RELATIVIZING CHAITIN'S HALTING PROBABILITY

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Abstract. As a natural example of a 1-random real, Chaitin proposed the halting probability Ω of a universal prefix-free machine. We can relativize this example by considering a universal prefix-free oracle machine U. Let Ω_U^A be the halting probability of U^A; this gives a natural uniform way of producing an A-random real for every $A \in 2^\omega$. It is this operator which is our primary object of study. We can draw an analogy between the jump operator from computability theory and this Omega operator. But unlike the jump, which is invariant (up to computable permutation) under the choice of an effective enumeration of the partial computable functions, Ω_U^A can be vastly different for different choices of U. Even for a fixed U, there are oracles $A =^* B$ such that Ω_U^A and Ω_U^B are 1-random relative to each other. We prove this and many other interesting properties of Omega operators. We investigate these operators from the perspective of analysis, computability theory, and of course, algorithmic randomness.

1. Introduction

We begin with a brief review of algorithmic randomness, focusing on Chaitin's halting probability Ω. For a more complete introduction, see Li and Vitanyi [15] or the upcoming monograph of Downey and Hirschfeldt [4].

A partial computable function $M\colon 2^{<\omega} \to 2^{<\omega}$ is called a *prefix-free machine* if whenever $\sigma, \tau \in \text{domain}(M)$, then σ is not a proper prefix of τ. There is a *universal* prefix-free machine, i.e., a prefix-free machine U such that for each prefix-free machine M there is a string $\tau \in 2^{<\omega}$ for which $(\forall\sigma)\ U(\tau\sigma) = M(\sigma)$ (or both $U(\tau\sigma)$ and $M(\sigma)$ diverge). We say that U *simulates* M by the prefix τ. The importance of prefix-free machines to algorithmic information theory is well established, originating independently in the seminal work of Levin [14] and Chaitin [2]. They modified Kolmogorov complexity to capture effective randomness for real numbers (an earlier approach is described in Levin [13]). For any prefix-free machine M, define $K_M(\sigma) = \min\{|\tau| \mid M(\tau) = \sigma\}$. If U is universal, then for each partial computable prefix-free M, there is a constant $c \in \omega$ such that $(\forall\sigma)\ K_U(\sigma) \le K_M(\sigma) + c$. We write K for K_U and call this *prefix-free Kolmogorov complexity*. Note that, up to a constant, K is independent of the choice of U. We say that $A \in 2^\omega$ is 1-*random* iff $(\forall n)\ K(A \upharpoonright n) > n - \mathcal{O}(1)$.

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Schnorr observed that this definition of randomness is equivalent to an earlier definition given by Martin-Löf [16] (see the next section).

Following Chaitin, if $M\colon 2^{<\omega} \to 2^{<\omega}$ is a prefix-free machine, we consider the probability Ω_M that M halts on (a prefix of) an infinite input string. Formally, $\Omega_M = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|}$. Such halting probabilities define reals which are the limits of monotonically increasing computable sequences of rationals. These are called *c.e.* (or *left computable*) reals. Furthermore, every c.e. real is the halting probability of some prefix-free machine.

Chaitin [2] proposed *the halting probability* $\Omega = \Omega_U$ as a natural example of a 1-random real, where U is any universal prefix-free machine. It is not hard to prove that Ω is 1-random; a straightforward generalization is proved in Proposition 3.1 below. Note that we call Ω *the* halting probability, even though the definition is machine dependent. This is akin to the situation in computability theory where *the* halting problem $\emptyset'$ also depends on the choice of universal machine. In that case, the machine dependence of $\emptyset'$ is entirely superficial; Myhill's theorem [17] states that it is always the same up to a computable permutation of the natural numbers. Here a similar situation occurs: any two versions of Ω are *Solovay* equivalent [26].

For $X, Y \in 2^\omega$, which we can think of as reals in $[0,1]$, we write $Y \leq_S X$ (Y is *Solovay reducible* to X) to mean that there is a $c \in \omega$ and a partial computable $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ such that if $q < X$, then $\varphi(q)\downarrow < Y$ and $Y - \varphi(q) < c(X - q)$. The idea is that given any sequence of rationals approximating X from below, we can generate a sequence of rationals approximating Y from below that converges no slower. We say that a c.e. real X is *Solovay complete* if $Y \leq_S X$ for every c.e. real Y. It is not difficult to prove that Ω is Solovay complete [26], which implies that Ω is well-defined up to Solovay equivalence.[1] Two further theorems should be mentioned.

Theorem 1.1 (Calude, Hertling, Khoussainov, Wang [1])**.** *If $A \in 2^\omega$ is a Solovay complete c.e. real, then $A = \Omega_U$ for some universal prefix-free machine U.*

Theorem 1.2 (Kučera and Slaman [10])**.** *Suppose that $X \in 2^\omega$ is a 1-random c.e. real. Then X is Solovay complete.*

Together, these results imply that the 1-random c.e. reals, the Solovay complete c.e. reals, and the possible values of Ω all coincide. We will consider the relativization of this class in Section 4.

Relativizing Ω. As we have already indicated, one can draw an analogy between the (measures of) domains of prefix-free machines in algorithmic randomness and the domains of partial computable functions in classical computability theory. Let us consider this analogy in detail.

[1] Solovay reducibility implies Turing reducibility on the c.e. reals, so the Turing degree of Ω is well-defined. Indeed, it is well known that $\Omega \equiv_T \emptyset'$.

(i) The domains of partial computable functions are exactly the c.e. sets, while the measures of the domains of prefix-free machines are exactly the c.e. reals.
(ii) The canonical example of a non-computable set is the halting problem $\emptyset'$, i.e., the domain of a universal partial computable function. The canonical example of a 1-random real is Ω, the halting probability of a universal prefix-free machine.
(iii) $\emptyset'$ is well-defined up to computable permutation, while Ω is well-defined up to Solovay equivalence.

How much further can this analogy be taken? Relativizing the definition of $\emptyset'$ gives the jump operator. If $A \in 2^\omega$, then A' is the domain of a universal A-computable machine. Myhill's theorem relativizes, so A' is well-defined up to computable permutation. Furthermore, if $A \equiv_T B$, then A' and B' differ by a computable permutation. *A fortiori*, the jump is well-defined on the Turing degrees. The jump operator plays an important role in computability theory; it gives a natural, uniform and degree invariant way to produce, for each $A \in 2^\omega$, a set A' with Turing degree strictly above A.

What happens, on the other hand, when the definition of Ω is relativized? In some ways, the situation is as nice as one would expect. First, note that for any oracle $A \in 2^\omega$ there is an A-computable prefix-free machine which is universal with respect to all such machines. We will find it convenient to use a *universal prefix-free oracle machine* $U^A\colon 2^{<\omega} \to 2^{<\omega}$, which essentially gives us a coherent choice of universal machines over all oracles (see Section 3). Let $\Omega_U^A = \sum_{U^A(\sigma)\downarrow} 2^{-|\sigma|}$ and $K^A(\sigma) = \min\{|\tau| \mid U^A(\tau) = \sigma\}$. By relativizing Chaitin's theorem, Ω_U^A is *A-random*; in other words, $(\forall n)\ K^A(\Omega_U^A \upharpoonright n) > n - \mathcal{O}(1)$. This much is well known. It is also clear that Ω_U^A is an A-c.e. real and well-defined up to A-Solovay equivalence. Furthermore, Theorems 1.1 and 1.2 can both be relativized (the latter requires care in the context of prefix-free oracle machines and is Theorem 4.3 below).

What goes wrong? One might hope for Ω_U^A to be well-defined, not just up to A-Solovay equivalence, but even up to Turing degree. Similarly, we might hope for Ω_U to be a degree invariant operator: in other words, if $A \equiv_T B$ then $\Omega_U^A \equiv_T \Omega_U^B$. Were this the case, Ω_U would provide a counterexample to a longstanding conjecture of Martin: it would induce an operator on the Turing degrees which is neither increasing nor constant on any cone. But as we show in Theorem 6.7, there are oracles $A =^* B$ (i.e., A and B agree except on a finite set) such that Ω_U^A and Ω_U^B are vastly different. In particular, we can ensure that Ω_U^A is a c.e. real while making Ω_U^B as random as we like. It follows easily that the Turing degree of Ω_U^A generally depends on the choice of U, and in fact, that the degree of randomness of Ω_U^A can vary drastically with this choice.

If U is a universal prefix-free oracle machine, then we call $\Omega_U\colon 2^\omega \to [0,1]$ an *Omega operator*. Basic properties of Omega operators are discussed in Sections 3 and 4. In Section 5, it is proved that the range of an Omega operator has positive measure and that every 2-random real is in the range of *some*

Omega operator. This is not true for every 1-random real. Section 6 turns to the question of degree non-invariance. We prove that every Omega operator maps a set of positive measure to a c.e. real. The preimage of any non-c.e. real has measure zero, so even for relativized halting probability the c.e. reals play a special role. We also prove that for any $Z \in 2^\omega$, every Omega operator maps a set of positive measure to the Z-random reals. It is now a simple consequence of *Kolmogorov's 0–1 law* (see next section) that there are reals $A =^* B$ such that Ω_U^A is a c.e. real and Ω_U^B is Z-random. Degree non-invariance is immediate.

In Section 7, we prove that $A \in 2^\omega$ is mapped to a c.e. real by *some* Omega operator iff Ω is A-random. Such an A is called *low for* Ω. (This property does not depend on the particular choice of Ω.) More interesting is the characterization in Section 8 of the reals $A \in 2^\omega$ which are mapped to c.e. reals by *every* Omega operator. These are proved to be the *K-trivial reals*: reals which have minimum prefix-free initial segment complexity. This class has been studied thoroughly in recent work [6, 19]. We prove that the K-trivial reals are the only reals for which the Turing degree of Ω_U^A does not depend on the choice of U.

In the final section, we consider the analytic behavior of Omega operators. We prove that Omega operators are lower semicontinuous but not continuous, and moreover, that they are continuous exactly at the 1-generic reals. We also produce an Omega operator which does not have a closed range. On the other hand, we prove that every non-2-random in the closure of the range of an Omega operator is actually in the range. As a consequence, there is an $A \in 2^\omega$ such that $\Omega_U^A = \sup(\operatorname{range} \Omega_U)$.

2. Preliminaries

We use "real" to denote a member of the Cantor space 2^ω equipped with the standard product topology. When convenient, we also think of reals as elements of $[0, 1]$. The basic clopen sets of Cantor space are of the form $[\sigma] = \{\sigma A \mid A \in 2^\omega\}$, where $\sigma \in 2^{<\omega}$. Every open set is of the form $[V] = \bigcup_{x \in V}[x]$, for some $V \subseteq 2^{<\omega}$. Let μ denote the standard Lebesgue measure on 2^ω; in particular, $\mu[\sigma] = 2^{-|\sigma|}$. For $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \preceq \tau$ to indicate that σ is a prefix of τ and $\sigma \prec \tau$ if it is a proper prefix. We write $\sigma \prec A$ to mean that $\sigma \in 2^{<\omega}$ is an initial segment of the real $A \in 2^\omega$. It is natural to identify a finite string $\sigma \in 2^{<\omega}$ with the dyadic rational having binary expansion $\sigma 0^\omega$.

