Some Recent Progress in Algorithmic Randomness

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1 Introduction

Recently there has been exciting progress in our understanding of algorithmic randomness for reals, its calibration, and its connection with classical measures of complexity such as degrees of unsolvability.

In this paper, I will give a biased review of (some of) this progress.

In particular, I will concentrate upon randomness for *reals*. In this paper "real" will mean a member of Cantor space 2^ω. This space is equipped with the topology where the basic clopen sets are $[\sigma] = \{\sigma\alpha : \alpha \in 2^\omega\}$. Such clopen sets have measure $2^{-|\sigma|}$. This space is measure-theoretically identical with the rational interval $(0, 1)$, without being homeomorphic spaces.

An important program which began in the early 20th Century was to give a proper mathematical foundation to notion of randomness. In terms of understanding this for probability theory, the work of Kolmogorov and others provides an adequate foundation.

However, another key direction is to attempt to answer this question via notion of randomness in terms of *algorithmic randomness*. Here we try to capture the nature of randomness in terms of algorithmic considerations. (This is implicit in the work on *Kollektivs* in the fundamental paper of von Mises [88].)

There are three basic approaches to algorithmic randomness. They are to characterize randomness in terms of algorithmic predictability ("a random real should have bits that are hard to predict"), algorithmic compressibility ("a random real should have segments that are hard to describe with short programs"), and measure theory ("a random real should pass all reasonable algorithmic statistical tests").

A classic example of the relationship between these three is given by the emergence of what is now called Martin-Löf randomness. For a real $\alpha = .a_1a_2 \cdots \in 2^\omega$, a consequence of the law of large numbers is that if α is to be random then $\lim_s \frac{a_1+\cdots+a_s}{s} = \frac{1}{2}$. Consider the null set of reals that *fail* such a test. Then Martin-Löf argued that a real α can only be random if it was not in such a null set. He argued that a random real should pass all such "effectively presented" statistical tests. Thus we define a Martin-Löf test as a computable collection

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$\{U_n : n \in \mathbb{N}\}$ of computably enumerable open sets such that $\mu(U_n) \leq 2^{-n}$. We say that α passes the test iff $\alpha \notin \cap_{n\in\mathbb{N}} U_n$. A real α is called Martin-Löf random or 1-random iff it passes all Martin-Löf tests. (Martin-Löf [77])

It turns out that there are *equivalent* definitions of this notion of randomness in terms of the other paradigms. For instance, Schnorr proved that α is 1-random iff there is a constant c such that for all n, $K(\alpha \upharpoonright n) \geq n - c$. Here K denotes prefix-free Kolmogorov complexity[1].

In this paper we will report on some recent work centered around a program attempting to understand when one real is more random than another and what consequences this has for computability theoretical-aspects of the real. For instance, what does it mean for a real to be "highly random"? If a real is "highly random" what can be said about its Turing degree? Should it be complicated or should it be relatively simple? Does it have high or low information content, etc.

We have seen that we can characterize 1-randomness in terms of initial segment. This suggests natural measures of relative complexity. One real is more random than another if, but some reasonable measure, the initial segment complexity of the first is at least as high as the second on initial segments. We will look at various methods of calibration by initial segment complexity such as those introduced by Solovay [112], Downey, Hirschfeldt, and Nies [30], Downey, Hirschfeldt, and LaForte [27], Downey [21].

These methods are related to lowness notions such as those of Kučera and Terwijn [64], Terwijn and Zambella [119], Nies [91, 93, 92], Downey, Griffiths and Reid [25]. Here a lowness notion is one that says the real does not help as an oracle. For instance, a real A is Martin-Löf low if the collection of reals 1-random *relative to A* are the *same* as the original 1-randoms. Miller and Yu [86] have demonstrated that this notion also gives rise to natural notions of calibration of randomness such as Miller and Yu's von Lambalgen reducibility. This is intimately related to our earlier work on initial segment complexity.

Finally, this is also related to higher level randomness notions going back to the work of Kurtz [66], Kautz [54], and Solovay [112], and other calibrations of randomness based on definitions along the lines of Schnorr.

These notions all have complex interrelationships, and relationships with classical computability notions such as relative computability and enumerability. For instance, a very exciting by-product of the program is a more-or-less natural

[1] We assume that the reader is somewhat with basic Kolmogorov complexity, and the notion of a prefix-free machine. The reader should recall that a prefix-free machine is a Turing machine M whose domain is a prefix-free set of strings; that is for all σ if $M(\sigma)\downarrow$, then for all strings τ, with $\sigma \prec \tau$, $M(\tau)\uparrow$. Of course, such machine have measurable domain, $\mu(\text{dom } (M)) = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|}$. Prefix-free machines are used in the algorithmic information theory of reals. There is a minimal universal such machine U, in the sense that for all M there is a constant c_M such that for all $K_U(\sigma) \leq K_M(\sigma) + c_M$. Here $K_D(\sigma)$ denotes the Kolmogorov complexity of a string σ relative to a machine D. That is the length of shortest string τ with $D(\tau) = \sigma$, and ∞ if no τ exists. We let $K(\sigma)$ denote $K_U(\sigma)$.

requirement-free solution to Post's problem, much along the lines of the Dekker deficiency set.

There is a large number of open problems in this area many of which are very basic. We will focus on some of these problems.

Given the space restrictions, I will not include any proofs. I refer the reader to Downey, Hirschfeldt, Nies and Terwijn [32] for an expanded version of this article, and to the forthcoming book of Downey and Hirschfeldt [26] for full details.

I will be concentrating upon algorithmic information theory for *reals* and hence be ignoring a lot of new combinatorial results, particularly from the Moscow school, on the Kolmogorov complexity of *strings.* Also ignored is the complexity theoretical work of Lutz and others relating effective Hausdorff and packing dimension to complexity classes. Here, we refer the reader to Ambos-Spies and E. Mayordomo [4] and Lutz [75]. Finally I will not talk about the recent work of Allender and others looking at efficient reductions to the collection of nonramdom strings. (See e.g. Allender et. al. [1])

2 Three approaches to randomness

Historically[2], there were three main approaches to the definition of an algorithmically random sequence. They are via what we call

(i) The measure-theoretical paradigm,
(ii) The incompressibility paradigm, and
(iii) The unpredictability paradigm.

2.1 The measure-theoretic paradigm, and stochasticity

The first author to address a possible "definition" of randomness was von Mises. In his remarkable paper [88], von Mises defined a notion of randomness based on such "admissible" stochastic properties, such a random real should have 1's and 0's equally likely, and noted that for any *countable* set of such properties a nonempty notion of randomness resulted. This was early in the 20th century and well before the development of the notion of a computable function. He did not have a canonical choice of such a countable set at hand. After the development of the notion of a computable function, Church made the connection with the theory of computability by suggesting that one should take all *computable* stochastic properties.

Jumping forward, Martin-Löf noted that sets coding such stochastic properties are a special kind of measure zero sets, and that a more general and smooth definition could be obtained by considering all effectively measure zero sets.

[2] In these notes, we will try to avoid discussion of the history of the evolution of the notion of algorithmic randomness. There is a thorough discussion in the monograph of Li and Vitányi [71], the paper of Zvonkin and Levin [129], and van Lambalgen's thesis [67].

The measure-theoretic paradigm is that random reals should be those with no *effectively rare* properties. If a property constitutes an effective null set, then a random real should not have such a property.

A collection of reals that is effectively enumerated is a Σ_1^0-class. We can represent any Σ_1^0-class of reals as $\{[\sigma] : \sigma \in W\}$ for some prefix-free c.e. set W. Now a "test" is a series $\{\mathcal{U}_i\}_{i\in\omega}$ of such Σ_1^0-classes that are shrinking in size. A real passes a test $\{\mathcal{U}_i\}_{i\in\omega}$ if it is not in the intersection $\bigcap_i \mathcal{U}_i$. The main idea is that a random real should pass all effective tests. This leads to the following definition.

Definition 1 (Martin-Löf [77]). *A set of reals $\mathcal{A} \subseteq 2^\omega$ is Martin-Löf null (or Σ_1-*null*) if there is a uniformly c.e. sequence $\{\mathcal{U}_i\}_{i\in\omega}$ of Σ_1^0-classes (called a* Martin-Löf test*) such that $\mu(\mathcal{U}_i) \leq 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i \mathcal{U}_i$. $A \in 2^\omega$ is Martin-Löf random, or Σ_1-*random*, or 1-random, if $\{A\}$ is not Σ_1-null.*

Solovay [112] observed that the definition of 1-randomness is equivalent to saying that a real α is 1-random iff for all c.e. collections of intervals $\{[\sigma_i] : i \in \mathbb{N}\}$, if $\sum_i 2^{-|\sigma_i|} < \infty$ then α is in at most finitely many $[\sigma_i]$.

One very interesting fact, due to Martin-Löf, is that there is a *universal* Martin-Löf test $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$. That is, a real α is Martin-Löf random iff $\alpha \notin \bigcap_n U_n$. To see this, given an enumeration $\{V_i^m : i, m \in \mathbb{N}\}$, of partial Martin-Löf tests, with $\mathcal{V}^m = \{V_i^m : i \in \mathbb{N}\}$, let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, with $U_n = \bigcup_k V_{n+k}^k$.

2.2 The incompressibility paradigm

The most celebrated approach to the notion of a random real is that essentially due to Kolmogorov [58]. Here we regard a *string* as random iff there is no short program to generate the string, meaning that the only way to generate the string is essentially to hardwire the string into the machine. (Consider, 101010 repeated 1000 times could be generated by a short program).

Plain Kolmogorov complexity Fix a universal Turing machine U. Given a string $\sigma \in 2^{<\omega}$, define the *plain Kolmogorov complexity* of σ as $C(\sigma) = \min\{|\tau| : U(\tau) = \sigma\}$. The basic facts concerning C are that : (i) the choice of U matters only an additive constant in the theory and (ii) for all σ, $C(\sigma) \leq |\sigma| + O(1)$.

We can define σ to be *k-random* if $C(\sigma) \geq |\sigma| - k$. An easy counting argument shows that random strings exist: Given length n, there are $\sum_{i=0}^{n-k-1} 2^{-i} = 2^{n-k} - 1$ programs of length $< n - k$, so there are $2^n - 2^{n-k} + 1$ k-random strings of length n. For every k, the set of k-random strings is an *immune* Π_1^0 set, i.e. it does not contain any infinite c.e. subsets. As a function, C is not computable. If $m(x) = \min\{C(y) : y \geq x\}$, then m is unbounded (because we run out of short programs), but m grows slower than any partial computable function.

We would like to extend the definition of randomness for finite strings to a definition for infinite strings. Naively, we could define a real A to be random if and only if for some k every $\sigma \preceq A$ would be k-random. However, Martin-Löf showed that such reals do not exist! This can be seen using the following argument of Katseff [53]. Suppose $A\restriction m$ is the n-th binary string. Then from the *length* of $A\restriction n$ we can recover $A\restriction m$, which is of length $\log n$. This gives

$$C(A\restriction n) \leq C(A\restriction_{m+1}^{n}) \leq n - \log n + O(1).$$

Namely first generate $C(A\restriction_{m+1}^{n})$. Compute n from its length $n - \log n$. Then recover $A\restriction m$.

The basic intuition is that the Kolmogorov complexity should be the length of the shortest string σ used to generate τ from the *bits* of σ. The *length* of σ seems to give an additional $\log n$ many bits of information. This idea is explicitly used to demonstrate that using plain Kolmogorov complexity we will always get complexity oscillations.

First Levin and then later Chaitin introduced methods to get around this problem, as we will see in section 2.2.

Prefix-free complexity Call a Turing machine T a *prefix machine* if $\mathrm{dom}(T) \subseteq 2^{<\omega}$ is prefix-free. It is easy to construct a *universal* prefix machine U:

$$U(\langle m, \sigma\rangle) = T_m(\sigma),$$

for an effective enumeration $T_0, T_1, \ldots$ of all prefix machines. Here $\langle m, \sigma\rangle$ denotes a prefix encoding of the pair (m, σ).

