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## *Contributors*



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# 1

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## *Parameterized Algorithms*

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### 1.1 Introduction

Since the advent of classical complexity theory in the early 1970's, the twin notions of *NP*-hardness and *NP*-completeness have been accepted as concrete measures of computational intractability. However, a verdict of *NP*-hardness does not do away with the need for solving hard computational problems, since the bulk of these problems are of both theoretical and practical importance.

The field of parameterized complexity theory and parameterized computation has developed rapidly over the past twenty years as a robust approach to dealing with hard computational problems arising from applications in diverse areas of science and industry. The parameterized paradigm augments classical complexity theory and computation, providing, on the one hand, systematic and practical algorithm design techniques for hard problems, and, on the other hand, more finely-grained complexity analysis and stronger computational lower bounds for natural computational problems.

The theory is based on the simple observation that many hard computational problems have certain aspects of their input, or expected solution, that vary only within a moderate range, at least for instances that are of practical importance. By exploiting such small associated parameters, many classically intractable problems can be efficiently solved.

Apparent parameterized intractability is established via a completeness program,

which parallels the traditional paradigm, but allows for stratification of problems into a far more richly-structured hierarchy of complexity classes.

A number of approaches have been proposed to deal with the central issue of computational intractability, including polynomial time approximation, randomization and heuristic algorithms. The parameterized paradigm is orthogonal to each of these earlier approaches, yet a range of fundamental connections has begun to emerge.

The aim of this chapter is to survey the current state of the art in the field of parameterized complexity, canvassing main techniques and important results. We concentrate on the distinctive algorithmic techniques that have emerged in the field, in particular those that lead to practical and useful algorithms for classically intractable problems.

While there are a large number of applications of these ideas in many diverse arenas, our plan is to present the ideas concentrating mainly on a small number of problems, particularly VERTEX COVER and some variants (defined in Section 1.2.) Our intention is to facilitate understanding of the principal techniques without the need for lengthy explanations of a diverse range of problems. Even so, space limitations mean that we cannot canvass all algorithmic techniques used in the field. The use of bounded variable integer linear programming and general graph modification techniques are two important omissions. Finally, we mention that implementations of many of the techniques that we will introduce have performed *much better* in practice than one might reasonably expect, but we have left out experimental discussions. We refer the reader to the recently published special issue of the Computer Journal [36], the monographs of Rolf Niedermeier [55] and Henning Fernau [45], and the ACM SIGACT News article by Guo and Niedermeier [46] for discussions on these points and for a tour of the wide range of applications.

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## 1.2 The Main Idea

It is generally accepted that solving an *NP*-hard problem will necessarily entail a combinatorial explosion of the search space. However, it is *not* necessarily the case that all instances of an *NP*-hard problem are equally hard to solve, hardness sometimes depends on the particular structure of a given instance, or of the expected solution. Instances of *NP*-hard problems arising from “real life” often exhibit more regular structure than the general problem description might, at first, suggest.

For example, suppose that we are concerned with solving computational problems to do with relational databases. Typically, a real life database will be huge, and the queries made to it will be relatively small. Moreover, real life queries will be questions that *people* actually ask. Hence, such queries will tend to be of low logical complexity. Thus, an algorithm that works very efficiently for small formulae with low logical depth might well be perfectly acceptable in practice. Alternatively,

suppose that we are concerned with computational problems where the focus is to recognize a particular substructure in the problem input. If the size of the expected substructure is small, then an algorithm that works very efficiently for small solutions may be acceptable in practice.

The main idea of parameterized complexity is to develop a framework that addresses complexity issues in this situation, where we know in advance that certain parameters of the problem at hand are likely to be bounded, and that this might significantly affect the complexity.

The basic insight that underpins parameterized complexity and parameterized computation arose from consideration of two well-known *NP*-complete problems for simple undirected graphs.

A *vertex cover* of  $G = (V, E)$  is a set of vertices  $V' \subseteq V$  that covers all edges: that is  $V' = \{v_1, \dots, v_k\}$  is a vertex cover for  $G$  iff, for every edge  $(u, v) \in E$ , either  $u \in V'$  or  $v \in V'$ . A *dominating set* of  $G = (V, E)$  is a set of vertices  $V' \subseteq V$  that covers all vertices: that is  $V' = \{v_1, \dots, v_k\}$  is a dominating set for  $G$  iff, for every vertex  $v \in V$ , either  $v \in V'$  or there is some  $u \in V'$  such that  $(u, v) \in E$ .

#### VERTEX COVER

*Instance:* A graph  $G = (V, E)$  and a positive integer  $k$ .

*Question:* Does  $G$  have a vertex cover of size at most  $k$ ?

#### DOMINATING SET

*Instance:* A graph  $G = (V, E)$  and a positive integer  $k$ .

*Question:* Does  $G$  have a dominating set of size at most  $k$ ?

Although both of these problems are, classically, *NP*-complete, the parameter  $k$  contributes to the complexity of these two problems in two qualitatively different ways.

**DOMINATING SET:** Essentially the only known algorithm for this problem is to try all possibilities. The brute force algorithm of trying all  $k$ -subsets runs in time  $O(n^{k+1})$  (we use  $n$  to denote  $|V|$  and  $m$  to denote  $|E|$ .)

**VERTEX COVER:** After many rounds of improvement, there is now an algorithm running in time  $O(1.286^k + kn)$  ([26]) for determining if a graph  $G = (V, E)$  has a vertex cover of size  $k$ . This has been implemented and is practical for  $n$  of unlimited size and  $k$  up to around 400 [66, 39].

The table below shows the contrast between these two kinds of complexity.

These observations are formalized in the framework of parameterized complexity theory [33, 34]. In classical complexity, a decision problem is specified by two items of information:

1. The input to the problem.
2. The question to be answered.

	$n = 50$	$n = 100$	$n = 150$
$k = 2$	625	2,500	5,625
$k = 3$	15,625	125,000	421,875
$k = 5$	390,625	6,250,000	31,640,625
$k = 10$	$1.9 \times 10^{12}$	$9.8 \times 10^{14}$	$3.7 \times 10^{16}$
$k = 20$	$1.8 \times 10^{26}$	$9.5 \times 10^{31}$	$2.1 \times 10^{35}$

The Ratio  $\frac{n^{k+1}}{2^{kn}}$  for Various Values of  $n$  and  $k$ .

In parameterized complexity, there are three parts to a problem specification:

1. The input to the problem.
2. The aspects of the input that constitute the parameter.
3. The question to be answered.

The notion of *fixed-parameter tractability* is the central concept of the theory. Intuitively, a problem is fixed-parameter tractable (FPT) if we can somehow confine any “bad” complexity behaviour to some limited aspect of the problem, the parameter.

More formally, we consider a *parameterized language* to be a subset  $L \subseteq \Sigma^* \times \Sigma^*$ . If  $L$  is a parameterized language and  $(I, k) \in L$  then we refer to  $I$  as the *main part* and  $k$  as the *parameter*.

**DEFINITION 1.1 Fixed Parameter Tractability (FPT)**

A parameterized language  $L \subseteq \Sigma^* \times \Sigma^*$  is fixed-parameter tractable if there is an algorithm (or a  $k$ -indexed collection of algorithms) that correctly decides, for input  $(I, k) \in \Sigma^* \times \Sigma^*$ , whether  $(I, k) \in L$  in time  $f(k) \cdot n^c$ , where  $n$  is the size of the main part of the input  $I$ ,  $k$  is the parameter,  $c$  is a constant (independent of both  $n$  and  $k$ ), and  $f$  is an arbitrary function dependent only on  $k$ .

Usually, the parameter  $k$  will be a positive integer, but it could be, for instance, a graph or an algebraic structure, or a combination of integer values bounding various aspects of the problem. The parameter will often bound the *size* of some part of the input instance or the solution. Alternatively, it can bound the *complexity* of the input instance in some well-defined sense. For example, in Sections 1.4.2 and 1.4.3 we introduce *width metrics* for graphs which precisely capture various notions of complexity in graphs. A single classical problem can often be parameterized in several natural ways, each leading to a separate parameterized problem.

In this chapter, we will concentrate mainly on the techniques for demonstrating parameterized *tractability*. There is also a very well developed theory of parameterized *intractability*, used to address problems like DOMINATING SET, which we introduce in Section 1.5, but for which space limitations preclude a deeper treatment. As we see in Section 1.5, there is a completeness and hardness theory, akin to that

of NP-completeness, that can be used to demonstrate parameterized intractability, based around a parameterized analog of NONDETERMINISTIC TURING MACHINE ACCEPTANCE.

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### 1.3 Practical FPT Algorithms

In this section we introduce the main practical techniques that have emerged in the field of FPT algorithm design. We focus first on two simple algorithmic strategies that are not part of the usual toolkit of *polynomial* algorithm design, but which have lead, in many important cases, to practical and useful algorithms for natural parameterized versions of NP-hard problems. These two techniques, (i) Kernelization and (ii) Depth-Bounded Search Trees, have formed the backbone of practical FPT algorithm design. In Section 1.3.3 we show how these two techniques can be profitably combined using the concept of *interleaving*. We also introduce *iterative compression*, a relatively new technique that has been successfully applied to a range of parameterized minimization problems, where the parameter is the size of the solution set.

#### 1.3.1 Kernelization

Kernelization is based on an old idea, that of pre-processing, or *reducing*, the input data of a computational problem. It often makes sense to try to eliminate those parts of the input data that are relatively easy to cope with, shrinking the given instance to some “hard core” that must be dealt with using a computationally expensive algorithm. In fact, this is the basis of many heuristic algorithms for NP-hard problems, in a variety of areas, that seem to work reasonably well in practice. In other words, it is something that many practitioners, faced with a real-world NP-hard problem, already do.

A compelling example of the effectiveness of data reduction, for a classically-posed NP-hard problem, is given by Weihe [67]. He considered the following problem in the context of the European railroad network: given a set of trains, select a set of stations such that every train passes through at least one of those stations and such that the number of selected stations is minimum. Weihe modeled this problem as a path cover by vertices in an undirected graph. Here, we formulate the problem as domination of one set of vertices by another in a bipartite graph.

##### TRAIN COVERING BY STATIONS

*Instance:* A bipartite graph  $G = (V_S \cup V_T, E)$ , where the set of vertices  $V_S$  represents railway stations and the set of vertices  $V_T$  represents trains.  $E$  contains an edge  $(s, t), s \in V_S, t \in V_T$ , iff the train  $t$  stops at the station  $s$ .

*Problem:* Find a minimum set  $V' \subseteq V_S$  such that  $V'$  covers  $V_T$ , that is, for every vertex  $t \in V_T$ , there is some  $s \in V'$  such that  $(s, t) \in E$ .

Weihe employed two simple data reduction rules for this problem. For our problem formulation they translate to the following:

REDUCTION RULE TCS1:

Let  $N(t)$  denote the neighbours of  $t$  in  $V_S$ . If  $N(t) \subseteq N(t')$  then remove  $t'$  and all adjacent edges of  $t'$  from  $G$ . If there is a station that covers  $t$ , then this station also covers  $t'$ .

REDUCTION RULE TCS2:

Let  $N(s)$  denote the neighbours of  $s$  in  $V_T$ . If  $N(s) \subseteq N(s')$  then remove  $s$  and all adjacent edges of  $s$  from  $G$ . If there is a train covered by  $s$ , then this train is also covered by  $s'$ .

In practice, exhaustive application of these two simple data reduction rules allowed for the problem to be solved in minutes, for a graph modeling the whole European train schedule, consisting of around  $1.6 \cdot 10^5$  vertices and  $1.6 \cdot 10^6$  edges.

This impressive performance begs the question: Why should data reduction be of more concrete use in the parameterized paradigm? The answer comes from the observation that a data reduction scheme for a parameterized problem can often give upper bounds on the size of the reduced instance in terms *solely of the parameter*. Once such a reduction scheme is established, a trivial FPT algorithm manifests as a brute-force search of the reduced instance. Thus, in the parameterized context, data reduction can often lead directly to an FPT algorithm to solve the problem. This contrasts with the classical context, where data reduction can clearly lead to a useful heuristic, but without any *provable performance guarantee*.

