A Δ^0_2 Set with No Infinite Low Subset in Either It or Its Complement

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Abstract

We construct the set of the title, answering a question of Cholak, Jockusch, and Slaman [1], and discuss its connections with the study of the proof-theoretic strength and effective content of versions of Ramsey's Theorem.

There is a constant and fruitful interplay between the fields of computability theory and reverse mathematics. In particular, it is often the case that results in computability theory have reverse-mathematical consequences, or have proofs that can be adapted to yield results in reverse mathematics. Conversely, questions in reverse mathematics can suggest interesting problems in computability theory. Examples of these phenomena abound in the study of the proof-theoretic strength and effective content of versions of Ramsey's Theorem. (See [1] for a discussion of the history of this area, as well as several new results.) In this paper we answer a computability-theoretic question posed in [1]. As we explain below, one consequence of our answer is that a potential approach to a problem in the reverse mathematics of Ramsey's Theorem will not work.

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We assume the reader is familiar with the basics of reverse mathematics and computability theory, including the method of organizing priority constructions on a tree. Standard references are [2] and [3], respectively. (Although the motivation for our main result comes from reverse mathematics, its proof is completely computability-theoretic.) We begin with a few definitions.

Definition.

- $[X]^2 = \{Y \subset X \mid |Y| = 2\}$.
- A 2-*coloring* of $[\mathbb{N}]^2$ is a function from $[\mathbb{N}]^2$ into $\{0, 1\}$.
- A 2-coloring C of $[\mathbb{N}]^2$ is *stable* if for each $x \in \mathbb{N}$ there exists a $y \in \mathbb{N}$ and a $c < 2$ such that $C(\{x, z\}) = c$ for all $z > y$.
- A set $H \subseteq \mathbb{N}$ is *homogeneous* for a 2-coloring C of $[\mathbb{N}]^2$ if C is constant on $[H]^2$.
- RT^2_2 is the statement in the language of second order arithmetic that says that each 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.
- SRT^2_2 is the statement in the language of second order arithmetic that says that each stable 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

In [1] (Theorem 3.1), Cholak, Jockusch, and Slaman show that for each computable coloring of $[\mathbb{N}]^2$ with finitely many colors there is an infinite low_2 homogeneous set. An adaptation of the proof of this result allows them to establish reverse-mathematical results concerning RT^2_2, such as that $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2 + \mathrm{RT}^2_2$ is conservative over $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2$ for Π^1_1 statements, where RCA_0 is the standard weak base theory for reverse mathematics (consisting of the ordered semiring axioms for the natural numbers, Δ^0_1-comprehension, and Σ^0_1-induction) and $\mathrm{I}\Sigma_2$ is the Σ^0_2-induction scheme.

Since stable colorings are in principle simpler than general colorings, a natural question is whether each computable stable 2-coloring of $[\mathbb{N}]^2$ has an infinite low homogeneous set. This question is asked in [1] (Question 13.9), where it is pointed out that it is equivalent to asking whether every Δ^0_2 set has an infinite low set entirely contained in either it or its complement, and also that a positive answer might lead to a proof that SRT^2_2 is strictly weaker than RT^2_2 over RCA_0, since there exist 2-colorings of $[\mathbb{N}]^2$ without infinite low homogeneous sets. In this paper, we give a negative answer to this question. Besides its independent computability-theoretic interest, this result shows that this particular approach to separating SRT^2_2 and RT^2_2 over RCA_0 will not work.

Before we proceed with the proof of our main result, we make a few remarks.

Another question in [1] (Question 13.10) is whether every Δ^0_3 set has an infinite low$_2$ set entirely contained in either it or its complement. The proof of our main result below is clearly relativizable, and hence shows that for any set B there exists a $\Delta^{0,B}_2$ set A such that neither A nor $\overline{A}$ has an infinite subset X with $X' \leqslant_{\mathrm{T}} B'$. Letting $B = \emptyset'$, this implies that there exists a Δ^0_3 set A such that neither A nor $\overline{A}$ has an infinite subset X with $X' \leqslant_{\mathrm{T}} \emptyset''$. Since for every low$_2$ set X it is the case that $X' \leqslant_{\mathrm{T}} \emptyset''$, the relativized version of our main result gives a negative answer to the above question.

The set A that we construct is ω-c.e., that is, it has a computable approximation whose number of mind-changes is bounded by a computable function. As pointed out by Cholak (personal communication), A could not be n-c.e. for any $n \in \omega$. (See Exercise III.3.10 in [3].)

For a degree $\mathbf{a}$, $\mathbf{a} >> \mathbf{0}'$ means that there exists a 2-valued $\mathbf{a}$-computable function such that $f(e) \neq \Phi^{\emptyset'}_e(e)$ for all $e \in \omega$. In [1] (Theorem 12.5) it is shown that there is a computable 2-coloring of $[\mathbb{N}]^2$ for which every infinite homogeneous set has jump of degree $>> \mathbf{0}'$. Jockusch (personal communication) asked whether such a coloring can be stable. He pointed out that, if the diagonalization used in the proof of our main result to show that $(U \subseteq A \vee U \subseteq \overline{A}) \wedge U \in \Delta^0_2 \wedge |U| = \omega \rightarrow U' \not\equiv_{\mathrm{T}} \emptyset'$ were effective, then it should be possible to adapt our proof to yield a positive answer to this question. However, as we shall see, it is an important feature of our proof that this diagonalization is not effective, and hence the question is still open.

Our notation is for the most part standard. For finite strings σ and τ, we will say that σ is to the left of τ, and write $\sigma <_L \tau$, if there exists an $n < |\sigma|, |\tau|$ such that $\sigma(m) = \tau(m)$ for all $m < n$ and $\sigma(n) < \tau(n)$. Here $|\sigma|$ is the length of σ and $\sigma(n)$ is the nth element of σ. We will denote the concatenation of σ and τ by $\sigma^\frown\tau$. Sets of size $n \in \omega$ will be called *n-sets*.

