INVARIANCE AND NONINVARIANCE IN THE LATTICE OF Π^0_1 CLASSES

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Abstract. We prove that there are two minimal Π^0_1 classes that are not automorphic.

1. Introduction

A (computably bounded) Π^0_1 class C can be defined as the set of infinite paths through a computable tree $T \subseteq 2^{<\omega}$. The study of Π^0_1 classes has a long and interesting history, and many applications. We refer the reader to the surveys Cenzer [1], Cenzer-Remmel [4], and Cenzer-Jockusch [3].

This paper continues the study of the lattice of Π^0_1 classes along the lines of Cenzer, Downey, Jockusch, and Shore [2]. In particular, we are interested in the class of Π^0_1 classes introduced in Downey [7], but first constructed under duality in Martin and Pour-El [13]. These are the *thin* classes, where an infinite class P is called thin if, for all Π^0_1 subclasses $P' \subseteq P$ there is a clopen set C such that $C \cap P = P'$. Of particular interest to us in the present paper, is the case where the thin class has a unique nonisolated (rank one) point in it, and hence every Π^0_1 subclass is either finite or cofinite in P. These were first introduced in [2] and are called *minimal* classes.

Thin classes more or less correspond to hyperhypersimple sets in the lattice of computably enumerable sets. This intuition was made clear in Cholak, Coles, Downey, and Herrmann [5], where it is proven that an infinite class P is thin iff the lattice of subclasses forms a Δ^0_2 Boolean algebra, and every Δ^0_2 Boolean algebra is isomorphic to the lattice of subclasses of some thin class. For example, minimal classes correspond

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to the Boolean algebra of finite and cofinite sets. This characterization can be viewed as the analog of Lachlan's result [12] that the collection of computably enumerable supersets of a hyperhypersimple set is a Σ^0_3 Boolean algebra, and every such Boolean algebra can be realized.

The main result of [5] is that if S and T are *perfect* thin classes, then there is an automorphism of the lattice of Π^0_1 classes taking S to T. Here the class is perfect iff the lattice of subclasses is isomorphic to the free Boolean algebra, or, equivalently the class has no isolated points.

The question we address in the present paper, is to determine whether the analogous result holds for minimal classes, a question raised in all of the surveys above and in [5]. Ultimately, the hope was that if the Boolean algebras of subclasses of a two thin classes are isomorphic, then the classes would be automorphic.

Unfortunately, our main result is to prove that this hope is forlorn even for minimal classes.

We need some definitions.

Definition 1. *We say that a minimal class S is* cohesively minimal *if it obeys the following property $\mathcal{P}$ below. (All quantifiers range over that lattice of Π^0_1 classes.)*

For all $F \supseteq S$ one of the following holds.

(i) There exists a finite $G \subseteq S$ and a clopen $C \subseteq F$, such that, for $\widehat{S} = S - G$, $\widehat{S} \subseteq C$.
(ii) There is a finite $G \subseteq S$, such that, for $\widehat{S} = S - G$, and for all isolated $I \subseteq \widehat{S}$, there is no clopen C such that

$$(I \subseteq C) \wedge (C \cap F = C).$$

Theorem 2. *There is a cohesively minimal Π^0_1 class.*

Theorem 3. *There is a minimal Π^0_1 class that is not cohesively minimal.*

Theorems 2 and 3 give us our main result:

Corollary 4. *There exist two minimal Π^0_1 classes that are not automorphic.*

Proof. It is enough to observe that property $\mathcal{P}$ is definable by an infinitary formula. To see this note that being isolated (i.e. an atom of the lattice of Π_1^0 classes) is elementarily definable and hence finite is definable via a nonelementary formula[1]. Being clopen is elementarily definable since it happens iff the class is complemented, as observed in Cholak, Coles, Downey, Herrmann [5]. This gives the result. $\square$

Actually, we prove a little more than Theorem 3. To get Theorem 3, we actually prove the following. Define a rank one class $[T]$ to be *tame* if its rank one point has degree computable from $\emptyset'$ and there is a Δ_2^0 function g such that

(i) $g(e) = \langle \sigma_{m(e)}, \phi_{m(e)} \rangle$ where ϕ_m is the characteristic function of an isolated point I_m of $[T]$, with σ_m an initial segment of I_m.
(ii) $\{I_{m(e)} : e \in \omega\}$ list all the isolated points of $[T]$, and
(iii) $\{\sigma_{m(e)} : e \in \omega\}$ form an antichain of strings.

For instance, any rank one class generated by $[T]$, where T has no dead ends is tame[2].

Theorem 5. *Suppose that S is a tame minimal class Then S does not satisfy $\mathcal{P}$.*

We remark that it would be enough to construct a thin class S satisfying an apparently weaker property than cohesive minimality. Say a class is weakly cohesive if in place of (i) in Definition 1, we had instead.

(i)' There exists a finite $G \subseteq S$ such that, for $\widehat{S} = S - G$,, for all isolated $I \subseteq \widehat{S}$, there is a clopen C such that

$$(I \subseteq C) \wedge (C \cap F = C).$$

This is because the proof of Theorem 5 demonstrates that classes with rank one points computable from $\emptyset'$ are not even weakly cohesively minimal. We do not know if the two properties are distinct.

[1] Cenzer and Jockusch have asked if the property of finiteness is elementarily definable.

[2] Suppose that we have ν as an initial segment of the rank one point, and the tree splits at ν. Then simply count which of $\nu\widehat{\ }0$ or $\nu\widehat{\ }1$ has the most split extensions to determine which way the rank one point goes. The σ_i are the shortest strings which are initial segments of the isolated points, with the property that they are not initial segments of any other isolated points. You can order them, first by length, and then lexicographically.

