COMPUTABILITY-THEORETIC AND PROOF-THEORETIC ASPECTS OF PARTIAL AND LINEAR ORDERINGS

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Abstract. Szpilrajn's Theorem states that any partial order $\mathcal{P} = \langle S, <_P \rangle$ has a linear extension $\mathcal{L} = \langle S, <_L \rangle$. This is a central result in the theory of partial orderings, allowing one to define, for instance, the dimension of a partial ordering. It is now natural to ask questions like "Does a well-partial ordering always have a well-ordered linear extension?" Variations of Szpilrajn's Theorem state, for various (but not for all) linear order types τ, that if $\mathcal{P}$ does not contain a subchain of order type τ, then we can choose $\mathcal{L}$ so that $\mathcal{L}$ also does not contain a subchain of order type τ. In particular, a well-partial ordering always has a well-ordered extension.

We show that several effective versions of variations of Szpilrajn's Theorem fail, and use this to narrow down their proof-theoretic strength in the spirit of reverse mathematics.

1. Introduction

The results of the present paper come from a fruitful interaction of combinatorics and logic. The context of the investigations is the attempt to understand the *effective* content and *proof-theoretical* content of classical mathematics. Before we

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discuss our results and their ramifications in detail, we give a brief outline of these two programs.

The study of the effective content of mathematics is that part of the work of mathematical logic that seeks to understand and classify the underlying algorithmics inherent in mathematics. Up until the beginning of the 20th century, virtually all mathematics was algorithmic, in the sense that if one claimed that a certain object existed, one gave a computable procedure to generate the object. It was Hilbert in his famous twenty-three problems who asked, essentially, if one could build a machine to generate all the theorems of, for example, Peano arithmetic. This consideration gave rise to Gödel's powerful incompleteness theorems and, indirectly, to computer science through the work of Turing, von Neumann, and others. Another classical example of such questions was Dehn's [De12] word, conjugacy, and isomorphism problems in finitely presented group theory, which led to the formation of the subject of combinatorial group theory.

In such studies one asks questions like "If one is (computably) *given* a structure $\mathcal{A}$, does this guarantee that one can effectively generate a related structure $\mathcal{B}$ of a particular kind?" A pretty example can be obtained from the work of Rabin, Fröhlich and Shepherdson, and Metakides and Nerode. Rabin [Ra60] demonstrated that if one is computably given a field $\langle F, \times, +, ^{-1}, 0, 1\rangle$ (so that F is a computable set coded by the natural numbers, upon which the normal field operations are computable) then one can effectively find a computable algebraic closure. Fröhlich and Shepherdson [FS56] showed that one can be given two computable algebraic closures of the same computable field which are not computably the same. This is interesting because the *usual* method of generating algebraic closures is to adjoin roots, and this necessarily specifies a unique computable closure. So, in particular, *Rabin's theorem must use a different method of constructing algebraic closures.* Metakides and Nerode [MN79] explained the phenomenon by proving that a computable field has a computably unique computable algebraic closure iff it has a *(separable) splitting algorithm*, which means, roughly speaking, that a computable field has a computably unique computable algebraic closure iff one can decide if a given polynomial over the field is irreducible, and hence perform the usual root adjoining process computably.

What does all this tell us? Firstly, we see that there is a demonstrably different way of constructing algebraic closures. This is typical: Clarifying levels of effectiveness involves far greater algebraic or analytic understanding of the structures under consideration. Secondly, we obtain a precise algorithmic equivalence between two processes: adjoining roots and constructing isomorphisms. So, aside from the intrinsic logical interest, we obtain significant insight into the classical algebra. We refer the reader to the *Handbook of Recursive Mathematics* [EGNR98] for more details.

Hand in hand with the above line of research is the attempt to understand the proof-theoretical strength of theorems of classical mathematics. One program here is the "reverse mathematics" of Friedman and Simpson. The idea is to ask whether, given a theorem, one can prove its equivalence to some axiomatic system, with the aim of determining what proof-theoretical resources are necessary for the theorems of mathematics. (A very old example of this line of investigation is Euclid's question of the necessity of the parallel axiom.)