Now we can define the *prefix-free complexity* of a string σ as $K(\sigma) = \min\{|\tau| : U(\tau) = \sigma\}$. Again, the choice of U matters only an additive constant in the theory. The prefix encoding $\widehat{\sigma} = 1^{|\sigma|}0\sigma$ gives $K(\sigma) \leq 2|\sigma| + O(1)$, and the prefix encoding $\widehat{|\sigma|}\sigma$ gives $K(\sigma) \leq |\sigma| + 2\log|\sigma| + O(1)$. Unlike C, K is *subadditive*: $K(x, y) := K(\langle x, y\rangle) \leq K(x) + K(y) + O(1)$.

The fundamental result about K is the following.

Theorem 1 (Counting Theorem, Chaitin [16]).

(i) $\max\{K(x) : |x| = n\} = n + K(n) + O(1)$.
(ii) $\big|\{x : |x| = n \wedge K(x) \leq n + K(n) - r\}\big| \leq 2^{n-r+O(1)}$, *where the constant* $O(1)$ *does not depend on* n *and* r.

We are now in a position to define randomness for reals in terms of initial segment complexity.

Definition 2 (Levin [70], Chaitin [16]). *A real* A *is* Levin-Chaitin random *(or Kolmogorov-Levin-Chaitin random) if there is a constant* c *such that* $K(A\restriction n) \geq n - c$ *for every* n.

Again we arrive at the same concept.

Theorem 2 (Schnorr, see Chaitin [16]). $A \in 2^\omega$ *is Martin-Löf random if and only if it is Levin-Chaitin random, i.e. there is a constant* c *such that* $K(A\restriction n) \geq n - c$ *for every* n.

The proof of Theorem 2 uses a basic tool of the area called Kraft's inequality which Chaitin observed was effective. Suppose that U is a prefix-free machine. Then the domain of U is measurable. Kraft's inequality is a kind of inverse to this.

Theorem 3 (Kraft [59] or Kraft-Chaitin). *Let $d_1, d_2, d_3, \ldots$ be a computable sequence of natural numbers, possibly with repetitions. Suppose that τ_i is a set of target strings, and we effectively enumerate the pairs (which we call Kraft-Chaitin axioms) $\{\langle d_i, \tau_i\rangle : i \in \mathbb{N}\}$. Then there is a prefix-free machine M and strings σ_i of length d_i, with $M(\sigma_i) = \tau_i$, if and only if $\sum_i 2^{-d_i} \leq 1$.*

Schnorr's Theorem, which can be interpreted to mean that *not only is a real Martin-Löf random iff its initial segment complexity is above n, but in fact its complexity will be "well above" n most of the time.*

Theorem 4 (Miller and Yu [86]).

(i) A real α is Martin-Löf random iff

$$\sum_{n\in\mathbb{N}} 2^{n-K(\alpha\restriction n)} < \infty.$$

(ii) If α is Martin-Löf random, then $\sum_{n\in\mathbb{N}} 2^{K(\alpha\restriction n)-n-K(n)} = \infty$.

A consequence of the above and some more analysis, is the following theorem. This theorem generalizes earlier results of Solovay (which only applied to computable functions f).

Theorem 5 (Miller and Yu [86]). *Suppose that f and g are arbitrary functions from $\mathbb{N}$ to $\mathbb{N}$, such that $\sum_{n\in\mathbb{N}} 2^{-f(n)} = \infty$ and $\sum_{n\in\mathbb{N}} 2^{-g(n)} < \infty$. Then*

(i) If α is random, $(\exists^\infty n)[K(\alpha \restriction n) > n + f(n)]$.
(ii) If α is random, $(\exists^\infty n)[K(\alpha \restriction n) > n + K(n) - g(n)]$.

A very longstanding question was whether there was a characterization of 1-randomness in terms of plain complexity C. It was known to Martin-Löf that if a real α had the property that

$$\exists^\infty n(C(\alpha \restriction n) > n - \mathbb{O}(1)),$$

then α was 1-random. Reals that infinitely often hit the maximal plain complexity are now called *Kolmogorov random.*[3] We know that Kolmogorov randomness and 1-randomness are different, although with probability 1, a real is 1-random

[3] There are some problems with terminology here. Kolmogorov did not actually construct or even name such reals, but he was the first more or less to define randomness for *strings* via initial segment plain complexity. The first person to actually construct what we are calling Kolmogorov random strings was Martin-Löf, whose name is already associated with 1-randomness. Schnorr was the first person to show that the notions of Kolmogorov randomness and Martin-Löf randomness were distinct. Again

iff it is Kolmogorov random. As we will later see, work of Nies, Stephan and Terwijn, and of Miller has shown that Kolmogorov randomness coincides with another natural notion of randomness, 2-randomness. In April, 2004, Joe Miller and Liang Yu finally solved the plain complexity question:

Theorem 6 (Miller and Yu [86]). *A real α is 1-random iff for all computable functions g with $\sum_{n\in\mathbb{N}} 2^{-g(n)} < \infty$ there is a constant d such that for all n,*

$$C(\alpha \upharpoonright n) > n - g(n) - d.$$

2.3 The unpredictability paradigm

Perhaps the most common intuition of randomness for a sequence such as coin tosses, is that the sequence is "unpredictable". In particular, a real $\alpha = .a_0a_1\ldots$ would be random if we could not predict any bits of it given others. One way to implement this idea is to use martingales. To formalize this intuition, Schnorr effectivized a notion of betting strategy introduced by Ville to describe Lebesgue measure.

Definition 3. *A* martingale *is a function $d : 2^{<\omega} \to \mathbb{R}^+$ that satisfies for every $\sigma \in 2^{<\omega}$ the averaging condition*

$$2d(\sigma) = d(\sigma 0) + d(\sigma 1). \tag{1}$$

Similarly, d is a supermartingale *if d satisfies*

$$2d(\sigma) \geq d(\sigma 0) + d(\sigma 1). \tag{2}$$

A (super)martingale d succeeds on *a set A if $\limsup_{n\to\infty} d(A\upharpoonright n) = \infty$. We say that d succeeds on, or* covers, *a class $\mathcal{A} \subseteq 2^\omega$ if d succeeds on every $A \in \mathcal{A}$. The* success set *$S[d]$ of d is the class of all sets on which d succeeds.*

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we can't use Schnorr randomness since Schnorr's name is associated with a randomness notion using tests of computable measure. Similar problems occur later with what we call strongly Chaitin random reals. These were never defined by Chaitin, nor constructed by him. They were first constructed by Solovay who has yet another well-known notion of randomness associated with him which is equivalent to 1-randomness. However, again Chaitin *did* look at the associated notion for *finite* strings, where he proved the fundamental lemma that $K(\sigma) \leq n + K(|\sigma|) + \mathbb{O}(1)$ which allows for the definition of the reals. It is also known that Loveland in his 1969 ACM paper proposed equivalent notions via uniform Kolmogorov complexity. Again, Loveland's name is commonly associated with yet another notion of complexity Kolmogorov-Loveland stochasticity which we look at in Section 9. Even there, there is a randomness notion Kolmogorov-Loveland randomness which is associated with Kolmogorov-Loveland nonmonotonic betting, and neither Kolmogorov nor Loveland considered such martingale-type operations, the randomness notion apparently first articulated by An. A. Muchnik, A. Semenov, and V. Uspensky in [83].

The reader should think of a martingale as a betting strategy. The function d is assigning a proportion of the capital to bet on the string σ. The success set is thus the collection of reals upon which we get to make infinite capital.

The following classical result shows how the concept of a martingale relates to measurability.

Theorem 7 (Ville [121]). *For any class $\mathcal{A} \subseteq 2^\omega$ the following statements are equivalent:*

(i) *$\mathcal{A}$ has Lebesgue measure zero,*
(ii) *There exists a martingale that succeeds on $\mathcal{A}$.*

Martingales will prove important when we look at refinements of classical Martin-Löf randomness. Martingales are the key to looking at measure and Hausdorff dimension in small classes such as polynomial time, a fact first realized by Lutz [73]. (See also Lutz [76].) Roughly speaking, the reason for this is that there is no analog of Kraft-Chaitin in such classes.

Schnorr's idea was to effectivize the notion above to look at *effective betting strategies.*

Definition 4 (Schnorr). *We will define a (super-)martingale d as being* effective*, Σ_1, or* computably enumerable *if $d(\sigma)$ is uniformly a c.e. real, and at every stage we have effective approximations to d in the sense that $d(\sigma) = \lim_s d_s(\sigma)$, with $d_s(\sigma)$ a uniformly computable increasing sequence of rationals.*

We remark that the reader might have expected that an effective martingale would be one with f a computable function rather than one with computable *approximations*. We will soon return to this very important point when we consider Schnorr and computable randomness.

Schnorr proved the following effective version of Ville's Theorem 7:

Theorem 8 (Schnorr [104], Satz 5.3). *$\mathcal{A} \subseteq 2^\omega$ is Martin-Löf null if and only if there is a Σ_1-supermartingale d such that $\mathcal{A} \subseteq S[d]$.*

Now Σ_1-supermartingales are Σ_1 objects, so it comes as no surprise that we can effectively list all of them in an enumeration $d_0, d_1, \ldots$ The sum $d = \sum_i 2^{-i} d_i$ is then again a Σ_1-supermartingale, and the success set $S[d]$ is the maximal Martin-Löf null set, so d is a *universal* Σ_1-supermartingale. Additionally, this is a *minimal* Σ_1-supermartingale, in the sense that for any other nontrivial Σ_1-supermartingale d', there is a c, such that for all σ, $cd'(\sigma) \geq d(\sigma)$. This result is also implicit in Levin's construction of a universal c.e. semimeasure (see Zvonkin and Levin [129]). We remark (see Downey and Griffiths [22]) that there is no minimal universal *martingale.*

Thus we see that 1-randomness is a mathematically robust notion in that it has 3 natural definitions. Reals which are random are effectively unpredictable, their segments are incompressible, and they avoid all effective null sets.

John Hitchcock made an interesting observation here. Computably enumerable (super-)martingales are monotonically non-decreasing in the amount you bet at any node σ. But this does not mean that the *side* the better might favour cant' change. To wit, suppose that the beginning of time we might have $f_0(\sigma) = 1$ so there is no preference. Then as we go along, we might raise $f_s(\sigma 0)$ to be above $f_s(\sigma 1)$ at some stage. Perhaps we set $f_s(\sigma 0) = 2$ whilst keeping $f_s(\sigma 1) = 1$. However, at some even later stage $t > s$ we *might* eventually raise $f_t(\sigma 1) > f_t(\sigma 0)$. For example, we might keep $f_t(\sigma 0) = 2$ but raise $f_t(\sigma 1) = 4$. Of course this has a consequential effect on a lower bound for the value of $f_t(\sigma)$, but the point is, *we can change our minds where we think we should play our capital. At some stage we might favour the left branch and at some later stage the right, or vise versa.* Suppose we define a *Hitchgale* to be a Σ_1-supermartingale f, where the *ratios* are Σ_1, so that such mind changes are now not possible. That is, we suppose that there is an additional computably enumerable function r such that for all s and all σ, and $i \in \{0, 1\}$,

$$r_s(\sigma\widehat{\ }i) = \frac{f_s(\sigma\widehat{\ }i)}{f_s(\sigma)}.$$

Note that in a Hitchgale, once we decide that we will be more than half of our capital on one side we are forced to do that henceforth. We say a real is Hitchcock random iff no Σ_1-Hitchgale succeeds upon it.

Question 1. **(Hitchcock)**[4] Is every Hitchcock random real Martin-Löf random?

Anticipating later definitions, we remark that if you replace supermartingale in the definition by martingale, then the definition coincides with computable randomness. Also Downey and Stephan have independently claimed that there are Hitchcock random reals that are not computably random, although no proofs have appeared.

3 Computably enumerable reals

In the same way that the domains of partial computable functions, the computably enumerable *sets*, are central players in classical computability theory, the *measures* of the domains of prefix-free machines are important in algorithmic information theory. These are precisely the reals α for which there exists a computable increasing sequence of rationals $\{q_i : i \in \mathbb{N}\}$ with $q_i \to \alpha$. In the literature, such reals are sometimes called *left computable*. An equivalent definition is that the left cut of α is a c.e. set of rationals.

A classical example of a computably enumerable real is the most famous explicit 1-random real, Chaitin's Ω:

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|} = \mu(\mathrm{dom}(U)),$$

where U is a universal prefix-free machine. This is called the *halting probability*.

[4] Of course, Hitchcock did not use the term "Hitchcock random," or "Hitchgale."