On the other hand, Cenzer and Remmel did show that sometimes minimal classes are automorphic. They gave an effectiveness condition based on the complexity of the representation of the lattices of subclasses that guaranteed automorphism of the classes. The condition is a special case of tameness. Specifically, (see [3], page 55) they defined a Π^0_1 class to be *standard* if its rank one point P is Δ^0_2 and for any $\nu\widehat{\ }i \prec P$, $\nu\widehat{\ }1-i$ is an initial segment of at most one isolated member of $[T]$. Clearly this condition is implied by tameness. We mildly generalize their result to prove:

Theorem 6 (essentially Cenzer and Remmel). *Suppose that S and T are tame minimal classes. Then S and T are automorphic.*

Theorem 6 was established jointly with Carl Jockusch, before we learned of the Cenzer-Remmel work. It has a nice corollary. Recall that the degree of a class C is the degree of the set of strings that are not initial segments of members of C. In [2], a minimal class is constructed as the paths through a tree with no nonextendible nodes, so the degree of the minimal class is $\mathbf{0}$, and hence the class is tame. It is routine to construct a tame minimal class that is noncomputable. (For instance, if a class has its rank one point of degree $\leq \emptyset'$ and the class has low degree, then it will be tame, because one can ask if a node σ on T has two or more isolated extensions, assuming that the node is *not* an initial segment of the rank one point, computably from $\emptyset'$, if the class is low.) This gives the following.

Theorem 7. *(i) Any two computable minimal classes are automorphic.*
(ii) Being computable is not definable in the lattice of Π^0_1 classes.

The question of exactly which Δ^0_2 Boolean algebras B have the property that for any two thin classes which have lattices of subclasses isomorphic to B, are in fact automorphic, is quite open. We have seen that, by [5], B being atomless is enough, and the proof will generalize to B having a finite number of atoms. By this paper, B being the Boolean algebra of finite and cofinite sets is not enough. We offer the following conjecture.

Conjecture 8. *The only Boolean algebras B, which have the property that any two thin classes with B as their lattices of subalgebras are automorphic, are ones with a finite number of atoms.*

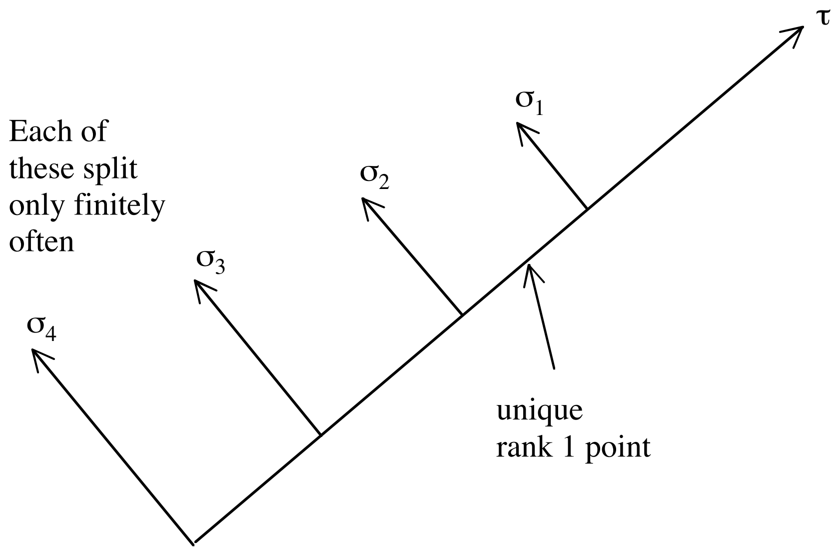

FIGURE 1. A rank one class

2. PRELIMINARIES

We let $[V_e] : e \in \omega$ and $[U_e] : e \in \omega$ be enumerations of the collection of Π^0_1 classes. (We use two notations for the same collection to avoid confusion between the thinness requirements and the cohesiveness requirements.) For simplicity, we choose the following representation of these classes. Let P_e denotes the e-th real time computable (i.e. $\sigma \in P_e$ is decidable in time $|\sigma|$) tree of strings. Then we will denote the stage s approximation to $[V_e]$ by $V_{e,s}$ and mean the strings of length s in P_e. Other notation more or less follows Soare [16].

3. PROOF OF THEOREM 2

The proof that there is a Π^0_1 class which is both minimal and has property P, is relatively delicate and uses some of the technology developed in Downey [6]. In particular, it will be a kind of "full approximation" argument. We will need to construct S so that it is homeomorphic to the paths through the rank one tree of figure 1.

The reader should keep in mind that we will be constructing a tree of strings $T = \lim_s T_s$, so that $S = [T]$ will be the desired class. At any stage s, T_s will consist of a collection of strings which have been irrevocably declared *terminal*, as well as a collection of strings of length

s we currently believe are reasonable as possible initial segments of $[T]$. In the transition from stage s to $s+1$ we will either terminate such a string or extend it to one (or more) of length $s+1$.

An important picture the reader should have in mind for the present construction is that at any stage s we will have a current "view" as to the length s initial segment of the (unique) rank one point in $[T]$. This is the only string we are allowed to split in the transition from stage s to $s+1$. The other strings on T_s of length s might be extended but they cannot be split *and* extended.

Now in the construction, we may change our mind as to where the rank one point really lies, and the guessed initial segment may prove wrong. Of course we may well have put extensions both splitting and extending that segment, based on the false belief that that segment was the correct one. The constructions will ensure that almost all such splittings will become terminal, so that all such "false splittings" with the "wrong guess" will die. (More on this later.)