Let $P(n)$ be a computable procedure that, given $n \in \omega$ as a parameter, lists pairs of the form $\langle e, X \rangle$, $e \in \omega$, $X \subset \omega$, $|X| < \omega$. For each $n \in \omega$, it is easy to find a total computable one-to-one function $\Phi_{j(n)}$ with the following property. Let $e = \Phi_{j(n)}(k)$ for some $k \in \omega$ and let $U \subseteq \omega$. Then $\Phi^U_e(e)\downarrow$ (or, in other words, $e \in U'$) if and only if $X \subseteq U$ for some X such that $\langle e, X \rangle$ is in the list produced by $P(n)$. In fact, the function $n \mapsto j(n)$ can be chosen to be computable. Thus, by Kleene's Recursion Theorem (Theorem II.3.1 in [3]), there is an $m \in \omega$ such that $\Phi_{j(m)} = \Phi_m$.

We think of our construction below as such a procedure $P(m)$, with m a fixed point as above. The pairs $\langle e, X \rangle$ enumerated by our construction will be called *axioms*. When we say that we choose an e for which to enumerate axioms for U' we mean that we let $e = \Phi_m(k)$ for the least k such that we have not yet chosen $\Phi_m(k)$ as a number for which to enumerate axioms. Thus we guarantee that $e \in U'$ if and only if we eventually enumerate an axiom $\langle e, X \rangle$ such that $X \subseteq U$.

Theorem. *There exists a Δ^0_2 set A such that neither A nor $\overline{A}$ has an infinite low subset.*

Proof. We need to satisfy the following requirements for all Σ^0_2 sets U and V and all partial computable binary functions Ψ and Θ:

$$\mathcal{R}_{U,\Psi} : U \subseteq A \wedge U \in \Delta^0_2 \wedge |U| = \omega \wedge \forall n(\lim_s \Psi(n, s) \text{ exists}) \to U' \not\equiv \lim_s \Psi(-, s)$$

and

$$\mathcal{S}_{V,\Theta} : V \subseteq \overline{A} \wedge V \in \Delta^0_2 \wedge |V| = \omega \wedge \forall n(\lim_s \Theta(n, s) \text{ exists}) \to V' \not\equiv \lim_s \Theta(-, s).$$

We begin by describing a strategy for satisfying a single requirement $\mathcal{R}_{U,\Psi}$; the analogous strategy could be used for satisfying a single $\mathcal{S}$-requirement. Of course, we could satisfy all $\mathcal{R}$-requirements simultaneously by letting $A = \emptyset$. However, we need a strategy that is more adaptable to the case in which strategies for satisfying $\mathcal{S}$-requirements are also present. Thus let us suppose that we begin with $A[0] = \omega$.

Our strategy begins by choosing an $e \in \omega$. Whenever a number x enters U, it enumerates the axiom $\langle e, \{x\} \rangle$ for U'. Whenever it sees that $\Psi(e, s)\downarrow= 1$ for some new number s, it puts every x for which it has enumerated an axiom $\langle e, \{x\} \rangle$ into $\overline{A}$.

Now if U is Δ^0_2 and infinite, and $\lim_s \Psi(e, s)$ exists and is not equal to 1, then eventually an axiom $\langle e, \{x\} \rangle$ for some $x \in U$ is enumerated, in which case $U'(e) = 1 \neq \lim_s \Psi(e, s)$. On the other hand, if $U \subseteq A$ and $\lim_s \Psi(e, s) = 1$ then for all axioms $\langle e, \{x\} \rangle$ that are enumerated by our strategy, x is eventually put into $\overline{A}$, which implies that $x \notin U$. Thus in this case $U'(e) = 0 \neq \lim_s \Psi(e, s)$.

The above strategy could be used for all $\mathcal{R}$-requirements simultaneously, since it only requires us to keep certain numbers out of A, and never to keep any numbers in A. However, strategies for the $\mathcal{S}$-requirements will want certain numbers to remain in A. This is the source of the conflict that must be resolved in this construction.

Let us consider then how we could satisfy two requirements of opposite kinds, $\mathcal{R}_{U,\Psi}$ and $\mathcal{S}_{V,\Theta}$, simultaneously. The basic idea is based on the observation that, if we enumerate an axiom $\langle e, X \rangle$ for U', where X is a finite set of numbers, then to guarantee that this axiom does not apply it is enough to guarantee that one of the elements of X

is not in U. Thus if $U \subseteq A$ then it is enough to guarantee that one of the elements of X is not in A.

Suppose that $\mathcal{R}_{U,\Psi}$ has been assigned stronger priority than $\mathcal{S}_{V,\Theta}$. The strategy $R_{U,\Psi}$ for satisfying $\mathcal{R}_{U,\Psi}$ acts much as before, but instead of enumerating an axiom involving a number as soon as the number enters U, it keeps two *bins* B_0 and B_1. Whenever at least one of $B_0 \cap U$ and $B_1 \cap U$ is currently empty and a number x that has not yet been put into either of the bins enters U, $R_{U,\Psi}$ puts x into B_i for the least i such that $B_i \cap U$ is currently empty, provided that x is larger than every number that had been mentioned in the construction by the last time (if any) that $R_{U,\Psi}$ put a number into $\overline{A}$. (The reason for this proviso will be explained below.) For each pair of numbers x_0, x_1 such that $x_i \in B_i$, $R_{U,\Psi}$ enumerates the axiom $\langle e, \{x_0, x_1\}\rangle$ for U'.

