m_0^s < m_1^s < \ldots.$ Note that the requirements $\mathbf{P}_e$ guarantee that A is nonrecursive. To insure as well that B is nonrecursive we may assume that C has low r.e. degree. This is enough since $\mathbf{deg}(A) \cup \mathbf{deg}(B) = \mathbf{deg}(M)$ and $\mathbf{deg}(M)$ is high.

Construction. *Stage $s+1$:*

Step 1. ($\mathbf{P}_e$)Let c be the least element enumerated in C at stage $s+1$. Let e be the least integer (if any) such that $W_e \cap A = \emptyset$ and there is an x satisfying

a) $x \in W_{e,s} \cap \overline{M}_s$,

b) $x \notin \{m_0^s, \ldots, m_e^s\}$ (priority),

c) $x > c$ (permitting).

Enumerate the least such x in A.

Step 2. ($\mathbf{Q}_i$)Find the least i (if any) and the least j for i such that $\sigma_{i,s}(m_i^s) < \sigma_{i,s}(m_j^s)$ and if x was enumerated in A at step 1, $m_j^s < x$. Enumerate m_i^s in B.

Lemma 0. *$A \leq_{\mathbf{T}} C$.*

Proof. If s is a stage such that $C_s[x] = C[x]$, then $A_s[x] = A[x]$.

Lemma 1. *For every e, $\mathbf{N}_e$ is satisfied.*

Proof. N_e is injured by each of $\mathbf{P}_0$,$\mathbf{P}_1$, ... $\mathbf{P}_{e-1}$ at most once each. It is injured thereafter by $Q_i, i \leq e$ at most 2^i times after it has ceased being injured by $\mathbf{Q}_j$, $j < i$. Thus $\mathbf{N}_e$ is injured only finitely often and is satisfied.

Lemma 2. *For every e, $\mathbf{Q}_e$ is satisfied.*

Proof. We show that for all $i, j \geq e$, if $m_i = \lim_s m_i^s$ and $m_j = \lim_s m_j^s$ then $\sigma_e(m_i) \geq \sigma_e(m_j)$. (This implies that $\mathbf{Q}_e$ is satisfied; since there are only finitely many e-states the sequence $\sigma_e(m_e) \geq \sigma_e(m_{e+1}) \geq \sigma_e(m_{e+2}) \ldots$ must have a limit.) If this is not true, there must be cofinitely many s such that $\sigma_{e,s}(m_i^s) < \sigma_{e,s}(m_j^s)$, $m_i^s = m_i$ and $m_j^s = m_j$. But at any such stage, m_j^s would be enumerated in B at step 2.

Lemma 3. *For every e, $\mathbf{P}_e$ is satisfied.*

Proof. Otherwise, let e be the least counterexample; we show that C is recursive contrary to hypothesis. Let σ be the e-state guaranteeed by Lemma 2; $e \in \sigma$ since $W_e = \overline{A}$. Let i_0 be such that $j \geq i_0$ implies that $m_j = \lim_s m_j^s$ has e-state σ. Let s_0 be such that $j \leq i_0 \Rightarrow m_j^{s_0} = m_j$ and $\sigma_{e,s}(m_{i_0}^{s_0}) = \sigma$.

Claim. Suppose that $s > s_0$ is such that $x = m_j^s$, $j \geq i_0$, and $\sigma_{e,s}(x) = \sigma$. Then for all $t \geq s$ there is $y \geq x$ such that $y = m_j^t$, $j \geq i_0$, and $\sigma_{e,t}(y) = \sigma$.

(The claim implies the lemma, since to answer $k \in C$, find $s \geq s_0$ such that there is $x \geq k$ such that x satisfies the hypotheses on x in the claim. Then by the claim, C does not permit below x after stage s or else x or some element $y \geq x$ is available at that stage to be enumerated in A at step 1.)

Proof of the claim. If $x \in \overline{M}$, then x satisfies the conclusion for all t. Otherwise, let t be such that x enters M at stage t (necessarily in step 2). then $x = m_i^t$ and there is $y = m_i^t, j > i$ such that $\sigma_{i,t}(x) < \sigma_{i,t}(y)$. Then $y > x$ since $j > i$ and $\sigma_{e,t}(y) = \sigma_{e,t}(x) = \sigma$ else $m_{i_0}^t$ would be enumerated in B instead of x. Thus y satisfies the conclusion at t. Now if $y \in \overline{M}$, then y satisfies the conclusion of the claim for all later t else we can repeat the above argument with y in place of x. □

Theorem 5. *For every high r.e. degree* $\mathbf{a}$, *there is a hemimaximal set A such that* $\mathbf{deg}(A) = \mathbf{a}$.

Proof. Our proof is a slight modification of Lachlan's version of Martin's proof (see [So2, Theorem XI.2.3]) that every high degree contains a maximal set. Given a high r.e. degree $\mathbf{a}$, there is an r.e. set D of degree $\mathbf{a}$, and an enumeration $\{D_s\}_{s\in\omega}$ of D such that if $\overline{D} = \{d_0 < d_1 < d_2 \dots\}$, $\overline{D}_s = \{d_0^s < d_1^s < \dots\}$, and $c_{\overline{D}}(x) = (\mu s)[d_x^s = d_x]$, then $c_{\overline{D}}$ dominates every total recursive function. We will construct disjoint r.e. sets A and B such that $M = A \cup B$ is maximal, $M \equiv_{\mathbf{T}} D$, B is nonrecursive, and $B \leq_{\mathbf{T}} A$. Thus A will be our desired hemimaximal set of degree $\mathbf{a}$. The requirements are the usual maximal set requirements:

$$\mathbf{Q}_e: \quad \overline{M} \text{ has almost constant } e\text{-state},$$

and the requirements to make B nonrecursive:

$$\mathbf{P}_e: \quad \overline{B} \neq W_e.$$

In addition we must insure that $B \leq_{\mathbf{T}} A$ and $A \equiv_{\mathbf{T}} D$. As in Theorem 4, $\overline{M}_s = \{m_0^s < m_1^s < \dots\}$.

CONSTRUCTION. *stage* $s+1$.

Step 1. (To satisfy $\mathbf{Q}_e$.) Given $e \leq i < j$, we say m_i^s is attracted to m_j^s for $\mathbf{Q}_e$ if

(10) $\qquad \sigma_{e-1,s}(m_i^s) = \sigma_{e-1,s}(m_j^s),$

(11) $\qquad \sigma_{e,s}(m_i^s) < \sigma_{e,s}(m_j^s)$, and

(12) $\qquad (\forall e' < e), [W_{e',s} \supseteq \{m_0^s, \dots, m_i^s\} \Rightarrow W_{e',s} \cap B_s \neq \emptyset].$

(The condition in (12) reflects the priority of $\mathbf{P}_{e'}$ over $\mathbf{Q}_e$ for $e' < e$.)
Choose the least i such that there are e, i, and j such that m_i^s is attracted to m_j^s for $\mathbf{Q}_e$ and such that

(13) $$(\exists y \leq m_i^s)[d_y^{s+1} \neq d_y^s].$$

Let e be least for i and j least for e. Enumerate each of $m_i^s, \dots m_{j-1}^s$ into A_{s+1}. (Thus $m_{i+k}^{s+1} = m_{j+k}^s$ for all $k \geq 0$.)