Turning to the details of the proof, we must meet the requirements below.

$$\begin{array}{rl} \mathcal{M}_e : & [U_e] \subseteq S \rightarrow \exists \text{ clopen } D(D \cap S = [U_e]. \\ \mathcal{R}_e : & S \subseteq [V_e] \rightarrow (i) \text{ or } (ii) \text{ below holds} \\ & (i)\exists \text{ finite } G \subset S(\exists \text{ clopen } C \subseteq [V_e](S - G \subseteq C \wedge [V_e] \cap C = C). \\ & (ii)\exists \text{ finite } G \subset S(\forall \text{ isolated } I \subset S - G \ \forall \text{ clopen } C \\ & (I \subseteq C \rightarrow (C \cap [V_e] \neq C)). \end{array}$$

Additionally, there will be implicit requirements asking that $[T]$ be have a unique rank one path.

We begin by reviewing the standard procedure for meeting the $\mathcal{M}_e$, thinness requirements. The basic idea is the following. Suppose that $[U_e] \subset S$. Then at some stage of the construction, for some string σ of length s in T_s, we will have $\sigma \notin U_{e,s}$. (That is, there can be no extension of σ in $[U_e]$.) Then we can win the requirement $\mathcal{M}_e$ by simply making sure that the rank one point in $[T]$ extends σ, and hence almost all members of S will extend σ. A simple priority argument will ensure that all the $\mathcal{M}_e$ get met.

Meeting the $\mathcal{R}_e$ requirements is much more difficult. Fix some e. At some stage, we have some belief as to some τ which we would like to be the initial segment of the rank one point of S. On the other hand, we will need to get the cohesiveness on the isolated paths of S. We do this by trying to verify (ii) of the definition of property P, on almost all paths.

Specifically, for the basic module, suppose that at some stage s_1 we had the belief that we should have τ as the initial segment of the rank one point, and 4 other strings $\sigma_1, \cdots, \sigma_4$, which we think ought to be initial segments of isolated paths in S, as in figure 1. We would like to know that there is no relevant clopen set for each of the σ_i, in terms of the last item of (ii) of the definition of property $\mathcal{P}$, and $\mathcal{R}_e$. The idea is the following.

We will (perhaps) temporarily abandon τ (and simply *directly* extend it as an isolated point till we return), and begin by making σ_1 the initial segment of the rank one point. This entails perhaps splitting strings above σ_1 in stages $t \geq s_1$ and directly extending σ_j for $j \neq 1$. We will do this until a stage s_2 is found where there exists a string γ with $\sigma_1 \prec \gamma$ and γ terminal in V_{e,s_2}. At this stage, we regard $\mathcal{R}_e$ as verified for σ_1, and will terminate all, save one, of the extensions of σ_1. See figure 2.

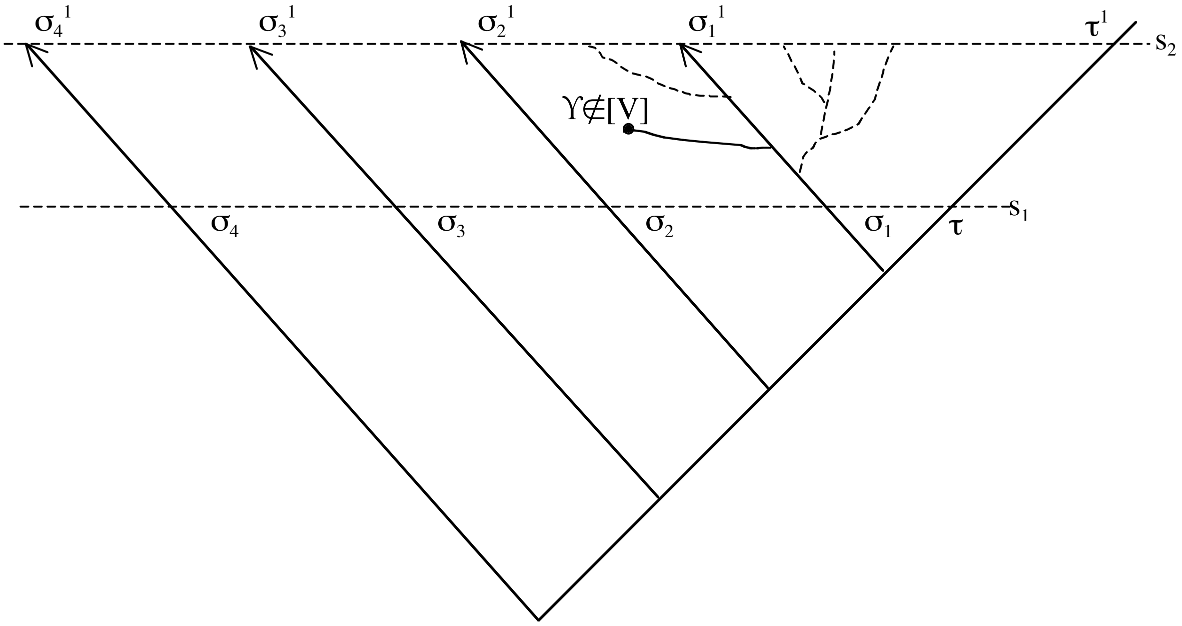

FIGURE 2. Termination of all but one extension

Remark 9. In the actual construction this termination is controlled by some e-state machinery. All of the strings extending σ_1 will be given

e-state of the form $\beta^\frown f$. (This is for the appropriate β.) When we abandon the cone above σ_1 as the place where most of the construction lives, we terminate all *but one* of the strings ν with e-states weaker than $\beta^\frown\infty$, which will terminate all, but one, of the ones with e-state $\beta^\frown f$. The one we do not abandon, we will raise the e-state to $\beta^\frown\infty$.