Here is a short proof that Ω is random. Let us use the recursion theorem to build a prefix-free machine M which is coded in U with coding constant e. Thus if we enumerate $M(\sigma) = \tau$, then in U the effect will be that U enumerates $U(1^e 0\sigma) = \tau$. At stage s, if we see $K_U(\Omega_s \upharpoonright n) < n - e$, say $U_s(\nu) = \Omega_s \upharpoonright n$ with $|\nu| < n - e$, then we will in M declare that $M_{s+1}(\nu) = \Omega_s \upharpoonright n$, causing $\Omega \upharpoonright n \neq \Omega_s \upharpoonright n$.

The analog of Ω in classical computability is the halting set $K = \{i : \varphi_i(i)\downarrow\}$. Of course we should really say that Ω_U is *a* halting probability, rather than *the* halting probability. But in classical computability we usually talk about *the* halting problem. The reason for this is that we can define an appropriate reducibility m-reducibility, and show that all halting sets are the same up to m-degree. Indeed they are all of the same 1-degree. In the next section we will show that all 1-random c.e. reals are the same up to an analytic version of m-reducibility we call Solovay reducibility.

There has been quite a bit of interest in computability theoretical aspects of computably enumerable reals. For instance, we can look at the degrees of sets A such that $\{q_i : i \in A\}$ *represents* α in the sense that $q_i \to \alpha$. It turns out that such computably enumerable A more or less correspond to (up to m-degree) splittings of the lower cut of α. We refer the reader here to Calude [10], Calude, Coles, Hertling, and Khoussainov, [12], Downey [21], Downey and Hirschfeldt [26].

Another way to generate such reals is via their *presentations*. A prefix free c.e. set A is said to *present* α if $\mu(A) = \alpha$. Downey and LaForte [35] constructed a c.e. noncomputable real with only computable presentations. Downey and LaForte also established that the degrees of presentations was essentially related to the *weak truth table* degrees below that of α. Downey and Terwijn [39] clarified this situation by showing that any appropriate "Σ^0_3 ideal" in the c.e. wtt degrees could be realized as the wtt degrees of presentations.

Authors have also studied classes more general than the c.e. reals. For instance, the field generated by the c.e. reals is in fact the collection of reals of the form $\alpha - \beta$ with α and β c.e. reals. (Ambos-Spies, Weihrauch and Zheng [7].) This field is thus called the field of d.c.e. reals. Recently the following was shown.

Theorem 9 (Ng Keng Meng [90], Raichev [100]). *The field of d.c.e. reals is real closed*[5].

The Turing degrees containing d.c.e. reals include the ω-c.e. degrees (in the sense of the Ershov Hierarchy) (Downey, Wu, Zheng [41]), as well as some other Δ^0_2 degrees, (Zheng [128]) but there are Δ^0_2 degrees with no d.c.e. reals (Downey, Wu, Zheng [41]). Finally Ho [48] observed that the reals which are the limits (not monotone) of computable sequences of rationals are exactly the Δ^0_2 reals. Zheng and Wu have explored other notions of convergence of reals. Little else is known.

[5] Actually Raichev showed something stronger. Under the rK-reducibility we later define, he showed that the collection of reals $\leq_{rK} \beta$ for any β is always real closed, and that this $\leq_{rK}$ result implies the d.c.e. one.

4 Solovay reducibility, and 1-random c.e. reals

In the last section we saw that the halting probability Ω is a c.e. real that it is that was 1-random. The analog of Ω in classical computability is the halting set $K = \{i : \varphi_i(i) \downarrow\}$. Of course we should really say that Ω_U is *a* halting probability, rather than *the* halting probability. But in classical computability we usually talk about *the* halting problem, and this situation is analogous. Of course underlying all of this is Myhill's Theorem which says that all halting problems are essentially the same. As we promised, in the present section, we will address this situation for c.e. reals and randomness.

Solovay [112] recognized the need for an analog of Myhill's Theorem for c.e. reals. He sought to introduce appropriate reducibilities to attempt to prove such an analog. Whilst Solovay's program was only recently realized by the joint work of several authors, Solovay reducibility will be our starting point.

Definition 5 (Solovay [112]). *We say that a real α is* Solovay reducible *to β (or β* dominates *α), $\alpha \leq_S \beta$ iff there is a constant c and a partial computable function f, so that for all $q \in \mathbb{Q}$, with $q < \beta$,*

$$c(\beta - q) > \alpha - f(q).$$

The intuition is that a sequence of rationals converging to β can be used to generate one converging to α at the same rate. The point is that if we have a c.e. sequence $\{q_n : n \in \omega\}$ of rationals converging to β then we know that $f(q_n) \downarrow$. Notice that if $r_n \to \alpha$ then for all m there is some k such that $\alpha > r_k > f(q_m)$. (The reals are not rational.) Noticing this yields the following characterization of Solovay reducibility.

Lemma 1 (Calude, Coles, Hertling, Khoussainov [12]). *For c.e. reals, $\alpha \leq_S \beta$ iff for all c.e. sequence or rationals $q_i \to \beta$ there exists a total computable g, and a constant c, such that, and c.e. rationals $r_i \to \alpha$, such that, for all m,*

$$c(\beta - q_m) > \alpha - r_{g(m)}.$$

Solovay [112] observed that this "analytic" version of m-reducibility was enough to establish many properties of Ω, for c.e. reals, in their sense that if $\Omega \leq_S \alpha$ then α is 1-random. The key property allowing us to see this is the following.

Lemma 2 (Solovay [112]). *$\leq_S$ is both a C- and a K- measure of relative randomness in the sense that for $I \in \{C, K\}$, we have*

$$\alpha \leq_S \beta \textit{ implies } (\exists c)(\forall n)[I(\alpha \upharpoonright n) \leq I(\beta \upharpoonright n) + c].$$

Using Kraft-Chaitin, Calude, Hertling, Khoussainov, and Wang [13] proved that if α is *Ω-like* (in the sense that $\Omega \leq_S \alpha$ and α is c.e.) then α is the halting probability of a universal prefix-free machine:

Theorem 10 (Calude, Hertling, Khoussainov, Wang [13]). *Let α be a c.e. real such that $\Omega \leq_S \alpha$. Then α is a halting probability. That is, there is a universal machine $\widehat{U}$ such that $\mu(\mathrm{dom}(\widehat{U})) = \alpha$.*

Kučera and Slaman finished the story by proving the following.

Theorem 11 (Kučera and Slaman [63]). *Suppose that α is 1-random and c.e. Then α is Solovay complete meaning that if β is a c.e. real, then $\beta \leq_S \alpha$.*

Theorem 11 gives great insight into the structure of c.e. random reals. Notice that to "qualify" as being random all we need is that the real α has $K(\alpha \upharpoonright n) \geq n - O(1)$ for all n. But of course $K(\sigma)$ can be near $|\sigma| + \log |\sigma|$. If fact we know that Ω must have oscillations above this. And, indeed by the work of Miller and Yu, all 1-random reals have such oscillations. The Kučera-Slaman theorem says that for c.e. random reals, there is only *one* and everything oscillates above n at essentially the same rate!

We were motivated to understand the structure of c.e. reals under the $\leq_S$. Naturally, the reducibility gives rise to equivalence classes, called *degrees*. Despite the many attractive features of the Solovay degrees of c.e. reals, their structure is largely unknown. Recently progress has been made.

Theorem 12 (Downey, Hirschfeldt and Nies [30]).
The Solovay degrees of c.e. reals

(i) form a distributive upper semilattice, where the operation of join is induced by $+$, arithmetic addition (or multiplication) (namely $[x] \vee [y] \equiv_S [x + y]$.)
(ii) are dense,
(iii) If $\mathbf{a}$ is incomplete and $\mathbf{b} <_S \mathbf{a}$, then there exist $\mathbf{a_1}|_\mathbf{S}\mathbf{a_2}$ such that $\mathbf{b} < \mathbf{a_1}, \mathbf{a_2}$, and $\mathbf{a} = \mathbf{a_1} \vee \mathbf{a_2}$. That is every incomplete degree splits over all lesser ones.
(iv) If $[\Omega] = \mathbf{a} \vee \mathbf{b}$ then either $[\Omega] = \mathbf{a}$ or $[\Omega] = \mathbf{b}$.

We point out that item (iv) above demonstrates that random c.e. reals are qualitatively different from all others in the sense that they cannot be split into two smaller c.e. reals. It is important to realize that this is only true of c.e. reals. The fact that $+$ induces the join operation reveals another definition of Solovay reducibility on c.e. reals.

Theorem 13 (Downey, Hirschfeldt, Nies [30]). *For c.e. reals, $\alpha \leq_S \beta$ iff there exists a c and a c.e. real γ such that*

$$c\beta = \alpha + \gamma.$$

Before we leave the Solovay degrees of c.e. reals, we note that the structure is very complicated.

Theorem 14 (Downey, Hirschfeldt, LaForte [28]). *The first-order theory of the Solovay degrees of c.e. reals is undecidable.*

The proof of Theorem 14 uses Nies' method of interpreting effectively dense boolean algebras, together with a technical construction of a certain class of (strongly) c.e. reals. Calude and Nies [14] have proven that the c.e. 1-random reals are all *wtt*-complete. Earlier Kučera [60] has used Arslanov's completeness criterion to show that all 1-random c.e. reals are Turing complete. This also follows from a result in Downey, Hirschfeldt, LaForte [27] that $\alpha \leq_S \beta$ *implies* $\alpha \leq_{wtt} \beta$. If we combine this with the result of Demuth below, and the fact that if a 1-random has c.e. degree then it is Turing complete, we see that *whilst* Ω *is wtt-complete, it is tt-incompatible with all c.e. sets.*

Theorem 15 (Demuth). *If* α *is 1-random and* $\beta \leq_{tt} \alpha$*, then there is a 1-random set* $\widehat{\beta} \equiv_T \beta$*.*

A proof of Demuth's Theorem can be found in Kautz [54], and Downey and Hirschfeldt [26]. Very little else is known about the Solovay degrees of c.e. reals.

A major philosophical question about c.e. reals is whether they are *red herrings*. The material in the present section would suggest that 1-random reals are computationally complex in the sense that, like the halting problem, they encode a lot of knowledge. Papers by Chaitin suggest that "Ω is the number of knowledge."

However, recent work will show that for more highly random reals, and indeed a "random" random real they are computationally very weak. One might think that therefore c.e. reals are red herrings in the sense that they are completely atypical.

The following result says that this is in another sense completely false.

Theorem 16 (Downey, Hirschfeldt, Miller, Nies [24]). *With probability 1, for a real* A *there is a set* B *such that* $A =^* \Omega^B$*.*

That is almost all random reals are in fact halting probabilities relative to *some* oracle! This seems very strange indeed. The answer lies in the fact that whilst Ω is a c.e. operator, it is not a CEA (computably enumerable in *and above*) one!

5 Other measures of relative randomness

A pre-ordering $\leq$ on reals is an initial segment K*-measure of relative randomness* if it satisfies the *Solovay property*:

$$\text{If } \beta \leq \alpha \text{ then } (\exists c)(\forall n)\big[K(\beta \upharpoonright n) \leq K(\alpha \upharpoonright n) + c\big].$$

Solovay reducibility is a C or K measure of relative randomness, but it is certainly not the only one. Moreover it is quite restrictive since it is too fine and uniform (as we see) and it really only makes sense for c.e. reals. It is quite easy to construct a real α and a *computable* set β with $\beta \not\leq_S \alpha$.

Downey, LaForte and Hirschfeldt [27] defined several new reducibilities which were measures of relative randomness.

Definition 6. *$\beta \leq_{sw} \alpha$ if there is a functional Γ such that $\Gamma^\alpha = \beta$ and the use of Γ is bounded by $x + c$ for some c.*

Again it is not difficult to prove that $\leq_{sw}$ is a measure of relative $C-$ or $K-$ complexity. This reducibility is quite close to one looked at by Soare, Nabutovsky and Weinberger see [109], the difference being that the constant must be 0 in their case. $\leq_{sw}$ agrees with $\leq_S$ on the c.e. *sets*. Furthermore if α is a c.e. real which is noncomputable, then there is a noncomputable strongly c.e. real $\beta \leq_{sw} \alpha$, and Downey, Hirschfeldt and LaForte [27] showed that this is *not* true in general, for $\leq_S$. By and large, however, sw-reducibility is very badly behaved, as witnessed by the next result.