Before prefix-free Kolmogorov complexity was used to characterize randomness, Martin-Löf [16] defined the random reals as those that pass all "effectively presented statistical tests". Each test is given as a presentation of the measure zero set of reals that *fail* the test. Formally, a *Martin-Löf test* is a computable sequence $\{V_i\}_{i\in\omega}$ of computably enumerable subsets of $2^{<\omega}$ such that $\mu([V_i]) \leq 2^{-i}$. A real $X \in 2^\omega$ *passes* the Martin-Löf test $\{V_i\}_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega}[V_i]$. A real which passes all Martin Löf tests is called *Martin-Löf random*, which Schnorr proved equivalent to being 1-random.

To capture stronger notions of randomness, take the sets $V_i \subseteq 2^{<\omega}$ to be uniformly c.e. relative to an oracle $A \in 2^\omega$. Then $\{V_i\}_{i\in\omega}$ is called an *A-Martin-Löf test* and, relativizing Schnorr's result, the A-random reals are exactly the reals which pass every such test. Of special interest are the $\emptyset^{(n-1)}$-random reals, which are called *n-random*.

Next we recall some of the results which are needed below. We repeatedly use the following elegant theorem of van Lambalgen [27] from the theory of algorithmic randomness (see [5] for a short proof).

van Lambalgen's theorem.

(i) *$A \oplus B$ is 1-random iff $A \in 2^\omega$ is 1-random and $B \in 2^\omega$ is A-random.*
(ii) *If $A \in 2^\omega$ is 1-random and $B \in 2^\omega$ is A-random, then A is B-random.*

We also require a few important theorems from classical measure theory.

The Lebesgue density theorem. *If $\mathcal{S} \subseteq 2^\omega$ is measurable, then for almost every $A \in \mathcal{S}$,*

$$\lim_{n\to\infty} 2^n \mu([A \upharpoonright n] \cap \mathcal{S}) = 1.$$

A proof of Lebesgue density can be found in [22]. We do not need the full strength of Lebesgue's theorem. Instead, we use the following corollary which says that if a class has positive measure then there is a neighborhood in which the local measure is arbitrarily close to one.

Corollary 2.1. *Let $\mathcal{S} \subseteq 2^\omega$ have positive measure. For every $\varepsilon > 0$, there is a $\sigma \in 2^{<\omega}$ such that $2^{|\sigma|}\mu([\sigma] \cap \mathcal{S}) \geq 1 - \varepsilon$.*

This corollary easily implies another result which we use below. Recall that for $X, Y \in 2^\omega$, we write $X =^* Y$ if X and Y agree on a cofinite set.

Kolmogorov's 0–1 law. *If $\mathcal{S} \subseteq 2^\omega$ is a measurable class closed under $=^*$, then $\mu(\mathcal{S})$ is either zero or one.*

Proof. Assume that $\mu\mathcal{S} > 0$. Take an $\varepsilon > 0$. By the Lebesgue density theorem, there is a $\sigma \in 2^{<\omega}$ such that $\mu([\sigma] \cap \mathcal{S}) \geq 2^{-|\sigma|}(1 - \varepsilon)$. But $\mathcal{S}$ is closed under $=^*$. So, for each τ with $|\tau| = |\sigma|$ we have $\mu([\tau] \cap \mathcal{S}) = \mu([\sigma] \cap \mathcal{S})$. Therefore, $\mu\mathcal{S} \geq 1 - \varepsilon$. But $\varepsilon > 0$ was arbitrary, hence $\mu\mathcal{S} = 1$. $\square$

Additionally, in Section 5 we use the theorem of Lusin that *analytic* sets (i.e., projections of Borel sets) are measurable. See Sacks [25] for details.[2]

K-trivial reals. We finish this section by reviewing an important class of reals: $A \in 2^\omega$ is called *K-trivial* if

$$(\forall n)\ K(A \upharpoonright n) \leq K(n) + \mathcal{O}(1).$$

The K-trivial reals are the central topic of Section 8 and are also useful elsewhere. In [19] it is proved that A is K-trivial iff A is *low for 1-randomness*,

[2] Sacks actually proves that Π^1_1 classes are measurable. But every analytic subset of 2^ω is a Σ^1_1 class relative to an appropriate oracle, so Lusin's theorem follows by relativization.

that is, each 1-random set is also 1-random relative to A. Another notion which turns out to be equivalent is due to Kučera [9]: A is a *base for* 1-*randomness* if $A \leq_T Z$ for some Z which is 1-random relative to A. By the Kučera–Gács theorem [12, 7], each set that is low for 1-randomness is a base for 1-randomness. In [8] it is shown that in fact each base for 1-randomness is K-trivial.

3. Omega operators

In this section we introduce *universal prefix-free oracle machines* and the primary objects of study in this paper: the *Omega operators*. These are a natural class of functions from 2^ω to $[0,1]$, each of which maps every oracle $A \in 2^\omega$ to an A-random A-c.e. real.

A partial computable oracle function $M^A\colon 2^{<\omega} \to 2^{<\omega}$ is a *prefix-free oracle machine* if M^A is prefix-free for every $A \in 2^\omega$. A prefix-free oracle machine U is *universal* if for every prefix-free oracle machine M there is a prefix $\rho_M \in 2^{<\omega}$ such that

$$(\forall A \in 2^\omega)(\forall \sigma \in 2^{<\omega})\; U^A(\rho_M \sigma) = M^A(\sigma).$$

In other words, U can simulate any prefix-free oracle machine by prepending an appropriate string to the input. Note that this condition is much stronger than the requirement that U^A is a universal A-computable prefix-free machine for all $A \in 2^\omega$. The existence of universal prefix-free oracle machines can be verified by a standard construction. It is not difficult to see that there is an effective enumeration $\{M_i\}_{i\in\omega}$ of prefix-free oracle machines. Given such an enumeration, we can define a universal prefix-free oracle machine U by $U^A(0^i 1\sigma) = M_i^A(\sigma)$.

For a prefix-free oracle machine M, let Ω_M^A be the halting probability of M^A. Formally, $\Omega_M^A = \sum_{M^A(\sigma)\downarrow} 2^{-|\sigma|}$. This defines an operator $\Omega_M\colon 2^\omega \to [0,1]$. If U is universal, then we call Ω_U an *Omega operator*. We will make frequent use of stage notation. In particular, we write $M^A(\sigma)[s]\downarrow$ to indicate that the prefix-free oracle machine M with oracle $A \in 2^\omega$ converges on $\sigma \in 2^{<\omega}$ by stage $s \in \omega$. Similarly, $\Omega_M^A[s] = \sum_{M^A(\sigma)[s]\downarrow} 2^{-|\sigma|}$.

Now that we have defined Omega operators, we make a few simple but important observations. Fix a universal prefix-free oracle machine U. The following proposition is a straightforward relativization of the 1-randomness of Ω.

Proposition 3.1. *There is a constant $b \in \omega$ such that, for each $A \in 2^\omega$, Ω_U^A is A-random with constant b, in other words, $(\forall n)\; K^A(\Omega_U^A \restriction n) \geq n - b$.*

Proof. We define a prefix-free oracle machine M as follows. For any $A \in 2^\omega$ and $\sigma \in 2^{<\omega}$, first calculate $\tau = U^A(\sigma)$. Then wait for a stage s such that $\Omega_U^A[s] \geq \tau - 2^{-|\tau|}$. If such an s is found, then let $M^A(\sigma)$ converge to a string longer than any in domain$(U^A[s])$. Note that the convergence of $M^A(\sigma)$ cannot already be taken into account in the calculation of $\Omega_U^A[s]$. Now assume that U simulates M by the prefix $\rho \in 2^{<\omega}$. So, either $\Omega_U^A < \tau - 2^{-|\tau|}$ or $\Omega_U^A \geq \Omega_U^A[s] + 2^{-|\rho\sigma|} \geq \tau - 2^{-|\tau|} + 2^{-|\rho\sigma|}$. Assume, for a contradiction, that there is an $n \in \omega$

such that $K^A(\Omega_U^A \upharpoonright n) < n - |\rho| - 1$. Letting σ be a minimal program for $\Omega_U^A \upharpoonright n$, so that $\tau = \Omega_U^A \upharpoonright n$, we have proved that either $\Omega_U^A - (\Omega_U^A \upharpoonright n) < -2^{-n}$, which is absurd, or $\Omega_U^A - (\Omega_U^A \upharpoonright n) \geq -2^{-n} + 2^{-|\rho\sigma|} > -2^{-n} + 2^{-n+1} = 2^{-n}$, which is also impossible. This is a contradiction, so $(\forall n)\ K^A(\Omega_U^A \upharpoonright n) \geq n - |\rho| - 1$. $\square$

It is clear that $(\forall A \in 2^\omega)(\forall \sigma \in 2^{<\omega})\ K(\sigma) \geq K^A(\sigma) - c$, for some $c \in \omega$ not depending on A. This proves that all reals in the range of Ω_U are 1-random with constant $b + c$. In other words, the range of Ω_U is contained in the closed set $\{X \mid (\forall n)\ K(X \upharpoonright n) \geq n - b - c\}$. In particular, every real in $(\operatorname{range} \Omega_U)^{\mathrm{c}}$, the closure of the range of Ω_U, is 1-random. We will discuss the range of Ω_U and its closure in more depth in Section 9.

Next we consider the complexity of Ω_U^A. Call $A \in 2^\omega$ an A-c.e. real if it is the limit of an increasing, A-computable sequence of rationals. The following observation is immediate.

Proposition 3.2. *Ω_U^A is an A-c.e. real.*

Every A-c.e. real is computable from A', hence $\Omega_U^A \leq_T A'$. Note that it is not usually the case that $\Omega_U^A \equiv_T A'$. To see this, let A be 1-random. By van Lambalgen's theorem, A is Ω_U^A-random. Hence $A \not\leq_T \Omega_U^A$. Therefore, $\Omega_U^A \equiv_T A'$ only on a set of measure zero. (We strengthen this in Theorem 8.3 below: $\Omega_U^A \equiv_T A'$ iff A is K-trivial, thus only for countably many choices of $A \in 2^\omega$.) On the other hand, the fact that $\Omega \equiv_T \emptyset'$ has a natural relativization in the following simple result.