Now we repeat this process with all the other σ_j for $j = 2, 3, 4$[3]. Suppose first that we repeat this cycle through all the σ_j infinitely often. It will follow that for all initial segments $\mu_i : i = 1, \cdots, 4$, of the isolated paths $P_1, \cdots, P_4$ beginning with $\sigma_1, \cdots, \sigma_4$, there is always some isolated path D_i extending μ_i with $D_i \not\subset [V_e]$, and hence it is not possible for there to be a clopen C containing P_i with $C \cap [V_e] = C$.

One the other hand, suppose that we eventually get stuck in the cycle on some σ_j. That will entail the rank one point extending $\sigma_j[t]$ and *above* $\sigma_j[t]$ for all stages $u \geq t$, every string of length u is in $V_{e,s}$. This means that above $\sigma_j[t]$, $[V_e]$ is clopen. Hence, with the exception of the finite number of isolated points we miss, $[V_e]$ is that clopen set, and hence we satisfy (i) of the requirement $\mathcal{R}_e$.

The above describes the situation for one requirement. The situation becomes more complex with two requirements, as we now see.

Suppose that we are considering two requirements $\mathcal{R}_e$, and $\mathcal{R}_h$, with $e < h$. Consider the situation where we are extending σ_1, pretending it is the initial segment of the rank one point, but now seeking verification of both $\mathcal{R}_e$ and $\mathcal{R}_h$.

Now we are imagining that we are seeking verification above σ_1 begun at stage s. Of course if neither of the requirements are verified then we are in good shape. However, if one of the requirements becomes verified, what should we do? It would be tempting to stay in the cone above σ_1 when we see $\mathcal{R}_e$ verified also waiting for $\mathcal{R}_h$. This is because the cone above σ_1 looks like a very nice place to build the rank one point from the point of view of $\mathcal{R}_h$. However, if we do this then we must realize that the other paths extending σ_j for $j \neq 1$ are never again verified for e, and neither are extensions of τ. This seems no problem as there are only a finite number of them, but the reader can see that this process could repeat with infinitely many other $\mathcal{R}_{f'}$, causing us to

[3]Strictly speaking in the construction we may not go in this exact order, but ensure that all are considered.

lose on $\mathcal{R}_e$, because we *fail* to e-verify infinitely many isolated points in $[T]$, and hence $[V_e]$ could be as capricious as it desired there.

Note that there is no problem if $\mathcal{R}_h$ is verified yet $\mathcal{R}_e$ is not, since the latter has higher priority than the former, so we should stay in the cone above σ_1.

The solution to the dilemma above is the following. Suppose that τ is the place we will be building an initial segment of the rank one point if the guess that e is always verified. The only nodes in the cone above τ can be assigned e-states $\beta^\frown\infty$.

Hence, if we abandon the cone above σ_1 for the sake of $\mathcal{R}_e$'s verification of the nodes σ_j, and τ, all such nodes will have e states too low, and hence it is reasonable to abandon them anyway.

However, if we are working in the nodes above the node τ (where we are agree that we will build the rank one point if the e-verification happens infinitely often) then this cancellation action is not reasonable since such a node ν_i, with $\tau \preceq \nu_i$, will have an h-state extending τ's e-state. Nevertheless, we will need to *temporarily* abandon ν_i for other nodes extending τ and the σ_j, for the sake of e-verification, while *acknowledging* that from $\mathcal{R}_h$'s point of view the cone above ν_i is the preferred place to build the rank one point. Hence, until we get verification of $\mathcal{R}_h$ above ν_i, we will cycle through the extensions of the other nodes σ_j and τ, verifying $\mathcal{R}_e$, but making sure that we will add no new (permanent) *splittings* above these nodes. Again this is all forced by the assignation of e states to all nodes on the tree T. All save one of the potential paths of the tree added in the cone above, say, σ_3 while we are waiting for ν_i, will have e-states $\beta^\frown f$, and hence will be canceled when we return. The nodes above ν_i will have h-state of the form $\beta^\frown\infty^\frown\gamma^\frown f$, for an appropriate $h-1$-state $\beta^\frown\infty^\frown\gamma$. Exactly one ν_i will be, like τ, the "preferred place" should the correct h-state be $\beta^\frown\infty^\frown\gamma^\frown\infty$. All save one of these wrong guess strings extending ν_i will be terminated, should we eventually f-verify ν_i, *and* e-verify all of the appropriate extensions of ν_i, at some stage $s' > s$. We will raise the state of the one remaining string. Should this be the preferred place for $\beta^\frown\infty^\frown\gamma^\frown\infty$, we will split the string at length $s'+1$ giving each extension the higher state, and picking the right one as the preferred place for the state $\beta^\frown\infty^\frown\gamma^\frown\infty^\frown 1^\frown\infty$. (This is the preferred place for the $\mathcal{R}_{h+1}$ on the assumption that the $\mathcal{M}_{h+1}$ fails to act, which is the meaning of the $^\frown 1$ in the state.) See Figure 3 for a typical situation.