Whenever $R_{U,\Psi}$ sees that $\Psi(e,s)\!\downarrow= 1$ for some new number s, it proceeds as follows. For each 2-set $X = \{x_0, x_1\}$ such that $x_i \in B_i$ and it has enumerated an axiom $\langle e, X\rangle$, $R_{U,\Psi}$ *claims* the elements of X and puts each x_i into $\overline{A}$ and into a *claim set* C_i. (We will say that the elements of X are *simultaneously claimed* by $R_{U,\Psi}$.) It then puts into a *neutral set* N every number that is not in either C_i and is less than or equal to the largest number seen in the construction so far. Finally, it empties B_0 and B_1.

Note that any number put into either of the bins from this point on will be larger than all numbers currently in $C_0 \cup C_1 \cup N$. This guarantees that C_0, C_1, and N are pairwise disjoint, and that, once a number enters $C_0 \cup C_1 \cup N$, it is never again claimed by $R_{U,\Psi}$.

Now $R_{U,\Psi}$ has two possible outcomes. Its finitary outcome is that it puts numbers into $\overline{A}$ only finitely often, its infinitary outcome that it does so infinitely often. By the same reasoning as before, if the finitary outcome is the correct one then $\mathcal{R}_{U,\Psi}$ is satisfied, while if the infinitary outcome is the correct one then $\mathcal{R}_{U,\Psi}$ is satisfied provided that, for each pair of numbers simultaneously claimed by $R_{U,\Psi}$, at least one of the numbers in the pair is eventually permanently in $\overline{A}$.

If $R_{U,\Psi}$ has finitary outcome then it does not care which numbers are in A, so the strategy $S^1_{V,\Theta}$ for satisfying $\mathcal{S}_{V,\Theta}$ below the finitary outcome of $R_{U,\Psi}$ has no real problems, and can act much as in the one-strategy case. Each time $R_{U,\Psi}$ puts numbers into $\overline{A}$, $S^1_{V,\Theta}$ is initialized, which means the following. First $S^1_{V,\Theta}$ relinquishes all its current claims. For any number x whose claim is thus relinquished, if x is also claimed by $R_{U,\Psi}$ then it is put into $\overline{A}$. Then $S^1_{V,\Theta}$ picks a new number e^1 for which to enumerate axioms, and makes sure that these new axioms only involve numbers greater than any number mentioned so far in the construction. Thus if the finitary outcome of $R_{U,\Psi}$ is indeed the correct one then $S^1_{V,\Theta}$ is eventually allowed to act as if $R_{U,\Psi}$ did not exist, while otherwise it has no permanent effect on which numbers are in A.

The strategy $S^0_{V,\Theta}$ for satisfying $\mathcal{S}_{V,\Theta}$ below the infinitary outcome of $R_{U,\Psi}$ must be more careful. It can put numbers into A, but it must do so in a way that does not injure $R_{U,\Psi}$. The key observation here is that if $R_{U,\Psi}$ has infinitary outcome then $C_0 \cup C_1 \cup N = \omega$, and hence if V is infinite then at least one of $V \cap (C_0 \cup N)$ and $V \cap (C_1 \cup N)$ is infinite. If $S^0_{V,\Theta}$ knew which of these sets is infinite then it could proceed as in the one-strategy case, but using only elements from that set. This would guarantee that, for some $i < 2$, each $x \in C_i$ would never be put back into A after being put into $\overline{A}$ by $R_{U,\Psi}$, which would be enough to guarantee the integrity of $R_{U,\Psi}$.

However, $S^0_{V,\Theta}$ does not have access to this information, so it must adopt a strategy that works in either case. To do this, $S^0_{V,\Theta}$ works with two numbers e^0_0 and e^0_1. For each $i = 0, 1$, whenever a number x enters $V \cap (C_i \cup N)$, if x is larger than every number that had been mentioned by the last time (if any) that $S^0_{V,\Theta}$ claimed a number then $S^0_{V,\Theta}$ enumerates the axiom $\langle e^0_i, \{x\}\rangle$ for V'. Whenever $S^0_{V,\Theta}$ sees that $\Theta(e^0_1, s)\downarrow= 1$ for some new number s, it claims every x for which it has enumerated an axiom $\langle e^0_1, \{x\}\rangle$ and puts it into A. Whenever $S^0_{V,\Theta}$ sees that $\Theta(e^0_0, s)\downarrow= 1$ for some new number s, it proceeds as follows. First it relinquishes all its claims on elements of C_1. For any such number x, if $R_{U,\Psi}$ has a claim on x then x is put into $\overline{A}$. Then $S^0_{V,\Theta}$ claims every x for which it has enumerated an axiom $\langle e^0_0, \{x\}\rangle$ and puts it into A. Finally, it selects a new value for e^0_1. (Note that it is the last permanent claim on x that decides whether $x \in A$.)

Now there are three possibilities. First suppose that $\Theta(e^0_0, s)\downarrow= 1$ for infinitely many s. Then $\lim_s \Theta(e^0_0, s) = 1$, if this limit exists. Furthermore, for every axiom $\langle e^0_0, \{x\}\rangle$ enumerated by $S^0_{V,\Theta}$, x is eventually put into A by $S^0_{V,\Theta}$, and it never leaves A after that. (Since x is already in $C_0 \cup N$ when it is claimed by $S^0_{V,\Theta}$, it is never later claimed by $R_{U,\Psi}$.) This means that if $V \subseteq \overline{A}$ then $V'(e^0_0) = 0 \neq \lim_s \Theta(e^0_0, s)$ (if this limit exists), and hence $\mathcal{S}_{V,\Theta}$ is satisfied.