One modern incarnation of this type of analysis comes from the fragment of mathematics living in second-order arithmetic. Second-order arithmetic is a system strong enough to encompass most of classical mathematics. Its underlying language is a two-sorted one with variables for numbers $(x, y, z, ...)$ and for sets of numbers $(X, Y, ...)$ with the usual logical connectives and quantifiers, together with the normal Peano axioms for number variables (e.g. $n+1 = m+1 \to m = n$), the induction scheme

$$(0 \in X \wedge \forall x(x \in X \to x+1 \in X)) \to \forall n(n \in X),$$

plus what are called *comprehension schemes*, which assert, roughly speaking, that if we specify an object X by a formula φ of a particular type, with X not occurring freely in φ, then the object exists. More formally, a comprehension scheme for a class of formulas is the collection of axioms stating that

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

for each formula φ in the given class such that X does not occur freely in φ.

The fundamental idea of reverse mathematics is to calibrate the proof-theoretical strength of a classical theorem by classifying how much comprehension is needed to establish the existence of the structures needed to prove the theorem. That is, we "reverse" the theorem to derive some sort of comprehension scheme. The calibrating measure is that of the allowable "logical complexity" of the φ's. Typically, this complexity might be the allowable quantifier depth and type of definition of φ when defined over some quantifier-free formula, although other measures are used as we will see below.

More precisely, a formula φ is called Σ^0_0, or Π^0_0, if it has no unbounded quantifiers. For example, the formula asserting the fact that "x is prime", i. e.,

$$\text{Prime}(x) \equiv \forall y \leq x \forall z \leq x(y \cdot z = x \to y = 1 \vee z = 1),$$

is an example of a Σ^0_0 formula. We now adjoin unbounded quantifiers and measure the complexity according to the number of alternations of quantifiers: we say that a formula φ is Σ^0_{n+1} iff there is a Π^0_n formula ψ such that $\varphi(x)$ holds iff $\exists y \psi(x, y)$ and similarly we define Π^0_{n+1} with the roles of Σ and Π reversed. Finally, a formula that is both Σ^0_n and Π^0_n is called Δ^0_n. The superscript "0" refers to the fact that there are no *set* quantifiers. Saying that a function is continuous is Π^0_3 with the normal ϵ-δ definition. If a formula is Σ^0_n or Π^0_n for some n we say that it is *arithmetical.* We obtain a similar hierarchy if we allow set quantification by putting a superscript "1" and measuring the number of alternations of set quantifiers over an arithmetical matrix. Thus, for instance, a formula φ of the form $\exists X \forall Y \psi(X, Y, n)$ with ψ arithmetical is said to be Σ^1_2, since it begins existentially and has one alternation of set quantifiers.

After all these definitions we can formulate the reverse mathematics program initiated by H. Friedman and Simpson. The goal of this program is to find the minimal "set-theoretical" axioms needed to prove theorems in "ordinary mathematics" by finding "set-theoretical" axioms in second-order arithmetic which not only prove the theorem in "ordinary mathematics" but such that the theorem can also prove the axioms (over some weaker axiom system, typically the axiom system RCA_0 of recursive (i. e., Δ^0_1-)comprehension, together with the basic axioms of a discretely ordered semiring, as above, and Σ^0_1-induction, as we define below.) In his address to the International Congress of Mathematicians, Friedman [Fr74] identified five systems of second-order arithmetic specified by starting with the basic axioms of a discretely ordered semiring and the Σ^0_1-induction scheme

$$(\varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1))) \to \forall n(\varphi(n))$$

where φ is Σ^0_1, and then classifying the types of *allowable comprehension.* The base system is called RCA_0 and allows only Δ^0_1 comprehension. The next system, WKL_0, includes the base system plus the comprehension scheme which says that every infinite binary tree has an infinite branch. The third system, ACA_0, allows for comprehension of sets described by arithmetic formulas. The fourth, ATR_0, is slightly technical to state, but is equivalent to the statement that any two countable well-orderings are comparable. And, finally, there is the system called Π^1_1-CA_0 which allows comprehension over Π^1_1 formulas. (Naturally, there are other possible systems such as Π^1_2-CA_0, which will, in fact, be relevant to our studies here.)