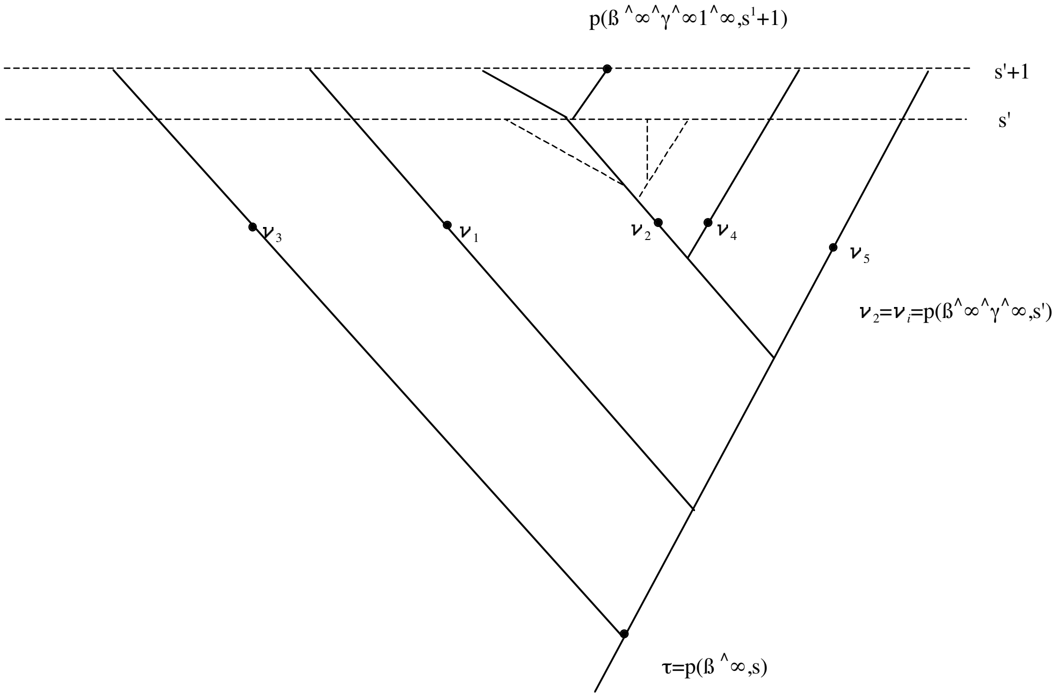

FIGURE 3. Preferred places – The function p refers to the "preferred places".

That is, suppose that we are working waiting for verification of $\mathcal{R}_e$, above, say, σ_3. Of course while we are waiting for verification of $\mathcal{R}_e$ we will put up splittings etc based on the belief that $\mathcal{R}_e$ is permanently stuck there. However should $\mathcal{R}_e$ be verified, then all but one of these paths will be terminated, and we will move to the next node, completely ignoring whether or not σ_3 is verified for $\mathcal{R}_f$.

Now should we return to some extension $\widehat{\nu}_i$, of ν_i to be $\mathcal{R}_e$, we will "restart" the verification process for $\mathcal{R}_f$ and ν_i, while working in the cone above $\widehat{\nu}_i$ for $\mathcal{R}_e$. The difference between the nodes above σ_3 and ν_i is that when we leave σ_3 we terminate almost all of them. When we leave ν_i, while $\mathcal{R}_f$ is not yet verified, we will not terminate them. Of course should it turn out that we eventually verify $\mathcal{R}_f$ above σ_1, then we may well decide to cancel all such extensions except one.

The only thing we must ensure is that there is some place on the tree where we are building splittings based upon the belief that *both* $\mathcal{R}_e$ and $\mathcal{R}_f$ are always verified. It is clear above such a node we should not terminate ever for the sake of $\mathcal{R}_e$ or $\mathcal{R}_f$. This is controlled by the nodes with the short states and the $\alpha^\frown\infty$ states.

In summary there will be a tree of strategies based on whether certain guesses are verified. At any stage for each guess there will be a cone above which is the *preferred place* to build the rank one point as far as that guess is concerned. The strings which are the bases of such cones will change from time to time, but we will argue that for the true path guesses, the bases come to a limit and will cohere.

In more detail, we will work with a priority tree $PT \subseteq \{\infty, f, 0, 1\}^{<\omega}$, such that if the length of α is even then α has two children $\infty <_L f$, and if the length of α is odd then α has two children $0 <_L 1$. (And this PT is a complete binary tree.) We extend this ordering as lexicographic ordering on the tree. Members of PT will be called *guesses*. To avoid confusion, we will use $\alpha, \beta, \gamma, \cdots$ (Greek letters from the beginning of the alphabet) for guesses, and $\sigma, \tau, \nu, \mu \cdots$ for members of $2^{<\infty}$, the nodes used to build the underlying tree T of the Π^0_1 class $[T]$. We assign $\mathcal{R}_e$ to the guesses α of length $2e$, and refer to ∞, f as the outcomes of α. We write $\mathcal{R}_\alpha$ as the version of $\mathcal{R}_e$ identified with guess α. Similarly we assign $\mathcal{M}_e$ to those β of length $2e+1$. All of this is more or less standard.

Now we can be more precise. For a guess α, $p(\alpha, s)$ will denote the current place, if any, we will build an initial segment of the rank one point should α be on the true path (TP). We ensure that for $\alpha \prec \beta \prec TP$, we will ensure that $p(\alpha, s), p(\beta, s)$ each have a limit with $p(\alpha) \prec p(\beta)$.

As usual we will use the phase initialize, which will mean that all parameters etc become undefined, strings terminated, etc. We will be using the following parameters in the construction.

$F(\alpha, s)$: a set of strings in T_s awaiting α-verification at stage s.

$\delta(\alpha, s)$: a split associated with α should $\alpha \prec TP$. Should $\alpha \prec TP$ then this will come to a limit and $\delta(\alpha)$ is an initial segment of the rank one point, and both $\delta(\alpha)^\frown 0, \delta^\frown 1$ will be initial segments of members of $[T]$.

