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MINIMAL DEGREES RECURSIVE IN 1-GENERIC DEGREES

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We work in the language of strings, which are functions from finite ordinals into $\{0, 1\}$. A string σ is an extension of a string τ, written $\sigma \geqslant \tau$, if $\sigma(x) = \tau(x)$ for all x in the domain of τ. We say that τ is a substring of σ and σ is a superstring of τ if σ extends τ. Let $\sigma > \tau$ denote proper extension. The length of a string σ, written $\text{lth}(\sigma)$, is the least number not in the domain of σ. We identify a set of natural numbers with its characteristic function. A string σ is then an initial segment of a set $G \subset \omega$ if $G(x) = \sigma(x)$ for all x in the domain of σ. We write this as $\sigma < G$. If σ is not an initial segment of G, then this is denoted $\sigma \nless G$. A set G is n-generic if it is Cohen generic with respect to arithmetical sentences with n blocks of alternating quantifiers. As characterized by Jockusch [5], this is equivalent to saying that every Σ_n collection S of strings either contains an initial segment of G or is disjoint from $\{\tau \mid \tau \geqslant \sigma\}$ for some $\sigma < G$.

A set M is of *minimal Turing degree* if it is not recursive and the only sets of strictly lower Turing degree are the recursive sets. The problem on the existence of a set of minimal Turing degree recursive in an n-generic set is of considerable interest, since the construction of an n-generic set, by way of searching for the least extension (of a string) in a given Σ_n collection of strings, is incompatible with the construction of a set of minimal degree, where successive subtrees are introduced at each stage, resulting in greater restrictions on the set of possible extensions of a given string. Jockusch [5] showed that if $n \geqslant 2$, then there is no set of minimal degree recursive in an n-generic set (or, equivalently, no n-generic degree bounds a minimal degree). Chong and Jockusch [3] showed that the same conclusion holds for 1-generic sets recursive in $\emptyset'$ as well. In this paper we prove the next theorem.

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Theorem 1. *There is a* 1*-generic degree bounding a minimal degree below* $\mathbf{0}'$. *Furthermore, the* 1*-generic degree may be chosen to lie below* $\mathbf{0}''$.

The key idea of the proof lies in the notion of Σ_1*-dense set of strings*. To define this, we first introduce some notation and terminology. Given a set of strings W, let $D(W)$ denote the set of strings obtained from W under downward closure. In other words, $\sigma \in D(W)$ if and only if σ is a substring of some string in W. If $M \subset \omega$, and W is a recursively enumerable (r.e.) sequence of strings, then W is *dense* in M if every initial segment of M is extended by some member of W (this is also written $M \subset D(W)$).

Definition 2. Let $M \subset \omega$. An infinite r.e. sequence Y of strings is said to be Σ_1*-dense* in M if

(a) No member of Y is an initial segment of M;

(b) For any r.e. W that is dense in M, some member of W extends a member of Y.

Observe that if Y is Σ_1-dense in M, then Y is dense in M. Theorem 1 follows (using Theorem 3) from the main result of this paper, as stated in the following theorem.

Theorem 4. *Let* $M \subset \omega$. *If there is no infinite r.e. sequence that is* Σ_1*-dense in* M, *then* M *is recursive in a* 1*-generic set.*

The significance of the notion of Σ_1-dense sets is apparent when one combines Theorem 4 with the key theorem of Chong and Downey [2], and obtains a complete characterization of degrees bounded by 1-generic degrees in terms of the existence of Σ_1-dense sets. Let $A \subset \omega$ be of Turing degree $\mathbf{a}$. We have

(**) $\mathbf{a}$ *is recursive in a* 1*-generic degree if and only if there is no infinite r.e. sequence* Σ_1*-dense in* A.

We will first construct a set M of minimal degree below $\mathbf{0}'$ with no Σ_1-dense set of strings (i.e., no infinite r.e. sequence of strings is Σ_1-dense in M) (Theorem 3). We then prove Theorem 4 and use it together with Theorem 3 to obtain a minimal degree not recursive in a 1-generic degree (Theorem 4 had been announced earlier in [1], but the sketched proof contained a gap that is corrected here).

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Theorem 3. *There exists a set* M *of minimal degree recursive in* $\emptyset'$ *with no* Σ_1*-dense set. Hence, if* Y *is r.e. and* $D(Y) \supset M$, *then either* Y *contains an initial segment of* M, *or there exists an infinite r.e. set* Y^* *such that* $D(Y^*) \supset M$ *and no string in* Y^* *extends a string in* Y.