Proposition 3.3. *$\Omega_U^A \oplus A \equiv_T A'$, for every $A \in 2^\omega$.*

Proof. It is clear that $\Omega_U^A \oplus A \leq_T A'$. For the other direction, define a prefix-free oracle machine M such that $M^A(0^n 1)\downarrow$ iff $n \in A'$, for all $A \in 2^\omega$ and $n \in \omega$. Assume that U simulates M by the prefix $\tau \in 2^{<\omega}$. To determine if $n \in A'$, search for a stage s such that $\Omega_U^A - \Omega_U^A[s] < 2^{-(|\tau|+n+1)}$. This can be done computably in $\Omega_U^A \oplus A$. Note that U^A cannot converge on a string of length $|\tau| + n + 1$ after stage s, so

$$n \in A' \iff M^A(0^n 1)\downarrow \iff U^A(\tau 0^n 1)\downarrow \iff U^A(\tau 0^n 1)[s]\downarrow .$$

Therefore, $A' \leq_T \Omega_U^A \oplus A$. $\square$

Recall that $B \in 2^\omega$ is called *generalized low* (GL_1) if $B' \leq_T B \oplus \emptyset'$.

Theorem 3.4 (Nies and Stephan)**.** *If a Δ_2^0 set $A \in 2^\omega$ is B-random, then B is GL_1.*

Proof. Let $f(n) = (\mu s)[(\forall t \geq s)\ \ A_t \upharpoonright n = A_s \upharpoonright n]$, so that $f \leq_T \emptyset'$. Let $\widehat{\mathcal{R}}_e$ be the interval $[A_s \upharpoonright e+1]$ when $\Phi_e^B(e)$ converges at s, where Φ_e is the eth Turing functional. Clearly, if $\mathcal{R}_i = \bigcup_{e \geq i} \widehat{\mathcal{R}}_e$, then $\{\mathcal{R}_i\}_{i \in \omega}$ is a Martin-Löf test relative to B. Since $A \notin \bigcap_r \mathcal{R}_i$, A is only in finitely many $\widehat{\mathcal{R}}_e$'s. So, for almost all e such that $\Phi_e^B(e)$ converges, $f(e) \geq (\mu s)\ \Phi_e^B(e)[s]\downarrow$. Hence $B' \leq_T B \oplus \emptyset'$. $\square$

In [20, Definition 3.1], the following notion is introduced: $B \in 2^\omega$ is *low for* Ω if Ω is B-random. It is shown that this property does not depend on the particular version of Ω used. We will see in Section 7 that the low for Ω reals are exactly those which can be mapped to a c.e. real by *some* Omega operator.

Applying Theorem 3.4 with $A = \Omega$, one obtains the following corollary.

Corollary 3.5 (Nies, Stephan, Terwijn [20])**.** *If $B \in 2^\omega$ is low for Ω, then B is generalized low.*

Finally, Theorem 3.4 implies that the class of low 1-random reals is closed under the action of every Omega operator.

Corollary 3.6. *If $A \in 2^\omega$ is Δ^0_2 and 1-random, then Ω^A_U is generalized low. If $A \in 2^\omega$ is a low 1-random, then Ω^A_U is low.*

Proof. Let $B = \Omega^A_U$. Clearly B is A-random, so by van Lambalgen's theorem, A is B-random and Theorem 3.4 applies. If in addition A is low, then Ω^A_U is Δ^0_2, hence low. □

4. On A-random A-c.e. reals

We can relativize Solovay reducibility as follows. For $A, X, Y \in 2^\omega$, we write $Y \leq^A_S X$ to mean that there is a $c \in \omega$ and a partial A-computable $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ such that if $q < X$, then $\varphi(q)\downarrow < Y$ and $Y - \varphi(q) < c(X - q)$. We say that $X \in 2^\omega$ is *A-Solovay complete* if $Y \leq^A_S X$ for every A-c.e. real $Y \in 2^\omega$.

Some basic facts about Solovay reducibility relativize easily. For example:

Proposition 4.1. *A-randomness is closed upward under $\leq^A_S$. In other words, if Y is A-random and $Y \leq^A_S X$, then X is also A-random.*

The relativization of Theorem 1.2 is also straightforward.

Theorem 4.2. *If X is an A-random A-c.e. real, then X is A-Solovay complete.*

On the other hand, a satisfactory relativization of Theorem 1.1 presents some difficulty. The direct relativization states that if $X \in 2^\omega$ is an A-c.e. real and A-Solovay complete, then there is an oracle machine M such that M^A is universal for A-computable prefix-free machines and $X = \Omega^A_M$. It is not hard to add the requirement that M be prefix-free for all oracles, but there is no reason that M should be universal for oracles other than A, let alone be a universal prefix-free oracle machine. However, with extra work we can satisfy this stronger requirement.

Theorem 4.3. *Suppose that X is an A-c.e. real and A-Solovay complete. Then there is a universal prefix-free oracle machine U such that $X = \Omega^A_U$.*

Proof. Let V be a universal prefix-free oracle machine. Because Ω^A_V is an A-c.e. real, we have $\Omega^A_V \leq^A_S X$. Choose $n \in \omega$ and a partial oracle-computable function $\varphi^B\colon \mathbb{Q} \to \mathbb{Q}$ such that 2^n and φ^A witness this Solovay reduction. In other words, if $q < \Omega^A_V$ is a rational, then $\varphi^A(q)\downarrow < \Omega^A_V$ and

$$\Omega^A_V - \varphi^A(q) < 2^n(X - q). \tag{1}$$

We also require n to be large enough that $2^{-n} \leq X \leq 1 - 2^{-n}$ (clearly, no computable real can be A-Solovay complete, so $X \neq 0, 1$).

We now define another universal prefix-free oracle machine U. To make U universal, let $U^B(0^n\sigma) = V^B(\sigma)$, for all $\sigma \in 2^{<\omega}$ and oracles $B \in 2^\omega$. For convenience, we preserve the stage of convergence; i.e., $U^B(0^n\sigma)[t]\downarrow$ iff $V^B(\sigma)[t]\downarrow$. The other strings in the domain of U are used to ensure that $\Omega_U^A = X$. Let $\psi^B\colon \omega \to \mathbb{Q}$ be a partial oracle-computable function such that $\{\psi^A(s)\}_{s\in\omega}$ is a nondecreasing sequence with limit X. Fix an oracle B. We add strings not extending 0^n to the domain of U in stages. For each s,

(i) Compute $q_s = \psi^B(s)$.
(ii) Compute $r_s = \varphi^B(q_s)$.
(iii) Search for a t_s such that $\Omega_V^B[t_s] \geq r_s$.
(iv) If possible, add enough strings (not extending 0^n) to the domain of U at stage t_s to make $\Omega_U^B[t_s] = q_s$.

Note that (if $B \neq A$) this procedure may get stuck in any of the first three steps. In this case, U^B will converge on only finitely many strings not extending 0^n. This completes the construction of U, which is clearly a universal prefix-free oracle machine.

It remains to verify that $\Omega_U^A = X$. By the definition of ψ, we have $q_s = \psi^A(s)\downarrow < X$, for each s. Therefore, $r_s = \varphi^A(q_s)\downarrow < \Omega_V^A$. So, there is a stage t_s such that $\Omega_V^A[t_s] \geq r_s$. Because $q_s < X \leq 1 - 2^{-n}$, there are enough strings available in step (iv) to ensure that $\Omega_U^A[t_s] \geq q_s$. But $\lim_s q_s = X$, so $\Omega_U^A \geq X$. Now assume, for a contradiction, that $\Omega_U^A > X$. Because the strings extending 0^n add at most $2^{-n} \leq X$ to Ω_U^A, there must be some s that causes too many strings to be added to the domain of U in step (iv). In other words, there is an s such that $\Omega_U^A[t_s] = q_s$ and

$$\Omega_U^A[t_s] + 2^{-n}(\Omega_V^A - \Omega_V^A[t_s]) > X.$$

So, $\Omega_V^A - \Omega_V^A[t_s] > 2^n(X - q_s)$. But in step (iii), we ensured that $\Omega_V^A[t_s] \geq r_s = \varphi^A(q_s)$. Therefore, $\Omega_V^A - \varphi^A(q_s) > 2^n(X - q_s)$, contradicting (1). This proves that $\Omega_U^A = X$, which completes the theorem. □

Combining Propositions 3.1 and 3.2 with Theorems 4.2 and 4.3, we get the following corollary.

Corollary 4.4. *For $A, X \in 2^\omega$, the following are equivalent:*

(i) *X is an A-c.e. real and A-random.*
(ii) *X is an A-c.e. real and A-Solovay complete.*
(iii) *$X = \Omega_U^A$ for some universal prefix-free oracle machine U.*

5. Reals in the range of some Omega operator

We proved in the last section that $X \in 2^\omega$ is in the range of *some* Omega operator iff there is an $A \in 2^\omega$ such that X is both A-random and an A-c.e. real. What restriction does this place on X? In this section, we show that every 2-random real is an A-random A-c.e. real for some $A \in 2^\omega$, but that not every 1-random real has this property. Furthermore, we prove that the range of every Omega operator has positive measure.

Theorem 5.1. *If $X \in 2^\omega$ is 2-random, then X is an A-random A-c.e. real for some $A \in 2^\omega$.*

Proof. Let $A = (1 - X + \Omega)/2$. Then $X = 1 - 2A + \Omega$ is an A-c.e. real. In particular, take a nondecreasing computable sequence $\{\Omega_s\}_{s\in\omega}$ of rationals limiting to Ω. Then X is the limit of $\{1 - 2(A \restriction s) + \Omega_s\}_{s\in\omega}$, a nondecreasing A-computable sequence of rationals. It remains to prove that X is A-random. Because X is 2-random it is Ω-random. Hence, by van Lambalgen's theorem, Ω is X-random. But then $A = (1 - X + \Omega)/2$ is X-random (because clearly, $\Omega \equiv_S^X (1 - X + \Omega)/2$). Therefore, applying van Lambalgen's theorem again, X is A-random. □

As was mentioned above, the previous theorem cannot be proved if X is only assumed to be 1-random.

Example 5.2. *$X = 1 - \Omega$ is not in the range of any Omega operator.*

Proof. The 1-random real $X = 1 - \Omega$ is a co-c.e. real, i.e., the limit of a decreasing computable sequence of rationals. Assume that X is an A-c.e. real for some $A \in 2^\omega$. Then A computes sequences limiting to X from both sides; hence $X \leq_T A$. Therefore, X is not an A-random A-c.e. real for any $A \in 2^\omega$. □

It would not be difficult to prove that $1 - \Omega$ cannot even be in the *closure* of the range of an Omega operator. In fact, a direct proof is unnecessary because this follows from Theorem 9.4 below.

There is more to be said about which reals can be in the range of an Omega operator. For example:

Question 1. If $X \geq_T \emptyset'$ is an A-random A-c.e. real for some $A \in 2^\omega$, then is X necessarily a c.e. real?

Consider a special case of the previous question: can $\Omega_U^A \geq_T \emptyset''$ for some $A \in 2^\omega$ (and some universal prefix-free oracle machine U)? Note that if such an A exists, then it must be *high* (i.e., $A' \geq_T \emptyset''$). Which leads us to ask:

Question 2. Is there a high $A \in 2^\omega$ such that $\Omega_U^A \geq_T \emptyset'$?