An important fact is that virtually all of classical "non-set-theoretical" mathematics can be carried out in Π^1_1-CA_0. It is a remarkable fact that almost all of the classical theorems of mathematics are equivalent to one of the five comprehension schemes above. Pursuing our field example, we note that Friedman, Simpson, and Smith [FSS83] re-interpreted and extended the computability results of Rabin-Fröhlich-Shepherdson-Metakides-Nerode mentioned above to show that the statement "Every countable field has an algebraic closure" is provable in RCA_0, whereas the uniqueness of the closure is equivalent to WKL_0.

The existence of a prime ideal in a countable commutative ring with 1 is equivalent to WKL_0, whereas the existence of a maximal ideal is equivalent to ACA_0 (again meaning that *another* construction of the prime ideal needs to be used than the usual one, which first uses Zorn's Lemma to construct a maximal ideal and then argues that it is prime). Finally, the existence of an Ulm resolution for a reduced abelian p-group is equivalent to ATR_0, and the existence of a decomposition of a countable abelian group into a maximal divisible subgroup and a reduced group is equivalent to Π^1_1-CA_0. We refer the reader to Simpson [Si98] for more details.

Again we can ask "What is the point of all this?" At one level, we can mention the greater insight one obtains from calibrating the precise resources needed to prove a theorem. We can, in some sense, quantify the intuition that some theorems are "harder" then others. A beautiful example of this is the work on "fast growing Ramsey functions". One result in this area is the celebrated Paris-Harrington version of the finite Ramsey Theorem [PH77], which is not provable in Peano Arithmetic. Here one shows that the theorem is equivalent to ACA_0, and hence *although the theorem is concerned only with finite sets, any proof must nevertheless use infinite sets.* An even more striking example of this phenomenon is the work of Friedman, Robertson, and Seymour [FRS87], who proved that the Graph Minor Theorem (even for graphs of bounded tree-width) is *not* provable in Π^1_1-CA_0 and hence the very complicated iterated minimal bad sequence arguments are, in some sense, necessary. (Actually, we remark that the original proof of the Graph Minor Theorem for graphs of bounded tree-width used Friedman's version of Kruskal's theorem with the "gap condition", which was specifically designed to construct stronger incompletenesses in Peano Arithmetic as part of the reverse mathematical program, so that the metamathematical considerations had a huge classical spin-off!)

Another use of reverse mathematics for classical mathematics is in *showing that reasonable classifications are not possible, or at least determining the level that any classification must have.* To make this more precise, let us turn to the area of this paper, in which we will be analyzing extensions of partial orderings to linear ones. Already, we know that this area should be full of metamathematical complexities because of the work of Slaman and Woodin [SW98]. They answered a question of Łoś, who had asked for *a classification of those partial orderings with a dense linear extension.* They showed that the collection is not Borel, that is, not Π^1_1, and hence admits *no reasonable classification.*

For the logician, we remark that from a model-theoretic point of view, partial and linear orderings are badly behaved: The existence of an infinite chain necessarily implies instability of the first-order theory. From a computability-theoretic point of view, however, partial and linear orderings are very interesting as they allow a wide variety of codings (see, e. g., Downey [Do98]).

In this paper, we prove some computability-theoretic results, as well as some corollaries for reverse mathematics, on partial and linear orderings.