Proof. We construct the set M using the method of splitting trees and full trees (cf. [6]), with an additional twist at each stage. Let M^s denote the initial segment of M obtained at stage s (we begin by setting M^0 to be the empty string and T_0^0 to be the full binary tree). Given a tree T, a reduction procedure e, and a string σ, let $\mathrm{Sp}(T, e, \sigma)$ denote the e-splitting subtree of T above σ. Suppose that for all $e \leqslant s$, T_e^s is either $\mathrm{Sp}(T_{e-1}^s, e, M^{s-1})$ (and nonempty), or is the full subtree of T_{e-1}^s above some string extending M^s. It is understood that if T_e^s is a full subtree of T_{e-1}^s, then T_e^s has no splitting pair of strings. We perform the following at stage $s+1$: let $e(s+1)$ be the least $e \neq e(s)$ (if the latter exists) and $e \leqslant s+1$ for which $\mathrm{Sp}(T_{e-1}^s, e, M^s)$ is empty. Let $T(s)$ be the full subtree of $T_{e(s+1)-1}^s$ above M^s, and let $n(s+1)$ be the least $n \leqslant s+1$ such that S_n, the nth recursively enumerable collection of strings, contains no initial segment of M^s, and such that there is a string σ in S_n extending M^s by some string τ lying on $T(s)$. Choose the least such τ and let it be M^{s+1}. Let $T_d^{s+1} = T_d^s$ if $d \leqslant e(s+1) - 1$ and let $T_{e(s+1)}^{s+1}$ be the full subtree of $T(s)$ above M^{s+1}. For $d > e(s+1)$, let T_d^{s+1} be successively defined to be the d-splitting subtree of T_{d-1}^{s+1} above M^{s+1}. This ends the construction at stage $s+1$.

Let $M = \bigcup_{s<\omega} M^s$.

Observe that by the choice of T_0^0 (which is the full binary tree), there are infinitely many stages s where $T(s)$ is infinite. Indeed there are infinitely many stages s such that $T(s)$ contains infinitely many initial segments of M. To see this, suppose that s_0 is given. Let $s_1 > s_0$ be chosen so that $e(s_1+1)$ is the least e such that $e = e(s+1)$ for some $s > s_0$. Then there is no $e < e(s_1+1)$ for which $e = e(s+1)$ for some $s > s_1$. This means that for all $e < e(s_1+1)$ and all $s > s_1$, either there is an e-splitting pair of strings extending M^s on T_e^s, or no e-splitting pair exists on $T_e^{s_1}$ $(= T_e^s)$ anywhere. By definition $T(s_1)$ is the full subtree of $T_{e(s_1+1)-1}^{s_1}$ above M^{s_1}. This implies that all M^s, $s > s_1$, will lie on $T_{e(s_1+1)}^{s_1+1}$ which is a subtree of $T(s_1)$. Hence infinitely many initial segments of M belong to $T(s_1)$.

By the Epstein–Posner principle [4], the set M is of minimal degree. One can furthermore verify that M is recursive in $\emptyset'$. We now show that M satisfies the condition of the theorem.

Let Y be a recursive sequence of strings such that $D(Y) \supset M$. We show that either Y contains an initial segment of M or there exists an infinite r.e. sequence Y^* such that $D(Y^*) \supset M$ and no string in Y^* extends a string in Y. Let $Y = S_n$. Let s_1 be a stage where we have dealt with S_m for all $m < n$. Let $s > s_1$, where $T(s)$ contains infinitely many initial segments of M. Now if there is a $\sigma \in Y$ extended by some $\tau \in T(s)$, then the construction chooses such a τ and ensures that $\tau < M$. Otherwise the set $T(s)$ is an infinite r.e. set of strings such that $D(T(s)) \supset M$ and no member of $T(s)$ extends a member of Y. We may therefore let $Y^* = T(s)$. This proves Theorem 3. □

Let M be a set of minimal degree recursive in $\emptyset'$ which has the property described in Theorem 3. We will show that M is recursive in a 1-generic set G. To do this we define a reduction procedure Φ (a recursive map) from strings to strings such that if $\tau > \sigma$, then $\Phi(\tau) \geqslant \Phi(\sigma)$ (notice that it follows that Φ is consistent). An infinite path G will be selected from the domain of Φ so that $\Phi(\sigma) < M$ for all $\sigma < G$, and $\text{lth}(\Phi(\sigma))$ is unbounded for $\sigma < G$. This gives $\Phi(G) = M$. We will write Y_i for the ith r.e. set of strings in the range of Φ, and S_i the ith r.e. set in the domain of Φ. To ensure that G is 1-generic, we take the following steps: If S_n is a Σ_1 collection of strings such that $\Phi(\sigma)$ is not an initial segment of M for all σ in S_n extending some initial segment σ_0 of G, and if $\Phi(\sigma)$ is defined for all such σ in S_n, then there is a $\sigma_1 \geqslant \sigma_0$ which is an initial segment of G such that no member of S_n extends σ_1. Specifically, given that for some $\sigma_0 < G$, the set