To illustrate the kernelization concept, we start with a simple data reduction scheme for the standard parameterized version of the *NP*-hard VERTEX COVER problem (introduced in Section 1.2.) As for subsequent examples given in this section, we paraphrase the treatment given in [49].

K-VERTEX COVER

*Instance:* A graph  $G = (V, E)$ .

*Parameter:* A positive integer  $k$ .

*Question:* Does  $G$  have a vertex cover of size  $\leq k$ ?

Vertices with no adjacent edges are irrelevant, both to the problem instance and to any solution. This leads to :

REDUCTION RULE VC1:

Remove all isolated vertices.

In order to cover an edge in  $E$ , one of its endpoints must be in the solution set. If one of these endpoints is a degree one vertex, then the other endpoint has the potential to cover more edges than the degree one vertex, leading to:

REDUCTION RULE VC2:

For any degree one vertex  $v$ , add its single neighbour  $u$  to the solution set and remove

$u$  and all of its incident edges from the graph.

The reduced instance thus consists of both a smaller graph and a smaller parameter,  $(G, k) \rightarrow (G', k-1)$ .

These two data reduction rules are applicable in any problem solving context. However, in the parameterized setting, where we are looking only for a small solution, with size bounded by parameter  $k$ , we can do more. Sam Buss [21] originally observed that, for a simple graph  $G$ , any vertex of degree greater than  $k$  must belong to every  $k$ -element vertex cover of  $G$  (otherwise all the neighbours of the vertex must be included, and there are more than  $k$  of these).

REDUCTION RULE VC3:

If there is a vertex  $v$  of degree at least  $k+1$ , add  $v$  to the solution set and remove  $v$  and all of its incident edges from the graph.

The reduced instance again consists of both a smaller graph and a smaller parameter,  $(G, k) \rightarrow (G', k-1)$ .

After exhaustively applying these three rules, we have a new, reduced, instance,  $(G', k')$ , where no vertex in the reduced graph has degree greater than  $k'$ , or less than two. Thus, any vertex remaining can cover at most  $k'$  edges in this reduced instance. Since the solution set can contain at most  $k'$  vertices, if the reduced graph is a YES instance, then it must have at most  $k'^2$  edges, and consequently at most  $k'^2$  vertices.

Thus, in a polynomial amount of time, we have reached a situation where we can either declare our original instance to be a NO instance, or, by means of a brute force search of the reduced instance, in time  $O(k'^{2k'})$ ,  $k' \leq k$ , decide whether our original instance admits a vertex cover of size at most  $k$ .

The important point is that the reduced instance can either be immediately declared a NO instance or, otherwise, has size bounded by a *function of the parameter*. We formalise this idea in terms of a *reduction to a problem kernel*, or *kernelization*.

### DEFINITION 1.2 Kernelization

Let  $L \subseteq \Sigma^* \times \Sigma^*$  be a parameterized language. Let  $\mathcal{L}$  be the corresponding parameterized problem, that is,  $\mathcal{L}$  consists of input pairs  $(I, k)$ , where  $I$  is the main part of the input and  $k$  is the parameter. A reduction to a problem kernel, or kernelization, comprises replacing an instance  $(I, k)$  by a reduced instance  $(I', k')$ , called a problem kernel, such that

- (i)  $k' \leq k$ ,
- (ii)  $|I'| \leq g(k)$ , for some function  $g$  depending only on  $k$ , and
- (iii)  $(I, k) \in L$  if and only if  $(I', k') \in L$ .

The reduction from  $(I, k)$  to  $(I', k')$  must be computable in time polynomial in  $|I|$ .

The kernelization for K-VERTEX COVER described above uses rules that examine

only *local* substructures of the input (a vertex and its neighbourhood.) For a range of problems, this approach proves adequate for producing a reasonably-sized problem kernel. Another possibility is to consider the *global* properties of a problem instance.

Chen *et al.* [26] have used this second approach in exploiting a well-known theorem of Nemhauser and Trotter [54] to construct a problem kernel for VERTEX COVER having at most  $2k$  vertices. This seems to be the best that one could hope for, since a problem kernel of size  $(2 - \varepsilon) \cdot k$ , with constant  $\varepsilon > 0$ , would imply a factor  $2 - \varepsilon$  polynomial-time approximation algorithm for VERTEX COVER. The existence of such an algorithm is a long-standing open question in the area of approximation algorithms for NP-hard problems.

**THEOREM 1.1 Nemhauser and Trotter (1975)**

For an  $n$ -vertex graph  $G = (V, E)$  with  $m$  edges, we can compute two disjoint sets  $C' \subseteq V$  and  $V' \subseteq V$ , in  $O(\sqrt{n} \cdot m)$  time, such that the following three properties hold:

- (i) There is a minimum size vertex cover of  $G$  that contains  $C'$ .
- (ii) A minimum vertex cover for the induced subgraph  $G[V']$  has size at least  $|V'|/2$ .
- (iii) If  $D \subseteq V'$  is a vertex cover of the induced subgraph  $G[V']$ , then  $C = D \cup C'$  is a vertex cover of  $G$ .

**THEOREM 1.2 Chen et al. (2001)**

Let  $(G = (V, E), k)$  be an instance of K-VERTEX COVER. In  $O(k \cdot |V| + k^3)$  time we can reduce this instance to a problem kernel  $(G = (V', E'), k')$  with  $|V'| \leq 2k$ .

The kernelization begins by applying the three reduction rules described above, VC1, VC2 and VC3, to produce a reduced instance  $(G', k')$ , where  $G'$  contains at most  $O(k^2)$  vertices and edges, and  $k' \leq k$ . This reduction takes  $O(k \cdot |V|)$  time.

For the resulting reduced instance  $(G', k')$  we compute the two sets  $C'$  and  $V'$  as described in Theorem 1.1. Determining the two sets  $C'$  and  $V'$  involves computation of a maximum matching on a graph constructed from  $G'$  and can be achieved in time  $O(\sqrt{k^2} \cdot k^2) = O(k^3)$ .

The set  $C'$  contains vertices that have to be in the vertex cover, so we define a new parameter  $k'' = k' - |C'|$ . Due to Theorem 1.1, we know that if  $|V'| > 2k''$  then there is no vertex cover of size  $k$  for the original graph  $G$ . Otherwise, we let the induced subgraph  $G[V']$  be the problem kernel, having size at most  $2k'' \leq 2k$ . By Theorem 1.1, the remaining vertices for a minimum vertex cover of  $G$  can be found by searching for a minimum vertex cover in  $G[V']$ .

Recently, a third alternative to both local and global data reduction schemes has been explored. In this third case, local rules are generalized to examine *arbitrarily large substructures*. Continuing with our running example, K-VERTEX COVER, we show that the local rule VC2, which entails the deletion of any degree-1 vertex and

the admission of its sole neighbour into the vertex cover, can be generalized to the *crown reduction rule*.

A *crown* in a graph  $G = (V, E)$  consists of an independent set  $I \subseteq V$  (no two vertices in  $I$  are connected by an edge) and a set  $H$  containing all vertices in  $V$  adjacent to  $I$ . A crown in  $G$  is formed by  $I \cup H$  iff there exists a size  $|H|$  maximum matching in the bipartite graph induced by the edges between  $I$  and  $H$ , that is, every vertex of  $H$  is matched. It is clear that degree-1 vertices in  $V$ , coupled with their sole neighbours, can be viewed as the most simple crowns in  $G$ .

If we find a crown  $I \cup H$  in  $G$ , then we need at least  $|H|$  vertices to cover all edges in the crown. Since all edges in the crown can be covered by admitting at most  $|H|$  vertices into the vertex cover, there is a minimum size vertex cover that contains all vertices in  $H$  and no vertices in  $I$ . These observations lead to the following reduction rule.

**REDUCTION RULE CR:**

For any crown  $I \cup H$  in  $G$ , add the set of vertices  $H$  to the solution set and remove  $I \cup H$  and all of the incident edges of  $I \cup H$  from  $G$ .

The reduced instance thus consists of a smaller graph and a smaller parameter,  $(G, k) \rightarrow (G', k - |H|)$ . For both instance and parameter the reduction may be significant.

We are now faced with two issues. How to find crowns efficiently? and how to bound the size of the problem kernel that eventuates?

In [3] it is shown that finding a crown in a graph  $G$  can be achieved in polynomial time by computing maximum matchings in  $G$ . The size of the reduced instance that results is bounded above via the following theorem.

**THEOREM 1.3 Abu-Khzam, Collins, Fellows, Langston, Suters, Symons (2004)**

*A graph that is crown-free and has a vertex cover of size at most  $k$  can contain at most  $3k$  vertices.*

Another strategy for employment of crown reductions makes use of the following lemma from [29].

**LEMMA 1.1 Chor, Fellows, Juedes (2004)**

*If a graph  $G = (V, E)$  has an independent set  $V' \subseteq V$  such that  $|N(V')| < |V'|$ , then a crown  $I \cup H$  with  $I \subseteq V'$  can be found in  $G$  in time  $O(n + m)$ .*

The following simple crown kernelization algorithm, given in [65], uses this strategy to produce either a correct NO answer, or a problem kernel of size at most  $4k$ , for  $K$ -VERTEX-COVER.

We start by computing a maximal matching  $M$  in  $G$ . Since we have to pick one vertex for each edge in the matching it follows that the size of a minimum vertex

cover of  $G$  is at least  $|M|/2$ . Thus, if  $|V(M)| > 2k$ , then we output NO. Otherwise, if  $|V(M)| \leq 2k$ , then there are two possibilities:

Since  $M$  is a maximal matching it must be the case that  $V(G) - V(M)$  is an independent set in  $G$ . If we assume that  $G$  does not contain any isolated vertices then each vertex in  $V(G) - V(M)$  must be adjacent to some vertex in  $V(M)$ . Thus, if  $|V(G) - V(M)| > 2k$  then, by Lemma 1.1, we can find a crown  $I \cup H$  in  $G$  in time  $O(n + m)$ . The reduced instance is  $(G[V - (I \cup H)], k - |H|)$ .

If  $|V(G) - V(M)| \leq 2k$  then  $|V(G)| = |V(M)| + |V(G) - V(M)| \leq 2k + 2k = 4k$  so  $G$  is the required problem kernel of size at most  $4k$ .

The three kernelizations given here for K-VERTEX-COVER make for a compelling argument in support of data reduction in the parameterized context. In comparison with polynomial approximation, kernelization achieves the conjectured best possible result for this particular problem. In [3, 4] Abu-Khzam et al. report on experiments solving large instances of the K-VERTEX COVER problem in the context of computational biology applications. A common problem in many of these applications involves finding the maximum clique in a graph. However, the K-CLIQUE problem is  $W[1]$ -hard and so not directly amenable to an FPT algorithmic approach. A graph  $G$  has a maximum clique of size  $k$  iff its complement graph  $\bar{G}$  has a minimum vertex cover of size  $n - k$ . Thus, one approach to the Clique problem is to solve the Vertex Cover problem on the complement graph with parameter  $n - k$ . Results from [3, 4] show that the kernelization techniques presented here for K-VERTEX COVER perform far better in practice than is suggested by the theory, both in terms of running time and in terms of the size of the kernels that can be achieved. Implementations combining kernelization with a *depth-bounded search tree* approach (see Sections 1.3.2 and 1.3.3) work effectively on real data for  $k$  up to around 1000 [3].