The starting point of our investigations is

Szpilrajn's Theorem (Szpilrajn [Sz30]). *Any partial order $\mathcal{P} = \langle S, <_P \rangle$ has a linear extension $\mathcal{L} = \langle S, <_L \rangle$.*

Given a property P of partial orderings, it is natural to ask whether $\mathcal{P}$ satisfying property P implies that $\mathcal{L}$ can be chosen to satisfy property P as well. Call a linear order type τ *extendible*[1] if any partial order $\mathcal{P} = \langle S, <_P \rangle$ which does not contain a subchain of order type τ has a linear extension $\mathcal{L} = \langle S, <_L \rangle$ which also does not contain a subchain of order type τ.

The extendibility of various linear order types was studied extensively by Bonnet, Corominas, Fraïssé, Jullien, and Pouzet in France, as well as independently by Galvin, Kostinsky, and McKenzie in the United States. A complete characterization of the countable extendible linear order types was obtained by Bonnet [Bo69]. In his thesis [Ju69], Jullien obtained a characterization of all countable *weakly extendible* linear order types τ (i. e., those τ such that any *countable* partial ordering $\mathcal{P} = \langle S, <_P \rangle$ not containing a chain of order type τ can be extended to a linear ordering $\mathcal{L} = \langle S, <_L \rangle$ not containing a chain of order type τ). (Interestingly enough, there are indeed countable linear order types which are weakly extendible but not extendible, e. g., $\omega + 1$.)

The easiest example of an infinite extendible linear order type is ω^* (i. e., the order type of ω under the reverse ordering). This is simply to say that any well-founded partial ordering can be extended to a well-ordering. (Of course, by symmetry, this is equivalent to saying that ω is extendible.)

By Bonnet [Bo69], and independently by Galvin and McKenzie (unpublished), the countable dense linear order type without endpoints, η, is also extendible: If we call a partial ordering without a densely ordered linear chain *scattered*, then "η is extendible" simply means that any scattered partial ordering can be extended to a scattered linear ordering. (We refer the reader to the survey papers by Bonnet and Pouzet [BP82] and by Downey [Do98] for more background on linear extensions of partial orderings, and on computability-theoretic aspects of linear orderings, respectively.)

[1] In the literature, this is sometimes referred to as "enforceable". Perhaps a better name might be "omittable".

In the present paper, we analyze the extendibility of the order types ω^*, η, and ζ (the order type of the integers) in computability-theoretic and proof-theoretic terms. In particular, we study the extendibility of these three order types along the lines of the program of *reverse mathematics* as outlined above. The axiom systems we will use here are WKL_0, ACA_0, ATR_0, and Π^1_2-CA_0.

We note that there is a strong connection between results in reverse mathematics and effective versions of classical theorems. E. g., loosely speaking, a classical theorem is provable in RCA_0 alone iff the classical theorem holds effectively. Since, in the below, we show that the extendibility of the three linear order types ω^*, η, and ζ fails effectively in a very strong sense, we establish lower bounds for the proof-theoretic strength of their extendibility.

Consider first the extendibility of ω^*, i. e., the fact that, classically, any well-founded partial ordering has a well-ordered linear extension. Surprisingly, this result holds effectively if the partial ordering is assumed to be *classically well-founded* (i. e., there is no infinite descending sequence) by Rosenstein and Kierstead, but *not* if the partial ordering is only assumed to be *computably well-founded* (i. e., there is no *computable* infinite descending sequence) by Rosenstein and Statman. On the other hand, by Rosenstein, any computable, computably well-founded partial ordering has a computably well-ordered linear extension which is computable in $\mathbf{0}'$, the Turing degree of the halting problem. (See Rosenstein [Ro84] for all these results, and Rosenstein [Ro82] for more background.)

We sharpen these results as follows:

Theorem 1.

(1) *"ω^* is extendible" is provable in ACA_0.*
(2) *"ω^* is extendible" proves WKL_0 over RCA_0.*
(3) *"ω^* is extendible" is not provable in WKL_0.*

The exact proof-theoretic strength of the extendibility of ω^* thus remains open. We remark that the *particular proof* of the extendibility of ω^* from ACA_0 which we present below reverses to ACA_0.