$$Y = \{\Phi(\sigma) \mid \sigma \in S_n \;\&\; \sigma > \sigma_0\}$$

contains no initial segment of M, Theorem 3 implies that there is a Y^* with $D(Y^*) \supset M$ for which no string in Y^* extends a string in Y. The idea then is to define Φ on some extension σ_1 of σ_0 so that for $\sigma \geqslant \sigma_1$, $\Phi(\sigma)$ takes only substrings of strings in Y^* as values, and σ_1 will be chosen to be an initial segment of G (of course we ensure that $\Phi(\sigma_1) < M$). In this way no extension σ of σ_1 will have $\Phi(\sigma)$ extend a string in Y, so that no such σ belongs to S_n. This allows one to avoid all recursively enumerable collections of strings which do not behave nicely under the map Φ. On the other hand, if for all $n < \omega$ there is a $\sigma > G|n$ in S_n such that $\Phi(\sigma) < M$, then we will arrange G so that $\sigma < G$ for at least one such σ. We have then ensured the 1-genericity of G with respect to the Σ_1 collection S_n. The final problem that one has to take care of is the possibility that an r.e. collection S_n may be dense in G, and yet no member of S_n is defined under Φ. Clearly in such a situation the above device is not applicable. Our construction of Φ and G will not allow this situation to happen.

Now since G may be selected only after Φ has been defined, it is not possible to determine recursively which S_n, and hence which corresponding Y, will have the property mentioned above relative to some $\sigma_0 < G$. And therefore it is not possible to determine which string σ should be chosen to make $\Phi(\sigma)$ a substring of an element of Y^*. For that matter, given a Y as above, an appropriate Y^* cannot be located recursively. On the other hand, the map Φ, by virtue of the fact that it will be a reduction procedure, has to be recursive by definition. The way to get around these difficulties is to assign each τ in the domain of Φ, ω many pairwise disjoint extension strings $\tau^{i,j}$ ($i \in \omega \cup \{\emptyset\}, j < 2$) such that $\tau^{i,j}$ is a potential candidate to be defined under Φ with the property that if $\tau' > \tau^{i,j}$, then $\Phi(\tau')$, if defined, is a substring of an element of Y_i. To make this precise, we introduce the notion of the *tagging* of a finite function to a string. The idea is that if a finite function f is tagged (permanently) to a string σ, then for all $\sigma' \geqslant \sigma$, $\Phi(\sigma')$, if defined, is a substring of a string in $Y_{f(x)}$ for all x in the domain of f. Then, if Y has the property described above, and $Y^* = Y_i$ with i in the range of f, we see that choosing G to pass through σ would be the correct strategy. However, the tagging of finite functions to strings cannot be done arbitrarily, even if recursive matching of these two types of objects is observed, since one has to be certain that the extension of a finite function f to a finite function g, in connection with the extension of a string τ to a string σ, where the functions are tagged respectively to the strings, allows one to solve the difficulties like those presented by Y above (i.e., the appropriate i for $Y_i = Y^*$ has to be in the range of the finite function g). It is not difficult to see that a diagonal argument rules out any possibility of a recursive tagging of finite functions to strings that will immediately meet our requirements. Our approach then is to allow different strings to be tagged with different functions at different stages, with the possibility that the tagging stabilizes at all sufficiently large stage. There are two basic conditions to satisfy: Firstly, for $\sigma < G$, if σ is eventually tagged with f (i.e., tagged with f at all sufficiently large stage), then $Y_{f(x)}$ has to be dense in M for all x in the domain of f. This is needed to ensure that Φ is total on G, since the idea is that all extensions of σ, if defined under Φ, are mapped to strings which are substrings of some member of $Y_{f(x)}$. If $Y_{f(x)}$ is not dense in M, Φ will then be partial or finite on G. Secondly, if S_n is dense in G, we want Φ to be defined, and equal to an initial segment of M, on at least an extension of some member of S_n. This is dictated by the 1-genericity of G. This means, among other things, that there is an extension of a member of S_n which is tagged with a finite function g such that $Y_{g(x)}$ is dense in M for all x in the domain of g. Now there is clearly a conflict between these two conditions. In particular, the first condition demands a permanent commitment—if σ is permanently tagged with g, then all extensions σ' of σ, in order to be defined under Φ, have to meet the criterion set out by g (i.e., the intended $\Phi(\sigma')$ has to be extended by some member of $Y_{g(x)}$ for each x in the domain of g; this entails waiting for witnesses in $Y_{g(x)}$ to appear before $\Phi(\sigma')$ is defined), whereas if σ' turns out to be in S_n, then certain commitments may have to be breached in order for $\Phi(\sigma')$ to be defined (one cannot wait forever for a condition of the first type to be satisfied before action is taken to satisfy a condition of the second type). To resolve this conflict, a priority ordering of the requirements is used. Under this ordering, functions may be *untagged* from strings, and replaced by a function of shorter length, for the sake of requirements of higher priority. The priority is arranged according to the index n of S_n, and the *rank* of a function that is tagged to a string. Thus given σ, the desire to define Φ on an extension of some $\tau \geqslant \sigma$, $\tau \in S_n$, has higher priority than the desire to honor the commitment of tagging a function of rank $\geqslant n$ to a string extending σ. The priority is reversed when n is greater than the length of the function. This will be clear from the construction given below.