Step 2. Let $z = (\mu y)[d_y^{s+1} \neq d_y^s]$. If some $x \leq m_{2z+3}^s$ was enumerated in A at step 1, do nothing more. Else enumerate m_{2z+2}^s in A_{s+1} and m_{2z+3}^s in B_{s+1} unless one of these two integers has a higher $2z$-state than both of m_{2z}^s and m_{2z+1}^s. In that case, enumerate m_{2z}^s in A_{s+1} and m_{2z+1}^s in B_{s+1}. (The purpose of the choice in step 2 is this: if some m_i^s for $i < 2z$ is attracted to at least one of m_{2z}^s, m_{2z+1}^s, m_{2z+2}^s, or m_{2z+3}^s, then it will be attracted to at least one of the two of these elements which remain in $\overline{M}$ after step 2).

Lemma 1. *For each e, $\lim_s m_e^s = m_e$ is finite.*

Proof. By induction on e. Suppose that for $i < e$ there is a stage s_0 such that $m_i = m_i^{s_0}$. Let $t \geq s_0$ be such that $d_e^t = d_e$. Then step 2 never causes m_e^s to change at any stage $s > t$. However, step 1 causes m_e^s to change after s_0 only to increase its e-state. This happens 2^e times at most.

Lemma 2. *$D \equiv_{\mathbf{T}} M$, $B \leq_{\mathbf{T}} A$.*

Proof. $D \leq_{\mathbf{T}} M$ by step 2; if $m_{2e+3}^s = m_{2e+3}$, then $d_e^s = d_e$.

$M \leq_{\mathbf{T}} D$ since if $d_e^s = d_e$, then $e \in M$ iff $e \in M_s$.

$B \leq_{\mathbf{T}} A$ since elements enter B only at step 2 and when such an integer enters B, a smaller integer enters A at the same stage.

Lemma 3. *For each e,*

(a) *$\mathbf{Q}_e$ is met,*

(b) *$W_e \supseteq \overline{M} \Rightarrow W_e \cap B \neq \emptyset$. (This implies $\mathbf{P}_e$.)*

Proof. The proof is by induction on e. Suppose then that the lemma is true for $i < e$. This implies, by (a), that there is an $(e-1)$-state σ_0 and i_0 such that for all $i \geq i_0$, $\sigma_{e-1}(m_i) = \sigma_0$. Let s_0 be such that if $i \leq i_0$, then $m_i^{s_0} = m_i$. By (b), we may also assume that for all $i < e$, either $W_{i,s_0} \cap B_{s_0} \neq \emptyset$ or $(\exists j \leq i_0)[m_j \notin W_i]$. The significance of this last assumption is that clause (12) of the definition of attraction for $\mathbf{Q}_e$ will hold for all m_i^s where $i \geq i_0$ and $s \geq s_0$.

Now assume for a contadiction that $W_e \cap \overline{M}$ and $\overline{W}_e \cap \overline{M}$ are both infinite; let $\sigma_1 = \sigma_0 \cup \{e\}$.

Define a recursive function h as follows. If $x \leq m_{i_0}$, let $h(x) = 0$. For $x > m_{i_0}$, define

$$\begin{aligned} h(x) = (\mu s)[x \in M_s \text{ or } \sigma_{e,s}(x) > \sigma_0 \text{ or} \\ (\exists i)[x = m_i^s \text{ and } \sigma_{e,s}(m_i^s) = \sigma_0 \text{ and } (\exists j > i)[\sigma_{e,s}(m_j^s) \geq \sigma_1]]] \end{aligned} \tag{13}$$

(If the last clause of (13) holds, then $x = m_i^s$ is attracted to m_j^s for $\mathbf{Q}_{e'}$ for some $e' \leq e$.)

Claim. If $h(x)$ is defined by the last clause of (13), $x \in \overline{M}$, and $\sigma_e(x) = \sigma_0$, then $d_x^{h(x)} = d_x$.

To see that the claim gives the desired contradiction, note that the conlusion implies that $c_{\overline{D}}(x) < h(x) + 1$. However there are infinitely many x satisfying the hypotheses of the claim; namely any $x \in \overline{W}_e \cap \overline{M}$ which is not one of m_0, m_1, $\dots$, m_{i_0}. Thus $c_{\overline{D}}(x)$ does not dominate the recursive function $h(x) + 1$.

Proof of claim. Fix x and assume that the hypothesis of the claim holds for x. We will show that

$$(\forall s \geq h(x))(\exists k)[x < m_k^s \text{ and } \sigma_1 \leq \sigma_{e,s}(m_k^s)] \tag{14}$$

Now (14) implies the conclusion of the claim since it implies that for all s, there is an $e' \leq e$ such that x is attracted to some $y > x$ for the sake of $\mathbf{Q}_{e'}$ at all stages $s \geq h(x)$. We prove (14) by induction on $s \geq h(x)$. For $s = h(x)$, (14) is satisfied with $k = j$. Suppose $s \geq h(x)$ and m_k^s satisfies (14) for s but not for $s+1$. There are two cases.

Case 1. $m_k^s \in M_{s+1}$ by step 1. Then there are p and q, $n < p < q$ such that step 1 applies with $i = p$ and $j = q$. Now $p \leq k$ and $k \neq q$ since $m_k^s \in M_{s+1}$. Also $x < m_p^s$ since $x \notin M_{s+1}$. Thus $\sigma_{e,s}(m_q^s) \geq \sigma_{e,s}(m_k^s)$ since otherwise we would have chosen $j = k$ in step 1. Thus (14) holds at $s+1$ via $m_q^s = m_p^{s+1}$.

Case 2. $m_k^s \in M_{s+1}$ by step 2. Then (14) still holds at $s+1$ by virtue of the remark made at the end of step 2 of the construction.

We turn now to the proof of (b). Suppose that $W_e \supseteq \overline{M}$ and $B \cap W_e = \emptyset$. Then, using (a) for $i \leq e$, there are i_1 and s_1 such that

$$(\forall i \geq i_1)[\sigma_e(m_i) = \sigma_1],$$
$$(\forall i \leq i_1)[m_i^{s_1} = m_i],$$
$$(\forall i \leq i_1)[\sigma_{e,s_1}(m_i) = \sigma_e(m_i)].$$

Similar to the proof of theorem 4, we have the following claim.

Claim. Suppose that j and s_2 are such that $2j+3 \geq i_1$, $s_2 \geq s_1$, and for all i, if $i_1 \leq i \leq 2j+3$, then $\sigma_{e,s_2}(m_i^{s_2}) = \sigma_1$, then, for all $s \geq s_2$, $m_{2j+3}^{s_2} = m_{2j+3}^s$.

Proof of claim. Suppose to the contrary that s is least and k least for s such that $i_1 \leq k \leq 2j+3$ and $m_k^s \neq m_k^{s+1}$. There are two cases.

Case 1. Step 1 applied at $s+1$ with $i = k$. Now m_k^s can only be attracted to m_j^s for some $\mathbf{Q}_{e'}$, $e' \leq e$ for the conclusion of (12) does not hold with e in place of e'. Thus, we have that $\sigma_{e,s+1}(m_k^{s+1}) > \sigma_1$. But then we can argue by induction on $t \geq s$ that for all t, there is j such that $i_1 < j \leq k$ for which $\sigma_{e,t}(a_j^t) > \sigma_1$. This contradicts the choice of i_1.

Case 2. Step 2 applied at $s+1$. In this case, m_k^s is enumerated in A_{s+1} and m_{k+1}^s is enumerated in B_{s+1}. Since k is even, $k+1 \leq 2j+3$. Thus, $m_{k+1}^s \in W_{e,s}$. This gives a contradiction since $W_e \cap B$ was assumed to be empty.