Consider next the extendibility of η, i. e., the fact that, classically, any scattered partial ordering has a scattered linear extension. By Rosenstein [Ro82, Ro84], any computable, computably scattered partial ordering has a computably scattered linear extension which is computable in $\mathbf{0}'$.

Theorem 2.

(1) *"η is extendible" is provable in Π^1_2-CA_0.*
(2) *"η is extendible" is not provable in WKL_0.*

In particular, our proof of Theorem 2 answers long-standing open questions from Rosenstein [Ro84] by the following

Theorem 2A. *There is a classically scattered, computable partial ordering such that every computable linear extension has a computable densely ordered subchain.*

However, the exact proof-theoretic strength of the extendibility of η is still wide open. We remark that the extendibility of η remains one of the very few candidates for a theorem equivalent to Π^1_2-CA_0 (and hence not provable in Π^1_1-CA_0), which might explain why its proof is strangely difficult!

We finally classify precisely the proof-theoretic strength of the extendibility of ζ, and add to the small collection of classical theorems equivalent to ATR_0 over RCA_0:

Theorem 3. *"ζ is extendible" is equivalent to ATR_0 over RCA_0.*

The rest of the paper is devoted to the proofs of these theorems. Henceforth, we assume that the reader is familiar with the rudiments of reverse mathematics, referring to Simpson [Si98] where necessary, and assume that the reader is familiar with the rudiments of computability theory, as found in an initial segment of Soare [So86] or Rogers [Ro67].

2. The proof of Theorem 1

To prove part (1) of Theorem 1, simply observe that the proof of Kierstead and Rosenstein [Ro84] (see also [Do98, p. 909]) can be used: Fix a partial ordering $\mathcal{P} = \langle N, <_P \rangle$. (Note here that we may assume without loss of generality that the universe of $\mathcal{P}$ is N, the set of integers in the sense of the model $\mathcal{N}$ of ACA_0.) We define a linear extension $\mathcal{L} = \langle N, <_L \rangle$ by approximations $\mathcal{L}_s = \mathcal{L} \upharpoonright [0, s]$ as follows: At stage 0, set $L_0 = \{0\}$. At stage $s+1$, fix the $<_L$-least element $a \in L_s$ such that $s+1 <_P a$ and let $s+1$ be the immediate $\mathcal{L}_{s+1}$-predecessor of a. (If a fails to exist, make $s+1$ the greatest element of $\mathcal{L}_{s+1}$. So the idea is to insert $s+1$ into $\mathcal{L}_s$ at the rightmost place consistent with $<_P$.)

Now suppose $\{a_s\}_{s \in N}$ is a $<_L$-descending sequence (coded in the model $\mathcal{N}$). By Ramsey's Theorem, we may assume without loss of generality that $\{a_s \mid s \in \mathcal{N}\}$ is a $<_P$-antichain. (Note that by Jockusch [Jo72], Ramsey's Theorem can be used inside the model $\mathcal{N}$.) We may also assume that $\{a_s\}_{s \in N}$ is $<$-increasing (in the usual ordering of N). We can now construct (inside the model $\mathcal{N}$) a subsequence $\{a_{j_t}\}_{t \in N}$ and a $<_P$-descending sequence $\{b_t\}_{t \in N}$ with $a_{j_{t+1}} <_P b_t < a_{j_t}$ for all t as follows: Set $a_{j_0} = a_0$. Given t, and since the a's form a $<_P$-antichain, for each $s > j_t$ there is a ($<$-least) element d_s with $a_s <_P d_s <_L a_{j_t}$. By the construction (since we always place elements "rightmost" in $\mathcal{L}$), we have $d_s < a_{j_t}$. Among these d_s, fix the d $<$-least such that $d = d_s$ for $\mathcal{N}$-infinitely many s, and thin out the sequence of a's to only contain a_s with $d_s = d$ for the definition of $b_{t'}$ $(t' > t)$. We then set $b_t = d$ and $a_{j_{t+1}} =$ some $a_s <_P d$. Using the minimality in the choice of d, we can then argue that $\{b_t\}_{t \in N}$ is a $<_P$-descending sequence, contradicting the well-foundedness of $\mathcal{P}$.