Observe that to say that $D(Y_i) \supset M$ is equivalent to saying that every initial segment of M is extended by a string in Y_i. Note also that one has to define Φ so that given $\Phi(\sigma) < M$, there are strings $\sigma' > \sigma$ such that $\Phi(\sigma) < \Phi(\sigma') < M$. This is crucial since making $\Phi(G) = M$ is our ultimate aim.

Theorem 4. *Let $M \subset \omega$ be a set with no Σ_1-dense set. Then M is recursive in a 1-generic set G.*

Proof. Let Y_i^s be Y_i computed s steps. We also let S_n^s be the r.e. set S_n computed s steps. Let K be the set of finite (partial) functions from a finite ordinal into $\omega \cup \{\emptyset\}$ (for our purposes here, $\omega = \{0, 1, \ldots\}$ and $\emptyset$ is a symbol denoting 'empty', to be interpreted differently from 0). If $f \in K$, then the *length* of f, written $\mathrm{lth}(f)$, is the least number not in the domain of f. The *rank* f, written $\mathrm{rk}(f)$, is the number of elements in the range of f not equal to $\emptyset$. Given a finite function f, we will be considering the sets $Y_{f(x)}$ for x in the domain of f such that $f(x) < \omega$. We will abbreviate this as *for all x in the domain of f*, ignoring places where $f(x) = \emptyset$.

We define a reduction procedure Φ by induction. Again we set Φ_s to be Φ computed s steps. An $f \in K$ may be *tagged to σ at stage s*. Intuitively, this happens when any ν that is set to be $\Phi_s(\sigma)$ has to be a substring of some string in $Y_{f(x)}^s$ for each x in the domain of f.

In the construction to follow, we shall introduce two lists of strings L_1 and L_2, which will be used as reference points for our actions during the construction. Initially, a string σ is put into L_2. It can (possibly) later enter L_1 or it can be removed from L_2, or both. L_1 is an r.e. set of strings. If a string σ is in L_1, then $\Phi(\sigma)$ is defined and furthermore the tag of σ will be *confirmed* as we shall see. Here the tag g of σ will be confirmed only if we see strings $\gamma \in Y_{g(x)}$ with γ extending $\Phi(\sigma)$ for all x in the domain of g. If σ has tag g at stage s but has tag $g' \neq g$ at stage $s+1$, then g will be a proper extension of g' (written $g > g'$) if

$$(\forall x < \mathrm{lth}(g'))[g'(x) = g(x)] \quad \text{and} \quad \mathrm{lth}(g') < \mathrm{lth}(g).$$

As for strings, we give a partial ordering for finite functions under $\leqslant$, which is interpreted to mean either $<$ or $=$.

In the construction to follow, a string σ in the domain of Φ can become *k-attended* at some stage s. This means that σ was at first k-unattended and we see some $\tau \in S_k^s$ with $\tau \leqslant \sigma$. The effect of k-attention will be to (possibly) drop the length of the tag of σ. This idea is crucial in finding a 1-generic set G with $\Phi(G) = M$.

Let L_1^s and L_2^s denote, respectively, strings which are in L_1 and L_2 at stage s. At each stage s of the construction we shall ensure that the following induction hypothesis is satisfied (as the reader should observe):

(a) For all strings σ, at stage s, there exist recursively ω many pairwise incompatible extensions $\tau_s^{i,j}(\sigma)$ of σ (for $i \in \omega \cup \{\emptyset\}$, $j < 2$) such that $\tau_s^{i,j}(\sigma)$ is incompatible with each $\gamma \in L_1^s \cup L_2^s$ if $\gamma \not\leqslant \sigma$.

(b) If $\gamma \in L_2^s$, then γ properly extends some $\sigma \in L_1^s$ and $\gamma = \tau_s^{i,j}$ for some $\tau_s^{i,j} = \tau_s^{i,j}(\hat{\sigma})$ with $\hat{\sigma} \in L_1^s$ and $\hat{\sigma} \geqslant \sigma$.

(c) If $\gamma \in L_2^s$ and $\eta \in L_1^s$, then $\gamma \not\leqslant \eta$.

(d) At the end of stage s, if $\gamma \in L_1^s$, then γ has countably infinitely many pairwise incompatible extensions in L_2^s.

(e) If $\sigma \in L_1^s$ has tag g at stage s, then for all γ in L_1^s with $\gamma \leqslant \sigma$, γ has tag g' with $g' \leqslant g$.