Now we consider a *specific* Omega operator. Let U be an arbitrary universal prefix-free oracle machine. Recall that analytic sets are measurable and that the image of an analytic set under any Borel operator—for example, Ω_U—is also analytic.

Theorem 5.3. *The range of Ω_U has positive measure. In fact, if $\mathcal{S} \subseteq 2^\omega$ is any analytic set whose downward closure under $\leq_T$ is 2^ω, then $\mu(\Omega_U[\mathcal{S}]) > 0$.*

Proof. Let $\mathcal{R} = \Omega_U[\mathcal{S}]$. Note that $\mathcal{R}$ is an analytic subset of 2^ω. Hence $\mu(\mathcal{R})$ is defined. Assume, for a contradiction, that $\mu(\mathcal{R}) = 0$. In particular, the outer measure of $\mathcal{R}$ is zero. This means that there is a nested sequence $\mathcal{U}_0 \supseteq$

$\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ of open subsets of 2^ω such that $\mathcal{R} \subseteq \mathcal{U}_n$ and $\mu(\mathcal{U}_n) \leq 2^{-n}$, for each $n \in \omega$. Take a set $B \in \mathcal{S}$ which codes $\{\mathcal{U}_n\}_{n\in\omega}$ in some effective way. Then $\{\mathcal{U}_n\}_{n\in\omega}$ is a B-Martin-Löf test, which implies that $\Omega_U^B \notin \bigcap_n \mathcal{U}_n$. But $\mathcal{R} \subseteq \bigcap_n \mathcal{U}_n$, so $\Omega_U^B \notin \mathcal{R} = \Omega_U[\mathcal{S}]$. This is a contradiction, so $\mu(\mathcal{R}) > 0$. □

The theorem implies that many null classes have Ω_U-images with positive measure, for example $\mathcal{S} = \{A \mid (\forall n)\ 2n \notin A\}$.

We finish with a simple consequence of Theorem 5.3.

Corollary 5.4. *For almost every $X \in 2^\omega$, there is an $A \in 2^\omega$ such that $X =^* \Omega_U^A$.*

Proof. Let $\mathcal{S} = \{X \mid (\exists A \in 2^\omega)\ X =^* \Omega_U^A\}$. Then $\mathcal{S}$ is Σ_1^1—hence measurable by Lusin's theorem—and closed under $=^*$. But $\mu(\mathcal{S}) \geq \mu(\operatorname{range}\Omega_U) > 0$. It follows from Kolmogorov's 0–1 law that $\mu(\mathcal{S}) = 1$. □

6. When Ω^A is a c.e. real

In this key section, we consider reals $A \in 2^\omega$ for which Ω_U^A is a c.e. real. Far from being a rare property, we will show that $\mu\{A \mid \Omega_U^A \text{ is a c.e. real}\} > 0$ for any fixed universal prefix-free oracle machine U. On the other hand, only a c.e. real can have an Ω_U-preimage with positive measure. So c.e. reals clearly play an important role in understanding Ω_U. Their main application here is in our proof that no Omega operator is degree invariant. Recall that we want to obtain reals $A =^* B$ such that Ω_U^A is a c.e. real while Ω_U^B is random relative to a given (arbitrarily complex) Z. We show that each of these outcomes occurs with positive measure in Propositions 6.4 and 6.5, respectively. Proposition 6.5 has no obvious connection to the c.e. reals, but in fact, Proposition 6.4—applied to a modification of the universal machine U—is used to prove it.

Theorem 6.1. *Let M be a prefix-free oracle machine. If $\mathcal{P} \subseteq 2^\omega$ is a nonempty Π_1^0 class, then there is a $\emptyset'$-c.e. real $A \in \mathcal{P}$ such that $\Omega_M^A = \inf\{\Omega_M^C \mid C \in \mathcal{P}\}$, which is a c.e. real.*

Proof. Let $\mathcal{P} \subseteq 2^\omega$ be a nonempty Π_1^0 class and let $X = \inf\{\Omega_M^C \mid C \in \mathcal{P}\}$. Note that X is a c.e. real because it is the limit of the nondecreasing computable sequence $X_s = \inf\{\Omega_M^C[s] \mid C \in \mathcal{P}_s\}$. We will prove that there is an $A \in \mathcal{P}$ such that $\Omega_M^A = X$. Choose a sequence $\{B_n\}_{n\in\omega}$ such that $B_n \in \mathcal{P}$ and $\Omega_M^{B_n} - X \leq 2^{-n}$ for each $n \in \omega$. By compactness, $\{B_n\}_{n\in\omega}$ has a convergent subsequence $\{A_n\}_{n\in\omega}$. Note that $\Omega_M^{A_n} - X \leq 2^{-n}$. Let $A = \lim A_n$. Because $\mathcal{P}$ is closed, $A \in \mathcal{P}$. Therefore, $\Omega_M^A \geq X$. Assume, for a contradiction, that Ω_M^A is strictly greater than X. Take $m \in \omega$ such that $\Omega_M^A - X > 2^{-m}$. Then $\Omega_M^A[s] - X > 2^{-m}$ for some $s \in \omega$. Let k be the use of $\Omega_M^A[s]$ (under the usual assumptions on the use of computations, we can take $k = s$). In particular, if $B \upharpoonright k = A \upharpoonright k$, then $\Omega_M^A[s] = \Omega_M^B[s]$. Now take $n > m$ large enough that $A_n \upharpoonright k = A \upharpoonright k$. Then

$$2^{-n} \geq \Omega_M^{A_n} - X \geq \Omega_M^{A_n}[s] - X = \Omega_M^A[s] - X > 2^{-m} \geq 2^{-n}.$$

This is a contradiction, proving that $\Omega_M^A = X$.

Finally, we must prove that A can be a $\emptyset'$-c.e. real. Let $\mathcal{S} = \{C \in \mathcal{P} \mid \Omega_M^C = X\}$. Note that $\mathcal{S} = \{C \in 2^\omega \mid (\forall s)\ (C \in \mathcal{P}_s \text{ and } \Omega_M^C[s] \leq X)\}$. The fact that $X \leq_T \emptyset'$ makes $\mathcal{S}$ a $\Pi_1^0[\emptyset']$ class. We proved above that $\mathcal{S}$ is nonempty, so $A = \min(\mathcal{S})$ is a $\emptyset'$-c.e. real satisfying the theorem. □

We now consider reals $X \in 2^\omega$ such that $\Omega_U^{-1}[X]$ has positive measure.

Lemma 6.2. *Let M be a prefix-free oracle machine. If $X \in 2^\omega$ is such that $\mu\{A \mid \Omega_M^A = X\} > 0$, then X is a c.e. real.*

Proof. By the Lebesgue density theorem, there is an $\sigma \in 2^{<\omega}$ such that $\mu\{A \succ \sigma \mid \Omega_M^A = X\} > 2^{-|\sigma|-1}$. In other words, Ω_M maps more than half of the extensions of σ to X. So, X is the limit of the nondecreasing computable sequence $\{X_s\}_{s\in\omega}$, where for each $s \in \omega$, we let X_s be the largest rational such that $\mu\{A \succ \sigma \mid \Omega_M^A[s] \geq X_s\} > 2^{-|\sigma|-1}$. □

For $X \in 2^\omega$, let $m_U(X) = \mu\{A \mid \Omega_U^A = X\}$. Define the *spectrum* of Ω_U to be $\mathrm{Spec}(\Omega_U) = \{X \mid m_U(X) > 0\}$. By the lemma, the spectrum is a set of 1-random c.e. reals. We prove that it is nonempty.

Kurtz [11] defined $Z \in 2^\omega$ to be *weakly n-random* if it is not contained in a Π_n^0 class which has measure zero. He proved that this randomness notion lies strictly between n-randomness and $(n-1)$-randomness. In particular, an n-random real cannot be contained in a null Π_n^0 class. We use this fact below.

Lemma 6.3. *Let $X \in 2^\omega$ be a c.e. real. Then $m_U(X) > 0$ iff there is a 1-random $A \in 2^\omega$ such that $\Omega_U^A = X$.*

Proof. If $m_U(X) > 0$, then there is clearly a 1-random $A \in 2^\omega$ such that $\Omega_U^A = X$, as the 1-random reals form a class of measure one. For the other direction, assume that $A \in 2^\omega$ is a 1-random real such that $\Omega_U^A = X$. By van Lambalgen's theorem, the fact that X is A-random implies that A is X-random. But $X \equiv_T \emptyset'$, because X is a 1-random c.e. real, so A is 2-random. Note that $\{B \mid \Omega_U^B = X\}$ is a Π_2^0 class containing this 2-random. Hence $m_U(X) > 0$. □

Proposition 6.4. $\mathrm{Spec}(\Omega_U) \neq \emptyset$.

Proof. Apply Theorem 6.1 to a nonempty Π_1^0 class containing only 1-random reals. This gives a 1-random $A \in 2^\omega$ such that $X = \Omega_U^A$ is a c.e. real. Hence by Lemma 6.3, $X \in \mathrm{Spec}(\Omega_U)$. □

We have proved that Ω_U maps a set of positive measure to the c.e. reals. One might speculate that almost every real is mapped to a c.e. real. We now prove that this is not the case. (However, in the next section we will see that almost every real can be mapped to a c.e. real by *some* Omega operator.)

Proposition 6.5. *There is an $\varepsilon > 0$ such that*

$$(\forall Z \in 2^\omega)\ \mu\{B \mid \Omega_U^B \text{ is } Z\text{-random}\} \geq \varepsilon.$$

Proof. Let M be a prefix-free oracle machine such that $\Omega_M^B = B$ for every $B \in \omega$. Define a universal prefix-free oracle machine V by $V^B(0\sigma) = U^B(\sigma)$ and $V^B(1\sigma) = M^B(\sigma)$, for all $\sigma \in 2^{<\omega}$. Then $\Omega_V^B = (\Omega_U^B + B)/2$. Apply Proposition 6.4 to V to get a c.e. real $X \in 2^\omega$ such that $\mathcal{S} = \{B \mid \Omega_V^B = X\}$ has positive measure. Let $\varepsilon = \mu\mathcal{S}$.

Now take $Z \in 2^\omega$. We can assume, without loss of generality, that $Z \geq_T \emptyset'$. Let $B \in \mathcal{S}$ be Z-random. Then $\Omega_U^B = 2\Omega_V^B - B = 2X - B$ must also be Z-random, because $X \leq_T Z$. Therefore,

$$\mu\{B \in \mathcal{S} \mid \Omega_U^B \text{ is } Z\text{-random}\} \geq \mu\{B \in \mathcal{S} \mid B \text{ is } Z\text{-random}\} = \mu\mathcal{S} = \varepsilon,$$

since the Z-random reals have measure 1.[3] □

These results tell us that the Σ_3^0 class of reals A such that Ω_U^A is c.e. has intermediate measure.