To prove part (2) of Theorem 1, note that the proof of Rosenstein and Statman [Ro84] (see also [Do98, p. 910]) can be adapted: Fix an infinite tree $T \subseteq 2^\omega$ (coded in the model $\mathcal{N}$ of $RCA_0 +$ "ω^* is extendible"), and view it as partially ordered by reverse inclusion (which we denote by $<_P$). It now suffices to show that any linear extension $<_L$ of $<_P$ has an infinite $<_L$-descending chain, since then "ω^* extendible" implies that T has an infinite $<_P$-descending chain, i. e., an infinite path.

So fix an arbitrary linear extension $<_L$ of $<_P$. We define an infinite $<_L$-descending sequence $\{c_s\}_{s \in N}$ as follows: Let c_0 be the root of T. Given s, let $C_s = \{c_0, \ldots, c_s\}$, and let D_s the set of immediate T-successors (i. e., immediate $<_P$-predecessors) of elements of C_s. Then we let c_{s+1} be the $<_L$-maximal element of $D_s - C_s$. It is now easy to check that the maximality condition in the choice of c_s ensures that they form a $<_L$-descending sequence as desired.

To prove part (3) of Theorem 1, we will show that, given a sequence $X_0 \leq_T X_1 \leq_T \ldots$ of uniformly low, uniformly Δ^0_2-sets, there is a computable partial ordering $\mathcal{P} = \langle S, <_P \rangle$ such that, for any infinite $<_P$-descending sequence $\{c_n\}_{n \in N}$, there is some i such that $X_i \oplus \{c_n\}_{n \in N}$ can compute the halting set K; and such that for any i, any X_i-computable linear extension $<_L$ of $<_P$ contains an infinite $<_L$-

descending chain Turing computable in $<_L$. Since, by Jockusch and Soare [JS72] and Simpson [Si98, Theorem VIII.2], there is a model of WKL_0 whose second-order part consists of all sets in the Turing ideal generated by a sequence $X_0 \leq_T X_1 \leq_T \dots$ of uniformly low, uniformly Δ^0_2-sets (which thus in particular does not contain the halting set K), this implies that WKL_0 does not imply the extendibility of ω^*.

The construction of $\mathcal{P}$ is a finite-injury priority argument. We construct $\mathcal{P}$ as the disjoint union of sub-partial orderings $\mathcal{P}_{i,e}$ (for $e, i \in \omega$) such that each $\mathcal{P}_{i,e}$ is a connected component of $\mathcal{P}$ (when viewed as a directed graph). Each $\mathcal{P}_{i,e}$ will be devoted to showing that if the eth binary X_i-computable relation $\mathcal{L}_{i,e} = L_e^{X_i}$ is a linear extension of $<_P$ then it has an infinite descending sequence inside $P_{i,e}$ (the domain of $\mathcal{P}_{i,e}$) which is computable in $\mathcal{L}_{i,e}$. At the same time, we have to show that any infinite $<_P$-descending sequence inside $P_{i,e}$ can compute the halting set K. (Note here that any $<_P$-descending sequence in $\mathcal{P}$ must be completely contained in a single $P_{i,e}$.)

We can thus fix indices i and e and concentrate on the construction of the subordering $\mathcal{P}_{i,e}$. (The constructions for the various i and e can be fit together using a computable partition of ω. Since if $\mathcal{L}_{i,e}$ is not a linear extension of $<_P$ on $P_{i,e}$, $\mathcal{P}_{i,e}$ may turn out to be finite, this computable partition cannot be fixed beforehand but must be constructed simultaneously with the components $\mathcal{P}_{i,e}$.)