(f) If $\sigma \in L_1^s$ has tag g at stage s and $\sigma \in L_1^{s+1}$ has tag g' at stage $s+1$, then $g' \leqslant g$; and if $g' \neq g$, then some $\sigma' \in L_1^{s+1}$ with $\sigma' \geqslant \sigma$ and σ' having tag g' is k-attended, where $k = rk(g')$.

Construction

Stage 0. Define $\Phi(\emptyset) = \emptyset$. Tag $\emptyset$ with the empty function and put $\emptyset$ in L_1^0. Now recursively choose countably many pairwise incompatible extensions $\tau^{i,j} = \tau_0^{i,j}(\emptyset)$ (for $i \in \omega \cup \{\emptyset\}$ and $j < 2$) of $\emptyset$. This is done in such a way that we first form infinitely many pairwise incompatible extensions τ^i and then split them (by extension) into the $\tau^{i,j}$ so that condition (a) is satisfied. Assign to each $\tau^{i,j}$ the tag $\{(0, i)\}$ (and so $\mathrm{rk}(\{(0, i)\}) = 1$).

Stage $s + 1$

Step 1 (k-attention). Find any string σ and $k \in \omega$ such that

(i) σ is not yet k-attended;

(ii) $\sigma \in S_k^s$; and

(iii) for some $\sigma' \in L_1^s$ with $\sigma' \leqslant \sigma$, σ' has tag g with $\mathrm{rk}(g) = k - 1$ at stage s.

For all such σ, in order of k and $\mathrm{lth}(\sigma)$, find the $\gamma \in L_1^s$ with $\gamma \leqslant \sigma$ of the greatest length, and the $\sigma' \in L_1^s$ of the least length satisfying (iii). For all $\rho \in L_1^s$ with $\sigma' \leqslant \rho \leqslant \gamma$ or $\gamma \leqslant \rho$ declare ρ to have tag g. For all $\eta \geqslant \sigma$, declare η to be k-attended. Remove all $\rho \in L_2^s$ with $\rho \leqslant \gamma$ or $\rho \geqslant \gamma$ from L_2^s and remove any tags from ρ. Adopt the first case below which applies to the situation:

Case 1: $\sigma \in L_1^s$. Do nothing in this step.

Case 2: $\sigma \notin L_1^s$. Find a (long) extension τ and σ incompatible with all strings not $\leqslant \gamma$ generated in the construction so far, chosen so that condition (a) is preserved. Define $\Phi(\tau) = \Phi(\gamma)$, declare τ to have tag g, and put τ in L_1^{s+1}.

Step 2 (g-confirmation). For any string η remaining in L_2^s after Step 1, it will be the case that η has some tag $g = g_\eta$ and $\eta = \tau_s^{i,j}(\hat{\sigma})$ for some $\hat{\sigma}$. If for all x in the domain of g, there is a $\rho = \rho(g(x), s) \in Y_{g(x)}^{s+1}$ such that $\Phi(\hat{\sigma}) * j \leqslant \rho$, declare η as g-confirmed (i.e., x-confirmed for all x in the domain of g) and put η into L_1^{s+1} after removing it from L_2^{s+1}. Define $\Phi(\eta) = \Phi(\sigma) * j$.

Step 3 (Creating many possible paths). Finally, for all strings σ in L_1^{s+1} with a tag g at the end of Step 2, create infinitely many pairwise incompatible extensions $\tau_{s+1}^{i,j} = \tau_{s+1}^{i,j}(\sigma)$ (for $i \in \omega \cup \{\emptyset\}$ and $j < 2$), chosen so that condition (a) is met and that the extensions are all incompatible with any string in $L_1^{s+1} \cup L_2^s$ (at the end of Step 2), except those which are substrings of σ. Tag each such $\tau_{s+1}^{i,j}$ with $f = g \cup \{\mathrm{lth}(g), i)\}$ and put it in L_2^{s+1}. Note that $\mathrm{rk}(f) = \mathrm{rk}(g) + 1$ at the end of this stage unless $i = \emptyset$ or is in the range of g. For each $\tau_{s+1}^{i,j}(\sigma)$ we have just created perform Step 2. That is, if for each x in the domain of f, there is a $\rho \in Y_{f(x)}^{s+1}$ such that $\rho \geqslant \Phi(\sigma) * j$, *immediately* put $\tau_{s+1}^{i,j}(\sigma)$ into L_1^{s+1} after removing it from L_2^{s+1}. We also set $\Phi(\tau_{s+1}^{i,j}) = \Phi(\sigma) * j$ and declare that $\tau_{s+1}^{i,j}$ is f-confirmed. □ End of Construction

Verification and construction of G

The first lemma is proved by a straightforward induction.

Lemma 5.
(i) *Φ is consistent and partial recursive.*
(ii) *Claims* (a)–(f) *hold for the construction.*

We shall now define a 1-generic set G with $\Phi(G) = M$. We do this in steps (distinguishing it from the construction above which was carried out in *stages*).