Now suppose that the first case above does not hold, which implies that e^0_1 has a final value, and suppose further that, for this final value, $\Theta(e^0_1, s)\downarrow= 1$ for infinitely many s. Then $\lim_s \Theta(e^0_1, s) = 1$, if this limit exists. Furthermore, for every axiom $\langle e^0_1, \{x\}\rangle$ enumerated by $S^0_{V,\Theta}$, x is eventually put into A by $S^0_{V,\Theta}$, and it never leaves A after that, for the same reason as above. This means that if $V \subseteq \overline{A}$ then $V'(e^0_1) = 0 \neq \lim_s \Theta(e^0_1, s)$ (if this limit exists), and hence $\mathcal{S}_{V,\Theta}$ is satisfied.

Finally, suppose that neither of the cases above holds. If V is not Δ^0_2 or is finite then $\mathcal{S}_{V,\Theta}$ is vacuously satisfied, so suppose that V is Δ^0_2 and infinite. Then, for each $i = 0, 1$

and the final value of e_i^0, $\lim_s \Theta(e_i^0, s) \neq 1$, if this limit exists. On the other hand, the fact that $C_0 \cup C_1 \cup N = \omega$ implies that, for some $i = 0, 1$, $V \cap (C_i \cup N)$ is Δ_2^0 and infinite. It is easy to check that, for some number $x \in V \cap (C_i \cup N)$ and the final value of e_i^0, the axiom $\langle e_i^0, \{x\} \rangle$ is enumerated by $S^0_{V,\Theta}$. Since $x \in V$, $V'(e_i^0) = 1 \neq \lim_s \Theta(e_i^0, s)$ (if this limit exists), and hence $\mathcal{S}_{V,\Theta}$ is satisfied.

In any case, we claim that the action of $S^0_{V,\Theta}$ does not injure $R_{U,\Psi}$. To see this, suppose that, at some point in the construction, $R_{U,\Psi}$ simultaneously claims a pair of numbers x_0 and x_1, putting them in C_0 and C_1, respectively. If the value of e_1^0 ever gets changed after this point then any claim $S^0_{V,\Theta}$ may have made on x_1 is relinquished and never reinstated, since $S^0_{V,\Theta}$ never enumerates an axiom $\langle e_1^0, \{x_1\} \rangle$ for the new value of e_1^0. Otherwise, $S^0_{V,\Theta}$ never claims x_0. In any case, for every pair of numbers simultaneously claimed by $R_{U,\Psi}$, at least one of the numbers in the pair is not permanently claimed by $S^0_{V,\Theta}$.

So we see that $S^0_{V,\Theta}$ succeeds in satisfying $\mathcal{S}_{V,\Theta}$ while allowing $R_{U,\Psi}$ to succeed in satisfying $\mathcal{R}_{U,\Psi}$.

Now let us consider the case of a single strategy $R_{U,\Psi}$ with a finite number of levels of S-strategies below it. $R_{U,\Psi}$ can act much as before but with a bin B_α and a corresponding claim set C_α for each $\alpha \in 2^n$, where n is such that, for every strategy $S^\sigma_{V,\Theta}$ below $R_{U,\Psi}$, $|\sigma| < n$. Each axiom enumerated by $R_{U,\Psi}$ is then of the form $\langle e, \{x_0, \dots, x_{2^n-1}\} \rangle$, with each x_i in a different B_α, and $R_{U,\Psi}$ claims 2^n many numbers at a time, it being enough for $R_{U,\Psi}$ to succeed that at least one element of each such 2^n-set be kept out of A.

Instead of taking numbers from $C_0 \cup N$ and $C_1 \cup N$ as before, a strategy $S^\sigma_{V,\Theta}$ below the infinitary outcome of $R_{U,\Psi}$ takes elements from the two sets $P_0 = \bigcup_{\alpha(|\sigma|)=0} C_\alpha \cup N$ and $P_1 = \bigcup_{\alpha(|\sigma|)=1} C_\alpha \cup N$. As before, we can guarantee that, for every 2^n-set X of numbers simultaneously claimed by $R_{U,\Psi}$, there is an $i = 0, 1$ such that the elements of $P_i \cap X$ are not permanently claimed by $S^\sigma_{V,\Theta}$, thus ensuring that, for some $\alpha \in 2^n$, the element of $C_\alpha \cap X$ is not permanently claimed by any S-strategy. (The mechanism of initialization can be used to ensure that, for each 2^n-set X of numbers simultaneously claimed by $R_{U,\Psi}$, at most one S-strategy at any given level can permanently claim elements of X.)

The situation for $R_{U,\Psi}$ with infinitely many S-strategies below it is not much different. It keeps bins B_α and corresponding claim sets for each $\alpha \in 2^{<\omega}$. At any point in the construction, which bins $R_{U,\Psi}$ fills is determined by the number of times it has had infinitary outcome. That is, if $R_{U,\Psi}$ has had infinitary outcome n many times then it fills the bins B_α for $\alpha \in 2^{n+1}$. If $R_{U,\Psi}$ has infinitary outcome once again then it stops filling these bins and starts filling the bins B_α for $\alpha \in 2^{n+2}$.

If $R_{U,\Psi}$ has infinitary outcome only finitely often then it eventually acts no differently from an R-strategy with finitely many levels of S-strategies below it. Since the S-strategies below the finitary outcome of $R_{U,\Psi}$ do not care what $R_{U,\Psi}$ does, they can succeed.

On the other hand, if $R_{U,\Psi}$ has infinitary outcome infinitely often then it succeeds as before. Furthermore, for any $n \in \omega$, every sufficiently large number is put into $\bigcup_{|\alpha|>n} C_\alpha \cup N$. Thus a strategy $S^\sigma_{V,\Theta}$ below the infinitary outcome of $R_{U,\Psi}$ can take elements from the two sets $P_0 = \bigcup_{|\alpha|>|\sigma| \wedge \alpha(|\sigma|)=0} C_\alpha \cup N$ and $P_1 = \bigcup_{|\alpha|>|\sigma| \wedge \alpha(|\sigma|)=1} C_\alpha \cup N$ and succeed as before.