$TP(s, t)$: a string in PT that appears correct at substage s of stage t, which we write as stage (s, t).

$\mu(s,t)$: a string in T_s that appears to be the initial segment of the rank one point in T of length s.

Q_t : the value of a parameter Q at the beginning of stage (s,t).

We now turn to the formal details, although they are straightforward in view of the notes above.

Definition 10. We say that $\mathcal{M}_\alpha$ (with $|\alpha| = 2e+1$) *requires attention* at guess α if s is a α-stage, and

(i) $\mathcal{M}_\alpha$ is not currently declared satisfied,
(ii) There is some string $\sigma \in T_s$ of length s (so it appears to be an initial segment of a member of $[T]$), such that $\sigma \notin U_{e,s}$, and $p(\alpha, s+1)_t \prec \sigma$.

The Construction

Stage 1 Set $F(\lambda, 0) = \{1, 0\}$, assign $1, 0$ to have each have state f. Set $p(f, 0) = 1$, and set $p(\infty, 0) = \lambda$. Set $\mu(0) = 1$.

Stage $s+1$ This will consist of up to $2s$ many substages. We declare s to be a λ-stage, where λ denotes the empty string. Let $p(\lambda, s) = \lambda$. We suppose that stage $(s, t-1)$ is an α-stage.

Substage $t = 2e$. (Attend $\mathcal{R}_\alpha$) $(T1)$: Fact: $F(\alpha, s+1)_t$ is currently defined.

Case 1 There is some $\sigma \in F(\alpha, s+1)_t$ such that, for all ν with $\sigma \prec \nu$, $|\nu| \leq s$, $\nu \in V_{e,s}$, for $e = e(\sigma)$. Declare that σ is α-verified.

Action For the rightmost such σ, set $p(\alpha^\frown f, s+1) = \sigma$. For all $\tau \in F(\alpha, s+1)_t$ which are already α-verified, by direct extension and by termination, if necessary[4] make sure that there is exactly one string of length $s+1$ extending τ and give it state $\alpha^\frown\infty$. Let $\mu(s+1, t) = \sigma (= p(\alpha^\frown f, s+1))$. Set $TP(s+1, t) = \alpha^\frown f$.

Subcase 1 If $|\sigma| = s$ then set $\delta(\alpha^\frown f, s+1) = \sigma$, and add $\sigma^\frown 0, \sigma^\frown 1$ to $T_{s+1,t}$. Give them both $e+2$ state $\alpha^\frown f^\frown 1^\frown\infty$ and set $p(\alpha^\frown f^\frown 1^\frown \infty, s+1) = \sigma^\frown 1$, and $p(\alpha^\frown f^\frown 1, s+1) = \sigma$. Set $\mu(s+1) = \sigma^\frown 0$, $TP(s+1) = \alpha^\frown f^\frown f$ and initialize, as appropriate, as strings of lower state, making

[4]Termination would be needed if τ had become verified since the last α stage.

sure that the ones of higher state are extended to length $s+1$ if they are not terminal.

Subcase 2 If $|\sigma| < s$, simply go to stage $(s+1, t+1)$ after initializing where appropriate strings of lower state, and extending strings to length $s+1$ outside of the cone above σ.

Case 2 Not Case 1.

Action Set $TP(s+1,t) = \alpha\hat{}\infty$. Thus we know that all the strings in $F(\alpha, s+1)_t$ are e verified, and hence the ∞ outcome looks correct. $(T2)$ Fact: $p(\alpha\hat{}\infty, s+1)$ is currently defined.

Initialize by termination all strings of lower e-state than $\alpha\hat{}\infty$, making sure to leave one extension of length s for any currently nonterminated strings with e-state $\alpha\hat{}\infty$. No matter which subcase below pertains, at the end of stage $s+1$, set $F(\alpha, s+2)$ to be the collection of strings of length $s+1$ (i. e. alive at the end of the stage) *extending* $p(\alpha\hat{}\infty, s)$.

Subcase 1 $p(\alpha\hat{}\infty\hat{}i\hat{}\infty, s+1)_t$ is currently defined for some $i \in \{0,1\}$. In this subcase, simply go to the next substage.

Subcase 2 Otherwise. In this Subcase, extend all of the remaining strings to have length $s+1$, and make the rightmost one have a split extension at the end, so that it equals $\nu\hat{}0$ with $\nu\hat{}1$ also on the tree. Give both strings $e+2$ state $\alpha\hat{}\infty\hat{}1\hat{}\infty$, and set $\delta(\alpha\hat{}\infty\hat{}1\hat{}\infty, s+1) = p(\alpha\hat{}\infty\hat{}1, s+1) = \nu$. Set $p(\alpha\hat{}\infty\hat{}1\hat{}\infty, s+1) = \nu$, and $\mu(s+1) = p(\alpha\hat{}\infty\hat{}1\hat{}\infty, s+1) = \nu\hat{}0$. Go to stage $s+2$.

Substage $t = 2e+1$**.** (Attend $\mathcal{M}_e$) Let $|\alpha| = 2e+1$. $(T3)$ Fact : We may assume that $\mu(s+1, t-1) = p(\alpha, s+1)_t$.

Subcase 1 $\mathcal{M}_\alpha$ is not currently declared satisfied, and we see any string σ through which it requires attention.