At Step 0 set $G_0 = \emptyset$ and choose $f_0 = \emptyset$. We proceed by induction and assume that by Step n we have $G_n = \sigma$ for some $\sigma \in L_1^{t_n}$, where $t = t_n$ is a stage in the construction such that

(g) σ has tag f_n at all stages $s \geqslant t$;
(h) $\mathrm{rk}(f_n) = n$;
(i) for all x in the domain of f_n, $D(Y_{f_n(x)}) \supset M$;
(j) $\mathrm{lth}(\Phi(\sigma)) \geqslant n$ and $\Phi(\sigma) < M$;

and for all $m < n$,

(k) either $(\exists \tau)(\tau \in S_m^{t_m} \;\&\; \tau \leqslant \sigma)$ or $(\forall \tau \in S_m)(\sigma \not\leqslant \tau)$.

To define G_{n+1} we proceed as follows.

Case A. If there exists $\gamma \in S_{n+1}$ enumerated after stage t such that $\gamma' \geqslant \gamma \geqslant \sigma$ and $\Phi(\gamma') < M$ for some γ', then the construction ensures that $\gamma' \in L_1$ (since only strings in L_1 are defined under Φ; notice also that $L_1 = \bigcup L_1^s$ is an r.e. set). As $\gamma > \sigma$, the tag of γ' is an extension of f_n. Since $\mathrm{rk}(f_n) = n$ if γ is the string with the prescribed property of the least length, and γ' the corresponding string of the least length for γ, then the construction ensures that the tag of γ' is $\leqslant f_n$ and hence must be f_n. In this case let $t' \geqslant t$ be the least stage where $\gamma' \in L_1^{t'}$ has tag f_n at all $s \geqslant t'$.

Now at stage t', the construction provides γ' with countably many pairwise incompatible extensions $\tau_{t'}^{i,j}$ all with tag $f_n \cup \{\mathrm{lth}(f_n), i)\}$. Let i_0 be any index with $i_0 \notin \mathrm{domain}(f_n)$ and $D(Y_{i_0}) \supset M$. Let $\sigma_{n+1}^j = \tau_{t'}^{i_0,j}$, and $f_{n+1} = f_n \cup \{(\mathrm{lth}(f_n), i_0)\}$. Then

(l) σ_{n+1}^j has tag f_{n+1} at stage t';
(m) $\mathrm{rk}(f_{n+1}) = n + 1$;
(n) for all x in the domain of f_{n+1}, $D(Y_{f_{n+1}(x)}) \supset M$;
(o) for all $m \leqslant n+1$, either $(\exists \tau)(\tau \in S_m^{t'} \;\&\; \tau \leqslant \sigma_{n+1}^j)$ or $(\forall \tau \in S_m)(\sigma_{n+1}^j \not\leqslant \rho)$.

Now the only way the tag of σ_{n+1}^j could be changed is due to the action of some S_m (through m-attention) for $m \leqslant n+1$. But no $m < n$ can interfere by induction hypothesis, and S_{n+1} has been dealt with by the choice of γ' and $\tau_{t'}^{i,j}$. It follows that (l) can be replaced by

(l′) σ_{n+1}^j has tag f_{n+1} for all stages $s \geqslant t'$.

Now let j be such that $\Phi(\gamma') * j < M$. Then $\sigma_{n+1} = \sigma_{n+1}^j$ gives $\sigma_{n+1} \in L_1$ and

(p) $\mathrm{lth}(\Phi(\sigma_{n+1})) \geqslant n + 1$.

Set $t_{n+1} = \max\{t', u\}$, where u is the stage we add σ_{n+1} to L_1. Thus in this case we set $G_{n+1} = \sigma_{n+1}$ and complete the induction (observe that $\Phi(\sigma_{n+1}) = \Phi(\gamma') * j$).

Case B. This is the remaining case. Here for all $\gamma \in S_{n+1}$, and all μ such that $\mu \geqslant \gamma > \sigma$, $\Phi(\mu)\downarrow$ implies that $\Phi(\mu) \nless M$. The first step of the construction evidently arranges that for $\gamma \in S_{n+1}$ extending σ and $(n+1)$-attended at $s > t$, there exists μ such that $\sigma < \gamma \leqslant \mu$, with $\Phi_s(\mu)\downarrow$, and μ, given tag f_n at the end of stage s', enters L_1. For those $\gamma \in S_{n+1}$ such that there exist $\mu \in L_1$ with tag f_n and $\sigma < \gamma \leqslant \mu$, let $\rho = \rho(\gamma)$ denote the string in L_1 of the greatest length (if exists) such that $\sigma < \rho \leqslant \gamma$. The construction ensures that $\Phi(\mu) = \Phi(\rho)$.

Let

$$S = (\gamma \in S_{n+1} \mid \rho(\gamma) \text{ is defined}\} \quad \text{and} \quad Y = \{\Phi(\rho(\gamma)) \mid \gamma \in S\}.$$

We need to consider two cases.