Theorem 17 (Downey, Hirschfeldt, LaForte [27]).

(i) The sw-degrees of c.e. reals do not form an upper semilattice.
(ii) sw and S are incomparable in the sense that there exist c.e. reals α, β, γ, δ with $\alpha \leq_S \beta$ and $\alpha \not\leq_{sw} \beta$, and $\gamma \leq_{sw} \delta$ and $\gamma \not\leq_S \delta$.

The next theorem gives significant insight into the relative behavior of 1-random c.e. reals. It says that whilst the initial segment complexity of all random c.e. reals is the same, there is no natural uniform way to get the *bits* of one version of Ω from another.

Theorem 18 (Yu and Ding [124]). *There does not exist an sw-complete c.e. real. Thus the analog of the Kučera-Slaman theorem cannot hold for sw-reducibility.*

There is something we can say about sw-reducibility and c.e. *sets*.

Theorem 19 (Downey and Hirschfeldt [26]). *Suppose that α is a c.e. real whose initial segment complexity is (for example) ϵn for some $\epsilon > 0$ and for almost all n. Then for any c.e. set B, $B \leq_{sw} \alpha$ with use $\alpha \upharpoonright m$ for $B(m)$. Thus, not only is Ω wtt-complete, but it is sw-complete for c.e. sets.*

Both S-reducibility and sw-reducibility are uniform in a way that relative initial-segment complexity is not. Motivated by this idea, Downey, Hirschfeldt, and LaForte introduced the following.

Definition 7. *Let α and β be reals. We say that β is* relative K-reducible *(rK-reducible) to α, and write $\beta \leq_{rK} \alpha$, if there exist a partial computable binary function f and a constant k such that for each n there is a $j \leq k$ for which $f(\alpha \upharpoonright n, j)\downarrow = \beta \upharpoonright n$.*

Theorem 20 (Downey, Hirschfeldt, LaForte [27]).

(i) $\leq_{rK}$ is a K- and C- measure of relative randomness.
(ii) Let α and β be c.e. reals. If $\beta \leq_S \alpha$ or $\beta \leq_{sw} \alpha$, then $\beta \leq_{rK} \alpha$.
(iii) A c.e. real α is rK-complete if and only if it is random.
(iv) If $\beta \leq_{rK} \alpha$ then $\beta \leq_T \alpha$.

The most interesting characterization of rK-reducibility (and the reason for its name) is given by the following result, which shows that there is a very natural sense in which rK-reducibility is an exact measure of relative randomness. Recall that the prefix-free complexity $K(\tau \mid \sigma)$ of τ *relative to* σ is the length of the shortest string μ such that $M^{\sigma}(\mu) \downarrow= \tau$, where M is a fixed self-delimiting universal computer. (Similarly for C.)

Theorem 21 (Downey, Hirschfeldt, LaForte [27]). *Let α and β be reals. Then $\beta \leq_{rK} \alpha$ if and only if there is a constant c such that $K(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leq c$ for all n. (And $C(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leq c$.)*

The rK-degrees have nice structural properties.

Theorem 22 (Downey, Hirschfeldt, LaForte [27]).

(i) The rK-degrees of c.e. reals form an upper semilattice with least degree that of the computable sets and highest degree that of Ω.
(ii) The join of the rK-degrees of the c.e. reals α and β is the rK-degree of $\alpha+\beta$.
(iii) For any rK-degrees $\mathbf{a} < \mathbf{b}$ of c.e. reals there is an rK-degree $\mathbf{c}$ of c.e. reals such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.
(iv) For any rK-degrees $\mathbf{a} < \mathbf{b} < \deg_{rK}(\Omega)$ of c.e. reals, there are rK-degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ of c.e. reals such that $\mathbf{a} < \mathbf{c_0}, \mathbf{c_1} < \mathbf{b}$ and $\mathbf{c_0} \vee \mathbf{c_1} = \mathbf{b}$.
(v) $deg_{rK}(\Omega)$ is join inaccessible in the uppersemilattice of c.e. rK-degrees.

We remark that we do not know if the rK-degrees are distributive. The theories of neither the sw- nor the rK- degrees has yet been proven undecidable whilst surely this must be the case.

5.1 The basic measures $\leq_K$ and $\leq_C$

Of course, our fundamental interest will be in the measures of relative complexity *defined* by the Solovay properties. Thus we can define e.g. $\alpha \leq_K \beta$ if there is a c such that for all n,

$$K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + c.$$

Note that $\leq_K$ is not really a *reducibility*, but simply a transitive pre-ordering measuring relative complexity. This is best seen by the following result.

Theorem 23 (Yu, Ding, Downey [126]). $|\{\alpha : \alpha \leq_K \Omega\}| = 2^{\aleph_0}$.

Joe Miller has proven that the K-degree of Ω, or any other 1-random real, is always countable. (This is consistent since there is no natural join operator.) Presumably this is also true for C, but we offer the following as a question:

Question 2. Is the C-degree of each 1-random real countable?

In spite of Theorem 23 the following question remains open.

Question 3. Is there a K- or a C- degree with uncountably many members?

In Theorem 23, we can replace Ω by any 1-random. Later we will see that $\leq_K$ does not imply $\leq_T$ even on the c.e. reals. For $\leq_C$ we have the following.

Theorem 24 (Stephan [114]). *Suppose that α and β are c.e. reals such that $\alpha \leq_C \beta$. Then $\alpha \leq_T \beta$.*

Stephan's Theorem generalizes an old result of Chaitin [16] which says that the sets $\leq_C 1^\omega$ are exactly the computable sets. (which generalizes an even older result of Loveland [72]).

Before we turn to the very interesting relationship of $\leq_K$ to $\leq_T$, we look at the structure of c.e. reals under $\leq_K$.

Theorem 25 (Downey, Hirschfeldt, LaForte [27]).

(i) The K-degrees of c.e. reals form an upper-semilattice with highest degree that of Ω.
(ii) The join of the K-degrees of the c.e. reals α and β is the K-degree of $\alpha + \beta$.
(iii) For any K-degrees $\mathbf{a} < \mathbf{b}$ of c.e. reals there is an K-degree $\mathbf{c}$ of c.e. reals such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.
(iv) For any K-degrees $\mathbf{a} < \mathbf{b} < \deg_K(\Omega)$ of c.e. reals, there are K-degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ of c.e. reals such that $\mathbf{a} < \mathbf{c_0}, \mathbf{c_1} < \mathbf{b}$ and $\mathbf{c_0} \vee \mathbf{c_1} = \mathbf{b}$.
(v) $deg_K(\Omega)$ is join inaccessible in the uppersemilattice of c.e. K-degrees.

The reader will note the obvious similarities between Theorems 12, 22, and 25. Downey and Hirschfeldt [26] have proven the following generalization.

Theorem 26 (Downey and Hirschfeldt [26]). *Let $\leq_Q$ be any measure of relative randomness which is Σ^0_3 and having the properties that the 0-degree contains the computable sets, + induces the join operation, and the top degree is that of Ω. The the following hold for $\leq_Q$.*

(i) The Q-degrees of c.e. reals form an uppersemilattice with highest degree that of Ω.
(ii) For any Q-degrees $\mathbf{a} < \mathbf{b}$ of c.e. reals there is an K-degree $\mathbf{c}$ of c.e. reals such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.
(iii) For any Q-degrees $\mathbf{a} < \mathbf{b} < \deg_Q(\Omega)$ of c.e. reals, there are Q-degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ of c.e. reals such that $\mathbf{a} < \mathbf{c_0}, \mathbf{c_1} < \mathbf{b}$ and $\mathbf{c_0} \vee \mathbf{c_1} = \mathbf{b}$.
(iv) $deg_Q(\Omega)$ is join inaccessible in the uppersemilattice of c.e. Q-degrees.

At a talk by the first author in Heidelberg, in May 2003, Alexander Shen pointed out that a natural measure of relative randomness would be to replace the constant in the definition by $O(\log n)$. That is, $\alpha \leq \beta$ iff for all n, $K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + O(\log n)$. The reason he suggests this is that, since all the reducibilities are within a log factor of one another it would be independent. We have not looked at this at all, but point at the paper Chernov et al. [18]. Notice that the complexity of the reducibility remains Σ^0_3.

Remark 1 (Monotone Complexity). Another measure not yet explored is the following. Levin introduced *monotone* complexity. Here a machine M is called monotone if its action is continuous: $M(\sigma)\downarrow$ and $M(\tau)\downarrow$ and $\sigma \prec \tau$ imply that $M(\sigma) \prec M(\tau)$. We can prove a characterization of 1-randomness via monotone complexity in the same way. α is 1-random iff its monotone complexity is $n - \mathbb{O}(1)$ on $\alpha \upharpoonright n$. This is, again, a Σ^0_3 measure the top degree is that of Ω and the bottom is the computable sets. It is not very discerning for calibrating the complexity of random reals since all random reals have the same degree! However, the structure of the monotone degrees of c.e. reals is more or less open.

The measures $\leq_C$ and $\leq_K$ are pretty difficult to deal with directly. In view of Theorem 23, it is not even clear whether there are uncountably many K-degrees. This was recently solved by showing that whilst the cardinality of the set of reals K below a given one can be large, the measure is small.

Theorem 27 (Yu, Ding, and Downey [126]). $\mu(\{\alpha : \alpha \leq_K \beta\}) = 0$. *Hence there are uncountably many K-degrees of random reals.*

Using Theorem 27, Yu and Ding established the following.

Theorem 28 (Yu and Ding [125]). *There are $2^{\aleph_0}$ many K-degrees of random reals.*

In [31], it is shown that for c.e. reals we have the following.

Theorem 29 (Downey, Hirschfeldt, Nies, and Stephan [31]). *On the K- and C-degrees of c.e. reals, $+$ is a join.*

Notice that since $\leq_K$ and $\leq_C$ are Σ^0_3, this implies that the degree structures restricted to the c.e. reals form a dense uppersemilattice.

Question 4. Are there minimal pairs of $\leq_K$ degrees? Is the structure a lattice on the c.e. reals?

Question 5. Given any α is there a random β with $\alpha \leq_K \beta$?

Question 6. Are there minimal K-degrees? The answer is surely no.

A truly fundamental question is the following which remains open in spite of the best efforts of several authors.

Question 7. Do there exist *random* α, β with $\alpha <_I \beta$ with *either* $I \in \{C, K\}$?

I regard this as a *central* question. The reason is that a *negative* answer says something very significant about initial segment complexity *characterizing* randomness. I find it hard to believe that the answer is "no."

There are a number of exciting recent results on $\leq_K$ and $\leq_C$ due to Liang Yu and Joe Miller that we will discuss in the next section.

6 Relativized randomness

A classical way to calibrate the level of randomness is to increase the complexity of the tests in terms of the arithmetical hierarchy. This material will be of particular interest when we study the relationship between randomness and Turing degrees. The basic result connecting 1-randomness to Turing reducibility is the celebrated result often attributed only to Gács, but actually first proved by Kučera.

Theorem 30 (Kučera [60], Gács [44]). *Every set is wtt-reducible to a Martin-Löf random set.*

The easiest proof of Theorem 30 is the recent one of Merkle and Mihailovic [81].

Other results on the Turing degrees of 1-random sets include Kučera's result [60] that all degrees above $\mathbf{0}'$ contain 1-random reals. Since the collection of reals with initial segment complexity $n - c$ for all n is a Π^0_1 class, there are 1-random reals of low Turing degree. Kučera [60] proved that Δ^0_2 1-random degrees are not closed upwards. For the Δ^0_2 degrees they are closed upwards at least in terms of the jump operator: using a new basis theorem for Π^0_1 classes with no computable members, Downey and Miller [36] proved that for every S which is computably enumerable in, and above $\emptyset'$, there is a Δ^0_2 1-random real A with $A' \equiv_T S$. (This had earlier been claimed in a remark following a construction of a high incomplete 1-random real in each Π^0_1 class of sufficient measure, by by Kučera in [61], Theorem 12 and Remark 8.)

Kučera also observed that 1-randomness was connected to the PA-degrees, the degrees of complete extensions of Peano arithmetic. The picture was recently clarified by Frank Stephan who proved the following.