Now let us consider how an S-strategy $S^\sigma_{V,\Theta}$ with several R-strategies above it can act. Let k be the number of R-strategies with infinitary outcome in σ. Instead of working with two numbers e^σ_0 and e^σ_1 as before, $S^\sigma_{V,\Theta}$ works with a number e^σ_γ for each $\gamma \in 2^k$.

More specifically, let $\tau_0, \ldots, \tau_{k-1}$ be a list in order of length of all strings contained in σ such that $R^{\tau_i}_{U,\Psi}$ is a strategy with infinitary outcome above $S^\sigma_{V,\Theta}$. Associated with each $R^{\tau_i}_{U,\Psi}$ are sets $C^{\tau_i}_\alpha$ and N^{τ_i} as described above. For each $\gamma \in 2^k$, let P_γ be the set of all $x \in \omega$ such that, for each $i < k$, either $x \in C^{\tau_i}_\alpha$ for some α such that $|\alpha| > |\sigma|$ and $\alpha(|\sigma|) = \gamma(i)$ or $x \in N^{\tau_i}$. It will also be convenient to have a claim set $C^{\sigma,\gamma}$ for each $\gamma \in 2^k$.

Whenever a number x enters $V \cap P_\gamma$, if x is larger than every number that had been mentioned by the last time (if any) that $S^\sigma_{V,\Theta}$ claimed a number then $S^\sigma_{V,\Theta}$ enumerates the axiom $\langle e^\sigma_\gamma, \{x\}\rangle$ for V'. Whenever $S^\sigma_{V,\Theta}$ sees that $\Theta(e^\sigma_\gamma, s)\downarrow = 1$ for some new number s, it acts as follows. First it relinquishes all claims on elements of $C^{\sigma,\delta}$, $\delta >_L \gamma$. For any such number x, if there is a current claim on x and the latest such claim is by an R-strategy then x is put into $\overline{A}$; otherwise, it is put into A. Then $S^\sigma_{V,\Theta}$ claims every x for which it has enumerated an axiom $\langle e^\sigma_\gamma, \{x\}\rangle$ and puts it into A and into $C^{\sigma,\gamma}$. Finally, it selects a new value for each e^σ_δ, $\delta >_L \gamma$.

It is now possible to argue much as before that $S^\sigma_{V,\Theta}$ can succeed in satisfying $\mathcal{S}_{V,\Theta}$ while at the same time ensuring the integrity of each R-strategy above it.

Of course, in general, a given strategy will have several strategies of the opposite kind above it and infinitely many such strategies below it, and thus will have to combine the two aspects of the construction described above, working with multiple e_γ's and also filling bins and keeping claim and neutral sets (one collection of such bins and sets for each γ) instead of directly claiming elements as they arrive.

This brings us to our full construction. We establish a priority ordering among our requirements and place strategies for them, along with these strategies' possible outcomes, on a tree in the usual manner. A strategy $R^\sigma_{U,\Psi}$ for $\mathcal{R}_{U,\Psi}$ (resp. $S^\sigma_{V,\Theta}$ for $\mathcal{S}_{V,\Theta}$) has as possible outcomes f (for *finite*) and each binary string of length equal to the number k_σ of S-strategies (resp. R-strategies) with outcome other than f in σ, with f being the rightmost outcome and the other outcomes, called the infinitary outcomes, ordered left to right by $<_L$.

Associated with $R^\sigma_{U,\Psi}$ or $S^\sigma_{V,\Theta}$ are sets $B^{\sigma,\gamma}_\alpha$, $C^{\sigma,\gamma}_\alpha$, and $N^{\sigma,\gamma}$, $\gamma \in 2^{k_\sigma}$, $\alpha \in 2^{<\omega}$, as explained above. Each of these sets begins the construction empty, as does A.

We run our construction on the tree of strategies as usual, with the strategies acting as described below. At any stage at which a strategy acts, all strategies to its right are initialized, which means that all of the sets associated with these strategies are emptied and all of their claims relinquished. For any number x that has a claim on it relinquished, if there is a current claim on x and the latest such claim is by an R-strategy then x is put into $\overline{A}$; otherwise, it is put into A.

We describe the action of a strategy $R^\sigma_{U,\Psi}$. A strategy $S^\sigma_{V,\Theta}$ would of course act in the same way, *mutatis mutandis*. For any set Y that is approximated during the construction, $Y[s]$ will denote the (finite) set of numbers that are in Y at the beginning of stage s. If, on the other hand, we wish to consider the numbers that are in Y at a specific point in the construction, we will speak of numbers currently in Y.

When it is first active, and following each initialization, $R^\sigma_{U,\Psi}$ chooses a new e^σ_γ for each $\gamma \in 2^{k_\sigma}$. At a stage s at which it is accessible (which will be called a *σ-stage*), $R^\sigma_{U,\Psi}$ acts as follows. Let t be the last stage at which $R^\sigma_{U,\Psi}$ had infinitary outcome, if such a stage exists, and let $t = 0$ otherwise.