However, since K-VERTEX COVER is considered to be the “success story” of parameterized computation, it is fair to ask whether or not the program works so well in general. There are by now a plethora of kernelization algorithms in the literature, solving a wide variety of problems with practical applications in diverse areas of science and industry. Many of these yield sufficiently small problem kernels to be of concrete practical use. In this regard, the benchmark is a *linear* kernel, where the size of the fully reduced instance is a (small) linear function of the parameter. Examples of parameterized problems having linear problem kernels include K-DOMINATING SET and K-CONNECTED VERTEX COVER restricted to *planar graphs* (for general graphs K-CONNECTED VERTEX COVER has so far only been shown to have an exponentially bounded problem kernel and K-DOMINATING SET is  $W[2]$ -hard so has no problem kernel bounded by any function of the parameter) and K-TREE BISECTION AND RECONNECTION (see Section 1.6 for details of this problem from computational biology). In some documented cases, for example see [50], even though the provable bound on the size of the kernel might be large, even an exponential function of the parameter, the underlying data reduction still performs very well in practice.

We conclude this section by noting the following two caveats regarding kernelization:

For some problems obtaining a problem kernel is trivial. The following example

is given in [65]. We consider the  $K$ -DOMINATING SET problem for cubic graphs, where all vertices have degree three. No vertex in such a graph can dominate more than four vertices, itself and three neighbours. Thus, we can safely answer NO whenever the input graph has more than  $4k$  vertices. Note that this problem kernel of  $4k$  vertices and at most  $6k$  edges is obtained without the application of *any* reduction rule at all. However, by the same argument we see that no cubic graph has a dominating set of size *less than*  $n/4$ . Thus, for any non-trivial problem instance, we have  $k \geq n/4$  and  $4k \geq n$ . The bound obtained for the size of the kernel is at least as large as the size of the instance itself.

In this case, a more sensible problem to consider arises from the idea of *bounding above the guarantee*, first introduced in [52]. Given a cubic graph  $G$  and parameter  $k$ , it makes more sense to ask if there is a dominating set of size  $n/4 + k$  for  $G$ . Now, the parameter contributes to the problem in a non-trivial fashion, since it is has become a bound on the distance of the solution from some guaranteed minimum.

Two of the three kernelizations given here for  $K$ -VERTEX-COVER result in problem kernels that we commonly call linear kernels, since the number of vertices in the fully reduced instance is a linear function of the parameter. A more accurate description is to say that they are *polynomial kernels*, since the number of graph edges in the reduced instance may be quadratic in the size of the parameter. Recent results [15] suggest that some parameterized problems likely won't admit any polynomial kernel (i.e any problem kernel whose size is an arbitrary polynomial function of the parameter), under reasonable complexity-theoretic hypotheses, even though they can be shown to be FPT using some of the not-quite-practical FPT methods we introduce later in Section 1.4. This suggests that, for such problems to have FPT algorithms with *provably* fast running times, these must be of rather unusual types.

### 1.3.2 Depth-bounded Search Trees

Many parameterized problems can be solved by the construction of a search tree whose *depth* depends only upon the parameter. The total size of the tree will necessarily be an exponential function of the parameter, to keep the size of the tree manageable the trick is to find efficient *branching rules* to successively apply to each node in the search tree.

Continuing with our running example, consider the following simple algorithm for the  $K$ -VERTEX COVER problem.

We construct a binary tree of height  $k$ . We begin by labelling the root of the tree with the empty set and the graph  $G = (V, E)$ . Now we pick any edge  $(u, v) \in E$ . In any vertex cover of  $G$  we must have either  $u$  or  $v$ , in order to cover the edge  $(u, v)$ , so we create children of the root node corresponding to these two possibilities. The first child is labeled with  $\{u\}$  and  $G - u$ , the second with  $\{v\}$  and  $G - v$ . The set of vertices labeling a node represents a possible vertex cover, and the graph labeling a node represents what remains to be covered in  $G$ . In the case of the first child we have determined that  $u$  will be in our possible vertex cover, so we delete  $u$  from  $G$ , together with all its incident edges, as these are all now covered by a vertex in our

possible vertex cover.

In general, for a node labeled with a set  $S$  of vertices and subgraph  $H$  of  $G$ , we arbitrarily choose an edge  $(u, v) \in E(H)$  and create the two child nodes labeled, respectively,  $S \cup \{u\}$ ,  $H - u$ , and  $S \cup \{v\}$ ,  $H - v$ . At each level in the search tree the size of the vertex sets that label nodes will increase by one. Any node that is labeled with a subgraph having no edges must also be labeled with a vertex set that covers all edges in  $G$ . Thus, if we create a node at height at most  $k$  in the tree that is labeled with a subgraph having no edges, then a vertex cover of size at most  $k$  has been found.

There is no need to explore the tree beyond height  $k$ , so this algorithm runs in time  $O(2^k \cdot n)$ .

In many cases, it is possible to significantly improve the  $f(k)$ , the function of the parameter that contributes exponentially to the running time, by shrinking the search tree. In the case of  $K$ -VERTEX COVER, Balasubramanian et al [11] observed that, if  $G$  has no vertex of degree three or more, then  $G$  consists of a collection of cycles. If such a  $G$  is sufficiently large, then this graph cannot have a size  $k$  vertex cover. Thus, at the expense of an additive constant factor (to be invoked when we encounter any subgraph in the search tree containing only vertices of degree at most two), we need consider only graphs containing vertices of degree three or greater.

We again construct a binary tree of height at most  $k$ . We begin by labelling the root of the tree with the empty set and the graph  $G$ . Now we pick any vertex  $v \in V$  of degree three or greater. In any vertex cover of  $G$  we must have either  $v$  or *all of its neighbours*, so we create children of the root node corresponding to these two possibilities. The first child is labeled with  $\{v\}$  and  $G - v$ , the second with  $\{w_1, w_2, \dots, w_p\}$ , the neighbours of  $v$ , and  $G - \{w_1, w_2, \dots, w_p\}$ . In the case of the first child, we are still looking for a size  $k - 1$  vertex cover, but in the case of the second child we need only look for a vertex cover of size  $k - p$ , where  $p$  is at least 3. Thus, the bound on the size of the search tree is now somewhat smaller than  $2^k$ .

Using a recurrence relation to determine a bound on the number of nodes in this new search tree, it can be shown that this algorithm runs in time  $O(5^{k/4} \cdot n)$ .

A third search tree algorithm for  $K$ -VERTEX COVER, given in [49] uses the following three branching rules:

BRANCHING RULE VC1:

If there is a degree one vertex  $v$  in  $G$ , with single neighbour  $u$ , then there is a minimum size cover that contains  $u$  (by the argument given for rule VC2 in Section 1.3.1.) Thus, we create a single child node labeled with  $\{u\}$  and  $G - u$ .

BRANCHING RULE VC2:

If there is a degree two vertex  $v$  in  $G$ , with neighbours  $w_1$  and  $w_2$ , then either both  $w_1$  and  $w_2$  are in a minimum size cover, or  $v$  together with *all other neighbours* of  $w_1$  and  $w_2$  are in a minimum size cover.

To see that this rule is correct, assume that there is a minimum size cover containing  $v$  and only one of its neighbours. Replacing  $v$  with its second neighbour would

then also yield a minimum size cover and this is the cover that will be constructed in the first branching case. Thus, if there is a cover that is smaller than the cover containing both  $w_1$  and  $w_2$  then this cover must contain  $v$  and neither  $w_1$  nor  $w_2$ . This implies that all other neighbours of  $w_1$  and  $w_2$  must be in this cover.

**BRANCHING RULE VC3:**

If there is a degree three vertex  $v$  in  $G$ , then either  $v$  or all of its neighbours are in a minimum size cover.

Using a recurrence relation, it can be shown that if there is a solution of size at most  $k$  then the size of the corresponding search tree has size bounded above by  $O(1.47^k)$ .

The basic method of finding efficient branching rules is to look for a *structure* in the problem input which gives rise to only a few alternatives, one of which must lead to an acceptable solution, if such a solution exists. In all of the examples given here for  $K$ -VERTEX COVER, this structure consists of a vertex and its one or two hop neighbourhood. The “smallest” search tree found so far for  $K$ -VERTEX COVER has size  $O(1.286^k)$  [26] and is achieved by more complex case analysis than that described here, although the structures identified still consist simply of small local neighbourhoods in the input graph.

We now briefly canvas two quite different examples of such structures which give rise to efficient branching rules for two unrelated parameterized problems. Space limitations mean that most of the problem details are left out, the intention is merely to demonstrate the nature of the possibilities for search tree algorithms.

For the CLOSEST STRING problem [51], we are given a set  $S = \{s_1, s_2, \dots, s_k\}$  of  $k$  strings, each of length  $l$ , over an alphabet  $\Sigma$ , and the task is to find a string whose Hamming distance is at most  $d$  from each of the  $s_i \in S$ . The structure that we identify is a *candidate string*,  $\hat{s}$ . At the root node of the search tree  $\hat{s}$  is simply one of the input strings. If any other string  $s_i \in S$  differs from  $\hat{s}$  in more than  $2d$  positions, then there is no solution for the problem. At each step we look for an input string  $s_i$  that differs from  $\hat{s}$  in more than  $d$  positions but less than  $2d$  positions. Choosing  $d + 1$  of these positions, we branch into  $d + 1$  subcases, in each subcase modifying one position in  $\hat{s}$  to match  $s_i$ .

For the MAXIMUM AGREEMENT FOREST problem [47], we are given two phylogenetic  $X$ -trees,  $T_1$  and  $T_2$ , each an unrooted binary tree with leaves labeled by a common set of species  $X$  and (unlabeled) interior nodes having degree exactly three. The (labeled) topologies of  $T_1$  and  $T_2$  may differ. The task is to find at most  $k$  edges that can be cut from  $T_1$  so that the resulting forest “agrees with” the topologies of both  $T_1$  and  $T_2$ . One structure that we can identify here is called a *minimal incompatible quartet*, essentially a set of four leaf labels,  $Q = \{l_1, l_2, l_3, l_4\}$ , that gives rise to two different topologies in the restriction of each of the trees to those leaves labeled by  $Q$ . Given any solution set of edges  $E$  from  $T_1$  that gives rise to an *agreement forest*  $F$ , we can obtain an “equivalent” set of edges  $E'$  that produces  $F$  by cutting at least one of a certain set of four edges induced by  $Q$  in  $T_1$ . Thus, after finding an

incompatible quartet, we branch into four subcases, in each subcase cutting one of these edges.

Finally, we note that search trees inherently allow for a parallel implementation: when branching into subcases, each branch can be further explored with no reference to other branches. This has proven of concrete use in practice for VERTEX COVER [28]. Along with the idea introduced in the next section, this is one of two powerful arguments in support of the use of the depth-bounded search tree approach to obtain *really practical* FPT algorithms.

### 1.3.3 Interleaving

It is often possible to combine the two methods outlined above. For example, for the  $K$ -VERTEX COVER problem, we can first reduce any instance to a problem kernel and then apply a search tree method to the kernel itself.

Niedermeier and Rossmanith [56] have developed the technique of *interleaving* depth-bounded search trees and kernelization. They show that applying kernelization repeatedly during the course of a search tree algorithm can significantly improve the overall time complexity in many cases.

Suppose we take any fixed-parameter algorithm that satisfies the following conditions: The algorithm works by first reducing an instance to a problem kernel, and then applying a depth-bounded search tree method to the kernel. Reducing any given instance to a problem kernel takes at most  $P(|I|)$  steps and results in a kernel of size at most  $q(k)$ , where both  $P$  and  $q$  are polynomially bounded. The expansion of a node in the search tree takes  $R(|I|)$  steps, where  $R$  is also bounded by some polynomial. The size of the search tree is bounded by  $O(\alpha^k)$ . The overall time complexity of such an algorithm running on instance  $(I, k)$  is

$$O(P(|I|) + R(q(k))\alpha^k).$$

The strategy developed in [56] is basically to apply kernelization at any step of the search tree algorithm where this will result in a significantly smaller problem instance. To expand a node in the search tree labelled by instance  $(I, k)$  we first check whether or not  $|I| > c \cdot q(k)$ , where  $c \geq 1$  is a constant whose optimal value will depend on the implementation details of the algorithm. If  $|I| > c \cdot q(k)$  then we apply the kernelization procedure to obtain a new instance  $(I', k')$ , with  $|I'| \leq q(k)$ , which is then expanded in place of  $(I, k)$ . A careful analysis of this approach shows that the overall time complexity is reduced to

$$O(P(|I|) + \alpha^k).$$

This really does make a difference. In [55] the 3-HITTING SET problem is given as an example. An instance  $(I, k)$  of this problem can be reduced to a kernel of size  $k^3$  in time  $O(|I|)$ , and the problem can be solved by employing a search tree of size  $2.27^k$ . Compare a running time of  $O(2.27^k \cdot k^3 + |I|)$  (without interleaving) with a running time of  $O(2.27^k + |I|)$  (with interleaving).