Action Set $TP(s+1) = \alpha\hat{}0$. Initialize all other requirements, as usual, and set $\mu(s+1) = \sigma\hat{}0$ and add this to T_s. Also add $\sigma\hat{}0$ to T_s. Set $p(\alpha\hat{}0, s+1) = p(\alpha\hat{}0\hat{}\infty, s+1) = \delta(\alpha\hat{}0, s+1) = \sigma$. Set $p(\alpha\hat{}0\hat{}f) = \sigma\hat{}0$. Give both strings $\sigma\hat{}i$ $e+2$ state $\alpha\hat{}0\hat{}\infty$. $\mathcal{M}_\sigma$ is declared satisfied. To then form T_{s+1}, as usual initialize terminate strings remembering to make sure that ones with states not lower than $\alpha\hat{}0$ are extended to have length $s+1$ by a single extension. $(T3)$ Fact:

Note that this cancellation will terminate all but $\sigma\widehat{\ }i$ in the cone above $p(\alpha, s+1)$ since all the strings there will have states too low. Go to stage $s+2$.

Case 2 $\mathcal{M}_\sigma$ is currently satisfied.

Action Set $TP(s+1,t) = \alpha\widehat{\ }0$ and set $\mu(s+1,t) = p(\alpha\widehat{\ }0, s+1)_t$. Go to substage $t+1$.

Case 3 Otherwise.

Action Set $TP(s+1,t) = \alpha\widehat{\ }1$ and $\mu(s+1,t) = p(\alpha\widehat{\ }1, s+1)_t$.. Go to substage $t+1$.

End of Construction

Verification This runs along the lines of the intuitive discussion. It is easy to verify by induction the statements $T1 - T4$ of the proof. Furthermore, we claim,

(i) There is a leftmost path TP of strings in PT visited infinitely often and its length (measured by lim inf) is unbounded.
(ii) For $\alpha \prec TP$, $\lim_s p(\alpha, s) = p(\alpha)$ exists, and there are at most finitely many paths in $[T]$ that do not extend initial segments of $p(\alpha)$.
(iii) For $\alpha \prec TP$, $|\alpha = 2e+1|$, $\mathcal{M}_\alpha$ requires attention at most finitely often, and is met.
(iv) for $\alpha \prec TP$, $\lim_s \delta(\alpha, s) = \delta(\alpha)$ exists, and both $\delta(\alpha)\widehat{\ }0$ and $\delta(\alpha)\widehat{\ }1$ are initial segments of paths in $[T]$.
(v) Suppose that $|\alpha| = 2e$ and $\alpha \prec TP$. If $\alpha\widehat{\ }\infty \prec TP$, then the initial segments of paths extending $P(\alpha)$ all have final $2e$-state $\alpha\widehat{\ }\infty$. Furthermore, all initial segments of paths in $p(\alpha)$ have extensions that are are e-verified at some stage.
(vi) Suppose that $\alpha| = 2e$ and $\alpha\widehat{\ }f \prec TP$. Then $\lim_s F(\alpha, s) = F(\alpha)$ exists and there is some node $\sigma \in F(\alpha)$ (the rightmost) which equals $p(\alpha\widehat{\ }f)$. Furthermore no node in the cone above σ is ever e-verified.

The proof of the items above is very straightforward, and proceeds by way of induction. Suppose that we have the lemma for all $\beta \preceq \alpha$. Further, suppose that we are at a stage s_0 where for all $s \geq s_0$ $TP(s) \not\leq_L \alpha$, and we assume that $\alpha \prec TP$. Go to the next α-stage s_1 after s, and we know that $p(\alpha, s_1)$ is set at or before that stage, since it is defined at α-stages. Note also that it is only reset through the action of β of stronger priority, and hence, by hypothesis it is now at a limit. Also $\delta(\alpha, s)$ is now at a limit.

Now suppose that $|\alpha| = 2e+1$. At every stage $s \geq s_0$ where we pass through $p(\alpha)$ we consider extensions of $p(\alpha, s)$ as see if they leave $[U_e]$. If we see this happen we reroute the construction through that extension, and declare the requirement $\mathcal{M}_\alpha$ as met. The outcome will change to $\alpha\widehat{\ }0$ and this will be the outcome each time we visit α, since this declaration cannot be initialized except through the agency of requirements of stronger priority than α, which won't happen by hypothesis. In this case, $\alpha\widehat{\ }0 \prec TP(s)$ for all α stages after the stage s_1 we declare $\mathcal{K}_\alpha$ as satisfied. Note that at that stage in the subcase 1 action of the construction for $\mathcal{M}_\alpha$, we will define $\delta(\alpha\widehat{\ }1, s_1) = \sigma$, put both the extensions on the tree, give the the high state $\alpha\widehat{\ }1$ and for the same reasons these actions will never be initialized. Thus because we always extend a string of a nonitialized state, we see that both of the extensions will have at least one path extending them in $[T]$. Also we have $p(\alpha, s_1) = p(\alpha)$, and similarly for $\delta(\alpha)$. Finally every string in the cone $p(\alpha)$ guessing $\alpha\widehat{\ }1$ is initialized, (save the ones that will be directly extended), and whilst we may put up splittings for β right of α, we know, by hypothesis that the cones $p(\beta, t)$ extending $p(\alpha)$ will be initialized, as $\beta \not\prec TP$. Note that every string in $P(\alpha\widehat{\ }1)$ will have as its initial segment a string with state $\alpha\widehat{\ }1$. Again, it is true that false splitting can be put up extending these strings, because of β right of α, but these will be initialized, and we only directly extend strings γ with state extending $\alpha\widehat{\ }1$. Thus in the limit all strings in the cone $p(\alpha\widehat{\ }1)$ have $2e+1$ state $\alpha\widehat{\ }1$.