Theorem 31 (Stephan [115]). *Suppose that* $\mathbf{a}$ *is PA-complete and 1-random. Then* $\mathbf{0}' \leq_T \mathbf{a}$.

Theorem 31 establishes that there are two varieties of randoms. The first variety are those that are computationally rich and can compute the halting problem. The second are those that are computationally feeble and cannot even compute a $\{0, 1\}$ valued fixed-point free function. As we see below, this means all 2-randoms (see below) are computationally weak.

The basic definition of 1-randomness can be generalized quite easily. For our purposes, we have the following noticed by many such as Solovay, Kurtz, and Martin-Löf:

Definition 8. *(i) A* Σ^0_n–test *is a computable collection* $\{V_n : n \in \mathbb{N}\}$ *of* Σ^0_n *classes such that* $\mu(V_k) \leq 2^{-k}$.

(ii) A real α *is* Σ^0_n*-random or n-random iff it passes all* Σ^0_n *tests.*

(iii) One can similarly define Π^0_n, Δ^0_n *etc tests and randomness.*

(iv) A real α *is called* arithmetically random *iff for any* n, α *is* n*-random.*

The reader might notice the subtle difference between the notion of Σ^0_1 test and Martin-Löf test, especially in the analogs used in Definition 8. The former is defined in terms of *classes* of *reals* and the latter in terms of c.e. *open sets* of *strings*. This is immaterial for $n = 1$ since every Σ^0_1 class C is equivalent to $\{[\sigma] : \sigma \in W\}$ for some c.e. set W. However, consider the case $n = 2$. Consider the Σ^0_2 class consisting of reals that are always zero from some point onwards. This Σ^0_2 class is *not* equivalent to one of the form $\{[\sigma] : \sigma \in W\}$ for some Σ^0_2 set W.

The use of open sets is basic in most arguments. Fortunately, this can be resurrected in the $n > 1$ cases also, as we now see.

Theorem 32 (Kurtz [66], Kautz [54]). *Let $q \in \mathbb{Q}$.*

(i) For S a Σ^0_n class, we can uniformly compute the index of a $\Sigma^{\emptyset^{(n-1)}}_1$ class which is also an open Σ^0_n class $U \supseteq S$ and $\mu(U) - \mu(S) < q$.
(ii) For T a Π^0_n class T, we can uniformly compute the index of a $\Pi^{\emptyset^{(n-1)}}_1$ class which is also a closed Π^0_n class $V \subseteq T$ and $\mu(T) - \mu(V) < q$.
(iii) For each Σ^0_n class S we can uniformly in $\emptyset^{(n)}$ compute a closed Π^0_{n-1} class $V \subseteq S$ such that $\mu(S) - \mu(V) < q$. Moreover, if $\mu(S)$ is a real computable from $\emptyset^{(n-1)}$ then the index for V can be found computably from $\emptyset^{(n-1)}$.
(iv) For a Π^0_n class T we can computably from $\emptyset^{(n)}$ obtain an open Σ^0_{n-1} class $U \supseteq T$ such that $\mu(U) - \mu(T) < q$. Moreover, if $\mu(S)$ is a real computable from $\emptyset^{(n-1)}$ then the index for U can be found computably from $\emptyset^{(n-1)}$.

Using the above, we can easily show that for instance, $n + 1$-randomness coincides with 1-randomness relative to $\emptyset^{(n)}$, a theorem of Kurtz. Let U be the standard universal prefix-free machine. Then U^X will be a universal machine relative to any X. (The question of relativization in this setting can be a bit vexed, and is discussed in the next section.) We then get the following natural $n + 1$-random sets.

$$\Omega^{(n)} =_{\mathrm{def}} \sum_{U^{\emptyset^{(n)}}(\sigma)\downarrow} 2^{-|\sigma|}.$$

Becher and Santiago have looked at other natural examples of n-random sets using versions of Post's Theorem.

We return to our comment about Kolmogorov random reals. Recall that we defined a real to be Kolmogorov random iff its plain complexity hit $n - \mathbb{O}(1)$ infinitely often. Whilst we have seen in section 2.2 that no real can have its C-complexity always high, the set of Kolmogorov random reals has measure 1. The next theorem shows that *Kolmogorov randomness is characterized by 2-randomness.* Yu, Ding, and Downey [126] proved that each 3-random set is Kolmogorov random. They also observed that no Kolmogorov random set is in Δ^0_2. This is also implied by Theorem 33 since 2-random sets cannot be Δ^0_2.

Theorem 33 (Nies, Stephan, Terwijn [96]). *The following are equivalent for any set Z:*

(I) *Z is 2-random*

(II) *Z is Kolmogorov random*

The most surprising implication (I) $\Rightarrow$ (II) in Theorem 33 was proven independently and earlier by Joseph Miller [84].

Another characterization of 2-randomness can be given if we consider sets that are low for Ω:

Definition 9. *A is* low for Ω *if Ω is Martin-Löf random relative to A.*

Theorem 34 (Nies, Stephan, Terwijn [96]). *A set A is 2-random if and only if A is 1-random and low for Ω.*

Theorem 34 can also be obtained from material of van Lambalgen [67]. There is an apparently difficult open question here. By Chaitin's Counting Theorem, we know that the highest K-complexity a string can have is $|\sigma| + K(|\sigma|) - \mathbb{O}(1)$. We define a real α to be *strongly Chaitin random* iff

$$\exists^{\infty} n(K(\alpha \upharpoonright n) > n + K(n) - \mathbb{O}(1).)$$

It is known that if α is 3-random then it is strongly Chaitin random, and that every strongly Chaitin real is already 2-random. The following is open.

Question 8. Does strong Chaitin randomness coincide with Kolmogorov randomness= 2-randomness?

There is a lot of interesting and unpublished material here about n-randomness. For $n \geq 2$, n-random reals seem to be "more typically random". They all have the property that $A \oplus \emptyset^{(n-1)} \equiv_T A^{(n-1)}$, (in particular $A \oplus \emptyset' \equiv_T A'$) and hence we see that randomness is a *lowness property*.

The initial segment complexity of $n+1$-random reals and in particular $\Omega^{(n)} = \Omega^{\emptyset^{(n)}}$, the natural $n+1$-random real.

Theorem 35 (Yu, Ding, Downey [126]). *For all c and $n < m$,*

$$(\exists^{\infty} k)\big[\,K(\Omega^{(n)} \upharpoonright k) < K(\Omega^{(m)} \upharpoonright k) - c\big].$$

For $n = 0, m = 1$ Theorem 35 was proven by Solovay [112], using totally different methods.

More powerful results have recently been obtained by Miller and Yu. We need the following classical theorem of van Lambalgen. (The easy direction of the following was also proven by Kurtz.)

Theorem 36 (van Lambalgen [67]). *B n-random and A is B-n-random iff $A \oplus B$ is n-random.*

Motivated by this theorem, Miller and Yu suggested the following.

Definition 10 (Miller and Yu [86]). *We say that* $\alpha \leq_{vL} \beta$*,* α *is* van Lambalgen[6] *reducible to* β *if for all* $x \in 2^\omega$*,* $\alpha \oplus x$ *is random implies* $\beta \oplus x$ *is random.*

Theorem 37 (Miller and Yu [86]). *For all* α, β*,*

(i) α *n-random and* $\alpha \leq_{vL} \beta$ *implies* β *is n-random.*
(ii) If $\alpha \oplus \beta$ *is random then* α *and* β *have no upper bound in the vL-degrees.*
(iii) If $\alpha \leq_T \beta$ *and* α *is 1-random, then* $\beta \leq_{vL} \alpha$*.*
(iv) There are random $\alpha \equiv_{vL} \beta$ *of different Turing degrees.*
(v) There are no maximal, minimal random vL-degrees, and no join.
(vi) If $\alpha \oplus \beta$ *is random then* $\alpha \oplus \beta <_{vL} \alpha, \beta$*.*
(vii) The Σ^0_1 *theory of the vL-degrees is decidable.*

Furthermore, Miller and Yu show that $\Omega^{(n)}$ and $\Omega^{(m)}$ have no upper bound in the vL degrees for $n \neq m$. One consequence of the above of the following very attractive result.

Corollary 1 (Miller and Yu [86]). *Suppose that the real* α *is n-random and that* $\beta \leq_T \alpha$ *is 1-random. Then* β *is n-random.*

All of this is filters through an interesting relationship between $\leq_{vL}$ and $\leq_C, \leq_K$.

Lemma 3 (Miller and Yu [86]). *For random* α, β*,*

(i) Suppose that $\alpha \leq_K \beta$*. Then* $\alpha \leq_{vL} \beta$*.*
(ii) Suppose that $\alpha \leq_C \beta$*. Then* $\alpha \leq_{vL} \beta$*.*

We state the following for $\leq_K$ but they hold equally for $\leq_C$.

Corollary 2 (Miller and Yu [86]).

(i) Suppose that $\alpha \leq_K \beta$*, and* α *is n-random. Then* β *is n-random.*
(ii) If $\alpha \oplus \beta$ *is 1-random, then* $\alpha|_K\beta$ *and have no upper bound in the K-degrees.*
(iii) For all $n \neq m$*, the K-degrees of* $\Omega^{(n)}$ *and* $\Omega^{(m)}$ *have no upper bound.*
(iv) If $\alpha \oplus \beta$ *is 1-random, then* $\alpha|_K\alpha \oplus \beta$*.*

Using new techniques and extensions of the above, Miller has proven the following.

Theorem 38 (Miller [85]).

(i) If α *is 1-random and* $\alpha \leq_{rK} \beta$*, then* $\beta \leq_{rK} \alpha$*. As a consequence,* $\alpha \equiv_T \beta$*.*
(ii) If α, β *are random, and* $\alpha \equiv_K \beta$*, then* $\alpha' \equiv_{tt} \beta'$*. As a consequence, every K-degree of a random real is countable.*

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[6] This is closely related to a relation introduced by Nies: He defined $A \leq_{LR} B$ if for all Z, Z is 1-B-random implies Z is 1-A-random. If A and B are both random then $A \leq_{LR} B$ iff $B \leq_{LR} A$. Reals with $A \leq_{LR} \emptyset$ are Martin-Löf low. $\leq_{vL}$ is really only interesting on the randoms, but there are a number of possible extensions which also make sense on the non-randoms. These have yet to be explored.

(iii) If $\alpha \leq_K \beta$, and α is 3-random, the $\beta \leq_T \alpha \oplus \emptyset'$. Furthermore, $\beta' \leq_{tt} \alpha'$.

Finally Miller has shown that if α is 3-random then its initial segment complexity is often very low indeed. We will call α *pseudo-low* if $(\exists^\infty n)[K(n) \leq K^\alpha(n) + O(1)]$. Thus in a pseudo-low real, the information in α is so useless that it cannot help to compress n. The following result echoes the theme articulated by Stephan that most random reals have little *usable* information in them.

Theorem 39 (Miller [85]).

(i) If α is 3-random it is pseudo-low.
(ii) If α is pseudo-low, then α is strongly Chaitin random in that

$$(\exists^\infty n)\big[K(\alpha \restriction n) \geq n + K(n) - O(1)\big].$$

7 *K*-antirandomness, Post's problem, and generalizing Kučera-Slaman

7.1 *K*-trivial reals

There have been really amazing developments relating $\leq_K$ and $\leq_T$.

We will call a real α *K-trivial* if $\alpha \leq_K \emptyset$. Solovay was the first to construct Δ_2^0 K-trivial reals. In [31], Downey, Hirschfeldt, Nies, and Stephan gave a new construction of a K-trivial real, and this time the real was strongly c.e. (Independently, this had been done by Kummer in an unpublished manuscript.) Earlier, Calude and Coles had indicated a construction of a K-trivial real. As we will later see this is a priority-free solution to Post's problem.

Theorem 40. (Downey, Hirschfeldt, Nies, and Stephan [31], Calude and Coles [11], Kummer (unpubl.) after Solovay [112]) *There is a noncomputable c.e. set A such that $K(A \restriction n) \leq K(n) + O(1)$.*

The proof of Downey et. al. is relatively easy and looks like the Dekker deficiency set construction. One remarkable fact about the K-trivials is that for any witnessing constant there are *few* of them.

