A Note on the Computability of Graph Minor Obstruction Sets for Monadic Second Order Ideals

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Abstract: The major results of Robertson and Seymour on graph well-quasi-ordering establish nonconstructively that many natural graph properties that constitute ideals in the minor or immersion orders are characterized by a finite set of forbidden substructures termed the *obstructions* for the property. This raises the question of what general kinds of information about an ideal are sufficient, or insufficient, to allow the obstruction set for the ideal to be effectively computed. It has been previously shown that it is not possible to compute the obstruction set for an ideal from a description of a Turing machine that recognizes the ideal. This result is significantly strengthened in the case of the minor ordering. It is shown that the obstruction set for an ideal in the minor order cannot be computed from a description of the ideal in monadic second-order logic.

Key Words: well-quasi-ordering, monadic second order logic, decidability
Category: F.1

1 Introduction

The celebrated results of Robertson and Seymour [RS83, RS85, RS94] prove the existence of finite obstruction sets for arbitrary minor and immersion order ideals, of which there are many natural examples. Planar graphs are famously a minor ideal for which the obstructions are $K_{3,3}$ and K_5 (Kuratowski's Theorem). These fundamental results are not effective, in the sense that knowing only a decision procedure for an ideal $\mathcal{F}$ does not provide enough information to be able to compute the obstruction set for $\mathcal{F}$ [FL89a]. In fact, this noncomputability result is essentially a straightforward corollary to Rice's Theorem, and has little specifically to do with graph minors.

We are naturally led to investigate what sorts of further information about an ideal might allow the obstruction set for the ideal to be systematically computed. There are a number of apparently difficult unresolved problems in this general area. For example, it is unknown whether the obstruction set for an arbitrary union of ideals $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ can be computed from the two corresponding obstruction sets $\mathcal{O}_1$ and $\mathcal{O}_2$, although this can be accomplished if at least one of these obstruction sets includes a tree [CDDFL97].

The main theorem of this paper significantly extends the negative result of [FL89a]. We prove the following for the minor ordering.

Theorem 1. There is no effective procedure to compute the obstruction set for a minor ideal $\mathcal{F}$ from a monadic second order (MSO) description of $\mathcal{F}$.

This should be contrasted with general positive results concerning the computation of obstruction sets. Fellows and Langston proved in [FL89b] that if we have access to the three pieces of information:

(i) A decision algorithm for $\mathcal{F}$.

(ii) A bound B on the maximum treewidth (or pathwidth) of the graphs in the set $\mathcal{O}$ of $\mathcal{F}$ obstructions.

(iii) A decision algorithm for a finite index congruence for $\mathcal{F}$.

Then $\mathcal{O}$ can be effectively computed.

Perhaps surprisingly, the algorithm of [FL89b] has been implemented and nontrivial, previously unknown obstruction sets for interesting ideals have been successfully computed [CD94, CDF95].

Since (i) and (iii) can be effectively derived from an MSO description of $\mathcal{F}$ [Co90a], our Theorem 1 shows that (ii) is essential in the earlier positive result of [FL89b]. Other work on the systematic computation of obstruction sets has appeared in [APS90, CDDFL97, GI91, Kin94, KL91, LA91, Lag93, Pr93].

2 Preliminaries

All of our discussion concerns finite simple graphs. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by a sequence of operations chosen from the list: (i) delete a vertex, (ii) delete an edge, (iii) contract an edge. (When applying the edge contraction operation, any multiple edges or loops that are formed are removed.) This defines the minor partial order on graphs, denoted $G \geq_m H$.

A graph H is *immersed* in a graph G if a graph isomorphic to H can be obtained from G by a sequence of operations chosen from the list: (1) delete a vertex, (ii) delete an edge, (iii) lift an edge. (The meaning of the *lift* operation is that a pair of edges uv and vw are replaced by a single edge uw.) This defines the immersion order, denoted $G \geq_i H$.

An *ideal* $\mathcal{J}$ in a partial order $(\mathcal{U}, \geq)$ is a subset of $\mathcal{U}$ such that if $X \in \mathcal{J}$ and $X \geq Y$ then $Y \in \mathcal{J}$. The *obstruction set* for $\mathcal{J}$ is the set of minimal elements of $\mathcal{U} - \mathcal{J}$.

A *filter* is a subset $\mathcal{J} \subseteq \mathcal{U}$ such that if $X \in \mathcal{J}$ and $X \leq Y$ then $Y \in \mathcal{J}$. The filter $\mathcal{J}_\mathcal{S}$ *generated* by a set $\mathcal{S} \subseteq \mathcal{U}$ is defined to be the set of all elements of $\mathcal{U}$ that are greater than or equal to some element of $\mathcal{S}$:

$$\mathcal{J}_\mathcal{S} = \{Y : \exists X \in \mathcal{S} \ \ Y \geq X\}$$

The syntax of the monadic second order (MSO) logic of graphs includes the usual logical connectives $\wedge, \vee, \neg$, variables for vertices, edges, sets of vertices and sets of edges, the quantifiers $\forall, \exists$ that can be applied to these variables, and the five binary relations:

(1) The membership relation $u \in U$ where u is a vertex variable and U is a

vertex set variable.
(2) The membership relation $d \in D$ where d is an edge variable and D is an edge set variable.
(3) The incidence relation $\iota(d, u)$ where d is an edge variable, u is a vertex variable, and the interpretation is that the edge d is incident on the vertex u.
(4) The adjacency relation $\alpha(u, v)$ where u and v are vertex variables, and the interpretation is that u and v are adjacent vertices.
(5) Equality for vertices, edges, sets of vertices and sets of edges.