Corollary 6.6. $0 < \mu\{A \mid \Omega_U^A$ *is a c.e. real*$\} < 1$.

The most important consequence of the work in this section is the following resoundingly negative answer to the question of whether Ω_U is degree invariant.

Theorem 6.7.

(i) *For all* $Z \in 2^\omega$*, there are* $A, B \in 2^\omega$ *such that* $A =^* B$*,* Ω_U^A *is a c.e. real and* Ω_U^B *is* Z*-random.*

(ii) *There are* $A, B \in 2^\omega$ *such that* $A =^* B$ *and* $\Omega_U^A \mid_T \Omega_U^B$ *(and in fact,* Ω_U^A *and* Ω_U^B *are* 1*-random relative to each other).*

Proof. (i) Let $\mathcal{S} = \{A \mid \Omega_U^A \text{ is a c.e. real}\}$ and $\mathcal{R} = \{B \mid \Omega_U^B \text{ is } Z\text{-random}\}$. By Propositions 6.4 and 6.5, respectively, both classes have positive measure. Let $\widehat{\mathcal{R}} = \{A \mid (\exists B \in \mathcal{R})\ A =^* B\}$. By Kolmogorov's 0–1 law, $\mu\widehat{\mathcal{R}} = 1$. Hence, there is an $A \in \mathcal{S} \cap \widehat{\mathcal{R}}$, completing the proof.

(ii) By part (i), there are $A, B \in 2^\omega$ such that $A =^* B$, Ω_U^A is a c.e. real and Ω_U^B is 2-random. Hence Ω_U^B is Ω_U^A-random and, by van Lambalgen's theorem, Ω_U^A is Ω_U^B-random. This implies that $\Omega_U^A \mid_T \Omega_U^B$. □

We close the section with two further observations on the spectrum.

Proposition 6.8. $\sup(\operatorname{range}\Omega_U) = \sup\{\Omega_U^A \mid A$ *is* 1*-random*$\} = \sup\operatorname{Spec}(\Omega_U)$.

Proof. Let $X = \sup(\operatorname{range}\Omega_U)$. Given a rational $q < X$, choose σ such that $\Omega_U^\sigma \geq q$. By the same proof as Proposition 6.4, there is a 1-random $A \succ \sigma$ such that Ω_U^A is a c.e. real. □

Proposition 6.9. *If* $p < q$ *are rationals and* $\mathcal{C} = \{A \in 2^\omega \mid \Omega_U^A \in [p, q]\}$ *has positive measure, then* $\operatorname{Spec}(\Omega_U) \cap [p, q] \neq \emptyset$.

[3] This simple construction shows more. Because $\Omega_U^B = 2X - B$ for $B \in \mathcal{S}$, we know that $\mu\{\Omega_U^B \mid B \in \mathcal{S}\} = \mu\{2X - B \mid B \in \mathcal{S}\} = \mu\mathcal{S} > 0$. Therefore, the range of Ω_U has a subset with positive measure. While this follows from the most basic case of Theorem 5.3, the new proof does not resort to Lusin's theorem on the measurability of analytic sets.

Proof. Note that $\mathcal{C}$ is the countable union of $[\sigma] \cap \mathcal{C}$ for every $\sigma \in 2^{<\omega}$ such that $\Omega^\sigma \geq p$. Because $\mu\mathcal{C} > 0$, for some such σ we have $\mu([\sigma] \cap \mathcal{C}) > 0$. But $[\sigma] \cap \mathcal{C} = \{A \succ \sigma \mid \Omega^A \leq q\}$ is a Π^0_1 class. Let $\mathcal{R} \subset 2^\omega$ be a Π^0_1 class containing only 1-randoms with $\mu\mathcal{R} > 1 - \mu([\sigma] \cap \mathcal{C})$. Then $\mathcal{R} \cap [\sigma] \cap \mathcal{C}$ is a nonempty Π^0_1 class containing only 1-randoms. Applying Theorem 6.1 to this class, there is a 1-random real $A \in \mathcal{C}$ such that $X = \Omega^A_U$ is a c.e. real. Then $X \in \mathrm{Spec}(\Omega_U) \cap [p, q]$, by Lemma 6.3 and the definition of $\mathcal{C}$. □

7. On the low for Ω reals

We turn the question of the last section around: for which oracles $A \in 2^\omega$ is there a universal prefix-free oracle machine U such that Ω^A_U is a c.e. real? We show that this is true for almost every A. Recall from Section 3 that if Ω is A-random for some—or equivalently any—version of Ω, then $A \in 2^\omega$ is said to be *low for* Ω.

Proposition 7.1. *$A \in 2^\omega$ is low for Ω iff there is a universal prefix-free oracle machine U such that Ω^A_U is a c.e. real.*

Proof. First assume that there is a universal prefix-free oracle machine U such that $X = \Omega^A_U$ is a c.e. real. Then $X \leq_S \Omega$, which means that $X \leq^A_S \Omega$. Both X and Ω are c.e. reals, hence they are A-c.e. reals. Applying Proposition 4.1, because X is A-random, Ω is also A-random. Therefore, A is low for Ω.

For the other direction, assume that $A \in 2^\omega$ is low for Ω. Then Ω is A-random and an A-c.e. real. By Corollary 4.4, $\Omega = \Omega^A_U$ for some universal prefix-free oracle machine U. □

It follows from the proof and Proposition 3.3 that if A is low for Ω, then $\Omega \oplus A \equiv_T A'$. Therefore $A' \equiv_T \emptyset' \oplus A$, giving a second proof of Corollary 3.5: low for Ω reals are GL_1.

Almost every real is low for Ω; in particular, every 2-random real is.

Proposition 7.2 (Nies, Stephan, Terwijn [20])**.** *A 1-random real $A \in 2^\omega$ is low for Ω iff A is 2-random.*

Proof. Assume that $A \in 2^\omega$ is 1-random. Recall that $\Omega \equiv_T \emptyset'$. So A is 2-random iff A is Ω-random iff Ω is A-random, where the last equivalence follows from van Lambalgen's theorem. □

More evidence for the ubiquity of low for Ω reals is the following basis theorem. It is an immediate corollary of Theorem 6.1 and Proposition 7.1.

Corollary 7.3 (The low for Ω basis theorem)**.** *Every nonempty Π^0_1 class contains a $\emptyset'$-c.e. real that is low for Ω.*

Every K-trivial real is low for 1-randomness, hence low for Ω. However, by the previous result applied to the Π^0_1 class of completions of Peano arithmetic, there is also a low for Ω real that is neither 2-random nor K-trivial.

Although it is a digression from our primary topic, we finish this section with a generalization of Corollary 7.3. The following result is a "low for X" basis theorem for every 1-random real $X \in 2^\omega$; it reduces to the corollary when we take $X = \Omega$. This result was found independently by Reimann and Slaman [24], for whom it is not a digression but a useful lemma.

Proposition 7.4. *For every 1-random $X \in 2^\omega$ and every nonempty Π^0_1 class $\mathcal{P} \subseteq 2^\omega$, there is an X-c.e. real $A \in \mathcal{P}$ such that X is A-random.*

Proof. Let $\mathcal{P} \subseteq 2^\omega$ be a nonempty Π^0_1 class. Our goal is to construct a Martin-Löf test $\{\mathcal{V}_i\}_{i\in\omega}$ such that if $X \in 2^\omega$ is not A-random for any $A \in \mathcal{P}$, then $X \in \bigcap_{i\in\omega} \mathcal{V}_i$. Fix a universal prefix-free oracle machine U. Whenever an $s \in \omega$ and $\sigma \in 2^{<\omega}$ are found such that

$$(\forall A \in \mathcal{P}_s)(\exists \tau \preceq \sigma)\ K^A_U(\tau) \leq |\tau| - i,$$

then put $[\sigma]$ into $\mathcal{V}_i$. Clearly, each $\mathcal{V}_i$ is a Σ^0_1 class. Fix an $A \in \mathcal{P}$ and note that $\mathcal{V}_i \subseteq \{X \mid (\exists n)\ K^A_U(X \upharpoonright n) \leq n - i\}$. Therefore $\mu(\mathcal{V}_i) \leq 2^{-i}$, so $\{\mathcal{V}_i\}_{i\in\omega}$ is a Martin-Löf test. Finally, assume that $X \in 2^\omega$ is not A-random for any $A \in \mathcal{P}$. By compactness, for every $i \in \omega$, there is an $\sigma \prec X$ such that $[\sigma] \subseteq \mathcal{V}_i$. Hence, $X \in \bigcap_{i\in\omega} \mathcal{V}_i$.

This proves that if $X \in 2^\omega$ is 1-random, then there is an $A \in \mathcal{P}$ such that X is A-random. We must still prove that A can be taken to be an X-c.e. real. For every $i \in \omega$, let $\mathcal{S}_i = \{A \in \mathcal{P} \mid (\forall n)\ K^A_U(X \upharpoonright n) > n - i\}$. Note that each $\mathcal{S}_i$ is a $\Pi^0_1[X]$ class. We proved above that $\mathcal{S}_i$ is nonempty, for large enough $i \in \omega$. So $A = \min(\mathcal{S}_i)$ is an X-c.e. real satisfying the theorem. □

8. Ω^A for K-trivial A

In the previous section, we considered the reals that can be mapped to c.e. reals by *some* Omega operator. Now we look at $A \in 2^\omega$ such that Ω^A_U is a c.e. real for *every* universal prefix-free oracle machine U. We will see that these are exactly the K-trivial reals.

The lemma below is a spinoff of the golden run construction from [19, Theorem 6.2]. It actually holds for any prefix free oracle machine M in place of U. That is, we do not use universality to prove the lemma.

Lemma 8.1. *Let U be a universal prefix-free oracle machine, and let $A \in 2^\omega$ be K-trivial. Then there is a computable sequence of stages $q(0) < q(1) < \cdots$ such that*

$$\widehat{S} = \sum\{\widehat{c}(x,r) \mid x \text{ is minimal s.t. } A_{q(r+1)}(x) \neq A_{q(r+2)}(x)\} < \infty, \tag{2}$$

where

$$\widehat{c}(x,r) = \sum\left\{2^{-|\sigma|}\ \middle|\ \begin{array}{c} U^A(\sigma)[q(r+1)]\downarrow\ \wedge \\ x < \mathrm{use}(U^A(\sigma)[q(r+1)]) \leq q(r) \end{array}\right\}.$$

Informally, $\widehat{c}(x,r)$ is the maximum amount that $\Omega^A_U[q(r+1)]$ can decrease by because of an $A(x)$ change after stage $q(r+1)$, provided we only count the $U^A(\sigma)$ computations with use $\leq q(r)$.

Proof. We refer to the proof of [19, Theorem 6.2] and use its notation. (For more details, see [18].) By [19, Lemma 6.6], choose a golden run $P_i(p,\alpha)$.