Case 1. If for each $\gamma \in 2^{k_\sigma}$ and each $u \in [t, s]$ it is the case that either $\Psi(e^\sigma_\gamma, u)[s]\uparrow$ or $\Psi(e^\sigma_\gamma, u)[s]\downarrow\neq 1$ then let n be one more than the number of times that $R^\sigma_{U,\Psi}$ has had an infinitary outcome before stage s. Let $\tau_0, \dots, \tau_{k_\sigma - 1}$ be a list in order of length of all strings contained in σ such that $S^{\tau_i}_{V,\Theta}$ is a strategy with infinitary outcome δ_i above $R^\sigma_{U,\Psi}$. For each $\gamma \in 2^{k_\sigma}$, let $P_{\gamma,s}$ be the set of all $x \in \omega$ such that, for each $i < k_\sigma$, either $x \in C^{\tau_i,\delta_i}_\beta[s]$ for some β such that $|\beta| > |\sigma|$ and $\beta(|\sigma|) = \gamma(i)$ or $x \in N^{\tau_i,\delta_i}[s]$.

For each $x \in U[s]$, taken in increasing order, if

1. x is greater than $|\sigma|$ and every number mentioned in the construction by the end of the last stage (if any) at which the construction was to the left of the f outcome of $R^\sigma_{U,\Psi}$,

2. x is not in any $B^{\sigma,\gamma}_{\alpha}[s]$, $\gamma \in 2^{k_\sigma}$, $\alpha \in 2^n$, and

3. $x \in P_{\gamma,s}$ for some $\gamma \in 2^{k_\sigma}$ for which $B^{\sigma,\gamma}_{\alpha} \cap U$ is currently empty for some $\alpha \in 2^n$,

then $R^{\sigma}_{U,\Psi}$ puts x into $B^{\sigma,\gamma}_{\alpha}$ for the leftmost $\gamma^\frown\alpha$, $\gamma \in 2^{k_\sigma}$, $\alpha \in 2^n$, such that $x \in P_{\gamma,s}$ and $B^{\sigma,\gamma}_{\alpha} \cap U$ is currently empty.

For each $\gamma \in 2^{k_\sigma}$ and each 2^n-set X such that each element of X is currently in some $B^{\sigma,\gamma}_{\alpha}$, $\alpha \in 2^n$, with no two elements in the same $B^{\sigma,\gamma}_{\alpha}$, $R^{\sigma}_{U,\Psi}$ enumerates the axiom $\langle e^{\sigma}_{\gamma}, X\rangle$ for U'.

Finally, $R^{\sigma}_{U,\Psi}$ ends its stage s action with outcome f.

Case 2. If case 1 does not hold then let $\gamma \in 2^{k_\sigma}$ be leftmost such that $\Psi(e^{\sigma}_{\gamma}, u)[s]\downarrow= 1$ for some $u \in [t, s]$. For each $\delta >_L \gamma$, $R^{\sigma}_{U,\Psi}$ acts as follows. First it removes all elements from each $C^{\sigma,\delta}_{\alpha}$, $\alpha \in 2^{<\omega}$. For any such element x, $R^{\sigma}_{U,\Psi}$ relinquishes its claim on x. If there is a current claim on x and the latest such claim is by an R-strategy then x is put into $\overline{A}$; otherwise, it is put into A. Then $R^{\sigma}_{U,\Psi}$ chooses a new e^{σ}_{δ}.

For each set X such that $\langle e^{\sigma}_{\gamma}, X\rangle$ has been enumerated by $R^{\sigma}_{U,\Psi}$ and each $x \in X$, if x is in $B^{\sigma,\gamma}_{\alpha}[s]$, $\alpha \in 2^{<\omega}$, then $R^{\sigma}_{U,\Psi}$ claims x and puts it into $C^{\sigma,\gamma}_{\alpha}$ and into $\overline{A}$. (As before, we say that the elements of X are simultaneously claimed by $R^{\sigma}_{U,\Psi}$.) Then $R^{\sigma}_{U,\Psi}$ puts into $N^{\sigma,\gamma}$ every number that is not currently in some $C^{\sigma,\gamma}_{\alpha}$, $\alpha \in 2^{<\omega}$, and is less than or equal to the greatest number seen in the construction so far but larger than every number mentioned in the construction by the end of the last stage (if any) at which the construction was to the left of the γ outcome of $R^{\sigma}_{U,\Psi}$.

Finally, $R^{\sigma}_{U,\Psi}$ empties each $B^{\sigma,\delta}_{\alpha}$, $\delta \geqslant_L \gamma$, $\alpha \in 2^{<\omega}$, and ends its stage s action with outcome γ.

This completes the construction; we now verify its correctness. As usual, the true path of the construction is the leftmost path visited infinitely often. We say that a strategy $R^{\sigma}_{U,\Psi}$ or $S^{\sigma}_{V,\Theta}$ is at *level* $|\sigma|$ of the tree of strategies.

We begin by showing that A is Δ^0_2. The following auxiliary lemma will also be useful later on.

Lemma 1. *Let $x \in \omega$. Only finitely many strategies can ever claim x, and each such strategy can claim it only finitely often.*

Proof. A strategy $R^{\sigma}_{U,\Psi}$ or $S^{\sigma}_{V,\Theta}$ will not claim x unless it first puts it into one of its bins $B^{\sigma,\gamma}_{\alpha}$, and this will not happen if $|\sigma| \geqslant x$. Since there are only finitely many strategies at each level of the tree of strategies, this proves the first part of the lemma. We now prove the second part.

Since a strategy can only make claims at stages at which it is accessible, each strategy to the left of the true path makes only finitely many claims. When a strategy is initialized at a stage s, all its bins are emptied and, henceforth, all numbers put into these bins, and thus all numbers claimed by the strategy after stage s, are greater than every number mentioned in the construction by stage s. Thus each strategy to the right of the true path can claim x only finitely often.

Now suppose that $R^{\sigma}_{U,\Psi}$ on the true path claims x at a stage s and let its outcome at that stage be γ. (The argument for S-strategies is the same.) Notice that $R^{\sigma}_{U,\Psi}$ cannot claim x at a stage t at which its outcome is γ unless $x \in B^{\sigma,\gamma}_{\alpha}[t]$ for some $\alpha \in 2^{<\omega}$.