Theorem 41 (Zambella [127]). *For each d there are $O(2^d)$ many reals α such that for all n, $K(\alpha \restriction n) \leq K(n) + d$.*

We remark that K-trivial reals are necessarily Turing incomplete, and not of high degree, and hence form a somewhat natural solution to Post's Problem.

Theorem 42 (Downey, Hirschfeldt, Nies, Stephan [31]). *If a real α is K-trivial then α is Turing incomplete. Indeed, α is not even high.*

The proof of Theorem 42, and its remarkable later improvements by Nies (which we discuss below) is tricky. It involves a kind of "pinball" or "decanter" construction, with what we call "quanta pushing." I think that there should be some easy proof. For example, by the work of Chaitin we know that every K-trivial real is Δ_2^0. Moreover, for a fixed d the K-trivial reals lie on a Δ_2^0 tree with exactly $O(2^d)$ many paths by Theorem 41. It seems that some oracle type basis result might be usable here to give an analysis of such trees.

Question 9. Is there a straightforward and easy proof that K-trivials are Turing incomplete?

We remark that Downey, Hirschfeldt, Nies, and Stephan [31] proved that the K-trivials are closed under *wtt*-reduction and $+$. The former will be improved to Turing reduction soon. As Nies remarks this gives the only "natural" Σ_3^0 ideal in the Turing degrees.

Note also that, since we have seen $\leq_C$ implies $\leq_T$ for c.e. reals, $\leq_K$ does not imply $\leq_C$.

Question 10. Does $\leq_C$ imply $\leq_K$ (for the c.e. reals)? We would suggest that the answer is no.

7.2 Lowness

We have already looked at relativized randomness. M. van Lambalgen and D. Zambella asked whether there exist noncomputable sets A such that every Σ_1-random set is already Σ_1-random relative to A. (The question was first stated in Zambella [127].) Note that every computable set is low for the Σ_1-random sets. The existence of nontrivial Martin-Löf low sets is given in the following.

Theorem 43 (Kučera and Terwijn [64]). *There exists a noncomputable c.e. set that is low for the Martin-Löf random sets.*

Downey, Hirschfeldt, Nies, and Stephan [31] noted that the construction of Theorem 43 bears much resemblance to their construction of a K-trivial set. The following definitive results shows that this is no coincidence.

Theorem 44 (Nies [92]). *A is low for the Martin-Löf random sets if and only if A is K-trivial.*

Notice that this means the class of K-trivials is closed under Turing reducibility, and earlier result of Nies and Hirschfeldt. Nies also shows that this class of K-antirandom sets consists of only *superlow* sets (that is $A' \equiv_{tt} \emptyset'$). The proofs are more intricate and nonuniform variations of the original Downey, Hirschfeldt, Nies, Stephan Theorem. Again we wonder if there might be easier proofs. Nies remarks that the proofs are in some sense intrinsically nonuniform. Nies, Stephan and Terwijn have shown that any c.e. set which keeps Ω random (i.e. low for Ω) is already K-trivial. The hypothesis that A be c.e. is essential since those authors also showed that each 2-random set is low for Ω, so the class of low for Ω sets has measure 1!

We mention last lowness results here. Andrei Muchnik (1998) worked on the class which as oracles do not help to reduce the prefix complexity of a string. He showed that $\mathcal{M}$ contains a non-computable set.

Definition 11. *A is* low for K *if $\forall y\ K(y) \leq K^A(y) + \mathcal{O}(1)$.*

Theorem 45 (Nies [92], Nies and Hirschfeldt [92]). *Each K-trivial real is low for K.*

The second says that K-antirandomness is somehow tied to computable enumerability.

Theorem 46 (Nies [92]). *For each K-trivial set A, there is a c.e. K-trivial set D such that $A \leq_{tt} D$.*

8 Schnorr, computable and Kurtz randomness

Having proved that a real is Martin-Löf random iff its segments are K-incompressible iff no computably enumerable martingale succeeds on the real, Schnorr argued that the result demonstrated a clear *failure* of the intuition behind the notion of Martin-Löf randomness. He argued that randomness should be concerned with defeating *computable* strategies or computably given tests rather than computably enumerable ones, since the latter are fundamentally asymmetric, in the same way that a computably enumerable set is semi-decidable rather than decidable.

Definition 12 (Schnorr randomness, Schnorr [104]).

(i) We say that a Martin-Löf test $\{V_n : n \in \mathbb{N}\}$ is a Schnorr test *if for all n, $\mu(V_n) = 2^{-n}$. In this case we call $\bigcap_n V_n$ a* Schnorr null set.
(ii) We say that a real α is Schnorr random *iff for all Schnorr tests, $\alpha \notin \bigcap_n V_n$.*

Definition 13 (computable randomness, Schnorr [104], Lutz [74]).

(i) A martingale f is called computable iff $f : 2^{<\omega} \mapsto \mathbb{R}^+ \cup \{0\}$ is a computable function with $f(\sigma)$ (the index of functions representing the effective convergence of) a computable real. (That is, we will be given indices for a computable sequence of rationals $\{q_i : i \in \mathbb{N}\}$ so that $f(\sigma) = \lim_s q_s$ and $|f(\sigma) - q_s| < 2^{-s}$.)
(ii) A real α is called computably random *iff no computable martingale succeeds on α.*

Schnorr observed that $\mathbb{R}^+ \cup \{0\}$ could be replaced by $\mathbb{Q}^+ \cup \{0\}$ in the definition of computable randomness, and 2^{-n} by any computable bound effectively shrinking to zero in the definition of Schnorr randomness.

The notion of computable randomness has inspired a number of notions in complexity theory since it naturally allows for time bounds. We refer the reader to Lutz [73, 74], Ambos-Spies and Kučera [3], and Ambos-Spies and Mayordomo [4].

It is known that Martin-Löf random implies computably random implies Schnorr random. That there are Martin-Löf random sets that are not computably random was proved by Schnorr [104], and that there are computably random sets that are not Schnorr random was proved by Wang [122]. In some sense these concepts were not really explored much after the work of Schnorr, perhaps for various technical reasons. For instance, lacking was a machine definition of either of them. Perhaps part of the reason for this was because the Martin-Löf randomness was *enough* for many results. Another reason was that the notion is much less tractable than Martin-Löf randomness. For instance, Schnorr proved that *there is no universal Schnorr test.* The cornerstone of Martin-Löf's version of randomness is that the three characterizations (i) machine incompressibility (ii) tests and (iii) martingales, all coincide, modulo some criticisms; and hence the notion of 1-randomness is mathematically very robust.

Schnorr gave the following martingale characterization of Schnorr randomness. The crucial notion is that of an order.

Definition 14 (Schnorr [104]). *An* order *is a nondecreasing unbounded function $h : \omega \to \omega$. (N.B. An "Ordnungsfunktion" in Schnorr's terminology is always computable whereas we prefer to leave the complexity of orders unspecified in general.) For a martingale d and an order h we define*

$$S_h[d] := \{X : \limsup_{n\to\infty} \frac{d(X\restriction n)}{h(n)} \geq 1\}.$$

Theorem 47 (Schnorr [104], Satz 9.4, 9.5). *$\mathcal{A} \subseteq 2^\omega$ is Schnorr null if and only if there are computable d and h such that $\mathcal{A} \subseteq S_h[d]$.*

It had been a longstanding open question to provide a machine characterization for Schnorr randomness. Terwijn [116] had made some progress in this area. Downey and Griffiths gave the following machine characterization of Schnorr randomness:

Definition 15 (Downey and Griffiths [22]). *A prefix free machine M is called* computable *iff*

$$\mu(\mathrm{dom} M) = \sum_{\sigma\in\mathrm{dom}(M)} 2^{-|\sigma|}$$

is a computable real.

Theorem 48 (Downey and Griffiths [22]). *A real α is Schnorr random iff for all computable machines M, there is a constant c such that, for all n, $K_M(\alpha \restriction n) \geq n - c$.*

The Kučera-Slaman theorem shows that all random c.e. reals are wtt-complete, being Solovay complete. (Kučera [60] was the first to prove that they are all Turing complete.) There also is a characterization of the Turing degrees of Schnorr random reals:

Theorem 49 (Downey and Griffiths [22]). *Let A be a Schnorr random c.e. real. Then A has high Turing degree.*

Later we will see a strong converse to this result by Nies, Stephan and Terwijn. The machine characterization of Martin-Löf randomness allows us to calibrate randomness via $\leq_K$, and we know that a c.e. real α is random iff $\beta \leq_K \alpha$ for all c.e. reals β.

Inspired by this we can calibrate the complexity of c.e. reals in terms of their Schnorr complexity.

Definition 16 (Downey and Griffiths [22]). *We say that $\alpha \leq_{Sch} \beta$ iff for all* computable *machines M, there is a constant c and computable machine $\hat{M}$ such that for all n,*

$$K_{\hat{M}}(\alpha \upharpoonright n) \leq K_M(\beta \upharpoonright n) + c.$$

Recently, Downey, Griffiths, and LaForte [23] proved that Schnorr trivials are quite different from K-trivials.

Theorem 50 (Downey, Griffiths, and LaForte [23]).

(i) There exist Turing complete c.e. sets A that are Schnorr trivial.
(ii) No Schnorr trivial set is wtt-complete.
(iii) There exist nonzero c.e. degrees containing no Schnorr trivial reals.

Little is known about $\leq_{Sch}$.

Extending earlier work of Downey, Griffith and LaForte [23] and Wang [122] we have the following.

Theorem 51 (Nies, Stephan, and Terwijn [96]). *For every set A, the following are equivalent.*

(I) *A is high.*
(II) *$\exists B \equiv_T A$, B is computably random but not Martin-Löf random.*
(III) *$\exists C \equiv_T A$, C is Schnorr random but not computably random.*

Furthermore, if A is a c.e. real then B and C can also be chosen to be c.e. reals.

There is also an (ugly) measure theoretical characterization of computable randomness:

Definition 17 (Downey, Griffiths, and LaForte [23]). *A Martin-Löf test $\{V_n\}$ is* computably graded *if there exists a computable map $f : 2^{<\omega} \times \omega \to \mathbb{R}$ such that, for any $n \in \omega$, $\sigma \in 2^{<\omega}$, and any finite prefix-free set of strings $\{\sigma_i\}_{i \leq I}$ with $\cup_{i=0}^{I}[\sigma_i] \subseteq [\tau]$, the following conditions are satisfied:*

1. $\mu(V_n \cap [\sigma]) \leq f(\sigma, n)$
2. $\Sigma_{i=0}^{I} f(\sigma_i, n) \leq 2^{-n}$
3. $\Sigma_{i=0}^{I} f(\sigma_i, n) \leq f(\tau, n)$

A real x withstands a computably graded test iff $x \notin \bigcap_n V_n$. The computably graded tests give an alternative to the martingale characterization of computable randomness:

Theorem 52 (Downey, Griffiths and LaForte [23]). *A real x is computably random iff it withstands all computably graded tests.*

A similar characterization was also found by Merkle, Mihailovic, and Slaman [82].

Question 11. (i) Is there any C-characterization of either Schnorr or computable randomness?
(ii) Is there any characterization using K- alone?

There are other variations on the notion of 1-randomness. In [66], Stuart Kurtz introduced a new notion of randomness which looks at the idea from another perspective. Namely, instead of thinking of a real as random if it *avoids* all effectively given null tests, Kurtz suggested that a real should be considered as random if it *obeys* every effectively given test of measure 1.

Definition 18 (Kurtz [66]).

(i) A Kurtz (positive) test *is a c.e. open set U such that $\mu(U) = 1$.*
(ii) A real is called Kurtz random *(or weakly 1-random) if for all Kurtz tests U, $\alpha \in U$.*

Kurtz originally called this notion *weak randomness*, and it is a weak notion in that, as shown by Wang[7] [122], the notion is not *stochastic.* In relativized form, Kurtz 2-randomness means being in every Σ_2^0 class of measure 1, is *equivalent* to passing every "generalized" Martin-Löf test $\{U_n : n \in \mathbb{N}\}$, where now we only know that $\mu(U_n) \to 0$, but we do not have a computable upper bound on the measure. There is an equivalent formulation of Kurtz randomness in terms of a null test. We remark that some care is needed here since it is *not* true that Kurtz 2-randomness is equivalent to being in every $\Sigma_1^{\emptyset'}$ (open) class of measure 1. In relativized form what is true is that for $n \geq 2$, Kurtz n-randomness is equivalent to Kurtz $\Sigma_2^{\emptyset^{(n-2)}}$ randomness.