Claim 8.2. *For each* stage s, *there is a* stage $t > s$ *such that, for all* $\sigma < s$, *if* $U^A(\sigma)[t] = y$ *with use* $w \leq s$, *then a run* $Q_{i-1,\sigma,y,w}$ *has returned by* t *and is not released yet, that is,* P_i *waits at (P2*$_\sigma$*).*

To see this, let $r \geq s$ be the least *stage* by which $A_r \upharpoonright s$ has settled. A run $Q_{i-1,\sigma,y,w}$ such that $w \leq r$ is never canceled after stage r, therefore it returns by the definiton of golden runs in [19, Lemma 6.6]. This proves the claim.

The least $t > s$ as in the claim can be determined effectively. Let $q(0) = 0$. If $s = q(r)$ has been defined, let $q(r+1)$ be the least t such that the condition of the claim holds. Let $g \in \mathbb{N}$ be the number such that $p/\alpha = 2^g$. We show that $\widehat{S} < 2^g$. Suppose x is minimal such that $A_{q(r+1)}(x) \neq A_{q(r+2)}(x)$. Then $A_{s-1}(x) \neq A_s(x)$ for some *stage* s with $q(r+1) < s \leq q(r+2)$. No later than s, the runs of procedures $Q_{i-1,\sigma,y,y+1}$ with $x \leq y < q(r)$ which are still waiting at (P2$_\sigma$) are released. This adds a weight of at least $\widehat{c}_U(x,r)$ to C_i. Thus $\widehat{S} < 2^g$, since otherwise the run of P_i reaches its goal. $\square$

The following proof uses an alternative characterization of 1-randomness due to Solovay [26]. A *Solovay test* is a computable sequence $\{I_r\}_{r\in\omega}$ of intervals with (dyadic) rational endpoints such that $\sum_{r\in\omega} |I_r|$ is finite. A real *passes* a Solovay test if it is in only finitely many of the intervals. It is not difficult to see that $X \in 2^\omega$ is 1-random iff it passes every Solovay test.

Theorem 8.3. *Let* U *be a universal prefix-free oracle machine. The following are equivalent for* $A \in 2^\omega$*:*

(i) *A is K-trivial.*
(ii) *A is Δ^0_2 and Ω^A_U is a c.e. real.*
(iii) $A \leq_T \Omega^A_U$.
(iv) $A' \equiv_T \Omega^A_U$.

Proof. (ii) $\Longrightarrow$ (iii) follows from the fact that each 1-random c.e. real is Turing complete. (iii) $\Longrightarrow$ (i) because A is a base for 1-randomness; see the end of Section 2. (iii) is equivalent to (iv) by Proposition 3.3.

(i) $\Longrightarrow$ (ii). Assume that A is K-trivial. As shown by Chaitin [3], A is Δ^0_2. We show that there is an $r_0 \in \omega$ and an effective sequence $\{\mu_r\}_{r\in\omega}$ of rationals such that $\Omega^A_U = \sup_{r\geq r_0} \mu_r$, and hence Ω^A_U is a c.e. real. Applying Lemma 8.1 to U, we obtain a computable sequence of stages $q(0) < q(1) < \cdots$ such that (2) holds. The desired sequence of rationals is

$$\mu_r = \sum\{2^{-|\sigma|} \mid U^A(\sigma)[q(r+1)]\downarrow \;\wedge\; \mathrm{use}(U^A(\sigma)[q(r+1)]) \leq q(r)\}.$$

Thus μ_r measures the computations existing at stage $q(r+1)$ whose use is at most $q(r)$. We define r_0 below; first we verify that $\Omega^A_U \leq \sup_{r\geq r_0} \mu_r$ for any $r_0 \in \omega$. Given $\sigma_1, \ldots, \sigma_m \in \mathrm{domain}(U^A)$, choose $r_1 \in \omega$ so that each computation $U^A(\sigma)$ has settled by stage $q(r_1)$, with use $\leq q(r_1)$. If $r \geq r_1$, then $\mu_r \geq \sum_{1\leq i\leq m} 2^{-|\sigma_i|}$. Therefore, $\Omega^A_U \leq \limsup_{r\in\omega} \mu_r \leq \sup_{r\geq r_0} \mu_r$.

Now define a Solovay test $\{I_r\}_{r\in\omega}$ as follows: if x is minimal such that $A_{q(r+1)}(x) \neq A_{q(r+2)}(x)$, then let

$$I_r = [\mu_r - \widehat{c}(x,r), \mu_r].$$

Then $\sum_{r\in\omega} |I_r|$ is finite by (2), so $\{I_r\}_{r\in\omega}$ is indeed a Solovay test. Also note that, by the comment after the lemma, $\min I_r \leq \max I_{r+1}$ for each $r \in \omega$.

Since Ω_U^A is 1-random, there is an $r_0 \in \omega$ such that $\Omega_U^A \notin I_r$ for all $r \geq r_0$. We show that $\mu_r \leq \Omega_U^A$ for each $r \geq r_0$. Fix $r \geq r_0$. Let $t \geq r$ be the first non-deficiency stage for the enumeration $t \mapsto A_{q(t+1)}$. That is, if x is minimal such that $A_{q(t+1)}(x) \neq A_{q(t+2)}(x)$, then

$$(\forall t' \geq t)(\forall y < x)\ A_{q(t'+1)}(y) = A_{q(t+1)}(y).$$

The quantity $\mu_t - \widehat{c}(x,t)$ measures the computations $U^A(\sigma)[q(t+1)]$ with use $\leq x$. These are stable from $q(t+1)$ on, so $\Omega_U^A \geq \min I_t$, and hence $\Omega_U^A > \max I_t$. Now $\Omega_U^A \notin I_u$ for $u \geq r_0$ and $\min I_u \leq \max I_{u+1}$ for any $u \in \omega$. Applying this to $u = t-1, \dots, u = r$, we obtain that $\Omega_U^A \geq \max I_r = \mu_r$. Therefore, $\Omega_U^A \geq \sup_{r \geq r_0} \mu_r$. □

One consequence of this theorem is the fact that Omega operators are degree invariant at least on the K-trivial reals. The next example shows that they need not be degree invariant anywhere else.

Example 8.4. *There is an Omega operator that is degree invariant only on K-trivial reals.*

Proof. Let M be a prefix-free oracle machine such that

$$\Omega_M^A = \begin{cases} A & \text{if } A(0) = 0 \\ 0 & \text{if } A(0) = 1. \end{cases}$$

For any $A \in 2^\omega$, define a real $\widehat{A}$ by $\widehat{A}(n) = A(n)$ iff $n \neq 0$. Let U be a universal prefix-free oracle machine. Define a universal prefix-free oracle machine V by $V^A(00\sigma) = U^A(\sigma)$, $V^A(01\sigma) = U^{\widehat{A}}(\sigma)$ and $V^A(1\sigma) = M^A(\sigma)$, for all $\sigma \in 2^{<\omega}$. Then $|\Omega_V^A - \Omega_V^{\widehat{A}}| = A/2$, for all $A \in 2^\omega$. Assume that $\Omega_V^{\widehat{A}} \leq_T \Omega_V^A$ for some $A \in 2^\omega$. Then $A \leq_T \Omega_V^A$, so A is a base for 1-randomness and hence K-trivial by [8]. If $\Omega_V^A \leq_T \Omega_V^{\widehat{A}}$, then again A is K-trivial. Therefore, if $A \in 2^\omega$ is not K-trivial, then $\Omega_V^A\ |_T\ \Omega_V^{\widehat{A}}$. □

The following corollary summarizes Theorem 8.3 and Example 8.4.

Corollary 8.5. *The following are equivalent for $A \in 2^\omega$:*
- (i) *A is K-trivial.*
- (ii) *Every Omega operator takes A to a c.e. real.*
- (iii) *Every Omega operator is degree invariant on $\mathbf{deg}_T(A)$.*

We have seen in Theorem 6.7 that no Omega operator is degree invariant. We have also seen that if $A \in 2^\omega$ is not K-trivial, then there are Omega operators that are not invariant on $\mathbf{deg}_T(A)$. Can these two results be combined?

Question 3. For a universal prefix-free oracle machine U and a real $A \in 2^\omega$ that is not K-trivial, is there a $B \equiv_T A$ such that $\Omega_U^B \not\equiv_T \Omega_U^A$?

Finally, a simple but interesting consequence of Example 8.4 is the following.

Corollary 8.6. *Every K-trivial is a d.c.e. real (i.e., the difference of two c.e. reals).*

Proof. Let V be the machine from Example 8.4. Assume that $A \in 2^\omega$ is K-trivial. Then Ω_V^A and $\Omega_V^{\widehat{A}}$ are both c.e. reals by Theorem 8.3. Therefore, $A = 2\,|\Omega_V^A - \Omega_V^{\widehat{A}}|$ is a d.c.e. real. □

It is known that the d.c.e. reals form a real closed field [21, 23]. The corollary gives us a nontrivial real closed subfield: the K-trivial reals. To see this, note that the K-trivial reals form an ideal in the Turing degrees ([6] for closure under $\oplus$ and [19] for downward closure). Because a zero of an odd degree polynomial can be computed relative to the coefficients, the K-trivial reals are also a real closed field.

9. Analytic behavior of Omega operators

In this section, we examine Omega operators from the perspective of analysis. Given a universal prefix-free oracle machine $U\colon 2^{<\omega} \to 2^{<\omega}$, we consider two questions:

(i) To what extent is Ω_U continuous?
(ii) How complex is the range of Ω_U?

To answer the first question, we observe that Ω_U is lower semicontinuous but not continuous. Furthermore, we prove that it is continuous exactly at 1-generic reals. Together with the semicontinuity, this implies that Ω_U can only achieve its supremum at a 1-generic. But must Ω_U actually achieve its supremum? This relates to the second question. Theorem 9.4 states that any real in $(\operatorname{range}\Omega_U)^{\mathrm{c}} \setminus \operatorname{range}(\Omega_U)$ must be 2-random. Because $X = \sup(\operatorname{range}\Omega_U)$ is a c.e. real—hence not 2-random, there is an $A \in 2^\omega$ such that $\Omega_U^A = X$.

It is natural to ask whether $\operatorname{range}(\Omega_U)$ is closed. In other words, is Theorem 9.4 vacuous? Example 9.6 demonstrates that for *some* choice of U, the range of Ω_U is not closed, and indeed, that $\mu(\operatorname{range}\Omega_U) < \mu((\operatorname{range}\Omega_U)^{\mathrm{c}})$. Whether this is the case for *all* universal prefix-free oracle machines is left open. Furthermore, we know of no nontrivial upper-bound on the complexity of $\operatorname{range}(\Omega_U)$, but we do observe that $(\operatorname{range}\Omega_U)^{\mathrm{c}}$ is a Π^0_3 class.