Note that, although the techniques of *kernelization* and *depth-bounded search tree* are simple algorithmic strategies, they are not part of the classical toolkit of

polynomial-time algorithm design since they both involve costs that are *exponential in the parameter*.

### 1.3.4 Iterative Compression

Iterative compression is a relatively new technique for obtaining FPT algorithms, first introduced in a paper by Reed, Smith and Vetta in 2004 [59]. Although currently only a small number of results are known, it seems to be applicable to a range of parameterized minimization problems, where the parameter is the size of the solution set. Most of the currently known iterative compression algorithms solve *feedback set problems* in graphs, problems where the task is to destroy certain cycles in the graph by deleting at most  $k$  vertices or edges. In particular, the  $K$ -GRAPH BIPARTISATION problem, where the task is to find a set of at most  $k$  vertices whose deletion destroys all odd-length cycles, has been shown to be FPT by means of iterative compression [59]. This had been a long-standing open problem in parameterized complexity theory.

To illustrate the concept, we again paraphrase the treatment given in [49]. The central idea is to employ a *compression routine*.

#### **DEFINITION 1.3** *Compression Routine*

A compression routine is an algorithm that, given a problem instance  $I$  and a solution of size  $k$ , either calculates a smaller solution or proves that the given solution is of minimum size.

Using such a routine we can find an optimal solution for a parameterized problem by inductively building up the problem structure and iteratively compressing intermediate solutions. The idea is that, if the compression routine is an FPT algorithm, then so is the whole algorithm. The manner by which the problem structure is inductively produced will be normally be straightforward, the trick is in finding an efficient compression routine. Continuing with our running example, we now describe an algorithm for  $K$ -VERTEX COVER based on iterative compression.

Given a problem instance  $(G = (V, E), k)$ , we build the graph  $G$  vertex by vertex. We start with an initial set of vertices  $V' = \emptyset$  and an initial solution  $C = \emptyset$ . At each step, we add a new vertex  $v$  to both  $V'$  and  $C$ ,  $V' \leftarrow V' \cup \{v\}$ ,  $C \leftarrow C \cup \{v\}$ . We then call the compression routine on the pair  $(G[V'], C)$ , where  $G[V']$  is the subgraph induced by  $V'$  in  $G$ , to obtain a new solution  $C'$ . If  $|C'| > k$  then we output NO, otherwise we set  $C \leftarrow C'$ .

If we successfully complete the  $n$ th step where  $V' = V$ , we output  $C$  with  $|C| \leq k$ . Note that  $C$  will be an optimal solution for  $G$ .

The compression routine takes a graph  $G$  and a vertex cover  $C$  for  $G$  and returns a smaller vertex cover for  $G$  if there is one, otherwise, it returns  $C$  unchanged. Each time the compression routine is used it is provided with an intermediate solution of size at most  $k + 1$ .

The implementation of the compression routine proceeds as follows. We consider

a smaller vertex cover  $C'$  as a *modification* of the larger vertex cover  $C$ . This modification retains some vertices  $Y \subseteq C$  while the other vertices  $S = C \setminus Y$  are replaced with  $|S| - 1$  new vertices from  $V \setminus C$ .

The idea is to try by brute force all  $2^{|C|}$  partitions of  $C$  into such sets  $Y$  and  $S$ . For each such partition, the vertices from  $Y$  along with all of their adjacent edges are deleted. In the resulting instance  $G' = G[V \setminus Y]$ , it remains to find an optimal vertex cover that is disjoint from  $S$ . Since we have decided to take no vertex from  $S$  into the vertex cover, we have to take that endpoint of each edge that is not in  $S$ . At least one endpoint of each edge in  $G'$  is in  $S$ , since  $S$  is a vertex cover for  $G'$ . If both endpoints of some edge in  $G'$  are in  $S$ , then this choice of  $S$  cannot lead to a vertex cover  $C'$  with  $S \cap C' = \emptyset$ . We can quickly find an optimal vertex cover for  $G'$  that is disjoint from  $S$  by taking every vertex that is not in  $S$  and has degree greater than zero. Together with  $Y$ , this gives a new vertex cover  $C'$  for  $G$ . For each choice of  $Y$  and  $S$ , this can be done in time  $O(m)$ , leading to  $O(2^{|C|}m) = O(2^k m)$  time overall for one call of the compression routine. With at most  $n$  iterations of the compression algorithm, we get an algorithm for  $K$ -VERTEX COVER running in time  $O(2^k mn)$ .

Note that, in general, a compression routine will have running time exponential in the size of the solution provided to it, it is therefore important that each intermediate solution considered has size bounded by some  $k' = f(k)$ , where  $k$  is the parameter value for the original problem.

The employment of an FPT compression routine in the manner described here for  $K$ -VERTEX COVER will work effectively for any parameterized minimization problem which is *monotone* in the sense that NO instances are closed under element addition. That is, given a problem instance  $(I, k)$  that is a NO instance, any problem instance  $(I', k)$  with  $I \subseteq I'$  is also a NO instance. If a problem is monotone in this sense then we can immediately answer NO if we encounter an intermediate solution that cannot be compressed to meet the original parameter bound. Note that many minimization problems are not monotone in this sense. For example, a NO instance  $(G = (V, E), k)$  for  $K$ -DOMINATING SET can be changed to a YES instance by means of the addition of a single vertex that is adjacent to all vertices in  $V$ .

Finally, as noted in [49], the employment of compression routines is not restricted to the mode detailed here. For example, we could start with a suboptimal solution, perhaps provided by some type of parameterized approximation algorithm as detailed in Section 1.6, and then repeatedly compress this solution until it is either “good enough” or we are not willing to invest more calculation time.

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## 1.4 Not-Quite-Practical FPT Algorithms

In this section we introduce two techniques that lead to “not-quite-practical” FPT algorithms, color-coding and dynamic programming on bounded width graph decompositions. Both of these techniques have potential for practical application and

have been extensively studied from a theoretical point of view. However, in contrast to the methods introduced in the previous section, these approaches have so far lead to only isolated practical implementations and experimental results.

We also introduce a series of *algorithmic meta-theorems*. These are based on results from descriptive complexity theory and topological graph theory and provide us with general FPT algorithms, sometimes non-constructive, pertaining to large classes of problems. We view these theorems not as an end in themselves, but as being useful “signposts” in the search for practically efficient fixed-parameter algorithms.

### 1.4.1 Color-coding

This technique is useful for problems that involve finding small subgraphs in a graph, such as paths and cycles. Introduced by Alon et al. [10], it can be used to derive seemingly efficient randomized FPT algorithms for several subgraph isomorphism problems.

We formulate a parameterized version of the SUBGRAPH ISOMORPHISM problem as follows:

**K-SUBGRAPH ISOMORPHISM**

*Instance:*  $G = (V, E)$  and a graph  $H = (V^H, E^H)$  with  $|V^H| = k$ .

*Parameter:* A positive integer  $k$ .

*Question:* Is  $H$  isomorphic to a subgraph in  $G$ ?

The idea is that, in order to find the desired set of vertices  $V'$  in  $G$ , such that  $G[V']$  is isomorphic to  $H$ , we randomly color all the vertices of  $G$  with  $k$  colors and expect that, with some high degree of probability, all vertices in  $V'$  will obtain different colors. In some special cases of the SUBGRAPH ISOMORPHISM problem, dependent on the nature of  $H$ , this will simplify the task of finding  $V'$ .

If we color  $G$  uniformly at random with  $k$  colors, a set of  $k$  distinct vertices will obtain different colors with probability  $(k!)/k^k$ . This probability is lower-bounded by  $e^{-k}$ , so we need to repeat the process  $e^k$  times to have sufficiently high probability of obtaining the required coloring.

We can derandomize this kind of algorithm using *hashing*, but at the cost of extending the running time. We need a list of colorings of the vertices in  $G$  such that, for each subset  $V' \subseteq V$  with  $|V'| = k$  there is at least one coloring in the list by which all vertices in  $V'$  obtain different colors. Formally, we require a  $k$ -perfect family of hash functions from  $\{1, 2, \dots, |V|\}$ , the set of vertices in  $G$ , onto  $\{1, 2, \dots, k\}$ , the set of colors.

#### **DEFINITION 1.4 $k$ -Perfect Hash Functions**

*A  $k$ -perfect family of hash functions is a family  $\mathcal{H}$  of functions from  $\{1, 2, \dots, n\}$  onto  $\{1, 2, \dots, k\}$  such that, for each  $S \subset \{1, 2, \dots, n\}$  with  $|S| = k$ , there exists an  $h \in \mathcal{H}$  such that  $h$  is bijective when restricted to  $S$ .*

By a variety of sophisticated methods, Alon et al. [10] have proved the following:

**THEOREM 1.4 Alon et al. (1995)**

Families of  $k$ -perfect hash functions from  $\{1, 2, \dots, n\}$  onto  $\{1, 2, \dots, k\}$  can be constructed which consist of  $2^{O(k)} \cdot \log n$  hash functions. For such a hash function,  $h$ , the value  $h(i)$ ,  $1 \leq i \leq n$ , can be computed in linear time.

We can color  $G$  using each of the hash functions from our  $k$ -perfect family in turn. If the desired set of vertices  $V'$  exists in  $G$ , then, for at least one of these colorings, all vertices in  $V'$  will obtain different colors as we require.

We now give a very simple example of this technique. The subgraph that we will look for is a cycle of length  $k$ . We use  $k$  colors. If a  $k$ -cycle exists in the graph, then there must be a coloring that assigns a different color to each vertex in the cycle.

For each colouring  $h$ , we check every ordering  $c_1, c_2, \dots, c_k$  of the  $k$  colours to decide whether or not it *realizes* a  $k$ -cycle. We first construct a directed graph  $G'$  as follows:

For each edge  $(u, v) \in E$ , if  $h(u) = c_i$  and  $h(v) = c_{i+1 \pmod{k}}$  for some  $i$ , then replace  $(u, v)$  with arc  $\langle u, v \rangle$ , otherwise delete  $(u, v)$ .

In  $G'$ , for each  $v$  with  $h(v) = c_1$ , we use a breadth first search to check for a cycle  $C$  from  $v$  to  $v$  of length  $k$ .

A deterministic algorithm will need to check  $2^{O(k)} \cdot \log |V|$  colorings, and, for each of these,  $k!$  orderings. We can decide if an ordering of colors realizes the  $k$ -cycle in time  $O(k \cdot |V|^2)$ . Thus, our algorithm is FPT, but, arguably, not practically efficient. The main drawback is that the  $2^{O(k)} \cdot \log |V|$  bound on the size of the family of hash functions hides a large constant in the  $O(k)$  exponent.

More interesting examples of applications of color-coding to subgraph isomorphism problems, based on dynamic programming, can be found in [10].

**1.4.2 Bounded Width Metrics**

Faced with intractable graph problems, many authors have turned to study of various restricted classes of graphs for which such problems can be solved efficiently. A number of graph *width metrics* naturally arise in this context which restrict the inherent complexity of a graph in various senses.

The idea here is that a useful width metric should admit efficient algorithms for many (generally) intractable problems on the class of graphs for which the width is small. This leads to consideration of these measures from a parameterized point of view. The corresponding naturally parameterized problem has the following form:

Let  $w(G)$  denote any measure of graph width.

*Instance:* A graph  $G = (V, E)$ .

*Parameter:* A positive integer  $k$ .