Subcase 1. $D(\Phi(S)) \not\Rightarrow M$. In this case we know that there is a proper extension λ of $\Phi(\sigma)$ such that for all $\gamma \in S$, $\Phi(\rho(\gamma)) \not> \lambda$ and $\lambda < M$. Now at each stage s we construct for each $\beta \in L_1$ infinitely many pairwise incompatible extensions of the form $\tau_s^{i,j}(\beta)$. In particular the extensions of β of the form $\tau_s^{\emptyset,j}(\beta)$ will add nothing to the tag of β (as far as the consideration of sets of the form $Y_{f(x)}$ is concerned, where f is the tag of β). Now for $\beta = \sigma$, let $s > t_n$ and suppose that $\tau_s^{\emptyset,j}(\sigma)$ is introduced at stage s with $\Phi(\sigma) * j < M$. Then one of the following holds.

(q) $\tau_s^{\emptyset,j}(\sigma)$ is $(n+1)$-attended at some stage $w \leqslant s$;

(r) $\tau_s^{\emptyset,j}(\sigma)$ is $(n+1)$-attended at some stage $s' > s$, and $\Phi_w(\tau_s^{\emptyset,j}(\sigma)) = \Phi(\sigma) * j$ is defined at some $w < s'$;

(s) $\tau_s^{\emptyset,j}(\sigma)$ is $(n+1)$-attended at some $s' > s$, and $\Phi(\tau_s^{\emptyset,j}(\sigma))$ is not defined by stage s'.

In the case of (q), $\tau_s^{\emptyset,j}(\sigma)$ has tag $f_n \cup \{(\text{lth}(f_n), \emptyset)\}$ at all stages after s, and $\Phi(\tau_s^{\emptyset,j}(\sigma)) = \Phi(\sigma) * j < M$. For (r), the string $\eta = \tau_s^{\emptyset,j}(\sigma)$ is in L_1 and so will have tag f_n at any stage greater than s'. Furthermore, for an appropriate k, $\tau_{s'}^{\emptyset,k}(\eta)$ will have tag $f_n \cup \{\text{lth}(f_n), \emptyset)\}$ at all stages after s', and $\Phi(\tau_{s'}^{\emptyset,j}(\eta)) = \Phi(\eta) * k < M$. In the case of (s), $\tau_s^{\emptyset,j}(\sigma)$ is removed from L_2. Let $\gamma' \in S_{n+1}$ be the string causing $\tau_s^{\emptyset,j}(\sigma)$ to be $(n+1)$-attended at $s' > s$. Then $\gamma' \leqslant \tau_s^{\emptyset,j}(\sigma)$ and Step 1 of the construction gives a $\mu \geqslant \gamma'$ such that $\Phi_{s'}(\mu) = \Phi(\sigma)$. Then $\tau_{s'}^{\emptyset,k'}(\mu)$ has tag $f_n \cup \{(\text{lth}(f_n), \emptyset)\}$, for $k' < 2$, at all stages after s', and in particular at some $w > s'$, we have $\Phi_w(\tau_{s'}^{\emptyset,k'}(\mu)) = \Phi(\sigma) * k' < M$ for an appropriate k'. In any case, we conclude that there is an extension ν of σ such that $\Phi(\nu) = \Phi(\sigma) * j < M$ and ν is tagged with either f_n or $f_n \cup \{(\text{lth}(f_n), \emptyset)\}$ at all sufficiently large stage after s. Similar comments apply to ν as they do to σ.

In this way we obtain by Step $n+1$ an extension τ^* of σ such that $\Phi(\tau^*) = \lambda$, and at all sufficiently large stage, the tag of τ^* is g', where $g' \geqslant f_n$ and $g'(x) = \emptyset$ for all x such that $\text{lth}(f_n) \leqslant x < \text{lth}(g')$. Note that for all $\gamma \in S_{n+1}$, either γ does not extend τ^*, or if it does, then it does not cause $(n+1)$-attention (else the $\gamma \geqslant \tau^*$ with the least length would have $\rho(\gamma)$ defined and $\geqslant \tau^*$, contradicting the choice of λ). Thus if there is a $\gamma \in S_{n+1}$ which extends τ^*, then some substring of τ^* is already enumerated in S_{n+1} by the stage γ is enumerated. Now let $i_0 \notin \mathrm{range}(f_n)$ be chosen so that $D(Y_{i_0}) \supset M$ and let $g'' = g' \cup \{(\mathrm{lth}(g'), i_0)\}$. Choose $G_{n+1} = \sigma_{n+1} > \tau^*$ to be of the form $\tau_u^{i_0,j'}(\tau^*)$, where u is the least stage such that $\Phi_u(\tau^*)$ is defined and in L_1, and where $\lambda * j' < M$ and $\tau_u^{i_0,j'}(\tau^*)$ has tag g'' at all $w \geqslant u$. Set $f_{n+1} = g''$, and let $t_{n+1} = u$. Then we get all of (g)–(k) (replacing n by $n+1$ at the appropriate places) for σ_{n+1} by the above observation.