Recall that a function $f\colon \mathfrak{X} \to \mathbb{R}$ is *lower semicontinuous* if $\{x \in \mathfrak{X} \mid f(x) > a\}$ is an open set for every $a \in \mathbb{R}$. Here $\mathfrak{X}$ is an arbitrary topological space. We claim that for any prefix-free oracle machine M, the function Ω_M is lower semicontinuous. Note that for any $A \in 2^\omega$,

$$(\forall \delta > 0)(\exists m)\ \Omega_M^A - \Omega_M^{A\upharpoonright m} \le \delta \tag{3}$$

and hence $(\forall X \succ A \upharpoonright m)\ \Omega_M^A - \Omega_M^X \le \delta$.

Proposition 9.1. *Ω_M is lower semicontinuous for every prefix-free oracle machine M.*

Proof. Take $a \in \mathbb{R}$ and assume that $\Omega_M^A > a$. Choose a real $\delta > 0$ such that $\Omega_M^A - \delta > a$. By the observation above, there is an $m \in \omega$ such that $X \succ A \upharpoonright m$ implies that $\Omega_M^A - \Omega_M^X \leq \delta$. Therefore, $\Omega_M^X \geq \Omega_M^A - \delta > a$. So $[A \upharpoonright m]$ is an open neighborhood of A contained in $\{X \mid \Omega_M^X > a\}$. But A was an arbitrary element of $\{X \mid \Omega_M^X > a\}$, proving that this set is open. □

Next we prove that Omega operators are not continuous and characterize their points of continuity. Recall that an open set $\mathcal{S} \subseteq 2^\omega$ is *dense* along $A \in 2^\omega$ if each initial segment of A has an extension in $\mathcal{S}$. We say that A is 1-*generic* if A is in every Σ_1^0 class $\mathcal{S}$ that is dense along A. We prove that Ω_U is continuous exactly on the 1-generics, for any universal prefix-free oracle machine U.

Theorem 9.2. *The following are equivalent for a set $A \in 2^\omega$:*

(i) *A is 1-generic.*
(ii) *If M is a prefix-free oracle machine, then Ω_M is continuous at A.*
(iii) *There is a universal prefix-free oracle machine U such that Ω_U is continuous at A.*

Proof. (i) $\Longrightarrow$ (ii). Let M be any prefix-free oracle machine. By (3), it suffices to show that

$$(\forall \varepsilon)(\exists n)(\forall X \succ A \upharpoonright n)\ \Omega_M^X \leq \Omega_M^A + \varepsilon.$$

Suppose this fails for a rational ε. Take a rational $r < \Omega_M^A$ such that $\Omega_M^A - r < \varepsilon$. The following Σ_1^0 class is dense along A:

$$\mathcal{S} = \{B \mid (\exists n)\ \Omega_M^B[n] \geq r + \varepsilon\}.$$

Thus $A \in S$. But this implies that $\Omega_M^A \geq r + \varepsilon > \Omega_M^A$, which is a contradiction.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i). Fix a universal prefix-free oracle machine U. We assume that A is not 1-generic and show that there is an $\varepsilon > 0$ such that

$$(\forall n)(\exists B \succ A \upharpoonright n)\ \Omega_U^B \geq \Omega_U^A + \varepsilon. \tag{4}$$

Take a Σ_1^0 class $\mathcal{S}$ that is dense along A but $A \notin \mathcal{S}$. Define a prefix-free oracle machine L^X as follows. When (some initial segment of) $X \in 2^\omega$ enters $\mathcal{S}$, then L^X converges on the empty string. Thus L^A is nowhere defined. Let $c \in \omega$ be the length of the coding prefix for L in U. We prove that $\varepsilon = 2^{-(c+1)}$ satisfies (4).

Choose m as in (3) for the given universal machine, where $\delta = 2^{-(c+1)}$. For each $n \geq m$, choose $B \succ A \upharpoonright n$ such that $B \in \mathcal{S}$. Since L^B converges on the empty string, $\Omega_U^B \geq \Omega_U^A - 2^{-(c+1)} + 2^c = \Omega_U^A + \varepsilon$. □

Let U be a universal prefix-free oracle machine.

Corollary 9.3. *If $\Omega_U^A = \sup(\operatorname{range} \Omega_U)$, then A is 1-generic.*

Proof. By the previous theorem, it suffices to prove that Ω_U is continuous at A. But note that the lower semicontinuity of Ω_U implies that

$$\{X \mid |\Omega_U^A - \Omega_U^X| < \varepsilon\} = \{X \mid \Omega_U^X > \Omega_U^A - \varepsilon\}$$

is open, for every $\varepsilon > 0$. Thus, A is 1-generic. □

The corollary above does not guarantee that the supremum is achieved. Surprisingly, it is. In fact, we can prove quite a bit more. One way to view the proof of the following theorem is that we are trying to prevent any real which is not 2-random from being in the closure of the range of Ω_U. If we fail for some $X \in 2^\omega$, then it will turn out that $X \in \mathrm{range}(\Omega_U)$. Note that this is a consequence of universality; it is easy to construct a prefix-free oracle machine $M \colon 2^{<\omega} \to 2^{<\omega}$ such that Ω_M does not achieve its supremum.

Theorem 9.4. *If $X \in (\mathrm{range}\,\Omega_U)^c \smallsetminus \mathrm{range}(\Omega_U)$, then X is 2-random.*

Proof. Assume that $X \in (\mathrm{range}\,\Omega_U)^c$ is not 2-random and let $\mathcal{R}_X = \Omega_U^{-1}[X] = \{A \mid \Omega_U^A = X\}$. For each rational $p \in [0,1]$, define $\mathcal{C}_p = \{A \mid \Omega_U^A \leq p\}$. Note that every $\mathcal{C}_p$ is closed (in fact, a Π_1^0 class). For every rational $q \in [0,1]$ such that $q < X$, we will define a closed set $\mathcal{B}_q \subseteq 2^\omega$ such that

$$\mathcal{R}_X = \bigcap_{q<X} \mathcal{B}_q \cap \bigcap_{p>X} \mathcal{C}_p, \tag{5}$$

where q and p range over the rationals. Furthermore, we will prove that every finite intersection of sets from $\{\mathcal{B}_q \mid q < X\}$ and $\{\mathcal{C}_p \mid p > X\}$ is nonempty. By compactness, this ensures that $\mathcal{R}_X$ is nonempty, and therefore, that $X \in \mathrm{range}(\Omega_U)$.

We would like to define $\mathcal{B}_q$ to be $\{A \mid \Omega_U^A \geq q\}$, which would obviously satisfy (5). The problem is that $\{A \mid \Omega_U^A \geq q\}$ is a Σ_1^0 class; $\mathcal{B}_q$ must be closed if we are to use compactness. The solution is to let $\mathcal{B}_q = \{A \mid \Omega_U^A[k] \geq q\}$ for some $k \in \omega$. Then $\mathcal{B}_q$ is closed (in fact, clopen) and, by choosing k appropriately, we will guarantee that Ω_U^A is bounded away from X for every $A \notin \mathcal{B}_q$.

For each rational $q \in [0,1]$, we build a prefix-free oracle machine M_q. For $A \in 2^\omega$ and $\sigma \in 2^{<\omega}$, define $M_q^A(\sigma)$ as follows.

(i) Wait for a stage $s \in \omega$ such that $\Omega_U^A[s] \geq q$.
(ii) Compute $\tau = U^{\emptyset'_s}(\sigma)$.
(iii) Wait for a stage $t \geq s$ such that $\Omega_U^A[t] \geq \tau$.

The computation may get stuck in any one of the three steps, in which case $M_q^A(\sigma)\uparrow$. Otherwise, let $M_q^A(\sigma)$ converge to a string longer than any in $\mathrm{domain}(U^A[t])$. The value to which $M_q^A(\sigma)$ converges is only relevant because it ensures that a U-simulation of M_q cannot converge before stage $t+1$.

We are ready to define $\mathcal{B}_q \subseteq 2^\omega$ for a rational $q \in [0,1]$ such that $q < X$. Assume that U simulates M_q by the prefix $\rho \in 2^{<\omega}$. Choose $\sigma, \tau \in 2^{<\omega}$ such that $U^{\emptyset'}(\sigma) = \tau \prec X$ and $|\tau| > |\rho\sigma|$. Such σ and τ exist because X is not 2-random. Choose $k_q \in \omega$ large enough that $U^{\emptyset'_s}(\sigma) = \tau$ for all $s \geq k_q$. Let $\mathcal{B}_q = \{A \mid \Omega_U^A[k_q] \geq q\}$.

We claim that the definition of $\mathcal{B}_q$ ensures that Ω_U^A is bounded away from X for any $A \notin \mathcal{B}_q$. Let $l_q = \min\{q, \tau\}$ and $r_q = \tau + 2^{-|\rho\sigma|}$. Clearly $l_q < X$. To see that $r_q > X$, note that $X - \tau \leq 2^{-|\tau|} < 2^{-|\rho\sigma|}$. Now assume that $A \notin \mathcal{B}_q$ and that $\Omega_U^A \geq l_q$. Thus $\Omega_U^A \geq q$ but $\Omega_U^A[k_q] < q$. This implies that the s found in step (i) of the definition of M_q is greater than k_q. Therefore, $U^{\emptyset'_s}(\sigma) = \tau$. But $\Omega_U^A \geq \tau$, so step (iii) eventually produces a $t \geq s$ such that $\Omega_U^A[t] \geq \tau$. This means that $M_q^A(\sigma)$ converges to a string longer than any in domain$(U^A[t])$, so $U^A(\rho\sigma)\downarrow$ sometime after stage t, which implies that $\Omega_U^A \geq \Omega_U^A[t] + 2^{-|\rho\sigma|} \geq \tau + 2^{-|\rho\sigma|} = r_q$. We have proved that

$$\Omega_U^A \in [l_q, r_q) \implies A \in \mathcal{B}_q. \tag{6}$$

Next we verify (5). Assume that $A \in \mathcal{R}_X$. We have just proved that $A \in \mathcal{B}_q$ for all rationals $q < X$. Also, it is clear that $A \in \mathcal{C}_p$ for all rationals $p > X$. Therefore, $\mathcal{R}_X \subseteq \bigcap_{q<X} \mathcal{B}_q \cap \bigcap_{p>X} \mathcal{C}_p$. For the other direction, assume that $A \in \bigcap_{q<X} \mathcal{B}_q \cap \bigcap_{p>X} \mathcal{C}_p$. Thus if $q < X$, then $\Omega_U^A \geq \Omega_U^A[k_q] \geq q$. Hence $\Omega_U^A \geq X$. On the other hand, if $p > X$, then $\Omega_U^A \leq p$. This implies that $\Omega_U^A \leq X$, and so $\Omega_U^A = X$. Therefore $A \in \mathcal{R}_X$, which proves (5).