*Question:* Is  $w(G) \leq k$ ?

One of the most successful measures in this context is the notion of *treewidth* which arose from the seminal work of Robertson and Seymour on graph minors and immersions [61, 62, 63]. Treewidth measures, in a precisely defined way, how *tree-*

like a graph is. The fundamental idea is that we can lift many results from trees to graphs that are tree-like. Related to treewidth is the notion of *pathwidth* which measures, in the same way, how *path-like* a graph is.

Many generally intractable problems become fixed-parameter tractable for the class of graphs that have bounded treewidth or bounded pathwidth, with the parameter being the treewidth or pathwidth of the input graph. Furthermore, treewidth and pathwidth generalize many other well-studied graph properties. For example, planar graphs with radius  $k$  have treewidth at most  $3k$ , series parallel multigraphs have treewidth two, chordal graphs (graphs having no induced cycles of length four or more) with maximum clique size  $k$  have treewidth at most  $k - 1$ , interval graphs  $G'$  with maximum clique size  $k$  have pathwidth at most  $k - 1$ .

A graph  $G$  has treewidth at most  $k$  if we can associate a tree  $T$  with  $G$  in which each node represents a subgraph of  $G$  having at most  $k + 1$  vertices, such that all vertices and edges of  $G$  are represented in at least one of the nodes of  $T$ , and for each vertex  $v$  in  $G$ , the nodes of  $T$  where  $v$  is represented form a subtree of  $T$ . Such a tree is called a *tree decomposition* of  $G$ , of width  $k$ .

**DEFINITION 1.5** [Tree decomposition and Treewidth]

Let  $G = (V, E)$  be a graph. A tree decomposition,  $TD$ , of  $G$  is a pair  $(T, \mathcal{X})$  where

1.  $T = (I, F)$  is a tree, and
2.  $\mathcal{X} = \{X_i \mid i \in I\}$  is a family of subsets of  $V$ , one for each node of  $T$ , such that

- (i)  $\bigcup_{i \in I} X_i = V$ ,
- (ii) for every edge  $\{v, w\} \in E$ , there is an  $i \in I$  with  $v \in X_i$  and  $w \in X_i$ , and
- (iii) for all  $i, j, k \in I$ , if  $j$  is on the path from  $i$  to  $k$  in  $T$ , then  $X_i \cap X_k \subseteq X_j$ .

The *width* of a tree decomposition  $((I, F), \{X_i \mid i \in I\})$  is  $\max_{i \in I} |X_i| - 1$ . The *treewidth* of a graph  $G$ , denoted by  $tw(G)$ , is the minimum width over all possible tree decompositions of  $G$ .

**DEFINITION 1.6** [Path decomposition and Pathwidth]

A path decomposition,  $PD$ , of a graph  $G$  is a tree decomposition  $(P, \mathcal{X})$  of  $G$  where  $P$  is simply a path (i.e. the nodes of  $P$  have degree at most two). The *pathwidth* of  $G$ , denoted by  $pw(G)$  is the minimum width over all possible path decompositions of  $G$ .

Any path decomposition of  $G$  is also a tree decomposition of  $G$ , so the pathwidth of  $G$  is at least equal to the treewidth of  $G$ . For many graphs, the pathwidth will be somewhat larger than the treewidth. For example, let  $B_k$  denote the complete binary tree of height  $k$ , having  $2^k - 1$  vertices, then  $tw(B_k) = 1$ , but  $pw(B_k) = k$ .

Graphs of treewidth and pathwidth at most  $k$  are also called *partial  $k$ -trees* and *partial  $k$ -paths*, respectively, as they are exactly the subgraphs of  $k$ -trees and  $k$ -paths. There are a number of other variations equivalent to the notions of treewidth and pathwidth (see, for example, [14].) For algorithmic purposes, the characterizations provided by the definitions given above tend to be the most useful.

The typical method employed to produce FPT algorithms for problems restricted to graphs of bounded treewidth (pathwidth) proceeds in two stages.

1. Find a bounded-width tree (path) decomposition of the input graph that exhibits the underlying tree (path) structure.
2. Perform dynamic programming on this decomposition to solve the problem.

The following Lemma encapsulates the two properties of tree decompositions on which the dynamic programming approach relies.

**LEMMA 1.2 Connected subtrees**

Let  $G = (V, E)$  be a graph and  $TD = (T, \mathcal{X})$  a tree decomposition of  $G$ .

- (i) For all  $v \in V$ , the set of nodes  $\{i \in I \mid v \in X_i\}$  forms a connected subtree of  $T$ .
- (ii) For each connected subgraph  $G'$  of  $G$ , the nodes in  $T$  which contain a vertex of  $G'$  induce a connected subtree of  $T$ .

In order for this approach to produce practically efficient FPT algorithms, as opposed to proving that problems are theoretically tractable, it is important to be able to produce the necessary decomposition reasonably efficiently.

Determining the treewidth or pathwidth of a graph is an *NP*-hard problem. However, polynomial time approximation algorithms have been found [16]. There is a polynomial time algorithm that, given a graph  $G$  with treewidth  $k$ , finds a tree decomposition of width at most  $O(k \cdot \log n)$  for  $G$ . There is a polynomial time algorithm that, given a graph  $G$  with pathwidth  $k$ , finds a path decomposition of width at most  $O(k \cdot \log^2 n)$  for  $G$ .

Bodlaender [13] gave the first *linear-time* FPT algorithms (i.e. linear in  $|G|$ ) for the constructive versions of both *K-TREewidth* and *K-PATHwidth*. Perkovic and Reed [58] have improved upon Bodlaender's work, although the  $f(k)$ 's involved mean that the algorithms given in both [13] and [58] are not workable in practice. However, there are far simpler FPT algorithms that produce tree and path decompositions having width at most a constant factor larger than the optimum [60, 63] (see Section 1.6.)

For some graph classes, the optimal treewidth and pathwidth, or good approximations of these, can be found using practically efficient polynomial time algorithms. Examples are chordal bipartite graphs, interval graphs, permutation graphs, circle graphs, [18] and co-graphs [19].

For planar graphs, Alber et al. [5, 7] have introduced the notion of a *layerwise separation property* pertaining to the underlying parameterized problem that one might hope to solve via a small-width tree decomposition. The *layerwise separation property* holds for all problems on planar graphs for which a linear problem kernel can be constructed.

For problems having this property, we can exploit the layer structure of planar graphs, along with knowledge about the structure of “YES” instances of the problem, in order to find small separators in the input graph such that each of the resulting components has small treewidth. Tree decompositions for each of the components are then merged with the separators to produce a small-width tree decomposition of the complete graph.

This approach leads to, for example, algorithms that solve  $K$ -VERTEX COVER and, more interestingly,  $K$ -DOMINATING SET, on planar graphs in time  $2^{O(\sqrt{k})} \cdot n$ . The algorithms work by constructing a tree decomposition of width  $O(\sqrt{k})$  for the kernelized graph, and then performing dynamic programming on this decomposition.

An algorithm that uses dynamic programming on a *tree* works by computing some value, or table of values, for each node in the tree. The important point is that the value for a node can be computed using only information directly associated with the node itself, along with values already computed for the children of the node.

Extending the idea of dynamic programming on *trees* to dynamic programming on *bounded-width tree decompositions* is really just a matter of having to construct more complicated tables of values. Instead of considering a single vertex at each node and how it interacts with the vertices at its child nodes, we now need to consider a reasonably small *set* of vertices represented at each node, and how this small set of vertices can interact with each of the small sets of vertices represented at its child nodes.

The most important factor in dynamic programming on tree decompositions is the size of the tables produced. The table size is usually  $O(c^k)$ , where  $k$  is the width of the tree decomposition and  $c$  depends on the combinatorics of the underlying problem that we are trying to solve. We can trade off different factors in the design of such algorithms. For example, a fast approximation algorithm that produces a tree decomposition of width  $3k$ , or even  $k^2$ , for a graph of treewidth  $k$  could be quite acceptable if  $c$  is small.

*Cliquewidth*, first introduced in [31], is another graph width metric that has more recently gained prominence in algorithm design. A graph that has cliquewidth  $k$  can be recursively constructed from single vertices with labels in  $[k] = \{1, 2, \dots, k\}$  using only the *composition operations* of graph union  $G = G_1 \cup G_2$ , vertex relabeling  $G = (G_1)_{i \rightarrow j}$ , and cross-product edge insertion between label sets  $G = (G_1)_{i \times j}$ .

The series of composition operations (called a *k-expression*) that produces such a cliquewidth- $k$  graph  $G$  gives rise to a decomposition of  $G$  into a tree of subgraphs of  $G$ . This decomposition then leads to a linear-time dynamic programming algorithm for many problems restricted to cliquewidth- $k$  graphs. However, in contrast with treewidth and pathwidth, there is no known FPT algorithm that constructs such a

decomposition for a given cliquewidth- $k$  graph. A polynomial time approximation algorithm has recently been presented in [57].

We note that, even though finding *exact* bounded width graph decompositions, for graphs with small width, does not yet appear to be feasible, in practice heuristics and approximations have proven to be quite effective. In addition, many graphs derived in practical applications are themselves constructed inductively, making them prime candidates for these methods. For examples of dynamic programming algorithms on various bounded width graph decompositions see, for example, [20].

### 1.4.3 Algorithmic Meta-theorems

Descriptive complexity theory relates the logical complexity of a problem description to its computational complexity. In this context there are some useful results that relate to fixed-parameter tractability. We can view these results not as an end in themselves, but as being useful “signposts” in the search for practically efficient fixed-parameter algorithms.

We will consider graph properties that can be defined in *first-order logic* and *monadic second-order logic*.

In first order logic we have an (unlimited) supply of *individual* variables, one for each vertex in the graph. Formulas of first-order logic (FO) are formed by the following rules:

1. *Atomic formulas*:  $x = y$  and  $R(x_1, \dots, x_k)$ , where  $R$  is a  $k$ -ary relation symbol and  $x, y, x_1, \dots, x_k$  are individual variables, are FO-formulas.
2. *Conjunction, Disjunction*: If  $\phi$  and  $\psi$  are FO-formulas, then  $\phi \wedge \psi$  is an FO-formula and  $\phi \vee \psi$  is an FO-formula.
3. *Negation*: If  $\phi$  is an FO-formula, then  $\neg\phi$  is an FO-formula.
4. *Quantification*: If  $\phi$  is an FO-formula and  $x$  is an individual variable, then  $\exists x \phi$  is an FO-formula and  $\forall x \phi$  is an FO-formula.

We can state that a graph has a clique of size  $k$  using an FO-formula. Here, the binary relation  $E(x_i, x_j)$  indicates the existence of an edge between vertices  $x_i$  and  $x_j$ .

$$\exists x_1 \dots x_k \bigwedge_{1 \leq i < j \leq k} E(x_i, x_j)$$

We can state that a graph has a dominating set of size  $k$  using an FO-formula,

$$\exists x_1 \dots x_k \forall y \bigvee_{1 \leq i \leq k} \left( E(x_i, y) \vee (x_i = y) \right)$$

In monadic second-order logic we have an (unlimited) supply of both individual variables, one for each vertex in the graph, and *set* variables, one for each subset of vertices in the graph. Formulas of monadic second-order logic (MSO) are formed by the rules for FO and the following additional rules:

1. *Additional atomic formulas:* For all set variables  $X$  and individual variables  $y$ ,  $Xy$  is an MSO-formula.
2. *Set quantification:* If  $\phi$  is an MSO-formula and  $X$  is a set variable, then  $\exists X \phi$  is an MSO-formula, and  $\forall X \phi$  is an MSO-formula.

We can state that a graph is  $k$ -colorable using an MSO-formula,

$$\exists X_1 \dots \exists X_k \left( \forall x \bigvee_{i=1}^k X_{ix} \wedge \forall x \forall y \left( E(x, y) \rightarrow \bigwedge_{i=1}^k \neg (X_{ix} \wedge X_{iy}) \right) \right)$$

The problems that we are interested in are special cases of the *model-checking problem*.