The claim implies part (b) of the lemma for since there are infinitely many j satisfying the hypotheses of the claim, the conclusion implies that M is recursive, contradicting Lemma 2. □

We remark that not just any maximal set M will have the property of M in Theorem 5, namely that there is a nontrivial splitting, A, B of M such that $A \equiv_{\mathbf{T}} M$. The methods of [DW] may be used to construct a maximal set M such that if A, B is a nontrivial splitting of M, then A and B form a minimal pair.

Theorem 6. *There is an r.e. set C such that if $A \equiv_{\mathbf{T}} C$, then A is not hemi-maximal.*

Proof. Let $(\Phi_e, \Gamma_e, U_e, V_e)_{e\in\omega}$ be an effective listing of all quadruples where Φ_e,Γ_e are recursive functionals and U_e,V_e are disjoint r.e. sets. Then the requirements on C amount to the following

$$\mathbf{R}_e: \quad \Phi_e(C) = U_e \text{ and } \Gamma_e(U_e) = C \text{ implies } U_e \cup V_e \text{ is not maximal.}$$

We will attempt to insure that $U_e \cup V_e$ is not maximal by enumerating an array $\{T_{e,i}\}_{i\in\omega}$ of disjoint, finite, r.e. sets such that $T_{e,i} \not\subseteq U_e \cup V_e$. This guarantees that not only is $U_e \cup V_e$ not maximal, it is not even hyperhypersimple. Thus the requirements $\mathbf{R}_e$ will be divided into the following:

$$\mathbf{R}_{e,i}: \quad \Phi_e(C) = U_e \text{ and } \Gamma_e(U_e) = C \text{ implies } T_{e,i} \cap \overline{U_e \cup V_e} \neq \emptyset.$$

We will assume that the requirements are ordered in some ω-sequence, thereby inducing a priority ordering on them. We first give the strategy for meeting a single requirement; it will be convenient in describing it to drop all subscripts. The requirement thus becomes:

$$\mathbf{R}: \Phi(C) = an\Gamma(U) = implieT\cap$$

To attack $\mathbf{R}$ we proceed as follows.

Step 1. Wait for a stage s such that there is an x with

1) $l_\Gamma(s) > x$,
2) $(\exists y \leq \gamma(x,s))\,[y \notin U_s \cup V_s \cup T_s]$, and
3) $l_\Phi(s) > \gamma(x,s)$.

(Such x and s must exist if the hypotheses of $\mathbf{R}$ are satisfied and $U \cup V$ is not cofinite.) Given x and s, our action is to enumerate into T_{s+1} all $y \leq \gamma(x,s)$ such that $y \notin U_s \cup V_s$ and to restrain from C all $z \leq \phi(\gamma(x,s),s)$.

Notice that after step 1, $\mathbf{R}$ is satisfied temporarily since $T_{s+1} \cap \overline{U_s \cup V_s} \neq \emptyset$. $\mathbf{R}$ will be satisfied forever (with the finite restraint imposed by step 1) unless there is a stage $t > s$ such that $(U_t \cup V_t) \supseteq T_t = T_{s+1}$. Now if any element, say z, of $T_{s+1} - (U_s \cup V_s)$ is enumerated into $U_t - U_s$, we have, by the restraints imposed on C at step 1, that

$$\Phi_t(C_t; z) = \Phi_s(C_s; z) = U_s(z) \neq U_t(z)$$

and this disagreement is preserved forever with finite restraint. Thus we may assume that each element enumerated in T at stage $s+1$ is later enumerated into V by stage t. Now at stage $t+1$ we perform

Step 2. Remove the restraint on C imposed by step 1. Enumerate x (the x of step 1) into C_{t+1}.

Step 2 wins requirement $\mathbf{R}$ forever since we have that

$$\Phi(U;x) = \Phi_s(U_s;x) = C_s(x) \neq C_{t+1}(x).$$

The first equality is the crucial one and is true since $U_s[\gamma(x,s)] = U[\gamma(x,s)]$ because U and V are disjoint sets, $U_t \cup V_t \supseteq \{x \mid x \leq \gamma(x,s)\}$, and $U_t = U_s$.

To see that the strategies for the various $\mathbf{R}_{e,i}$ cohere, notice that each $\mathbf{R}_{e,i}$ imposes only finitely much restraint on C and thus $\mathbf{R}_{e,i}$ may be restarted for the sake of $\mathbf{R}_{e',i'}$ of higher priority as in standard arguments of Friedberg-Muchnik type. The only restraints on the sets $T_{e,i}$ are to make $T_{e,i}$ disjoint from $T_{e,j}$ if $i \neq j$ and it is clear that this can be done. We will omit the details of combining the strategies for meeting the $R_{e,i}$ since this is a straightforward application of the finite injury priority method. $\square$

The strategy for meeting requirement $\mathbf{R}$ of the previous theorem can be combined with other techniques to provide various strenghthenings of the result. For example, we can also insure that $\mathbf{c}$ contains no hemi-r-maximal set. Another example is

Theorem 7. $\overline{\mathbf{H}}$ *is dense in the low r.e. degrees. (i.e., if* $\mathbf{a}, \mathbf{b} \in \mathbf{L}_1$*, then there is* $\mathbf{c}$ *such that* $\mathbf{a} \leq \mathbf{c} \leq \mathbf{b}$ *and* $\mathbf{c}$ *contains no hemimaximal set.)*

Proof. We have added two requirements on C to those of the previous theorem. First, we require that $\mathbf{deg}(C) \leq \mathbf{b}$. This is easy to arrange via the standard technique of permitting. Notice that step 2 of the construction of Theorem 6 , the only step requiring enumeration in C, does not have to be performed at stage $t+1$. In fact, at any stage $v \geq t+1$, the situation still exists for diagonalization and if x is enumerated in C at stage v, $\mathbf{R}$ is satisfied. This is precisely the characteristic of a construction necessary for combination with permitting.

The second additional requirement on C is that $\mathbf{deg}(C) \geq \mathbf{a}$. Let A be any fixed set of degree $\mathbf{a}$. Requirement $\mathbf{R}$ now reads:

$$\mathbf{R}: \qquad \Phi(C \oplus A) = U \text{ and } \Gamma(U) = C \text{ implies } T \cap \overline{U \cup V} \neq \emptyset.$$

Given that $\mathbf{a}$ is low, there is a standard technique (due to Robinson [R]) for meeting a requirement of the form $\mathbf{R}$. For a complete description and examples of the use of this technique we refer the reader to Soare [So2,Chapter 12, Theorem 3.1]. Here we give an informal description. The principal difficulty in meeting this new version of requirement $\mathbf{R}$ is that we now longer have complete control in step 1 to restrain the computation $\Phi_s(C_s \oplus A_s) = U_s$ since we have no power to restrain A. Robinson showed that if C is low, there is a way to "certify" computations as being "A-correct" in such a way that incorrect computations are certified only finitely often. Specifically, if A is low,

$$\{j \mid (\exists n)[n \in W_j \text{ and } D_n \cap A = \emptyset]\} \leq_{\mathbf{T}} \emptyset'. \tag{15}$$

Here D_n is the finite set with canonical index n. (15) guarantees the existence of a recursive function g such that

$$\lim_s g(x,s) = \begin{cases} 1 & \text{if } (\exists n)[n \in W_j \text{ and } D_n \cap A = \emptyset] \\ 0 & \text{otherwise.} \end{cases}$$

We use g as follows. Suppose we wish to perform step 1 in the above construction. Then we enumerate n in an otherwise empty r.e. set W where D_n is the set of numbers used negatively by A in the computation. By the recursion theorem, we assume that we know the index j for W. We then search for a stage $t > s$ such that $D_n \cap A_t \neq \emptyset$ or $g(j,t) = 1$. In the former case, we do not believe the computation and do not start an attack. In the latter case, we say that the computation is "certified", and start the attack as above. Now it may be the case that the computation was not A-correct; we then repeat the process using the same set W. The point is that by (15), we will not certify infinitely many incorrect computations so that finally we will be justified in believing a computation and then the argument for $\mathbf{R}$ procedes just as before. A more detailed description of a use of lowness is given in the next section in Theorem 9.