At stage s, each $B^{\sigma,\gamma}_{\alpha}$ is emptied. From that point on, no number that has appeared in the construction by stage u can enter any $B^{\sigma,\gamma}_{\alpha}$, which means that x is never again in any $B^{\sigma,\gamma}_{\alpha}$, and hence is never again claimed at a stage at which $R^{\sigma}_{U,\Psi}$ has outcome γ. Thus $R^{\sigma}_{U,\Psi}$ can claim x at most once for each of its finitely many outcomes. $\square$

Since the only times a number is put into A or $\overline{A}$ are when it is claimed and when a claim on it is relinquished, we may conclude the following.

Corollary 2. *A is Δ^0_2.*

Notice that the proof of Lemma 1 implies that A is in fact ω-c.e..

Now let $U \subseteq A$ be Δ^0_2 and infinite and let $R^{\sigma}_{U,\Psi}$ be the strategy for $\mathcal{R}_{U,\Psi}$ on the true path of the construction. We will argue that, whatever its true outcome, $R^{\sigma}_{U,\Psi}$ succeeds in satisfying $\mathcal{R}_{U,\Psi}$. (The argument for S-strategies is of course symmetric.)

Lemma 3. *If the true outcome of $R^{\sigma}_{U,\Psi}$ is f then $\mathcal{R}^{\sigma}_{U,\Psi}$ is satisfied.*

Proof. The assumption that the true outcome of $R^{\sigma}_{U,\Psi}$ is f implies that, for all $\gamma \in 2^{k_\sigma}$, e^{σ}_{γ} has a final value for which $\lim_s \Psi(e^{\sigma}_{\gamma}, s) \neq 1$. Thus to show that $\mathcal{R}_{U,\Psi}$ is satisfied it is enough to show that, for some $\gamma \in 2^{k_\sigma}$ and the final value of e^{σ}_{γ}, $U'(e^{\sigma}_{\gamma}) = 1$.

Let n be one more than the number of times that $R^{\sigma}_{U,\Psi}$ has an infinitary outcome. Let t be the last stage at which the construction is to the left of the f outcome of $\mathcal{R}^{\sigma}_{U,\Psi}$, if such a stage exists, and let $t = 0$ otherwise. Let $\tau_0, \dots, \tau_{k_\sigma - 1}$ be a list in order of length of all strings contained in σ such that $S^{\tau_i}_{V,\Theta}$ is a strategy with infinitary outcome δ_i above $R^{\sigma}_{U,\Psi}$. For each $\gamma \in 2^{k_\sigma}$, let P_γ be the set of all $x \in \omega$ such that, for each $i < k_\sigma$, either $x \in C^{\tau_i,\delta_i}_{\beta}$ for some β such that $|\beta| > |\sigma|$ and $\beta(|\sigma|) = \gamma(i)$ or $x \in N^{\tau_i,\delta_i}$. (So P_γ is the limit of the sets $P_{\gamma,s}$ mentioned in the description of the action of $R^{\sigma}_{U,\Psi}$.)

For each $i < k_\sigma$, every sufficiently large number is either in $C^{\tau_i,\delta_i}_{\beta}$ for some β such that $|\beta| > |\sigma|$ or in N^{τ_i,δ_i}, so every such number is in P_γ for some $\gamma \in 2^{k_\sigma}$. Furthermore, it is not hard to check that each P_γ is Δ^0_2. Thus the assumption that U is Δ^0_2 and infinite implies that, for some $\gamma \in 2^{k_\sigma}$, $U \cap P_\gamma$ is Δ^0_2 and infinite. We claim that this in turn implies that there is a σ-stage $s > t$ such that every $(B^{\sigma,\gamma}_\alpha \cap U)[s]$, $\alpha \in 2^n$, contains a number x_α that does not leave U after stage s. Before verifying this claim, we note that this suffices to establish the lemma. Indeed, assume the claim is valid. Let $X = \{x_\alpha \mid \alpha \in 2^n\}$. At stage s, $R^\sigma_{U,\Psi}$ enumerates the axiom $\langle e^\sigma_\gamma, X\rangle$. Since $X \subset U$, this means that $U'(e^\sigma_\gamma) = 1$.

So we are left with justifying the claim made in the previous paragraph. If $t > 0$ then let m be the least number not mentioned in the construction by the end of stage t, and otherwise let $m = 0$. If an element enters some $B^{\sigma,\delta}_\alpha \cap U$, $\alpha \in 2^n$, $\delta \in 2^{k_\sigma}$, and never leaves it, then, from that point on, no new elements can enter $B^{\sigma,\delta}_\alpha \cap U$, so there is a stage $u > t$ after which every element entering some $B^{\sigma,\delta}_\alpha \cap U$ eventually leaves it.

Since $U \cap P_\gamma$ is Δ^0_2 and infinite, and, by the definition of t, no elements ever leave any of the $B^{\sigma,\delta}_\alpha$ after stage t, this implies that there exists a σ-stage $v > u$ and an $x \in (U \cap P_\gamma)[v]$, $x \geqslant m$, such that x does not leave $U \cap P_\gamma$ during or after stage v and, for all $w \geqslant v$, x is the least element of $(U \cap P_\gamma)[w]$ that is greater than or equal to m and is not in any of the $B^{\sigma,\delta}_\alpha[w]$, $\alpha \in 2^{<\omega}$, $\delta \in 2^{k_\sigma}$.