Let  $\Phi$  be a class of formulas (logic), and let  $\mathcal{D}$  be a class of finite relational structures. The model-checking problem for  $\Phi$  on  $\mathcal{D}$  is the following problem.

*Instance:* A structure  $\mathcal{A} \in \mathcal{D}$ , and a sentence (no free variables)  $\phi \in \Phi$ .

*Question:* Does  $\mathcal{A}$  satisfy  $\phi$ ?

The model-checking problems for FO and MSO are PSPACE-complete in general. However, as the following results show, if we restrict the class of input structures then in some cases these model-checking problems become tractable.

The most well-known result, paraphrased here, is due to Courcelle [30].

**THEOREM 1.5 Courcelle 1990**

*The model-checking problem for MSO restricted to graphs of bounded treewidth is linear-time fixed-parameter tractable.*

Detleef Seese [64] has proved a converse to Courcelle's theorem.

**THEOREM 1.6 Seese 1991**

*Suppose that  $\mathcal{F}$  is any family of graphs for which the model-checking problem for MSO is decidable, then there is a number  $n$  such that, for all  $G \in \mathcal{F}$ , the treewidth of  $G$  is less than  $n$ .*

Courcelle's theorem tells us that if we can define the problem that we are trying to solve as a model-checking problem, and we can define the particular graph property that we are interested in as an MSO-formula, then there is an FPT algorithm that solves the problem for input graphs of bounded treewidth. The theorem by itself doesn't tell us how the algorithm works.

The automata-theoretic proof of Courcelle's theorem given by Abrahamson and Fellows [2] provides a generic algorithm that relies on dynamic programming over labelled trees (see [35] for details of this approach.) However, this generic algorithm is really just further proof of *theoretical* tractability. The importance of the theorem is that it provides a powerful engine for demonstrating that a large class of problems is FPT. If we can couch a problem in the correct manner then we know that it is

“worth looking” for an efficient FPT algorithm that works on graphs of bounded treewidth.

The next result concerns classes of graphs having *bounded local treewidth*. The local tree width of a graph  $G$  is defined via the following function.

$$ltw(G, r) = \max \{tw(N_r(v)) \mid v \in V(G)\}$$

where  $N_r(v)$  is the neighbourhood of radius  $r$  about  $v$  (including  $v$ .)

A class of graphs  $\mathcal{F}$  has *bounded local treewidth* if there is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that, for all  $G \in \mathcal{F}$  and  $r \geq 1$ ,  $ltw(G, r) \leq f(r)$ . The concept is a relaxation of bounded treewidth for classes of graphs. Instead of requiring that the treewidth of a graph overall is bounded by some constant, we require that, for each vertex in the graph, the treewidth of each neighbourhood of radius  $r$  about that vertex is bounded by some uniform function of  $r$ .

Examples of classes of graphs that have bounded local treewidth include graphs of bounded treewidth (naturally), graphs of *bounded degree*, *planar* graphs, and graphs of *bounded genus*.

Frick and Grohe [44] have proved the following theorem.

**THEOREM 1.7 Frick and Grohe 1999**

*Parameterized problems that can be described as model-checking problems for FO are fixed-parameter tractable on classes of graphs of bounded local treewidth.*

This theorem tells us, for example, that parameterized versions of problems such as DOMINATING SET, INDEPENDENT SET, or SUBGRAPH ISOMORPHISM are FPT on planar graphs, or on graphs of bounded degree. As with Courcelle’s theorem, it provides us with a powerful engine for demonstrating that a large class of problems is FPT, but leaves us with the job of finding practically efficient FPT algorithms for these problems.

The last meta-theorem that we will present has a somewhat different flavour. We first need to introduce some ideas from topological graph theory.

A graph  $H$  is a *minor* of a graph  $G$  iff there exists some subgraph,  $G^H$  of  $G$ , such that  $H$  can be obtained from  $G^H$  by a series of *edge contractions*.

We define an edge contraction as follows. Let  $e = (u, v)$  be an edge of the graph  $G^H$ . By  $G^H/e$  we denote the graph obtained from  $G^H$  by *contracting* the edge  $e$  into a new vertex  $v_e$  which becomes adjacent to all former neighbours of  $u$  and of  $v$ .  $H$  can be obtained from  $G^H$  by a series of edge contractions iff there are graphs  $G_0, \dots, G_n$  and edges  $e_i \in G_i$  such that  $G_0 = G^H$ ,  $G_n \simeq H$ , and  $G_{i+1} = G_i/e_i$  for all  $i < n$ .

Note that every subgraph of a graph  $G$  is also a minor of  $G$ . In particular, every graph is its own minor.

A class of graphs  $\mathcal{F}$  is *minor-closed* if, for every graph  $G \in \mathcal{F}$ , every minor of  $G$  is also contained in  $\mathcal{F}$ . A very simple example is the class of graphs with no edges. Another example is the class of acyclic graphs. A more interesting example is the following:

Let us say that a graph  $G = (V, E)$  is *within  $k$  vertices* of a class of graphs  $\mathcal{F}$  if there is a set  $V' \subseteq V$ , with  $|V'| \leq k$ , such that  $G[V - V'] \in \mathcal{F}$ . If  $\mathcal{F}$  is any minor-closed class of graphs, then, for every  $k \geq 1$ , the class of graphs within  $k$  vertices of  $\mathcal{F}$ ,  $\mathcal{W}_k(\mathcal{F})$ , is also minor-closed.

Note that for each integer  $k \geq 1$ , the class of graphs of treewidth or pathwidth at most  $k$  is minor-closed.

Let  $\mathcal{F}$  be a class of graphs which is closed under taking of minors, and let  $H$  be a graph that is not in  $\mathcal{F}$ . Each graph  $G$  which has  $H$  as a minor is not in  $\mathcal{F}$ , otherwise  $H$  would be in  $\mathcal{F}$ . We call  $H$  a *forbidden minor* of  $\mathcal{F}$ . A *minimal* forbidden minor of  $\mathcal{F}$  is a forbidden minor of  $\mathcal{F}$  for which each proper minor is in  $\mathcal{F}$ . The set of all minimal forbidden minors of  $\mathcal{F}$  is called the *obstruction set* of  $\mathcal{F}$ .

In a long series of papers, collectively entitled “Graph Minors”, Robertson and Seymour [63] have essentially proved that any minor-closed class of graphs  $\mathcal{F}$  must have a *finite* obstruction set. Robertson and Seymour have also shown that, for a fixed graph  $H$ , it can be determined whether  $H$  is a minor of a graph  $G$  in time  $O(f(|H|) \cdot |G|^3)$ .

We can now derive the following theorem:

**THEOREM 1.8 Minor-closed membership**

*If  $\mathcal{F}$  is a minor-closed class of graphs then membership of a graph  $G$  in  $\mathcal{F}$  can be determined in time  $O(f(k) \cdot |G|^3)$ , where  $k$  is the collective size of the graphs in the obstruction set for  $\mathcal{F}$ .*

This meta-theorem tells us that if we can define a graph problem via membership in a minor-closed class of graphs  $\mathcal{F}$ , then the problem is FPT, with the parameter being the collective size of the graphs in the obstruction set for  $\mathcal{F}$ . However, it is important to note two major difficulties that we now face. Firstly, for a given minor-closed class we have a proof of the *existence* of a finite obstruction set, but no effective method for *obtaining* the obstruction set. Secondly, the minor testing algorithm has very large hidden constants (around  $2^{500}$ ), and the sizes of obstruction sets in many cases are known to be very large.

Thus, again we have a theorem that provides us with a powerful engine for demonstrating that a large class of problems is, in fact, FPT, but the problem of finding practically efficient FPT algorithms for such problems remains open.

## 1.5 Parameterized Intractability

The question arises: What do we do with a problem for which we know of no FPT algorithm? Good examples are the problems DOMINATING SET or INDIDENT SET for which we know of no algorithm significantly better than trying all possibilities. For a fixed  $k$ , this takes time  $\Omega(n^{k+1})$ . Of course, we would *like* to prove that there is *no* FPT algorithm for such a problem, but, as with classical complexity, the best we can do is to formulate some sort of completeness/hardness program. Showing that  $K$ -DOMINATING SET is not FPT would also show, as a corollary, that  $P \neq NP$ .

Any completeness programme needs three things. First, it needs a notion of easiness, which we have: FPT. Second, it needs a notion of reduction, and third, it needs some core problem which we believe to be intractable.

Following naturally from the concept of fixed-parameter tractability is an appropriate notion of reducibility that expresses the fact that two parameterized problems have comparable parameterized complexity. That is, if problem (language)  $A$  reduces to problem (language)  $B$ , and problem  $B$  is fixed-parameter tractable, then so too is problem  $A$ .

### DEFINITION 1.7 Parameterized Transformation

A parameterized transformation\* from a parameterized language  $L$  to a parameterized language  $L'$  (symbolically  $L \leq_{FPT} L'$ ) is an algorithm that computes, from input consisting of a pair  $(I, k)$ , a pair  $(I', k')$  such that:

1.  $(I, k) \in L$  if and only if  $(I', k') \in L'$ ,
2.  $k' = g(k)$  is a computable function only of  $k$ , and
3. the computation is accomplished in time  $f(k)n^c$ , where  $n$  is the size of the main part of the input  $I$ ,  $k$  is the parameter,  $c$  is a constant (independent of both  $n$  and  $k$ ), and  $f$  is an arbitrary function dependent only on  $k$ .

If  $A \leq_{FPT} B$  and  $B \leq_{FPT} A$ , then we say that  $A$  and  $B$  are *FPT-equivalent*.

Now we have two ingredients: easiness and reductions. We need the final component for our program to establish the apparent parameterized intractability of computational problems: the identification of a “core” problem to reduce from.

In classical NP-completeness this is the heart of the Cook-Levin Theorem: the argument that a nondeterministic Turing machine is such an opaque object that it does not seem reasonable that we can determine in polynomial time if it has an accepting path from amongst the exponentially many possible paths. The idea of

\*Strictly speaking, this is a parameterized *many-one* reduction as an analog of the classical Karp reduction. Other variations such as parameterized Turing reductions are possible. The function  $g$  can be arbitrary, rather than computable, for other non-uniform versions. We give the reduction most commonly met in practice.

Downey and Fellows, introduced in the fundamental papers [33, 34], was to look at the following parameterized version of nondeterministic Turing machine acceptance.

**SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE**

*Instance:* A nondeterministic Turing machine (of arbitrary fanout)  $M$ .

*Parameter:* A positive integer  $k$ .

*Question:* Does  $M$  have a computation path accepting the empty string in at most  $k$  steps?

In the same sense that  $NP$ -completeness of the  $q(n)$ -STEP NON-DETERMINISTIC TURING MACHINE ACCEPTANCE, where  $q(n)$  is a polynomial in the size of the input, provides us with very strong evidence that no  $NP$ -complete problem is likely to be solvable in polynomial time, using SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE as a hardness core provides us with very strong evidence that no parameterized language  $L$ , for which SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE $_{\leq FPT} L$ , is likely to be fixed-parameter tractable. That is, if we accept the idea for the basis of  $NP$ -completeness, then we should also accept that the SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE problem is not solvable in time  $O(|M|^c)$  for any fixed  $c$ . Our intuition would again be that all paths would need to be tried.

We remark that the hypothesis “SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE is not FPT” is somewhat stronger than  $P \neq NP$ . Furthermore, connections between this hypothesis and classical complexity have recently become apparent. If SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE is FPT then we know that the EXPONENTIAL TIME HYPOTHESIS, which states that  $n$ -variable 3SAT is not in subexponential time ( $\text{DTIME}(2^{o(n)})$ ), fails. See Impagliazzo, Paturi and Zane [48], Cai and Juedes [24], and Estivill-Castro, Downey, Fellows, Prieto-Rodriguez and Rosamond [41].