There are no additional conflicts imposed by combining this lowness strategy and permitting. As these techniques, separately and in combination, are now quite standard and there are no surprises in their combination to meet all the requirements $\mathbf{R}_{e,i}$, we omit further details. $\square$

By employing techniques of Ambos-Spies and Fejer, [ASF,Theorem 3.3], we can extend Theorem 7 to the following.

Theorem 8. *There is an interval* $[\mathbf{a}, \mathbf{b}]$ *of low r.e. degrees containing no hemimaximal r.e. set. In fact, every interval of low r.e. degrees contains an interval with no hemimaximal set.*

3. Orbits containing complete sets. The question of just what sets are automorphic to complete sets is still open. Soare has conjectured that every r.e. set can be taken to a complete set by an automorphism. (Since creativeness is invariant under automorphisms, not every set is automorphic to an incomplete set.) Harrington (unpublished) has shown that not every r.e. set can be taken to a complete set by an automorphism Φ which is Σ_3-presented (i.e., the function f such that $\Phi(W_e) = W_{f(e)}$ is Σ_3). The significance of this is that all automorphisms constructed using Soare's machinery, such as that of section 1 of the current paper, are Σ_3-presented. Since there are complete hemimaximal sets, every hemimaximal set is automorphic to a complete set. Using the hemimaximal sets as a base and exploiting the idea of classifying r.e. sets by the types of nontrivial splittings that they have, we were able to extend this result to a much larger class of r.e. sets.

Definition 8. An r.e. set A is *halfhemimaximal* if there is a nontrivial splitting A_0, A_1 of A such that A_0 is hemimaximal.

More generally, if P is any property of r.e. sets, A is *halfP* if there is a nontrivial splitting A_0, A_1 of A such that A_0 has property P.

The importance of halfhemimaximality for this program is the following theorem.

Theorem 9. *If A is halfhemimaximal, then there is a $B \equiv_{\mathbf{T}} \emptyset'$ and $\Phi \in \mathbf{Aut}(\mathcal{E})$ such that $\Phi(A) = B$.*

Proof. Let A_0, A_1 be a nontrivial splitting of A with A_0 hemimaximal. Let B_0 be a complete hemimaximal set (which exists by Theorem 5). Let $\Phi \in \mathbf{Aut}(\mathcal{E})$ be such that $\Phi(A_0) = B_0$. Now $B = \Phi(A)$ is our desired complete set since $\emptyset' \equiv_{\mathbf{T}} B_0 \leq_{\mathbf{T}} B$. (The last inequality is since $B = B_0 \cup \Phi(A_1)$ and B_0 and $\Phi(A_1)$ are disjoint). □

Though we will show below that many sets are halfhemimaximal, it is easy to see from the following theorem that there are sets which are not.

Theorem 10. *If A is halfhemimaximal, then A has a maximal superset.*

Proof. Let A_0, A_1 be a nontrivial splitting of A and B_0, B_1 a nontrivial splitting of a maximal set B with $B_0 = A_0$. We claim that $A \subseteq^* B$. Otherwise, since B is maximal, $\overline{A} \subseteq^* B$. Thus $\overline{B}_0 =^* A_1 \cup B_1$ and so B_0 is recursive contradicting its being a part of a nontrivial splitting. □

Corollary 1. *Every non–low$_2$ degree contains a set which is not halfhemimaximal.*

Proof. Shoenfield[Sh] has shown that every nonlow$_2$ degree contains a set with no maximal superset. The result then follows directly from the theorem. □

We can improve Theorem 10 and its corollary with a little more work.

Theorem 11. *Every nonzero r.e. degree contains a nonhalfhemimaximal set.*

Proof. Suppose that we are given a nonrecursive set B. We will construct A nonhalfhemimaximal such that $B \equiv_{\mathbf{T}} A$. Let g be a 1-1 recursive function such that $g(\omega) = B$. Let $\{F_x\}_{x \in \omega}$ be a recursive sequence of disjoint finite sets such that $\cup\{F_x \mid x \in \omega\} = \omega$ and $\mid F_x \mid = x + 2$ for every x. We will insure that $A \equiv_{\mathbf{T}} B$ in the following way. At stage s of the construction, we will enumerate exactly one element into A chosen from $F_{g(s)}$. Thus $\mid A \cap F_x \mid \leq 1$ and $\mid A \cap F_x \mid = 1$ iff $x \in B$. It is easy to see from this that $A \equiv_{\mathbf{T}} B$.

The requirements that A be nonhalfhemimaximal can be put as follows. Let $(U_e, V_e)_{e \in \omega}$ be a recursive listing of all pairs of disjoint r.e. sets. Then the requirements are

$$\mathbf{N}_e : \quad U_e \subseteq A \text{ and } U_e \cup V_e \supseteq A \quad \Rightarrow \quad U_e \cup V_e \text{ is not maximal or } U_e \text{ is recursive.}$$

(The requirements are enough by Theroem 10. In fact they show something slightly stronger about A than only nonhalfhemimaximality.) We may assume in the light of the hypotheses on $\mathbf{N}_e$ that no element of U_e is enumerated in U_e before it is enumerated in A.

Construction *stage s.*

Let $x = g(s)$. We must enumerate one element of F_x into A. Choose the least element z from F_x such that for all $e \leq x$, $\mid F_x \cap V_{e,s} \mid \geq x+1$ implies that $z \in V_{e,s}$. This is possible since $\mid F_x \mid = x+2$ and the condition on z requires it only to be in the intersection of at most $x+1$ subsets of F_x each of cardinality at least $x+1$. This intersection is nonempty.

To complete the verification we need only show that $\mathbf{N}_e$ is satisfied for each e. So suppose that the hypotheses of $\mathbf{N}_e$ are satisfied and that $U_e \cup V_e$ is maximal. We will show that U_e is recursive. Since $U_e \cup V_e$ is maximal, there is an integer x_0 such that for all $x \geq x_0$, $\mid F_x \cap (U_e \cup V_e) \mid \geq x+1$ (see [Ro,Chapter 12, Theorem XIII]). Let z be fixed such that $z \in F_x$, $x \geq x_0$. We show how to decide if $z \in U_e$. Let s be a stage such that $\mid F_x \cap V_{e,s} \mid \geq n+1$ or $\mid F_x \cap A_s \mid = 1$. One or the other must happen since if $F_x \cap A = \emptyset$ then $\mid F_x \cap V_e \mid = x+1$. In the former case, $z \notin U_e$ since if z is later enumerated in A, $z \in V_e$. In the latter case, if $z \in A$ we can enumerate U_e and V_e until z appears in one or the other (this must happen since $A \subseteq U_e \cup V_e$) and if $z \notin A$ then $z \notin U_e$. $\square$

Despite Theorem 11, there are however large natural classes of r.e. sets consisting entirely of halfhemimaximal sets.