Let $\alpha \in 2^n$. If $(B^{\sigma,\gamma}_\alpha \cap U)[w]$ is empty for some σ-stage $w \geqslant v$ then x is put into some $B^{\sigma,\delta}_\beta$ at stage w, contradicting the choice of x, so $(B^{\sigma,\gamma}_\alpha \cap U)[w]$ is nonempty for every σ-stage $w \geqslant v$. But no new elements are added to $B^{\sigma,\gamma}_\alpha$ at a stage w unless w is a σ-stage and $(B^{\sigma,\gamma}_\alpha \cap U)[w]$ is empty, so this means that, for every σ-stage $w \geqslant v$, $(B^{\sigma,\gamma}_\alpha \cap U)[w]$ contains one of the finitely many elements that entered $B^{\sigma,\gamma}_\alpha$ before stage v. Since $B^{\sigma,\gamma}_\alpha \cap U$ is Δ^0_2, this means that there exists an $s_\alpha > t$ and an $x_\alpha \in (B^{\sigma,\gamma}_\alpha \cap U)[s_\alpha]$ such that x_α does not leave U after stage s_α. Letting $s = \max\{s_\alpha \mid \alpha \in 2^n\}$ completes the proof. □

Lemma 4. *If the true outcome of $R^\sigma_{U,\Psi}$ is infinitary then $\mathcal{R}^\sigma_{U,\Psi}$ is satisfied.*

Proof. Let γ be the true outcome of $R^\sigma_{U,\Psi}$. We need to show that, for the final value of e^σ_γ (which exists) and every axiom $\langle e^\sigma_\gamma, X\rangle$ enumerated by $R^\sigma_{U,\Psi}$, at least one element of X is not in A. It will then follow that $U'(e^\sigma_\gamma) = 0$, while the assumption that the true outcome of $R^\sigma_{U,\Psi}$ is γ implies that $\lim_s \Psi(e^\sigma_\gamma, s) = 1$, if this limit exists.

It is enough to show that, if s is a stage by which the construction has stopped moving to the left of the γ outcome of $R^\sigma_{U,\Psi}$, $R^\sigma_{U,\Psi}$ has outcome γ at stage s, and $R^\sigma_{U,\Psi}$ simultaneously claims the elements of a set X at stage s, then at least one $x \in X$ is not permanently claimed by any S-strategy during or after stage s. By Lemma 1, this implies that there is a stage after which no S-strategy ever has a claim on x. Since $R^\sigma_{U,\Psi}$'s claim on x is never relinquished, this means that $x \in \overline{A}$.

If either $S^\tau_{V,\Theta}$ is to the left of the γ outcome of $R^\sigma_{U,\Psi}$ or $R^\sigma_{U,\Psi}$ is below the f outcome of $S^\tau_{V,\Theta}$ then $S^\tau_{V,\Theta}$ makes no claims after stage s. If $S^\tau_{V,\Theta}$ is to the right of the true path then it makes no permanent claims.

If $R^\sigma_{U,\Psi}$ is below an infinitary outcome δ of $S^\tau_{V,\Theta}$ then any number claimed by $R^\sigma_{U,\Psi}$ at stage s must be in $(C^{\tau,\delta}_\alpha \cup N^{\tau,\delta})[s]$, so if $S^\tau_{V,\Theta}$ claims a number during or after stage s then this number must be greater than all numbers claimed by $R^\sigma_{U,\Psi}$ at stage s, and hence greater than the elements of X.

Thus it is enough to consider the S-strategies below the γ outcome of $R^\sigma_{U,\Psi}$. Let n be such that $|X| = 2^n$. Note that each element of X is in a different $C^{\sigma,\gamma}_\beta$, $\beta \in 2^n$.

Suppose that $S^\tau_{V,\Theta}$ is below the γ outcome of $R^\sigma_{U,\Psi}$ and let i be such that $R^\sigma_{U,\Psi}$ is the $(i+1)$th R-strategy with infinitary outcome above $S^\tau_{V,\Theta}$. If $|\tau| \geqslant n$ then $S^\tau_{V,\Theta}$ can never claim an element of X, so assume that $|\tau| < n$.

If $S^\tau_{V,\Theta}$ permanently claims $x \in X$ at some stage $t \geqslant s$ then let δ be the outcome of $S^\tau_{V,\Theta}$ at stage t. For each possible outcome $\varepsilon \geqslant_L \delta$ of $S^\tau_{V,\Theta}$, each $B^{\tau,\varepsilon}_\alpha$ is emptied at stage t, from which it follows that no element of X is ever claimed by $S^\tau_{V,\Theta}$ at a stage after stage t at which its outcome is ε. Furthermore, $S^\tau_{V,\Theta}$ can never have outcome to the left of δ after stage t, since otherwise its claim on x would be relinquished. Thus we conclude that the only stage greater than or equal to s at which $S^\tau_{V,\Theta}$ permanently claims elements of X is t, and hence every element of X permanently claimed by $S^\tau_{V,\Theta}$ is in $C^{\sigma,\gamma}_\beta$ for some $\beta \in 2^n$ such that $\beta(|\tau|) = \delta(i)$.

But if the construction ever moves to the left of $S^\tau_{V,\Theta}$ after stage s then $S^\tau_{V,\Theta}$ cannot permanently claim any elements of X, so for each $m < n$ there is at most one S-strategy at level m that permanently claims elements of X after stage s. Thus for each $m < n$ there is a $j_m \in \{0, 1\}$ such that every element of X permanently claimed by a level m S-strategy is in $C^{\sigma,\gamma}_\beta$ for some $\beta \in 2^n$ such that $\beta(m) = j_m$. Let $\alpha \in 2^n$ be such that $\alpha(m) = 1 - j_m$ for all $m < n$. Then the element of X in $C^{\sigma,\gamma}_\alpha$ is not permanently claimed by any S-strategy during or after stage s. $\square$

Of course, Lemmas 3 and 4 also hold for S-strategies, with symmetric proofs. Combining these with Corollary 2 yields the theorem. $\blacksquare$

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