The class of problems FPT-equivalent to SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE is called  $W[1]$ , for reasons discussed below. The parameterized analog of the classical Cook-Levin theorem (that CNFSAT is  $NP$ -complete) uses the following parameterized version of 3SAT:

**WEIGHTED CNF SAT**

*Instance:* A CNF formula  $X$ .

*Parameter:* A positive integer  $k$ .

*Question:* Does  $X$  have a satisfying assignment of weight  $k$ ? Here the *weight* of an assignment is the Hamming weight, that is, the number of literals set to be true.

Similarly, we can define WEIGHTED  $n$ CNF SAT, where the clauses have only  $n$  variables and  $n$  is some number fixed in advance. WEIGHTED  $n$ CNF SAT, for any fixed  $n \geq 2$ , is complete for  $W[1]$ .

***THEOREM 1.9*** *Downey and Fellows [34] and Cai, Chen, Downey*

**and Fellows [22]**

WEIGHTED  $n$ CNF SAT  $\equiv_{FPT}$  SHORT NON-DETERMINISTIC TURING MACHINE ACCEPTANCE.

The original theorems and classes were first characterized in terms of boolean circuits of a certain structure. These characterizations lend themselves to easier *membership* proofs, we define them here for completeness<sup>†</sup>.

We consider a 3CNF formula as a circuit consisting of one input (of unbounded fanout) for each variable, possibly inverters below the variable, and structurally a large *and* of small *or*'s (of size 3) with a single output line. We can similarly consider a 4CNF formula to be a large *and* of small *or*'s where "small" is defined to be 4. More generally, it is convenient to consider the model of a *decision circuit*. This is a circuit consisting of large and small gates with a single output line, and no restriction on the fanout of gates. For such a circuit, the *depth* is the maximum number of gates on any path from the input variables to the output line, and the *weft* is the "large gate depth." More precisely, the weft is defined to be the maximum number of large gates on any path from the input variables to the output line, where a gate is called large if its fanin exceeds some pre-determined bound.

The *weight* of an assignment to the input variables of a decision circuit is the Hamming weight, the number of variables made true by the assignment.

Let  $\mathcal{F} = \{C_1, \dots, C_n, \dots\}$  be a family of decision circuits. Associated with  $\mathcal{F}$  is a basic parameterized language

$$L_{\mathcal{F}} = \{\langle C_i, k \rangle : C_i \text{ has a weight } k \text{ satisfying assignment}\}.$$

We will denote by  $L_{\mathcal{F}(t,h)}$  the parameterized language associated with the family of weft  $t$ , depth  $h$ , decision circuits.

**DEFINITION 1.8**  $W[1]$  **Downey and Fellows [33]** *We define a language  $L$  to be in the class  $W[1]$  iff there is a parameterized transformation from  $L$  to  $L_{\mathcal{F}(1,h)}$ , for some  $h$ .*

We remark that, since publication of the original papers of Downey and Fellows, hundreds of problems have been shown to be  $W[1]$ -complete and many have been shown to be  $W[1]$  hard. We refer the reader to Downey and Fellows [35] for a list of examples, as of 1998, and to Flum and Grohe [43] for some more recent examples.

Notice that, in Theorem 1.9, we did *not* say that WEIGHTED CNF SAT is  $W[1]$ -complete. The reason for this is that we do not believe that it is!

Classically, using a padding argument, we know that CNF SAT  $\equiv_m^P$  3CNF SAT. However, the classical reduction *doesn't* define a parameterized transformation from WEIGHTED CNF SAT to WEIGHTED 3CNF SAT, it is not structure-preserving

<sup>†</sup>Other approaches to characterization of parameterized hardness classes have been proposed, notably that of Flum and Grohe [43] using finite model theory. We refer the reader to [43].

enough to ensure that parameters map to parameters. In fact, it is conjectured [33] that there is *no* parameterized transformation at all from WEIGHTED CNF SAT to WEIGHTED 3CNF SAT. If the conjecture is correct, then WEIGHTED CNF SAT is *not* in the class  $W[1]$ .

The point here is that parameterized reductions are more refined than classical ones, and hence we believe that we get a wider variety of apparent hardness behaviour when intractable problems are classified according to this more fine grained analysis.

We can view an input formula  $X$  for WEIGHTED CNF SAT as a product of sums. Extending this reasoning, we can define WEIGHTED  $t$ -NORMALIZED SAT as the weighted satisfiability problem for a formula  $X$  where  $X$  is a product of sums of products of sums ... with  $t$  alternations. We can define WEIGHTED SAT to be the weighted satisfiability problem for a formula  $X$  that is unrestricted.

**DEFINITION 1.9**  $W[t]$  For each  $t \geq 1$ , we define a language  $L$  to be in the class  $W[t]$  iff there is a parameterized transformation from  $L$  to  $L_{\mathcal{F}(t,h)}$  for some  $h$ .

In [33] Downey and Fellows show that, for all  $t \geq 1$ , WEIGHTED  $t$ -NORMALIZED SAT is complete for  $W[t]$ . Thus,  $W[1]$  is the collection of parameterized languages  $FPT$ -equivalent to WEIGHTED 3CNF SAT,  $W[2]$  is the collection of parameterized languages  $FPT$ -equivalent to WEIGHTED CNF SAT, and for each  $t \geq 2$ ,  $W[t]$  is the collection of parameterized languages  $FPT$ -equivalent to WEIGHTED  $t$ -NORMALIZED SAT.

These classes form part of the basic hierarchy of parameterized problems below.

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \dots \subseteq W[t] \subseteq W[SAT] \subseteq W[P] \subseteq AW[P] \subseteq XP$$

This sequence is commonly termed the  $W$ -hierarchy. The complexity class  $W[1]$  can be viewed as the parameterized analog of  $NP$ , since it *suffices* to establish likely parameterized intractability. We remark that a number of natural problems have been found at various levels of this hierarchy. For example, DOMINATING SET is complete for the level  $W[2]$ .

The classes  $W[SAT]$ ,  $W[P]$  and the  $AW$  classes were introduced by Abrahamson, Downey and Fellows in [1]. The class  $W[SAT]$  is the collection of parameterized languages  $FPT$ -equivalent to WEIGHTED SAT. The class  $W[P]$  is the collection of parameterized languages  $FPT$ -equivalent to WEIGHTED CIRCUIT SAT, the weighted satisfiability problem for a decision circuit  $C$  that is unrestricted. A standard translation of Turing machines into circuits shows that  $K$ -WEIGHTED CIRCUIT SAT is the same as the problem of deciding whether or not a deterministic Turing machine accepts an input of weight  $k$ . It is conjectured that the containment  $W[SAT] \subseteq W[P]$  is proper [35].

$AW[P]$  captures the notion of *alternation*.  $AW[P]$  is the collection of parameterized languages  $FPT$ -equivalent to PARAMETERIZED QUANTIFIED CIRCUIT SATISFIA-

BILITY, the weighted satisfiability problem for an unrestricted decision circuit that applies *alternating quantifiers* to the inputs, defined here.

#### PARAMETERIZED QUANTIFIED CIRCUIT SAT

*Instance:* A decision circuit  $C$  whose inputs correspond to a sequence  $s_1, \dots, s_r$  of pairwise disjoint sets of variables.

*Parameter:*  $r, k_1, \dots, k_n$ .

*Question:* Is it the case that there exists a size  $k_1$  subset  $t_1$  of  $s_1$ , such that for every size  $k_2$  subset  $t_2$  of  $s_2$ , there exists a size  $k_3$  subset  $t_3$  of  $s_3$ , such that ... (alternating quantifiers) such that, when  $t_1 \cup t_2 \cup \dots \cup t_r$  are set to true, and all other variables are set to false,  $C$  is satisfied?

Many parameterized analogs of game problems are complete for the *AW* classes, such as the parameterized analog of GEOGRAPHY.

The class *XP*, introduced in [35], is the collection of parameterized languages  $L$  such that the  $k$ th slice of  $L$  (the instances of  $L$  having parameter  $k$ ) is complete for  $DTIME(n^k)$ . *XP* is provably distinct from *FPT* and seems to be the parameterized class corresponding to the classical class *EXP* (exponential time). It is conjectured that all of the containments here are proper, but all that is currently known is that *FPT* is a proper subset of *XP*.

If we compare classical and parameterized complexity it is evident that the framework provided by parameterized complexity theory allows for more finely-grained complexity analysis of computational problems. It is deeply connected with algorithmic heuristics and exact algorithms in practice. We refer the reader to either the survey by Flum and Grohe [42], or those in two recent issues of *The Computer Journal* [36] for further insight.

We can consider many different parameterizations of a single classical problem, each of which leads to either a tractable, or (likely) intractable, version in the parameterized setting. This allows for an extended dialog with the problem at hand. This idea towards the solution of algorithmic problems is explored in, for example, [38].

The reader may note that parameterized complexity is addressing intractability *within polynomial time*. In this vein, the parameterized framework can be used to demonstrate that many classical problems that admit a PTAS don't, in fact, admit any PTAS with a practical running time, unless  $W[1] = FPT$  (see end of Section 1.6.) It has been used to show that resolution is not automatizable unless  $W[P] = FPT$  (Alekhovich and Razborov [8], Eickmeyer, Grohe and Grüber [40].) It can also be used to show that the large hidden constants (various towers of two's) in the running times of generic algorithms obtained through the use of algorithmic metatheorems (Section 1.4.3) cannot be improved upon (see [43].)

We finally note that there are alternative useful parameterized complexity hierarchies, such as the *A* and *M*-hierarchies, see e.g. Flum and Grohe [43].

Rather than further pursuing parameterized intractability and the rich area of parameterized structural complexity theory, we have concentrated this survey on the collection of distinctive techniques that has been developed for fixed-parameter tractable algorithm design. It would take a survey of comparable length to comprehensively

tackle the area of parameterized intractability. We will simply refer the reader to Downey and Fellows [35], Flum and Grohe [43], or to survey articles such as Downey [32] for further details.

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## 1.6 Parameterized Approximation Algorithms

We close this chapter with discussion of a relatively new topic, parameterized approximation, introduced independently by three papers presented at IWPEC 2006 (The 3rd International Workshop on Parameterized Complexity and Exact Algorithms) [37], [27], [23].

There are various ways in which parameterized complexity and parameterized computation can interact with approximation algorithms. The interplay between the two fields is covered comprehensively in [53]. Here, we will consider only the most natural extension of parameterized complexity in this direction. We first need to introduce some definitions, we follow those given in [53].

Each input instance to an *NP*-optimization problem has associated with it a set of feasible solutions. A cost measure is defined for each of these feasible solutions. The task in solving the optimization problem is to find a feasible solution where the measure is as good as possible.

Formally, we define an *NP*-optimization problem as a 4-tuple  $(I, sol, cost, goal)$  where

- $I$  is the set of instances.
- For an instance  $x \in I$ ,  $sol(x)$  is the set of feasible solutions for  $x$ . The length of each  $y \in sol(x)$  is polynomially bounded in  $|x|$ , and it can be decided in time polynomial in  $|x|$  whether  $y \in sol(x)$  holds for given  $x$  and  $y$ .
- Given an instance  $x$  and a feasible solution  $y$ ,  $cost(x, y)$  is a polynomial-time computable positive integer.
- $goal$  is either *max* or *min*.

The cost of an optimum solution for instance  $x$  is denoted by  $opt(x)$ .

$$opt(x) = goal\{cost(x, y') \mid y' \in sol(x)\}$$

If  $y$  is a solution for instance  $x$  then the *performance ratio* of  $y$  is defined as

$$R(x, y) = \begin{cases} cost(x, y)/opt(x) & \text{if } goal = \textit{min}, \\ opt(x)/cost(x, y) & \text{if } goal = \textit{max}. \end{cases}$$

For a real number  $c > 1$ , we say that an algorithm is a *c-approximation* algorithm if it always produces a solution with performance ratio at most  $c$ .

One obvious parameter of an optimization problem instance is the optimum cost. This leads to a *standard parameterization* of an optimization problem  $X$ , where we define the corresponding parameterized decision problem  $X_{\leq}$  as

$X_{\leq}$

*Instance:* An instance  $x$  of  $X$ .