Theorem 12. *The following are halfhemimaximal:*

(a) *every low$_2$ simple set,*
(b) *every semilow$_{1.5}$ simple set,*
(c) *every d-simple set with a maximal superset.*

Proof. Robinson [R] proved that every low r.e. set has a maximal superset. Lachlan [L] and Bennison-Soare [BSo] extended this result and the technique used in proving it to low$_2$ and semilow$_{1.5}$ sets respectively. We will show how to extend Robinson's method to prove

(d) every low simple r.e. set is halfhemimaximal.

The same device we use to prove (d) can be used to extend the Lachlan and Bennison-Soare methods to prove (a) and (b).

Suppose then that A is low and simple. We will construct $B \supseteq A$, B maximal, which witnesses that A is halfhemimaximal. We briefly review the Robinson construction of B. Our exposition follows Soare [So2]. We have a set of movable markers Λ_e, $e \in N$. The position of Λ_e at the end of stage s, $\Lambda_{e,s}$, is the e^{th} element of $\overline{B}_s$. Thus, since we will always have $A_s \subseteq B_s$, we make B coinfinite by meeting for every e the requirements

$$\mathbf{R}_e : \qquad \lim_s \Lambda_{e,s} < \infty.$$

The usual strategy for making B maximal is to move $\Lambda_{e,s}$ to maximize its e-state. this guarantees that the following requirements are met:

$$\mathbf{Q}_e : \quad \text{almost all elements of } \overline{B} \text{ have the same } e\text{-state.}$$

In the usual construction of a maximal set (as in Theorem 4), at stage s we move Λ_e to a new element of the complement of B_s only if that new element has higher e-state than $\Lambda_{e,s}$. This requires us to move Λ_e only finitely often (2^e times at most after $\Lambda_0, \Lambda_1, \dots \Lambda_{e-1}$ have ceased moving). Further, it guarantees $\mathbf{Q}_e$ since the e-state of the final positions of the markers $\Lambda_e, \Lambda_{e+1}, \dots, \Lambda_{e+k}, \dots$ are nonincreasing in k.

This strategy conflicts with the requirement $A \subseteq B$ in the following way. Λ_0 for instance is moving so as to maximize its 0-state. Thus, if $(\exists x)[x \in \overline{B}_s$ and $x \in W_{0,s}]$, $\Lambda_{0,s}$ will move to x if $\Lambda_{0,s} \notin W_{0,s}$. However, later x may be enumerated in A causing Λ_0 to move again. We are thus tempted to move Λ_0 infinitely often if $W_0 \backslash A$ is infinite but $W_0 - A = \emptyset$.

To avoid this problem, we need to use the lowness of A to give us advice on which elements to which Λ_e might move are actually elements of A. The lowness of A guarantees that there is a recursive function f such that

$$\lim_s f(e,s) = \begin{cases} 1 & \text{if } W_e \cap A \neq \emptyset \\ 0 & \text{if } W_e \cap A = \emptyset \end{cases}$$

The oracle f is used in the following way. Suppose that we are attempting to move Λ_e to x to raise its e-state to σ. We then enumerate x in a certain test set U_σ. By the recursion theorem, we may assume that we know an index for U_σ, say $U_\sigma = W_e$. Using f, we locate a stage $t \geq s$ such that

$$U_\sigma \subseteq A_t \quad \text{or} \quad f(e,t) = 1.$$

By the properties of f, t must exist. Now only in the latter case do we move Λ_e to x. Of course it may still be that x is later enumerated in A; we then perform the same procedure with another x' which is in e-state σ. The point is that Λ_e will not be moved infinitely often by this process in an attempt to find an element in e-state σ. For, if Λ_e is moved infinitely often to an element in e-state σ, we must have that $\lim_s f(e,s) = 1$ and so $U_\sigma \cap \overline{A} \neq \emptyset$. But any element of $U_\sigma \cap \overline{A}$ will remain as Λ_e forever. Of course the above argument presumes that no higher priority Λ_i or $\sigma' > \sigma$ intervenes. In that case, we say that U_σ is injured and we begin anew with a "fresh" (empty) version of U_σ.

To make A halfhemimaximal, it suffices to enumerate B_0, B_1 such that $B_0 \subseteq A$, $B_0 \cup B_1 = B$, $B_0 \cap B_1 = \emptyset$, and B_0 nonrecursive. To see that this is enough, define $A_0 = B_0$ and $A_1 = B_1 \cap A$. Obviously A_0, A_1 split A and A_0 is nonrecursive. A_1 is nonrecursive since otherwise $B - A = B_1 - A_1$ is r.e. contradicting the simplicity of A. ($B - A$ is infinite since no low r.e. set is maximal.) Finally, B_1 is nonrecursive since $B_0 \leq_{\mathbf{T}} A$ and $B \equiv_{\mathbf{T}} B_0 \oplus B_1$. (The former implies that B_0 is low, and thus the latter implies B_1 is nonrecursive since B is high.)

We will define B_0 and B_1 as follows.

$$B_0 = \{x \mid \exists s(x \in A_{s+1} - B_s)\},$$
$$B_1 = \{x \mid \exists s(x \in B_{s+1} - A_{s+1})\}$$

Informally, B_0 is the set of those x which are enumerated in A before we have decided to enumerate them in B. B_0, B_1 satisfy all of the above requirements except perhaps the requirement that B_0 is nonrecursive. Thus we have the following requirements on B_0:

$$\mathbf{S}_e : \quad B_0 \neq \overline{W}_e$$

$\mathbf{S}_e$ will function as a negative requirement on B as follows. The ideal way for $\mathbf{S}_e$ to be satisfied is for an element of W_e to be enumerated in A before we enumerate it in B. This element is then in $B_0 \cap W_e$ and $\mathbf{S}_e$ is satisfied forever. Thus our strategy will be to restrain elements of W_e from B. If we restrain an infinite r.e. set W of such elements from B, we must have that $W \cap A \neq \emptyset$ by the simplicity of A giving us the desired witness to $W_e \cap B_0 \neq \emptyset$. This restraint is accomplished in the following construction by giving requirement $\mathbf{S}_e$ priority over all but $\mathbf{N}_0, \mathbf{N}_1, \ldots, \mathbf{N}_e$. Thus, we are entitled to enumerate a marker position, $\Lambda_{i,s}$ for $i \geq e$ into B only if by so doing it will increase its e-state (rather than only its i-state as is usual) if $W_{e,s} \cap B_{0,s} = \emptyset$ and $\Lambda_{i,s} \in W_{e,s}$. We now give the details of the construction.

We shall assume that $W_0 = A$. During the course of the construction, we will speed up the enumeration of A. Our enumeration at the end of stage s will be denoted A_s. We will always have $W_{0,s} \subseteq A_s$. Our construction will have for each stage $s+1$, substages e for $0 \leq e \leq s+1$. During stage $s+1$, A_{s+1} will denote the result in our enumeration of A at the beginning of the current substage. Given U an r.e. set, to *test* U at stage s is to do the following. Compute an index e of U from the construction. (This uses the Recursion Theorem.)Find $t \geq s$ such that

$$U_s \subseteq W_{0,t} \cup A_s \quad \text{or} \quad f(e,t) = 1.$$

Such a t always exist since $\lim_s f(e,s) = 1$ if $W_e \cap \overline{A} \neq \emptyset$. In the latter case, ($f(e,t) = 1$) we say that the test *succeeds*. For all sets U which we test in the construction, we will have exactly one element x in $U_s - A_s$. If the test fails (necessarily because $x \in W_{0,t}$), we will enumerate x in A_s. For each e and e-state σ, we will define an r.e. set $U_{e,\sigma}$. These r.e. sets will serve as test sets for the marker Λ_e. These sets $U_{e,\sigma}$ will be enumerated uniformly in e, σ and so we will be entitled to test them in the sense described above. Furthermore, from time to time we shall reset $U_{e,\sigma}$; to reset $U_{e,\sigma}$ at stage s means to replace $U_{e,\sigma}$ by a "fresh" empty version of $U_{e,\sigma}$. When $U_{e,\sigma}$ is referred to in the construction below, the current version is meant.