Some authors would argue that Kurtz n+1 randomness should be called *strong* Martin-Löf n-randomness, and that perhaps Kurtz (1-)randomness should not be regarded as a randomness notion at all; rather a genericity notion.

Definition 19 (Wang [122]). *A* Kurtz null test *is a collection $\{V_n : n \in \mathbb{N}\}$ of c.e. open sets, such that*

[7] Comments in both Kurtz' and Kautz' theses indicate that this fact was probably known to them, but not stated.

(i) $\mu(V_n) \leq 2^{-n}$, *and*

(ii) *There is a computable function* $f : \mathbb{N} \mapsto (2^{<\omega})^{<\omega}$ *such that* $f(n)$ *is a canonical index for a finite set of* σ*'s, say,* $\sigma_1, \ldots, \sigma_n$ *and* $V_n = \{[\sigma_1], \ldots, [\sigma_n]\}$.

Theorem 53 (Wang [122], after Kurtz [66]). *A real* α *is Kurtz random iff it passes all Kurtz null tests.*

There is a martingale definition of Kurtz randomness:

Theorem 54 (Wang [122]). *A real* α *is Kurtz random iff there is no computable martingale* F *and nondecreasing computable unbounded function* h*, such that for* almost all n,

$$F(\alpha \upharpoonright n) > h(n).$$

Thus Schnorr randomness implies Kurtz randomness. In spite of the fact that Jockusch proved that all Kurtz random reals are bi-immune, degree-theoretical they have nice behaviour. Extending an earlier theorem of Kurtz we have the following.

Theorem 55 (Downey, Griffiths, Reid [25]). *Let* $\mathbf{a} \neq \mathbf{0}$ *be a computably enumerable degree. Then* $\mathbf{a}$ *contains a Kurtz random computably enumerable real.*

Question 12. Characterize the Kurtz random degrees.

Downey, Griffiths, and Reid were able to give a nice machine characterization of Kurtz randomness in the style of Theorem 48.

Definition 20 (Downey, Griffiths, and Reid [25]). *A prefix-free machine* M *is called* computably layered *if there is a related computable function*

$$f : \omega \to (2^{<\omega})^{<\omega}$$

such that

- $\bigcup_i f(i) = \mathrm{dom} M$.
- *If* $\gamma \in f(i+1)$*, then* $(\exists \tau \in f(i))$*, such that* $M(\tau) \preceq M(\gamma)$.
- *If* $\gamma \in f(i)$*, then* $|M(\gamma)| = |\gamma| + i + 1$.

The idea of a computably layered machine is that each layer of the *domain*, $f(i)$, provides a layer of the *range*, and the range elements just become more refined as i increases.

Theorem 56 (Downey, Griffiths, and Reid [25]). α *is Kurtz random iff for all computably layered machines* M,

$$(\exists d)(\forall n)(K_M(\alpha \upharpoonright n) \geq n - d).$$

Interestingly, there is yet another machine characterization; this time in terms of computable machines.

Theorem 48 says that a real x is *Schnorr random* iff for all computable machines M, there is a constant c such that for all n, $K_M(x \upharpoonright n) \geq n - c$. Computable machines also characterize Kurtz randomness according to the following theorem:

Theorem 57 (**[Downey, Griffiths, and Reid [25]).** *α is* not *Kurtz random iff there is a computable machine M and a computable function $f : \omega \to \omega$ such that:*

$$(\forall d)(K_M(\alpha \upharpoonright f(d)) < f(d) - d).$$

There are other possible notions we could look at. For instance, we say that a real is *finitely random* if it avoids all Martin-Löf tests $\{U_n : n \in \mathbb{N}\}$ which are finite, and *computably finitely random* iff additionally there is a computable function g such that $|U_n| < g(n)$. These notions have been explored by Downey, Miller and Reimann [37]. They showed that, for instance, the Δ^0_2 reals which are finitely random are exactly the Martin-Löf random reals; something that fails in general. Also all non-low$_2$ c.e. degrees bound computably finitely random c.e. reals, which all have "array noncomputable degrees."

The point here is that one can come up with a number of meaningful notions of randomness which have interesting relationships with the degrees.

Question 13. Is there a notion of randomness capturing, say, any given double jump class?

Lowness questions are very interesting here. For a notion like Schnorr randomness there are two possible lowness notions. The first is that X is low for *tests* meaning that for any X-Schnorr test $\{U_n^X : n \in \mathbb{N}\}$, there is a Schnorr test $\{\widehat{U}_n : n \in \mathbb{N}\}$, such that $\cap_n \widehat{U}_n \supseteq \cap_n U_n^X$. The second notion is that the collection of Schnorr randoms relative to X is the same as the collection of Schnorr randoms.

Terwijn and Zambella [119] proved that there are uncountably many reals that are low for Schnorr tests and they live in the hyperimmune free degrees.

Definition 21 (**Terwijn and Zambella [119]).** *A set X is* computably traceable *if there is a computable function p (called a* bound*, such that, for all functions $g \leq_T X$, there is a computable function h where, for all n,*

(i) $|D_{h(n)}| < p(n)$,
(ii) $g(n) \in D_{h(n)}$.

Theorem 58 (**Terwijn and Zambella [119]).** *A degree* **a** *is low for Schnorr tests iff it is computably traceable.*

Here is the situation for the low-for-Schnorr-randomness notion.

Theorem 59. [Kjos-Hanssen and Nies [56], Bedregal and Nies [9]] *A degree* **a** *is low for Schnorr randoms iff it is low for Schnorr tests.*

Nies also answered the question for computable randomness.

Theorem 60 (**Nies [93]).** *A degree* **a** *is low for computably random reals iff it is computable.*

For other randomness notions the questions are still open. Downey, Griffiths and Reid [25] showed that if a degree **a** is low for Kurtz tests then it is hyperimmune-free and if the degree is computably traceable, it is low for Kurtz tests.

Question 14. (i) Suppose that **a** is low for Kurtz tests. Is **a** Schnorr low?
(ii) Is low for Kurtz tests the same as being low for Kurtz random?

Notice that such questions can be asked for many other randomness notions such as polynomial time randomness, etc.

Actually relativization here is a vexed issue since there is no oracle A where Kurtz 1-A-random implies Martin-Löf randomness. Joe Miller has suggested that the correct thing to study would be relativization only for Kurtz-n-randomness for $n \geq 2$.

9 Selection

Actually, there is a possible answer to Schnorr's criticisms of Martin-Löf randomness using computable predictions which has attracted a lot of interest recently.

Before we discuss this possible answer, we first return to von Mises vision of "selection rules." The basic idea is that a selection rule is one that given some bits of a real predicts another bit. A *selection rule* is a partial function $r : 2^{<\omega} \mapsto \{yes, no\}$. The subsequence of a sequence A selected by a selection rule r is that with $r(A \upharpoonright n) = yes$. The sequence of selected places are those i such that $r(A \upharpoonright i) = yes$. Then for a given selection rule r and a given real A, we generate a sequence $n_0, n_1, \ldots$ of selected places, and we say that a real is *stochastic* with respect to admissible selection rules iff for any such selection rule, either the sequence of selected places is finite, or

$$\lim_{n\to\infty} \frac{\{i \leq n : A(n_i) = 1\}}{n} = \frac{1}{2}.$$

We say that a real is *von-Mises-Church-Wald stochastic* iff it is stochastic for all partial computable selection rules.

Of course, if α is Martin-Löf random it is easily seen to be von-Mises-Church-Wald stochastic. The converse is well-known to fail (von Lambalgen [67]). Recently Merkle showed that being stochastic has some consequences.

Theorem 61 (Merkle [79]). *Suppose that α is von-Mises-Church-Wald stochastic. Then for all c there exist infinitely many n such that*

$$C(\alpha \upharpoonright n) > c \log n.$$

We remark that if only computable selections are allowed then the resulting stochasticity notion is called Church randomness since it follows a suggestion of Church [19]. Evidently, every computably random real is Church stochastic.

A strengthening of the notion of computable randomness is to additionally consider partial computable martingales. As best I can determine, these were first considered by Ambos-Spies and Yongge Wang. (see Ambos-Spies [2], just before the bibliography.) They are also discussed in Terwijn's Thesis [116] were they are attributed to Fortnow, Freivalds, Gasarch, Kummer, Kurtz, Smith, and Stephan [43].

Theorem 62 (Merkle [79]). *Let $\mathcal{F}$ be the collection of all computable functions $f : \mathbb{N} \mapsto \mathbb{N}$ that are nondecreasing and unbounded. Then*[8]

(i) There is a computably random real α such that for all $f \in \mathcal{F}$, and almost all n,

$$C(\alpha \upharpoonright n|n) \leq f(n).$$

(ii) There is a partially computably random real β such that for all $f \in \mathcal{F}$, and almost all n,

$$C(\beta \upharpoonright n|n) \leq f(n) \log n.$$

It follows that there is a computably random set with $C(\alpha \upharpoonright n) < 2 \log n + c$, and hence we can obtain the following result of Ambos-Spies.

Corollary 3 (Ambos-Spies [2]). *There is a computably random real that is not von-Mises-Church-Wald stochastic.*

It is also known from the thesis of von Lambalgen that there are von-Mises-Church-Wald stochastic reals that are not computably random. It is known that computable randomness imples computable stochasticity where now in the definition of von-Mises-Church-Wald stochasticity we only allow computable selections. Ambos-Spies, Mayorodomo, Wang and Zheng [5] have shown that stochasticity can be viewed as a kind of randomness for *restricted* kind of martingales.

Now we will change the rules and begin to consider nonmonotonic *selection.* We are given a real α. Now we are allowed to select the places in some nonmonotonic fashion. The result is another notion of stochasticity called Kolmogorov-Loveland stochasticity. Similarly there is a martingale-betting game associated with this notion of selection. To wit, the betting game will select a sequence $n_0, n_1, \dots$ of distinct places where we are allowed to bet on whether the place will have value 1. The place n_{i+1} depends solely upon the previous scanned bits $\alpha(n_0), \alpha(n_1) \dots \alpha(n_i)$. Thus the game is defined by *pairs* (r_i, b_i) where the game receives as information $(r_0, b_0) \dots (r_i, b_i)$ about the places r_j and bets b_j, than outputs (r_{i+1}, b_{i+1}). r_{i+1} is the next position to bet on, and b_{i+1} is the portion of the current capital we will bet. (More details can be found in Muchnik, Semenov, Uspensky [83] where this notion was first articulated, and in Merkle et. al. [80].)

The crucial point is that unlike a martingale we are not constrained to bet on position 1, then 2, then 3 etc. Here we can bet *nonmonotonically.* We call

[8] Actually, Merkle proved these for monotone complexity rather than C, which is stronger.

a real *Kolmogorov-Loveland*[9] *random* if no nonmonotonic computable betting strategy succeeds on it.

Question 15. **(An. A. Muchnik, Semenov, Uspensky [83])** Is Kolmogorov-Loveland randomness the same as Martin-Löf randomness?

We note that the answer is known for *stochasticity.*

Theorem 63 (Shen [107]). *There is a Kolmogorov-Loveland stochastic real that is not Martin-Löf random.*

If the answer to Question 15 is yes then this is a strong counter argument to Schnorr's in that it is a computable notion of randomness which has the same strength as Martin-Löf randomness. The question seems hard. Merkle et. al. have made the best progress so far on this question. They have proven that the answer is "close to" yes, at least in terms of Hausdorff dimension and density.

Theorem 64 (Merkle et. al. [80]). *For any Kolmogorov-Loveland-random sequence R and any $\delta > 0$ there is a (monotonic) computable subsequence S of R of density at least $1 - \delta$ that is 1-random.*

Theorem 65 (Merkle et. al. [80]).

(i) Under any computable splitting, one of the halves of a Kolmogorov-Loveland-random real is 1-random.
(ii) If the real is Δ^0_2, then both halves are 1-random

Merkle et. al. also show that this result does *not* imply that the real is 1-random.