*Parameter:* A positive integer  $k$ .

*Question:* Is  $\text{opt}(x) \leq k$ ?

We can define  $X_{\geq}$  analogously. For many problems, if we can solve  $X_{\leq}$ , or  $X_{\geq}$ , via an FPT algorithm, then we can also actually find an optimum solution for any instance  $x$  by repeatedly applying the algorithm to slightly modified versions of  $x$ . This strategy is known as *polynomial-time self-reducibility*. In some cases, an FPT algorithm that solves  $X_{\leq}$ , or  $X_{\geq}$ , will construct an optimal solution simply as a side effect of deciding the answer to the question.

Many of the standard problems in the parameterized complexity literature are standard parameterizations of optimization problems, for example,  $\kappa$ -VERTEX COVER,  $\kappa$ -CLIQUE,  $\kappa$ -DOMINATING SET,  $\kappa$ -INDEPENDENT SET. If such a standard parameterization is fixed-parameter tractable then this means that we have an efficient algorithm for exactly determining the optimum for those instances of the corresponding optimization problem where the optimum is small. A  $W[1]$ -hardness result for a standard parameterization of an optimization problem shows that such an algorithm is unlikely to exist. In this case, we can ask the question: Is it possible to efficiently *approximate* the optimum as long as it is small?

We use the following definition for an *FPT-approximation algorithm* proposed by Chen et al. [27].

**DEFINITION 1.10 Standard FPT-approximation algorithm** *Let*  $X = (I, \text{sol}, \text{cost}, \text{goal})$  *be an optimization problem. A standard FPT-approximation algorithm with performance ratio*  $c$  *for*  $X$  *is an algorithm that, given input*  $(x, k)$  *satisfying*

$$\begin{cases} \text{opt}(x) \leq k & \text{if goal} = \text{min}, \\ \text{opt}(x) \geq k & \text{if goal} = \text{max}, \end{cases} \quad (1.1)$$

*computes a*  $y \in \text{sol}(x)$  *in time*  $f(k) \cdot |x|^{O(1)}$  *such that*

$$\begin{cases} \text{cost}(x, y) \leq k \cdot c & \text{if goal} = \text{min}, \\ \text{cost}(x, y) \geq k/c & \text{if goal} = \text{max}. \end{cases} \quad (1.2)$$

*For inputs not satisfying (1.1) the output can be arbitrary.*

One example of this type of approximability is given by the  $\kappa$ -SUBTREE PRUNE AND REGRAFT problem. The input to this problem is a pair of phylogenetic  $X$ -trees,  $T_1$  and  $T_2$ , each an unrooted binary tree with leaves labeled by a common set of species  $X$  and (unlabeled) interior nodes having degree exactly three. The (labeled)

topologies of  $T_1$  and  $T_2$  may differ. The task is to find at most  $k$  *subtree prune and regraft* operations that will transform  $T_1$  into  $T_2$ .

A single *subtree prune and regraft* (SPR) operation on  $T_1$  begins by “pruning” a subtree of  $T_1$  by detaching an edge  $e = (u, v)$  in  $T_1$  from one of its (non-leaf) endpoints, say  $u$ . The vertex  $u$  and its two remaining adjacent edges,  $(u, w)$  and  $(u, x)$  are then *contracted* into a single edge,  $(w, x)$ . Let  $T_u$  be the phylogenetic tree, previously containing  $u$ , obtained from  $T_1$  by this process. We create a new vertex  $u'$  by subdividing an edge in  $T_u$ . We then create a new tree  $T'$  by adding an edge  $f$  between  $u'$  and  $v$ . We say that  $T'$  has been obtained from  $T_u$  by a single *subtree prune and regraft* (SPR) operation.

A related operation on the phylogenetic tree  $T_1$  is the *tree bisection and reconnection* (TBR) operation. A single *tree bisection and reconnection* (TBR) operation begins by detaching an edge  $e = (u, v)$  in  $T_1$  from both of its endpoints,  $u$  and  $v$ . Contractions are then applied to either one or both of  $u$  and  $v$  to create two new phylogenetic trees (note that a contraction is necessary only in the case of a non-leaf vertex.) Let  $T_u$  and  $T_v$  be the phylogenetic trees, previously containing, respectively,  $u$  and  $v$ , obtained from  $T_1$  by this process. We create new vertices  $u'$  and  $v'$  by subdividing edges in  $T_u$  and  $T_v$ , respectively. We then create a new tree  $T'$  by adding an edge  $f$  between  $u'$  and  $v'$ . We say that  $T'$  has been obtained from  $T_1$  by a single *tree bisection and reconnection* (TBR) operation. Note that the effect of a single TBR operation can be achieved by the application of either one or two SPR operations and that every SPR operation is also a TBR operation.

In [47] it is shown that the K-TREE BISECTION AND RECONNECTION problem can be solved via an FPT algorithm for the equivalent K-MAXIMUM AGREEMENT FOREST problem, running in time  $O(k^4 \cdot n^5)$ . The algorithm employed in [47] uses a kernelization strategy from [9] as a pre-processing step, followed by a search tree strategy. Neither of the techniques used in [47] is applicable in the case of the K-SUBTREE PRUNE AND REGRAFT problem. It has long been conjectured that the K-SUBTREE PRUNE AND REGRAFT problem is fixed-parameter tractable, however, proof of this conjecture is a long-standing open problem in the parameterized complexity community.

The algorithm given in [47] can easily be adapted to give either a NO answer to the K-TREE BISECTION AND RECONNECTION problem or, otherwise, a minimal set of TBR operations that transforms  $T_1$  into  $T_2$ . Given two phylogenetic trees,  $T_1$  and  $T_2$ , if there is a set  $S$  of at most  $k$  SPR operations that transforms  $T_1$  into  $T_2$  then  $S$  is also a set of at most  $k$  TBR operations transforming  $T_1$  into  $T_2$ . Thus, in this case, the algorithm given in [47] will return a solution  $S'$  consisting of at most  $k$  TBR operations. This set  $S'$  will translate into a set of at most  $2k$  SPR operations transforming  $T_1$  into  $T_2$ . If there is no size  $k$  set of SPR operations that transforms  $T_1$  into  $T_2$ , then the algorithm given in [47] might still return a solution  $S'$  consisting of at most  $k$  TBR operations and again, this set  $S'$  will translate into a set of at most  $2k$  SPR operations transforming  $T_1$  into  $T_2$ . However, in this case, there is no guarantee of success.

This is currently the only example (known to us) of a problem that is not proven to be FPT but that does have a standard FPT-approximation algorithm, although there

are other examples of FPT-approximation algorithms appearing in the literature. For example, as recorded in [53],  $\kappa$ -TREEWIDTH is FPT, in fact for every  $k$ , if a tree decomposition of a given graph  $G = (V, E)$  of width  $k$  exists, then this can be computed in time linear in  $|V|$  [13]. The algorithm given in [13] is rather complex and not at all workable in practice. However, there are far simpler FPT algorithms that produce tree decompositions having width at most a constant factor larger than the optimum [60, 63].

In the negative, for some parameterized problems, it is possible to show that there is *no* standard FPT-approximation algorithm for any performance ratio, be it a constant ratio, or, otherwise, some ratio that is a *function of the parameter  $k$* .

Given a graph  $G = (V, E)$ , an *independent dominating set*  $V'$  is an *independent* set of vertices  $V' \subseteq V$  such that  $V'$  is a dominating set for  $G$ . The corresponding optimization problem is the MINIMUM INDEPENDENT DOMINATING SET problem, where the goal is to minimize  $|V'|$ . Downey et al. [37] prove that the standard parameterization of this problem is *completely inapproximable*.

**THEOREM 1.10 Downey, Fellows, McCartin**

*If  $\kappa$ -INDEPENDENT DOMINATING SET has a standard fpt-approximation algorithm with performance ratio function  $f(k)$ , for some computable function  $f(k)$ , then  $W[2] = FPT$ .*

This particular problem is not *monotone*, where monotone here means that, if we extend a feasible solution with additional vertices, then it remains feasible. Clearly, we can arbitrarily add vertices to  $V'$  such that  $V'$  remains a dominating set, but such a  $V'$  may no longer be independent. Thus, we can manufacture an instance where the optimum is  $k$  and where every feasible solution also has size  $k$ . It would be more interesting to obtain inapproximability results for the monotone problem  $\kappa$ -DOMINATING SET.

Important progress on parameterized approximation, at the level of  $W[P]$ , has recently been achieved by Eikmeyer, Grohe and Grüber [40]. They looked at *multiplicative* approximation ratios, proving the following:

**THEOREM 1.11 Eikmeyer, Grohe and Grüber [40]**

*All known “natural”  $W[P]$  complete problems (including all the ones from [35],[1]) have no FPT approximation algorithms with approximation ratio  $\exp(\log^\gamma k)$  for some constant  $\gamma \in (0, 1)$  ( $\gamma$  depending on the problem) unless  $W[P] = FPT$ .*

One illustration of an application of Theorem 1.11 is the problem MIN LINEAR INEQUALITY BY DELETION which has parameter  $k$  and asks “Does the deletion of  $k$  elements from a system of linear inequalities result in a system that is solvable?”

We end this discussion with a final negative result, which illustrates a direct connection between parameterized complexity and classical approximation algorithms.

We say that a problem  $X$  admits a *polynomial-time approximation scheme (PTAS)* if there is an algorithm  $A$  such that, for every instance  $x$  of  $X$  and every  $\varepsilon > 0$ ,  $A$  produces a  $(1 + \varepsilon)$ -approximate solution,  $y \in \text{sol}(x)$ , in time  $|x|^{f(1/\varepsilon)}$ , for some arbitrary computable function  $f$ .

Such an algorithm  $A$  runs in polynomial time for every fixed value of  $\varepsilon$ , but if  $\varepsilon$  is small then the exponent of the polynomial  $|x|^{f(1/\varepsilon)}$  can be very large. Two restricted classes of approximation schemes have been defined that avoid this problem. An *efficient polynomial-time approximation scheme (EPTAS)* is a PTAS with running time of the form  $f(1/\varepsilon) \cdot |x|^{O(1)}$ . A *fully polynomial-time approximation scheme (FPTAS)* is a PTAS with running time of the form  $(1/\varepsilon)^{O(1)} \cdot |x|^{O(1)}$ .

Parameterized complexity affords evidence to show that, in some cases, an EPTAS will not be forthcoming. The following theorem has been proposed independently by Bazgan [12] and Cesati and Trevisan [25].

**THEOREM 1.12** *Bazgan (1995), Cesati and Trevisan (1997)*

*If an optimization problem  $X$  admits an EPTAS, then the standard parameterization of  $X$  is FPT.*

We can use the contra-positive of Theorem 1.12 to show that an EPTAS likely does not exist for a particular  $NP$ -optimization problem.

**COROLLARY 1.1**

*If the standard parameterization of an optimization problem is  $W[1]$ -hard, then the optimization problem does not have an EPTAS (unless  $FPT = W[1]$ .)*

Note that the converse of Theorem 1.12 is not true. For example,  $K$ -VERTEX COVER is FPT, as we have repeatedly shown throughout this article, but MINIMUM VERTEX COVER is APX-hard, implying that it doesn't have any type of PTAS at all (unless  $P = NP$ .)

## 1.7 Conclusions

Our aim in this chapter has been to introduce the reader to the distinctive algorithmic techniques of parameterized complexity, in the hope that these might find a useful place in the repertoire of a widening circle of algorithm designers. We have endeavoured to strike a balance between high-level, generalized, descriptions and technical details. Space limitations have inevitably meant that many technical details have been omitted. There are many aspects of the parameterized paradigm which we have not canvassed at all. In this regard, we enthusiastically refer the reader to the recently published collection of parameterized complexity survey papers in the

Computer Journal [24], the monographs [43], [45] and [55], as well as the original parameterized complexity text [35], for more extensive coverage of parameterized complexity theory and parameterized algorithms.

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