Construction.

Stage 0. Define $\Lambda_{e,s} = e$ for all e.

Stage $s+1$. *Substage* e, $0 \leq e \leq s$.

If $e = 0$ enumerate all of $W_{0,s+1}$ into A_{s+1}. Substage e occurs only if $\Lambda_{0,s}, \ldots, \Lambda_{e-1,s}$ have not moved during this stage. In this case, we determine whether Λ_e wants to move during this stage (to some x for the sake of some σ). There are two cases. Let $z = \Lambda_{e,s}$.

Case 1. $z \in A_{s+1}$. Then Λ_e wants to move at $s+1$. σ and x are determined as follows. In order of decreasing σ, σ an e-state, we search for a σ for which the following procedure results in a successful test of $U_{e,\sigma}$. Given σ, enumerate in the current version of $U_{e,\sigma}$ the least $x > z$ such that $x \notin B_s \cup A_{s+1}$ and $\sigma_{e,s}(x) = \sigma$. If no such x exists, proceed to the next σ else test $U_{e,\sigma}$. If the test succeeds, we have found σ and x; otherwise we choose the next least x satisfying $x > z$ and $\sigma_{e,s}(x) = \sigma$, enumerate it in $U_{e,\sigma}$ and test $U_{e,\sigma}$. If σ is not the empty e-state, this procedure must eventually end with a successful test or no further x in e-state σ. If σ is the empty e-state this procedure must end with a successful test since otherwise a cofinite set is enumerated for $U_{e,\sigma}$, and so $U_{e,\sigma} \cap \overline{A} \neq \emptyset$ but $\lim_s f(i,s) = 0$ where i is the index of $U_{e,\sigma}$. This contradicts the properties of f.

Case 2. $z \notin A_s$. Let $\sigma' = \sigma_{e,s}(z)$. For each $\sigma > \sigma'$, we do the following. We first determine whether z is restrained by some $\mathbf{S}_i$ for σ. Let j be the least integer such that $j \in \sigma - \sigma'$. Then z is restrained for σ by $\mathbf{S}_i$ if $i < j$, $i \in \sigma'$ and $W_{i,s} \cap A_s = \emptyset$. If z is not restrained for σ we perform the procedure described in case 1 for determining whether σ and x satisfy the condition of Λ_e wanting to move to x for the sake of σ. This test will be finite since for each such σ there are only finitely many x in state σ at stage $s+1$. This case may terminate without Λ_e wanting to move to any x.

If $\Lambda_{e,s}$ wants to move to x for either of the above two reasons, we move Λ_e by enumerating into B_{s+1} all y such that $\Lambda_{e,s} \leq y < x$. (Thus $x = \Lambda_{e,s+1}$.)

Substage $s+1$. At substage $s+1$, we enumerate all of A_{s+1} into B_{s+1}.

Lemma 1. $\lim_s \Lambda_{e,s}$ *exists, and so* $\overline{B}$ *is infinite.*

Proof. The proof is by induction on e. Let s_0 be the least stage such that $\Lambda_{i,s_0} = \lim_s \Lambda_{i,s}$ for all $i < e$. At s_0, $U_{e,\sigma}$ is reset for every σ. Assume for a contradiction that Λ_e moves infinitely often; each time Λ_e moves after s_0 it moves because it wants to. Let σ be the maximal e-state such that Λ_e wants to move for the sake of σ infinitely often. Let $s_1 \geq s_0$ be the least stage such that after s_1, Λ_e never wants to move for any $\sigma' > \sigma$. Then $U_{e,\sigma}$ is reset at s_1. Now for each stage $s \geq s_1$ at which Λ_e moves for the sake of σ to x, there is $t > s$ such that $f(i,t) = 1$ where i is the index of the current (final) version of $U_{e,\sigma}$. Thus, $\lim_s f(i,s) = 1$ so that $U_{e,\sigma} \cap \overline{A} \neq \emptyset$. However by the construction, there is at any stage s at most one element of $U_{e,\sigma,s} - A_s$ and that element becomes the marker position at the end of stage s. Thus, there is $s \geq s_1$ such that $\Lambda_{e,s} \in U_{e,\sigma} - A$ and that element remains the marker position of Λ_e forever. This contradiction establishes the lemma.

Lemma 2. *For every* e, $W_e \neq \overline{B}_0$, *only finitely many elements of* $\overline{B}$ *are permanently restrained from* B *by* $\mathbf{S}_e$, *and there is an* $(e+1)$*-state* σ *such that almost every element* $x \in \overline{B}$ *has* $(e+1)$*-state* σ.

Proof. The proof is by induction on e. Suppose then that the lemma is true for $i < e$. then there is an e-state σ such that almost all elements of $\overline{B}$ are in e-state σ. Therefore, either $e \in \sigma$ or $e \notin \sigma$. In the former case, obviously $W_e \neq \overline{B}$ and only finitely many elements of $\overline{B}$ are permanently restrained from B by $\mathbf{S}_e$. (Any element restrained by $\mathbf{S}_e$ is an element of W_e). Suppose then that $e \in \sigma$. Let i_0 and s_0 be such that

$$(\forall i > i_0)[\Lambda_i \text{ has final } e\text{-state } \sigma],$$
$$(\forall i \leq i_0)[\Lambda_{i,s_o} = \lim_s \Lambda_{i,s}],$$
$$(\forall i \leq i_0)[\sigma_{i,s_0}(\Lambda_{i,s_0}) = \sigma_i(\Lambda_{i,s_0})].$$

Define an r.e. set W as follows:

$$W = \{x \mid (\exists i \geq i_0)(\exists s \geq s_0)[x = \Lambda_{i,s} \text{ and } (\forall j)[i_0 \leq j \leq i \Rightarrow \sigma_{e,s}(\Lambda_{i,s}) = \sigma]\}.$$