Theorem 66 (Merkle et. al. [80]). *There is a non-random (not even computably random) real such that under any computable splitting both halves are 1-random.*

Theorem 67 (Merkle et. al. [80]). *All Kolmogorov-Loveland-stochastic reals α have Σ^0_1 Hausdorff dimension 1. That is,*

$$\liminf_n \frac{K(\alpha \upharpoonright n)}{n} \to 1.$$

Theorem 67 strengthens an earlier result or Muchnik [83] that a sequence cannot be Kolmogorov-Loveland-stochastic if there is some $\alpha < 1$ such that we can effectively compute infinitely many n such that the length n prefix of the sequence can be compressed (according to C or K) to a string of length $\leq \alpha n$.

We remark that there is also a lot of interest in the realtionship between these ideas and partial computable randomness. Recently Nies and Stephan

[9] We remind the reader of the earlier footnote regarding naming of terms. Perhaps this notion should be Muchnik-Semenov-Uspensky randomness or more simply non-monotonic randomness.

have proven that partial computable random reals are *not* closed under computable permutations of $\mathbb{N}$. Miller and Nies call a real permutation random iff any computaable permutation of them remains partial computably random. The question is, are they distinct from Kolmogorov-Loveland random reals?

Mostly these concepts remain to be explored as do the higher analogs and their degrees. Progress seems hard.

10 Ω^A and Post's Problem

Relativization is a strange business in the setting of, say, universal prefix-free machines. The reason for this is that we are dealing with c.e. operators, but not CEA (computably enumerable in *and above*) operators. Indeed, if A is not GL_1, then no A-random set is Turing above A (Kučera [62]). In particular, whatever Ω^Ω is, $\Omega \not\leq_T \Omega^\Omega$.

One hope that we had was that perhaps for some choice of universal machine, Ω would be a degree-invariant operator. Being degree-invariant would entail $A \equiv_T B$ implies $\Omega^A \equiv_T \Omega^B$. Should this be possible, Ω would be a degree invariant operator which was not a jump or iterate of a jump; thereby resolving the longstanding conjecture of Martin.

As we will see this program to disprove Martin's conjecture fails dismally. However, there is a lot of very interesting information we can prove about Ω as an operator.

Clearly any reasonable definition must have that Ω^X is a X-c.e. real which is X-random. Also we want it to be relatively machine independent. Thus we want that a universal machine U for which U^X is universal (i.e. minimal and prefix-free) *and has coding locations for all machines: that is, for any prefix-free* U_e^X*, there is a string* σ_e *such that* $U_e^X(\nu) = U^X(\sigma_e \nu)$, for all oracles X. *Henceforth this will always be what we mean by "universal".*

For example, we recall the theorem of Kučera and Slaman that if α is $1-$random and c.e. , then α is Solovay complete (Theorem 11). When combined with Theorem 10 we conclude that the only $1-$random c.e. reals are versions of Ω.

The following theorem shows that in spite of the fact that the original proof does not seem to relativize, there is a way of doing so.

Theorem 68 (Downey, Hirschfeldt, Miller, Nies [24]). *Suppose that* α *is* X*-c.e. and* X*-Solovay complete. Then there is a universal* U *with* $\alpha = \Omega_U^X$.

As we mentioned earlier, in Theorem 16, the range of the Ω-operator is wide indeed:

Theorem 69 (Downey, Hirschfeldt, Miller, Nies [24]). *Suppose that* α *is 2-random. Then there is a universal* U *and an oracle* X *such that* $\alpha = \Omega_U^X$.

Of course not all random reals are Ω^X for some X. Consider the real $\alpha = 1 - \Omega$.

Question 16. Suppose that $\mathbf{a} > \mathbf{0}'$. Can $\mathbf{a}$ contain a real α which is Ω^X for some X?

How far Ω fails to be degree invariant is revealed by the following theorem.

Theorem 70 (Downey, Hirschfeldt, Miller, Nies [24]). *For all universal U, there exist $A =^* B$ such that $\Omega^A|_T\Omega^B$.*

There are many other interesting results from [24], but in the interests of space, I will only mention a few more.

Theorem 71 (Downey, Hirschfeldt, Miller, Nies [24]). *Suppose that A is Δ^0_2. Then A is K-trivial iff Ω^A is a c.e. real.*

A consequence of this result is that any K-trivial real is a d.c.e. real.

The final results look at Ω as an operator acting on 2^ω.

Theorem 72 (Downey, Hirschfeldt, Miller, Nies [24]). *The following are equivalent.*

(i) A is 1-generic
(ii) Ω^A_M is continuous at A for every prefix-free oracle machine M.
(iii) Ω^A_U is continuous on A for some universal U.

As a corollary if $\Omega^A = \sup(\mathrm{ra}\ (\Omega^A_U))$, then A is 1-generic. (Here ra and sup denote the range and supremum of the relevant operators and sets.) Finally we have the following.

Theorem 73 (Downey, Hirschfeldt, Miller, Nies [24]). *If $X \in \overline{\mathrm{ra}\ (\Omega_U)} - \mathrm{ra}\ (\Omega_U)$ for some universal U, then X is 2-random.*

11 Hausdorff dimension and partial randomness

There has been a lot of very nice work on efectivizing dimension beginning with Schnorr, and more recently by Ambos-Spies and his group in Heidelberg, and Lutz' group in Iowa. Suppose that $\Omega = .a_1a_2\ldots$. Then we would expect that $\tilde{\Omega} = a_10a_20\ldots$ would be something like "$\frac{1}{2}$-random." We would like to make this more precise. A classical refinement of the notion of a measure zero set, is given by dimensions such as packing and Hausdorff dimensions. Recall from Definition 14 the null sets of the form $S_h[d]$. Schnorr also addressed null sets of the form $S_h[d]$ with $h(n) = 2^{\varepsilon n}$, $\varepsilon \in (0,1]$, of *exponential* order. Schnorr did not mention Hausdorff dimension, but one can arrive at that by looking at exponential orders.

The most well-cited approach is due to Lutz. Lutz constructivized Hausdorff dimension in [75, 76], using what he called *s-gales*.

Definition 22. *Let $s \in [0,\infty)$. An* s-gale *is a function $d : 2^{<\omega} \to \mathbb{R}^+$ that satisfies the averaging condition*

$$2^s d(w) = d(w0) + d(w1) \tag{3}$$

for every $w \in 2^{<\omega}$. Similarly, d is an s-supergale *if (3) holds with $\geq$ instead of equality. The success set $S[d]$ is defined exactly as was done for martingales in section 2.3.*

The idea behind an s-gale is that inflation is at work. In a martingale, nothing bad happens if you decide not to favour $\sigma 0$ or $\sigma 1$ with your bet at σ. However, in the case of an s-gale, not betting means that you will definitely lose money. Though the following is really a Theorem of Lutz, we will take it as our definition of effective Hausdorff dimension.

Definition 23. *For a complexity class $\mathcal{C}$, $\mathcal{A}$ has* $\mathcal{C}$-dimension *α if*

$$\alpha = \inf\{s \in \mathbb{Q} : \exists d \in \mathcal{C} \ (\ d \text{ supermartingale and } \mathcal{A} \subseteq S_{2^{(1-s)n}}[d]\ )\}.$$

For Σ_1-dimension, $\dim(\beta)$, we have the following characterization.

Theorem 74 (Lutz [75, 76]). *For a real β,*

$$dim\ (\beta) = \liminf_n \frac{I(\beta \upharpoonright n)}{n},$$

for I either C or K.

Thus, a real has Σ_1-dimension 1 iff it is somehow a smoothed version of a 1-random real. For instance, if you take a very sparse set within a 1-random or even dimension 1 real, then you can code at will in that sparse set. Thus degrees containing dimension 1 sets are closed upwards. Note that this will also mean that there are dimension 1 degrees with no random sets.

The relations between the various notions are as follows:

$$\begin{array}{ccc}
\Delta^0_2\text{-random} & & \\
\Downarrow & & \\
\text{Schnorr } \Delta^0_2\text{-random} & \Longrightarrow & \Delta^0_2\text{-dimension } 1 \\
\Downarrow & & \Downarrow \\
\Sigma_1\text{-random} & \Longrightarrow & \Sigma_1\text{-dimension } 1 \\
\Downarrow & & \\
\text{computably random} & & \Big\Downarrow \\
\Downarrow & & \\
\text{Schnorr random} & \Longrightarrow & \text{computable dimension } 1
\end{array}$$

No other implications hold than the ones indicated.

At this stage I would like to remark that using complexity theory, Lutz has shown that it is possible to give a meaningful dimension to an individual *string*!

It is possible to show that our "stretched" Ω, $\tilde{\Omega}$, has dimension $\frac{1}{2}$. In fact, Tadaki [120] used an s-variation of measure on prefix free V:

$$\mu_s(V) = \sum_{\sigma \in V} 2^{-s|\sigma|},$$

to allow him to define s-Martin-Löf randomness, and show that $\tilde{\Omega}$ is $\frac{1}{2}$-random. There is a difficulty concerning the triad or definitions for Martin-Löf randomness when s-Martin-Löf randomness is used in place of randomness. The problem is that if you put $[\sigma 0]$ into some normal test set (a c.e. open set) and then later want to put the whole cone $[\sigma]$ in, then we only need to put $[\sigma 1]$ into the test. However, when you use s-tests, this does not work. These considerations lead to a new notion of test which allows for the characterization of s-Marrtin-Löf random. We refer the reader to Calude, Staiger and Terwijn [15].

This leads to a longstanding and fundamental question.

Question 17. Suppose a real β has positive Σ_1-dimension. Does it T-bound a 1-random real?

Other notions of effective Hausdorff dimension are less well studied. Terwijn in his thesis had a lot of nice material. One case study is on Schnorr and computable dimensions. Recently, Downey and Reimann [38] proved that Schnorr dimension has a machine characterization in terms of computable machines, akin to the characterization of Schnorr randomness of Downey and Griffiths.

Authors such as Lutz have also studied *packing dimension.* Without going through the classical definition, the characterization boils down to looking at, in the Σ_1-case, $\limsup_n \frac{I(\alpha \restriction n)}{n}$. Thus limsup replaces liminf. This notion is mostly unexplored for other complexity classes, as is other dimensions such as box dimension and the like. Some of these are discussed in Jan Reimann's Thesis from Heidelberg.

Here is one example on computable packing dimension of computable enumerable *sets*. Computably enumerable sets are relatively simple in terms of their initial segment complexity. Barzdins' showed that they have complexity $\leq 2\log n + \mathbb{O}(1)$, and Kummer proved the following lovely classification result. (The "array computable degrees" are a proper subclass of the low_2 c.e. degrees.)

Theorem 75 (Kummer's Gap Theorem, Kummer [65]). *Exactly one of the following (i) or (ii) holds.*

(i) A c.e. degree $\mathbf{a}$ *contains a c.e. set* A *such that* $\exists^\infty n(C(A \restriction n) \geq 2\log n - \mathbb{O}(1))$ *iff* $\mathbf{a}$ *is array noncomputable.*
(ii) If A *is c.e. and of array computable degree, then for every unbounded, non-decreasing, total computable function* f *there is a constant* c *such that*

$$C(A \restriction n) \leq \log n + f(n) + c.$$

(iii) Hence each c.e. degree $\mathbf{a}$ *has the following gap phenomenon:*

(iiia) *Either, there is a c.e. set A in* **a** *such that $C(A \upharpoonright n) \geq 2\log n - O(1)$ for infinitely many n.*
(iiib) *Or there is* no *c.e. set $A \in$* **a** *and $\epsilon > 0$ such that $(\exists^\infty n)[C(A \upharpoonright n) \geq (1+\epsilon)\log n - \mathcal{O}(1)]$.*

Barzdins' upper bound fails dismally when you try to show its Schnorr dimension is something like $2\log n$. It is the totality of the domain of the prefix-free machine that causes problems.

Theorem 76 (Downey and Reimann [38]). *Suppose that α is a computable real < 1. Then there is a computably enumerable set whose computable (and hence Schnorr) packing dimension is above α.*

The point here is in some sense the *reason* that the Σ_1-dimension is of a c.e. set so small is because the measuring devise is simply too powerful. It is unknown if we can have a c.e. set of Schnorr packing dimension 1. It is unknown if Theorem 76 holds for Schnorr Hausdorff dimension. Merkle and Reimann have shown that it fails for computable Hausdorff dimension. Little else is known.

Degree theoretical aspects of, say, Σ_n dimension are not yet explored as are reducibilities like $\leq_C$ on them.

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