It remains to prove that $\mathcal{R}_X$ is nonempty. Let Q be a finite set of rationals less than X and P a finite set of rationals greater than X. Define $l = \max\{l_q \mid q \in Q\}$ and $r = \min(P \cup \{r_q \mid q \in Q\})$. Note that $X \in (l, r)$. Because $X \in (\operatorname{range} \Omega_U)^{\mathrm{c}}$, there is an $A \in 2^\omega$ such that $\Omega_U^A \in (l, r)$. From (6) it follows that $A \in \mathcal{B}_q$ for all $q \in Q$. Clearly, $A \in \mathcal{C}_p$ for every $p \in P$. Hence $\bigcap_{q\in Q} \mathcal{B}_q \cap \bigcap_{p \in P} \mathcal{C}_p$ is nonempty. By compactness, $\mathcal{R}_X$ is nonempty. □

If $X \in \operatorname{range}(\Omega_U)$ is not 2-random, then an examination of the construction gives an upper-bound on the complexity of $\Omega_U^{-1}[X]$. The Π_1^0 classes $\mathcal{C}_p$ can be computed uniformly. The $\mathcal{B}_q$ are also Π_1^0 classes and can be found uniformly in $X \oplus \emptyset'$. Therefore, $\Omega_U^{-1}[X] = \bigcap_{q<X} \mathcal{B}_q \cap \bigcap_{p>X} \mathcal{C}_p$ is a nonempty $\Pi_1^0[X \oplus \emptyset']$ class.

The following corollary gives an interesting special case of Theorem 9.4. It is not hard to prove that there is an $A \in 2^\omega$ such that $\Omega_U^A = \inf(\operatorname{range} \Omega_U)$ (see Theorem 6.1). It is much less obvious that Ω_U achieves its supremum.

Corollary 9.5. *There is an $A \in 2^\omega$ such that $\Omega_U^A = \sup(\operatorname{range} \Omega_U)$.*

Proof. Note that $\sup(\operatorname{range} \Omega_U)$ is a c.e. real, hence not 2-random. So, the corollary is immediate from Theorem 9.4. □

No 1-generic is 1-random, so $\mu\{A \mid \Omega_U^A = \sup(\operatorname{range} \Omega_U)\} = 0$. Therefore, $\sup(\operatorname{range} \Omega_U)$ is an example of a c.e. real in the range of Ω_U which is not in $\operatorname{Spec}(\Omega_U)$.

One might ask whether Theorem 9.4 is vacuous. In other words, is the range of Ω_U actually closed? We can construct a specific universal prefix-free oracle machine such that it is not. The construction is somewhat similar to the proof of Theorem 5.3. In that case, we avoid a measure zero set by using an oracle that codes a relativized Martin-Löf test covering that set. Now we will avoid a measure zero *closed* set by using a natural number to code a finite open cover with sufficiently small measure.

Example 9.6. *There is a universal prefix-free oracle machine V such that*
$$\mu(\operatorname{range}\Omega_V) < \mu((\operatorname{range}\Omega_V)^c).$$

Proof. Let U be a universal prefix-free oracle machine. Let M be a prefix-free oracle machine such that
$$\Omega_M^A = \begin{cases} 1 & \text{if } |A| > 1 \\ 0 & \text{otherwise.} \end{cases}$$
Define a universal prefix-free oracle machine V by $V^A(0\sigma) = U^A(\sigma)$ and $V^A(1\sigma) = M^A(\sigma)$, for all $\sigma \in 2^{<\omega}$. This definition ensures that $\Omega_V^A \leq 1/2$ iff $|A| \leq 1$. Therefore $\mu(\operatorname{range}(\Omega_V) \cap [0,1/2]) = 0$. We will prove that $\mu((\operatorname{range}\Omega_V)^{\mathrm{c}} \cap [0,1/2]) > 0$.

Let $\{\mathcal{O}_i\}_{i\in\omega}$ be an effective enumeration of all finite unions of open intervals with dyadic rational endpoints. We construct a prefix-free oracle machine N. By the recursion theorem for prefix-free oracle machines, we may assume in advance that we know the prefix ρ by which V simulates N.[4] Given an oracle $A \in 2^\omega$, find the least $n \in \omega$ such that $A(n) = 1$. Intuitively, N^A will try to prevent Ω_V^A from being in $\mathcal{O}_n$. Whenever a stage $s \in \omega$ occurs such that $\Omega_V^A[s] \in \mathcal{O}_n$ and $(\forall\sigma \in 2^{<\omega})\ V^A(\rho\sigma)[s] = N^A(\sigma)[s]$, then N^A acts as follows. Let ε be the least number such that $\Omega_V^A[s]+\varepsilon \notin \mathcal{O}_n$ and note that ε is necessarily a dyadic rational. If possible, N^A converges on additional strings with total measure $2^{|\rho|}\varepsilon$. This would ensure that $\Omega_V^A \geq \Omega_V^A[s] + \varepsilon$. If $\mu\mathcal{O}_n \leq 2^{-|\rho|}$, then N^A cannot run out of room in its domain and we have $\Omega_V^A \notin \mathcal{O}_n$.

Assume, for the sake of contradiction, that $\mu((\operatorname{range}\Omega_V)^{\mathrm{c}} \cap [0,1/2]) = 0$. Then there is an open cover of $(\operatorname{range}\Omega_V)^{\mathrm{c}} \cap [0,1/2]$ with measure less than $2^{-|\rho|}$. We may assume that all intervals in this cover have dyadic rational endpoints. Because $(\operatorname{range}\Omega_V)^{\mathrm{c}} \cap [0,1/2]$ is compact, there is a finite subcover $\mathcal{O}_n$. But $\mu\mathcal{O}_n < 2^{-|\rho|}$ implies that $\Omega_V^{0^n10^\omega} \notin \mathcal{O}_n$. This is a contradiction, so $\mu((\operatorname{range}\Omega_V)^{\mathrm{c}} \cap [0,1/2]) > 0$. ☐

Note that the proof above shows that if U is a universal prefix-free oracle machine and $\mathcal{A} = \{\Omega_U^{0^n10^\omega}\}_{n\in\omega}$, then $\mathcal{A}^{\mathrm{c}}$ has positive measure and $\mathcal{A}^{\mathrm{c}} \smallsetminus \mathcal{A}$ contains only 2-randoms.

Having constructed a *specific* Omega operator whose range is not closed, it is natural to ask if this is always the case.

Question 4. Is it true for every universal prefix-free oracle machine U that $\operatorname{range}(\Omega_U)$ is not closed?

In the other direction, we have no nontrivial upper-bound on the complexity of the range of Ω_U.

[4]The recursion theorem for prefix-free oracle machines is a straightforward application of the relativized recursion theorem.

Question 5. Is range(Ω_U) an arithmetical class (or even Borel) for a universal prefix-free oracle machine U?

Related to both questions, note that $(\operatorname{range}\Omega_U)^{\mathrm{c}}$ is an arithmetical class.

Proposition 9.7. *$(\operatorname{range}\Omega_U)^c$ is a Π^0_3 class.*

Proof. It is easy to verify that $a \in (\operatorname{range}\Omega_U)^{\mathrm{c}}$ iff

$$(\forall \varepsilon > 0)(\exists \sigma \in 2^{<\omega}) \left[\begin{array}{c} \Omega_U^\sigma[|\sigma|] > a - \varepsilon \wedge \\ (\forall n \geq |\sigma|)(\exists \tau \succ \sigma)\ |\tau| = n \wedge \Omega_U^\tau[n] < a + \varepsilon \end{array} \right],$$

where ε ranges over rational numbers. This is a Π^0_3 definition because the final existential quantifier is bounded. □

References

[1] Cristian S. Calude, Peter H. Hertling, Bakhadyr Khoussainov, and Yongge Wang. Recursively enumerable reals and Chaitin Ω numbers. In *STACS 98 (Paris, 1998)*, volume 1373 of *Lecture Notes in Comput. Sci.*, pages 596–606. Springer, Berlin, 1998.

[2] Gregory J. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.

[3] Gregory J. Chaitin. Algorithmic information theory. *IBM Journal of Research and Development*, 21:350–359, 496, 1977.

[4] Rod Downey and Denis R. Hirschfeldt. *Algorithmic randomness and complexity.* Springer-Verlag, Berlin. To appear.

[5] Rod Downey, Denis R. Hirschfeldt, André Nies, and Sebastiaan A. Terwijn. Calibrating randomness. To appear in *Bull. Symb. Logic.*

[6] Rod G. Downey, Denis R. Hirschfeldt, André Nies, and Frank Stephan. Trivial reals. In *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 103–131, Singapore, 2003. Singapore Univ. Press.

[7] Péter Gács. Every sequence is reducible to a random one. *Inform. and Control*, 70:186–192, 1986.

[8] Denis R. Hirschfeldt, André Nies, and Frank Stephan. Using random sets as oracles. To appear.

[9] Antonín Kučera. On relative randomness. *Ann. Pure Appl. Logic*, 63(1):61–67, 1993. 9th International Congress of Logic, Methodology and Philosophy of Science (Uppsala, 1991).

[10] Antonín Kučera and Theodore A. Slaman. Randomness and recursive enumerability. *SIAM J. Comput.*, 31(1):199–211 (electronic), 2001.

[11] Stuart Kurtz. *Randomness and genericity in the degrees of unsolvability.* Ph.D. Dissertation, University of Illinois, Urbana, 1981.

[12] Antonín Kučera. Measure, Π^0_1-classes and complete extensions of PA. In H.-D. Ebbinghaus, G. H. Müller, and G. E. Sacks, editors, *Recursion Theory Week*, volume 1141 of *Lect. Notes in Math.*, pages 245–259. Springer, Berlin, 1985.

[13] Leonid A. Levin. The concept of a random sequence. *Dokl. Akad. Nauk SSSR*, 212:548–550, 1973.

[14] Leonid A. Levin. Laws on the conservation (zero increase) of information, and questions on the foundations of probability theory. *Problemy Peredači Informacii*, 10(3):30–35, 1974.

[15] Ming Li and Paul Vitányi. *An introduction to Kolmogorov complexity and its applications.* Graduate Texts in Computer Science. Springer-Verlag, New York, second edition, 1997.

[16] Per Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

[17] John Myhill. Creative sets. *Z. Math. Logik Grundlagen Math.*, 1:97–108, 1955.

[18] André Nies. Computability and randomness. To appear.

[19] André Nies. Lowness properties and randomness. To appear in Adv. in Math.

[20] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. To appear.

[21] Ng Keng Meng. Master Thesis, National University of Singapore. To appear.

[22] John C. Oxtoby. *Measure and category*, volume 2 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1980. A survey of the analogies between topological and measure spaces.

[23] Alexander Raichev. Relative randomness and real closed fields. To appear in J. Symbolic Logic.

[24] Jan Reimann and Theodore A. Slaman. Randomness, entropy, and reducibility. To appear.

[25] Gerald E. Sacks. *Higher recursion theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.

[26] Robert M. Solovay. Draft of paper (or series of papers) on Chaitin's work. unpublished notes, 215 pages, May 1975.

[27] Michiel van Lambalgen. The axiomatization of randomness. *J. Symbolic Logic*, 55(3):1143–1167, 1990.

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