It is obvious that W is r.e. and that $\overline{B} \subseteq^* W$. Suppose that $W_e = \overline{B}_0$. This implies that no element of W enters A before it enters B. We claim however that no element of W is enumerated in B before it is enumerated in A. This would provide the desired contradiction since W is infinite and A is simple. Suppose then to the contrary that x is the least element of W that enters B, say at stage s. Then $x = \Lambda_{i,s}$ for some $i > i_0$, $\sigma_{e,s}(x) \geq \sigma$, and x enters B because it wants to (else there is some smaller element of W entering B at stage s). Since x wants to enter B, necessarily by Case 2, there is $y = \Lambda_{j,s}$, such that $j > i$ and $\sigma_{i,s}(x) < \sigma_{j,s}(y)$. But x is restrained from B at stage s by $\mathbf{S}_e$ so in fact $\sigma_{e,s}(x) < \sigma_{i,s}(y)$. But then Λ_{i_0} would want to move to y at substage i_0 of stage s (or else y would be enumerated in A) since $\sigma_{e,s}(\Lambda_{i_0,s}) = \sigma < \sigma_{e,s}(y)$. Thus x never wants to move. This contradiction establishes that $W_e \neq B_0$. Notice that we have proved that if $\overline{B} \subseteq^* W_e$, then $W_e \cap B_0 \neq \emptyset$. Thus in this case too only finitely many elements are restrained from B by $\mathbf{S}_e$. To see that the final clause of Lemma 2 is correct, let σ_0 and σ_1 be the two $(e+1)$-states which ca appea infinitel ofte i B; $\sigma_0 = \sigma$ and $\sigma_1 = \sigma \cup \{e+1\}$. Suppose that $x = \lim_s \Lambda_{i,s}$, $y = \lim_s \Lambda_{j,s}$, where $i < j$ and $\sigma_0 = \sigma_{e+1}(x) < \sigma_{e+1}(y) = \sigma_1$ and x and y are not permanently restrained from B by $\mathbf{S}_0, \ldots \mathbf{S}_e$. Then Λ_i would want to move to y at infinitely many stages and so would move. Thus we cannot have infinitely many such pairs x and y and so there is an $(e+1)$-state (either σ_0 or σ_1) in which almost every element of $\overline{B}$ resides.

Turning now to the proof of part (c) of the theorem, recall the definition of d-simple.

Definition 9. An r.e. set A is *d-simple* if for every r.e. set X there is an r.e. set $Y \subseteq X$ such that $X \cap \overline{A} = Y \cap \overline{A}$ and if Z is any r.e. set with $Z - X$ infinite, then $(Z - Y) \cap A \neq \emptyset$.

So suppose A is d-simple and has a maximal superset M. Let X in the above definition be M. Given the resulting Y, let $M_1 = Y \backslash A$ and $M_0 = A - (Y \backslash A)$. Since $Y \cap \overline{A} = M \cap \overline{A}$, M_0, M_1 form a splitting of M. To see that M_0 is nonrecursive, suppose that $Z = \overline{M}_0$. Then $Z - M$ is infinite so $(Z - Y) \cap A \neq \emptyset$. But $(Z - Y) \cap A \subseteq M_0$. This, of course is a contradiction.

To complete the proof, there are two cases.

Case 1. M_1 is nonrecursive. Then M_0, M_1 form a nontrivial splitting of M and thus $M_0 \cap A$ and $M_1 \cap A$ form a nontrivial splitting of A with $M_0 \cap A = M_0$ hemimaximal witnessing that A is halfhemimaximal.

Case 2. M_1 is recursive. Then let B_0, B_1 be a nontrivial splitting of M_0. Let $M_0' = B_0$ and $M_1' = M_1 \cup B_1$. M_0', M_1' form a nontrivial splitting of M and as in Case 1, $M_0' \cap A$ and $M_1' \cap A$ witness that A is halfhemimaximal. $\square$

4. Orbits of hemiP sets. Various strong (but false) conjectures might be made after examining the result of section 1 that hemimaximal sets form an orbit. We had hoped that the following might be true.

Conjecture 1. (False) If P is a property of r.e. sets such that $\mathcal{P} = \{A \mid A \text{ has } P\}$ forms an orbit, then the hemiP sets form an orbit.

That conjecture 1 is false is easy to see. Perhaps the most striking counterexample is the property of creativeness. The counterexample relies on the following very easy theorem.

Theorem 13. *If K is a creative set and C is any r.e. set, then $K \oplus C (= \{2x \mid x \in K\} \cup \{2x+1 \mid x \in C\}$ is creative.*

Proof. We have $K \leq_1 K \oplus C$ by $f(x) = 2x$. Thus $K \oplus C$ is 1-complete and hence creative. □

Now the creative sets form an orbit. However if C is any nonrecursive r.e. set, $A_0 = \{2x \mid x \in K\}$ and $A_1 = \{2x+1 \mid x \in C\}$ form a nontrivial splitting of $K \oplus C$. It is obvious that not all sets A_1 which arise in this way are automorphic. For instance, A_1 may be maximal in an infinite, coinfinite recursive set (if C is maximal) or nowhere simple (if, say, C is nowhere simple).

One promising possibility motivated by Theorem 2 is to weaken Conjecture 1 by requiring that the splittings be Friedberg splittings and the property P imply simplicity. Thus, we weaken Conjecture 1 to

Conjecture 2. Let $\mathcal{C}$ be a class of r.e. simple sets which forms an orbit. Then

$$\{A \mid A \text{ is half of a Friedberg splitting of a set in } \mathcal{C}\}$$

forms an orbit.

We do not have much evidence in either direction for Conjecture 2. Obviously the problem is that we do not know many orbits. However we are able to establish Conjecture 2 for the next easiest case to the hemimaximal sets.

Definition 10. An r.e. set Q is k-quasimaximal if Q is the intersection of exactly k maximal sets which are pairwise infinitely different.

Soare showed, by applying his result on maximal sets, that the k-quasimaximal sets form an orbit. We establish Conjecture 2 for the quasimaximal sets by proving the following.

Theorem 14. *Let A and B be k-quasimaximal and A_0, A_1 and B_0, B_1 be Friedberg splittings of A and B respectively. Then there is $\Phi \in \mathbf{Aut}(\mathcal{E})$ such that $\Phi(A_0) = B_0$.*

Proof. We give the proof only for $k = 2$. The case $k > 2$ is similar. Let M_0, M_1 be maximal sets such that $M_0 \cap M_1 = A$; we may assume that $M_0 \cup M_1 = N$. Similarly, let N_0, N_1 be maximal sets such that $N_0 \cup N_1 = N$ and $\mathbf{N}_0 \cap N_1 = B$. Let R and S be recursive sets such that $\overline{M}_0 \subseteq R$, $\overline{M}_1 \subseteq \overline{R}$, $\overline{N}_0 \subseteq S$, and $\overline{N}_1 \subseteq \overline{S}$. (These r.e. sets exist by applying the r.e reduction principle to the pairs M_0, M_1 and N_0, N_1.)

Now observe that $R \cap A_0$ and $R \cap A_1$ are nonrecursive. For if $R \cap A_0$ is recursive, then $R - A_0$ is recursive so that $R - A$ is r.e. by the definition of Friedberg splitting. However, $R - A$ is infinite and A is simple. Thus we have that $R \cap A_0$ and $R \cap A_1$ form a nontrivial splitting of $R \cap A$ and similarly $S \cap B_0$ and $S \cap B_1$ form a nontrivial splitting of $S \cap B$. Now within the recursive set R, $R \cap A$ is maximal. ($\overline{A} \cap R = \overline{M}_0$ and the latter is cohesive.) Similarly $S \cap B$ is maximal in S. Thus by Theorem 3, there is an automorphism Φ_0 of $\mathcal{E}$ restricted to R which maps R to S and $R \cap A_i$ to $S \cap B_i$ for $i = 0, 1$. Likewise, there is an automorphism Φ_1 of the r.e. sets restricted to $\overline{R}$ which maps $\overline{R} \cap A_i$ to $\overline{S} \cap B_i$ for $i = 0, 1$. Piecing together Φ_0 and Φ_1 gives the desired automorphism. □

Corollary 1. *The class of Friedberg splittings of k-quasimaximal sets form an orbit for each k.*

Proof. We have show that if A and B are halves of Friedberg splittings of 2-quasimaximal sets then there is a Φ such that $\Phi(A) = B$. Conversely, if A is half of a Friedberg splitting of a 2-quasimaximal sets and $\Phi(A) = B$, then B must be half of a Friedberg splitting of a 2-quasimaximal set since this is an elementary property and so must be preserved under automorphism. □

Corollary 2. *There are four different automorphism types of hemi-2-quasimaximal sets.*

Proof. Let M_0, M_1 be maximal sets such that $M_0 \cup M_1 = N$ and $M_0 \cap M_1 = A$. Let A_0, A_1 be a notrivial splitting of A. Let R be a recursive set such that $\overline{M}_0 \subseteq R$ and $\overline{M}_1 \subseteq \overline{R}$ and further that all of $R \cap A_0$, $R \cap A_1$, $\overline{R} \cap A_0$, and $\overline{R} \cap A_1$ are infinite. Now one of the following is true:

(16) $\qquad R \cap A_0$ and $\overline{R} \cap A_1$ are nonrecursive, or

(17) $\qquad R \cap A_1$ and $\overline{R} \cap A_0$ are nonrecursive.