If ϕ is a well-formed formula of MSO, then the set of finite graphs that are models of ϕ is denoted $\mathcal{F}(\phi)$. That a graph G is a model of ϕ is denoted $G \models \phi$. The basic reference on MSO graph properties is [Co90a].

3 The Main Result

We will use the following lemma due to Courcelle concerning the description in MSO logic of generated filters in the minor order.

Lemma 1 [Co92]. Given an MSO formula ϕ, we can effectively produce an MSO formula ϕ' such that $\mathcal{F}(\phi')$ is the filter generated in the minor order by $\mathcal{F}(\phi)$. □

Proof Sketch. A graph G has a graph H as a minor if and only if if is possible to identify a set of disjoint connected subgraphs G_v of G indexed by the vertex set of H, such that if u and v are adjacent vertices of H, then the corresponding subgraphs G_u and G_v of G are "adjacent" in the sense that there are vetices $x \in G_u$ and $y \in G_v$ such that $xy \in E(G)$. The formula ϕ' can be constructed by expressing in M2O the statements:
(1) There exists a set of edges that forms a forest in G.
(2) The exists a set V_0 of roots for the trees of the forest of (1), with one root for each tree.
(3) There exists a set E_0 of edges between the trees of the forest, in the sense of "adjacency" described above.

The formula ϕ' consists of this preface, followed by ϕ modified by some substitutions and restrictions:
(4) Quantification is restricted to V_0 and E_0. (These are in some sense the "virtual" vertices and edges of the minor H of G that the preface asserts to exist. We are now concerned with expressing that this H is a model of ϕ.)
(5) Incidence and adjacency terms in ϕ are replaced by statements concerning suitable paths in the forest of (1). (In other words, incidence and adjacency statements about the virtual vertices and edges, V_0 and E_0, are interpreted with the means provided by (1), (2) and (3).) □

The following proposition is a corollary of a theorem of Trakhtenbrot [Tr50] on the undecidability of the first order logic of graphs (see [Co90b] for a discussion). We remark that Seese has shown that undecidability still holds even for planar graphs [Se75, Se91].

Proposition 1 ([Tr50, Se91]). Given an MSO formula ϕ, there is no algorithm to decide if there is a finite graph G such that $G \models \phi$. □

We can now prove our main result.

Theorem 1. There is no effective procedure to compute the obstruction set for a minor ideal $\mathcal{F}$ from a monadic second order description of $\mathcal{F}$.

Proof. We argue that if there were such a procedure, then we could solve the problem of determining whether a formula ϕ of MSO has a finite model, contradicting Proposition 1. Let ϕ' denote the formula computed from ϕ by the effective procedure of Lemma 1. Let $\psi = \neg\phi'$. We note the following.
• If no finite graph is a model of ϕ, then the set of models of ϕ' is empty, and every finite graph is a model of ψ. Thus ψ describes an ideal for which the set of obstructions is the empty set.
• If ϕ has a finite model, then ϕ' describes a nontrivial filter and ψ describes a nontrivial ideal complementary to ϕ' for which the set of obstructions is nonempty.

By computing the obstruction set for the ideal described by ψ we can therefore determine, on the basis of whether this obstruction set is empty or nonempty, whether any finite graph is a model of ϕ. □

4 Summary Discussion

The central question that forms the context of this work is: what sorts of information about lower ideals allow obstruction sets to be effectively computed? The main previous results in this area are the following:
(1) The obstruction set for an ideal cannot be effectively computed from the description of a Turing machine that recognizes the ideal [FL89a].
(2) Obstruction sets can be computed from the three pieces of information: (i) a Turing machine that recognizes the ideal, (ii) a bound on the maximum obstruction treewidth, and (iii) a finite-index congruence for the ideal [FL89b].
These results hold for both the minor and immersion orders.

In this paper we have strengthened the negative result (1) for the minor order, since (i) and (iii) can be derived from an MSO description of an ideal in either the minor or immersion orders.

Does our main theorem also hold for the immersion order? Our proof can be adapted to the immersion order if an analog of Lemma 1 for the immersion order can be proved. The difficulty is in finding a "virtual" representation of the vertices and edges of an immersed graph, without knowing in advance how many vertices and edges there are. (Given this information, the job becomes much easier, since the edge representations can then be quantified by separate sets.) It is conceivable, of course, that an analog of Lemma 1 does *not* hold for the immersion order, even though for any ϕ, we know (nonconstructively) that a ϕ' describing the filter generated by $\mathcal{F}(\phi)$ in the immersion order exists.

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