Subcase 2. $D(\Phi(S)) \supset M$. This is where we use the fact that M has no Σ_1-dense sets. Now by hypothesis on M, there exists an r.e. set of strings Y_{i_0} such that $D(Y_{i_0}) \supset M$ and such that

(*) For all $\alpha \in Y_{i_0}$ and $\beta \in Y$, $\alpha \not\geqslant \beta$.

Consider then the extension $\tau_s^{i_0,j} = \tau_s^{i_0,j}(\sigma)$ with $\Phi(\sigma) * j < M$, for $s \geqslant t_n + 1$. Let u be a stage where for each $i \in \{f_n(x) \mid x \in \mathrm{domain}(f_n)\}$ and $i \neq \emptyset$, some string μ occurs in Y_i with $\mu > \Phi(\sigma) * j$. Then at stage $u+1$, $\tau = \tau_{u+1}^{i_0,j}$ is put immediately into L_1^{u+1}. Note that τ has tag $g' = f_n \cup \{(\mathrm{lth}(f_n), i_0)\}$. We claim that this is so at all stages $s \geqslant u+1$. The only way this could not be so is if some γ occurs in S_{n+1} with either $\gamma \geqslant \tau$ or $\sigma \leqslant \gamma \leqslant \tau$. In the latter case we would have $\Phi(\tau) < M$, with $\tau \geqslant \gamma \geqslant \sigma$ for some $\gamma \in S_{n+1}$, and would therefore contradict the fact that we are in Case B.

We therefore finish by showing that there is no $\gamma \geqslant \tau$ with $\gamma \in S_{n+1}$, thus simultaneously giving (k) for $m = n+1$ and $\sigma_{n+1} = \tau$, and showing that the tag of τ is g' for all stages after $u+1$. Hence suppose that $\gamma \geqslant \tau$ occurs with $\gamma \in S_{n+1}$. Choose a γ of the least length. Then at the least stage $s > u+1$ where this takes place, we will $(n+1)$-attend γ and so $\rho(\gamma)$ will be defined. But $\rho(\gamma) \geqslant \tau$ and hence $\Phi(\rho(\gamma)) \geqslant \Phi(\tau)$. Using $\rho(\gamma) \geqslant \tau$, we see that at the stage v where $\Phi_v(\rho(\gamma))$ was defined, it must be the case that the tag g'' of $\rho(\gamma)$ was an extension of g' and hence, in particular, $\rho(\gamma)$ would have been i_0-confirmed at stage v. It follows that there must have been a string $\delta \in Y_{i_0}$ such that $\delta > \Phi(\rho(\gamma))$. This specifically contradicts the choice of Y_{i_0}. Let $G_{n+1} = \sigma_{n+1} = \tau$, $f_{n+1} = g'$, and choose t_{n+1} to be the stage $u+1$ above.

The proof of Theorem 4 is complete. □

It is of interest to determine the complexity of a 1-generic set G bounding a set of minimal degree below $\mathbf{0}'$. It is not difficult to see from our construction that G is recursive in $M \oplus \emptyset''$, and in particular if $M \leqslant_T \emptyset'$, then $G \leqslant_T \emptyset''$. As in [1] it is possible to modify Theorem 3 to construct many minimal degrees recursive in 1-generic degrees. Now when viewed together with the result that no 1-generic degree recursive in $\mathbf{0}'$ bounds a minimal degree [3], our result indicates that so far as the jump hierarchy is concerned, $\mathbf{0}'$ is the best upper limit for arithmetical 1-generic degrees not to lie above minimal degrees. It will be an interesting program to investigate the order relationship below $\mathbf{0}^{(2)}$ between 1-generic degrees and minimal degrees. In particular how far can one extend the results of [3] beyond $\mathbf{0}'$? For example, does there exist an $\mathbf{a} > \mathbf{0}'$ such that no 1-generic degree recursive in it bounds a minimal degree? On the other hand, it is known that there is a minimal degree below $\mathbf{0}'$ not bounded by any minimal degree [2]. Thus the interplay between 1-generic degrees and minimal degrees appears to be fairly complex. Ultimately, one would like to obtain a possibly degree-theoretic characterization of minimal degrees below $\mathbf{0}'$ which are recursive in 1-generic degrees.

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Notes added in proof

(a) M. Kumabe has independently proved Theorem 1 (A 1-generic degree which bounds a minimal degree, J. Symbolic Logic 55 (1990) 733–743). His method is different from ours.

(b) Our characterization (**) and the proof of Theorem 4 actually produce the following basis result: A minimal degree below $\mathbf{0}'$ is bounded by a 1-generic degree if and only if it is bounded by one that is below $\mathbf{0}''$. Furthermore, the latter can be obtained uniformly.

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