For if both of the above fail then one of $\overline{M}_0$, $\overline{M}_1$, A_0, or A_1 is recursive. We will suppose that (16) is true. Of the remaining pair of sets, there are four possibilities.

(1) $R \cap A_1$ and $\overline{R} \cap A_0$ are nonrecursive,
(2) $R \cap A_1$ and $\overline{R} \cap A_0$ are recursive,
(3) $R \cap A_1$ is nonrecursive and $\overline{R} \cap A_0$is recursive, and
(4) $R \cap A_1$ is recursive and $\overline{R} \cap A_0$ is nonrecursive.

All four possibilities can occur; note that (3) and (4) are symmetric.

Case 1. We have already seen in the Theorem that any sets A_0 and A_1 which arise in this way are of a single orbit.

Case 2. In this case, we see that $R \cap A_0$ is maximal in $R - (R \cap A_1)$ which is a recursive set. Since all such sets are automorphic by Soare's theorem, all such sets A_0 arising is this way are automorphic. A_1 is also of this type. This class is distinct from case (1) since $R - (R \cap A_1)$ witnesses that A_0, A_1 do not form a Friedberg splitting of A.

Case 3 (and Case 4 by symmetry). In this case, $S = \overline{R} - A_0$ is recursive and so $S \cup A_1 = S \cup (A_1 \cap R)$ is nonrecursive. This implies that $S \cup A_1$ and A_0 form a nontrivial splitting of M_1. Thus A_0 is hemimaximal. This determines the automorphism type of A_0 by Theorem 3. This class is distinct from those of cases 1 and 2 simce in neither of those cases is A_0 hemimaximal (this can easily be checked). Now A_1 is not hemimaximal but it is easy to show again that all such A_1 are automorphic (by piecing together automorphisms on R and $\overline{R}$ as in the Theorem 13). This gives the fourth and final automorphism type.

Summarizing, there are four automorphism types of hemi-2-quasimaximal sets:

(1) half of a Friedberg splitting of a 2-quasimaximal set,
(2) maximal in an infinite-coinfinite recursive set R,
(3) hemimaximal, and
(4) not hemimaximal but half of a nontrivial splitting of a 2-quasimaximal set for which the other half is hemimaximal. □

The above classification for hemi-2-quasimaximal sets can be extended to hemi-k-quasimaximal sets for $k > 2$ without much difficulty (but much detail).

As further evidence for Conjecture 2, we state without proof

Theorem 15. *Let $\mathcal{C}$ be a class of hyperhypersimple sets for which the lattices of supersets are pairwise Σ_3-isomorphic. Then every pair of sets in*

$$\{A \mid A \textit{ is half of a Friedberg splitting of a set in } \mathcal{C}\}$$

are automorphic.

The proof of Theorem 15 is a combination of our technique of Theorem 3 and the proof of Maass that any two sets in $\mathcal{C}$ are automorphic.

5. Further remarks and open questions. Theorem 3 justified our hope of finding orbits through splittings. While Theorem 13 gave a negative result along these lines, we still have some hope that Conjecture 2 or some conjecture like it might be true. A test question in this program is the following:

Question 1. Let A and B be promptly simple and low. Let A_0, A_1 and B_0, B_1 be Friedberg splittings of A and B. Is there $\Phi \in \mathbf{Aut}(\mathcal{E})$ such that $\Phi(A_0) = B_0$? (Maass has shown that there is $\Phi \in \mathbf{Aut}(\mathcal{E})$ with $\Phi(A) = B$.)

Notice that even Theorem 13 does not refute Conjecture 2 for nonsimple sets since the splittings produced there are not Friedberg splittings.

Question 2. What are the automorphism types among Friedberg splittings of a creative set?

The hemimaximal sets form an interesting degree-theoretic class. Left open by the results of section 2 is

Question 3. Are there degrees containing no hemimaximal set which are not low?

Concerning the halfhemimaximal sets, a question which we have been unable to answer is

Question 4. Is K halfhemimaximal?

We have shown that creative sets are halfhemi-2-quasimaximal.

We have introduced the properties hemimaximal and halfhemimaximal in this paper because of their use in finding orbits of $\mathbf{Aut}(\mathcal{E})$. However, for various properties P, the hemiP and halfP sets might prove interesting in their own right. For instance, in contrast with Theorem 11, we have been able to show that although there are nonhalfhemisimple sets, there are degrees containing only halfhemisimple sets.

References

[[**ASF**]]A[redacted]bos Spies and P. Fejer, *Degree theoretic splitting properties of recursively enumerable sets* (to appear).

[[**BS**]]B[redacted]nnison and R.I. Soare, *Some lowness properties and computational complexity sequences*, Theoretical Computer Science **6** (1978), 233-254.

[[**D**]]R[redacted]Downey, *Nowhere simplicity in matroids*, J. Austral. Math. Soc. (Ser A) **35** (1983), 28-45.

[[**DW**]]D[redacted]wney and L. Welch, *Splitting properties of r.e. sets and degrees*, J. Symbolic Logic **51** (1986), 88-109.

[[**L**]]A[redacted] Lachlan, *Degrees of recursively enumerable sets which have no maximal supersets*, J. Symbolic Logic **33** (1868), 431-443.

[[**M**]]W[redacted] Maass, *Characterization of recursively enumerable sets with supersets isomorphic to all recursively enumerable sets*, Trans. A. M. S. **279** (1983), 311–336.

[[**MS**]]M[redacted]aass and M. Stob, *The intervals of the lattice of Recursively enumerable sets determined by major subsets*, Annals of Pure and Applied Logic **24** (1983), 189-212.

[[**R**]]R[redacted]W. Robinson, *The inclusion lattice and degrees of unsolvability of the recursively enumerable sets*, Ph. D. dissertation, Cornell University (1966).

[[**Ro**]] H. [redacted]ogers, Jr., *Theory of recursive functions and effective computability*, McGraw-Hill, New York.

[[**Sh**]] J. [redacted]. Shoenfield, *Degrees of classes of r.e. sets*, J. Symbolic Logic **41** (1976), 695–696.

[[**So1**]] R. I. [redacted]oare, *Automorphisms of the lattice of recursively enumerable sets, Part I: Maximal sets*, Ann. of Math. **100** (1974), 80–120.

[[**So2**]] R. [redacted] Soare, *Recursively Enumerable Sets and Degrees*, Springer–Verlag, Berlin, Heidelberg, New York, Tokyo, 1986.

[[**St**]] M[redacted] Stob, *Invariance of properties under automorphisms of the lattice of recursively enumerable sets*, Pacific J. of Math. **100** (1982), 445–471.