The argument for $\alpha\widehat{\ }1$ is even easier. things are now at their limits. Hence we have the above for the case $|\alpha| = 2e+1$.

Finally consider the case that $|\alpha| = 2e$. The argument is very similar. One can verify that at the first α stage after s_0 we will set $p(\alpha\widehat{\ }\infty, s)$ if not before, and by priorities $p(\alpha\widehat{\ }\infty, s) = p(\alpha\widehat{\ }\infty)$ is not at its limit. one sees that either we play this outcome infinitely often, and when we do is initialize lower priority stuff, of $\alpha\widehat{\ }f \prec TP$ because we get stuck verifying some $\sigma \in F(\alpha)$, in which case $p(\alpha\widehat{\ }f, s)$ comes to a limit. The argument that everything comes to a limit in the cone and the $\delta(\alpha\widehat{\ }i, s)$ come to a limit is virtually identical. The fact that we always reset $F(\alpha, s)$ to all the live strings in the cone above $p(\alpha\widehat{\ }\infty)$

each time we play $\alpha\widehat{\ }\infty$ means that all potential initial segments of isolated points are systematically examined and the theorem follows.

4. Proof of Theorem 5

This argument is much more straightforward. Let $[T]$ be the relevant tame class presented by the computable tree $T = \lim_s T_s$. We suppose that the unique rank one point P is Δ_2. By the definition of tameness, it follows that there is a Δ_2 collection $I_0, I_1, \cdots$ of all of isolated members of $[T]$ ordered using initial segments $\sigma_i = \lim_s \sigma_{i,s} \prec I_i$, ($\sigma_i$ is *not* an initial segment of P) $\{\sigma_i : i \in \omega\}$ forming an antichain of strings. The enumeration, of I_j assumes that, after some stage, $s(j)$, I_j is constructed as an ever increasing string in a uniformly computable fashion.

It is therefore easy to construct a class $[V]$ which demonstrates the failure of property $\mathcal{P}$. $[V]$ will be constructed as a collection of paths through a computable tree $[V]$. On the even I_{2e}, we will, in a stage by stage basis, put all the extensions of $\sigma_{2e,s}$ of length s, which have not been terminated into V_s. On I_{2e+1}, for each initial segment τ with $\sigma_{2e+1,s} \preceq \tau \prec I_{2e+1,s} \restriction s$, (the current approximation to I_{2e+1}) if $|\tau|$ is divisible by 4, terminate all extensions of $\tau\widehat{\ }i-1$ given $\tau\widehat{\ }i \preceq I_{2e+1,s} \restriction s$. If $|\tau|$ is not divisible by 4, put all extensions of $\tau\widehat{\ }i-1$ of length s, not extending strings already terminated at a previous stage, into V_s.

For each j, if $\sigma_{j,s+1}$ changes (we change our mind on I_j), terminate all strings in V_{s+1} extending $\sigma_{j,s}$ of length s, except ones extending strings in T_s of length s. Finally at every stage, terminate all strings of length s not extending some $\sigma_{i,s}$ that are not initial segments of T_s.

The point is that after some stage I_j becomes stable and from that point on, if j has even length, we will add every string extending some initial segment of I_j, (hence $[V]$ is clopen there, and contains I), or j is odd and hence from some point onwards, we force any $U \subseteq V$ containing I_j to miss out on the paths extending $\tau\widehat{\ }i-1$ for infinitely many $\tau\widehat{\ }i \prec I_j$.

Hence $[T]$ does not satisfy property $\mathcal{P}$, or even the weak cohesiveness.

5. **Proof of Theorem 6.**

Let $[T_1]$ and $[T_2]$ be two tame minimal classes. By tameness, as in the previous section, we can have Δ^0_2 lists of the isolated points of the respective classes

$$I^j_0, I^j_1, \cdots \text{ for } j \in \{1,2\},$$

with corresponding strings

$$\sigma^j_0, \sigma^j_1, \cdots,$$

together with a Δ^0_2 collection of strings τ^j_i for $j \in \{1,2\}$, such that

$$\lim_s \tau^j_{i,s} \prec \lim_s \tau^j_{i+1,s} \prec P_j,$$

where P_j denotes the rank one point in $[T_j]$.

From [5], we recall that a *basis* for $\overline{[T_j]}$ is a computably enumerable set of strings $\gamma^j_i : i \in \omega$ generating the filter of strings with no extension in $[T]$, and which form an antichain. Given a Π^0_1 class one can easily construct such a basis.

The extension lemma (Theorem 8.1 of [5]) interpreted in the present setting of minimal classes, says the following:

Theorem 11 ([5], Theorem 8.1, special case)**.** *Suppose that there is a computable permutation p and computably enumerable bases γ^j_i as above, such that additionally*

$$\forall k \exists s \forall i > s((\sigma^1_k \prec \gamma^1_i \textit{ iff } \sigma^2_k \prec \gamma^2_{p(i)}) \wedge (\tau^1_k \prec \gamma^1_i \textit{ iff } \tau^2_k \prec \gamma^2_{p(i)})).$$

Then $[T_1]$ is automorphic to $[T_2]$.

It is now clear how to finish the proof. We assume that we are given two computable bases as above, and we define the p so that we try to force $\sigma^1_k \prec \gamma^1_i$ iff $\sigma^2_k \prec \gamma^2_{p(i)}$ and $\tau^1_k \prec \gamma^1_i$ iff $\tau^2_k \prec \gamma^2_{p(i)}$, noting that there is no conflict in these two for almost all strings. We do this in a stage by stage fashion, and use the approximations to the σ^j_k at each stage. The fact that the approximations are Δ^0_2 means that it all settles down, giving the result.

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