Remark. Since we are assuming that X_i is low (uniformly in i), we may assume that $L_e^{X_i}$ is either total (and thus can be approximated effectively à la Shoenfield's Limit Lemma), or else $L_e^{X_i}$ is finite. For simplicity, we will suppress the details of this approximation.

We will now construct a partial ordering $\mathcal{P}$ such that

(1) if $\mathcal{L}_{i,e} \restriction P_{i,e}$ is a linear extension $<_L$ of $<_P$ on $P_{i,e}$, then we have an infinite $\mathcal{L}_{i,e}$-computable $<_L$-descending sequence $\{c_n\}_{n\in N}$ of elements of $P_{i,e}$, and
(2) from the join of X_i and of any infinite set of elements of $P_{i,e}$, each of which has infinitely many elements $<_P$-below it, we can compute K.

We first illustrate our construction by showing how to code whether $0 \in K$ into any infinite $<_P$-descending chain in $P_{i,e}$ while at the same time, computably in $<_L$, fixing an element $c_0 >_P$ infinitely many elements of $P_{i,e}$. We start with seven elements $a, a_1, a_2, \dots, a_6$, declaring

$$a >_P a_1 >_P a_2 >_P a_3 \text{ and } a >_P a_4 >_P a_5 >_P a_6$$

with no other comparabilities. We call a_2 and a_5 the 0-*critical elements* and wait for $\mathcal{L}_{i,e}$ to decide whether or not $a_2 >_L a_5$. If $a_2 >_L a_5$ then we let a_1 be the first element c_0 of our $<_L$-descending chain, and we build the rest of $P_{i,e}$ in the interval (a_1, a_2) until 0 enters K; when 0 enters K then we switch to building the rest of $P_{i,e}$ in the interval (a_5, a_6). Symmetrically, if $a_5 >_L a_2$ then we let a_4 be the first element c_0 of our $<_L$-descending chain, and we build the rest of $P_{i,e}$ in the interval (a_4, a_5) until 0 enters K; when 0 enters K then we switch to building the rest of $P_{i,e}$ in the interval (a_2, a_3). If $<_L$ eventually decides whether or not $a_2 >_L a_5$, then, since, if $0 \in K$, we will eventually see 0 enter K, only one of the four intervals $[a_2, a_1)$, $[a_3, a_2)$, $[a_5, a_4)$, and $[a_6, a_5)$ will be infinite (we will call this interval the 0-*active interval*), and we will have put only elements $<_L$-above this interval in our $<_L$-descending chain. (If $\mathcal{L}_{i,e}$ does not converge on whether or not $a_2 >_L a_5$ then $P_{i,e}$ will be finite.) Note that from $<_L$ we can compute the first element c_0

of our $<_L$-descending chain. And any infinite $<_P$-descending chain must contain either elements $\leq_P a_1$ or elements $\leq_P a_4$ (but not both), and so from this and the $<_L$-ordering of the 0-critical elements, we can compute whether $0 \in K$.

The full construction simply nests the above: The previous paragraph describes the definition of the 0-active interval. Given the n-active interval $[b, c)$, we create six new elements $b_1, \dots, b_6$ in it, declaring

$$b >_P b_1 >_P b_2 >_P b_3 \text{ and } b >_P b_4 >_P b_5 >_P b_6$$

with no other comparabilities in the interval $[b, c)$. We call b_2 and b_5 the $(n+1)$-*critical elements* and wait for $\mathcal{L}_{e,i}$ to decide whether or not $b_2 >_L b_5$. If $b_2 >_L b_5$ then we let b_1 be the element c_{n+1} of our $<_L$-descending chain, and we build the rest of the interval $[b, c)$ in the interval (b_1, b_2) until $n+1$ enters K; when $n+1$ enters K then we switch to building the rest of the interval $[b, c)$ in the interval (b_5, b_6). Symmetrically, if $b_5 >_L b_2$ then we let b_4 be the element c_{n+1} of our $<_L$-descending chain, and we build the rest of the interval $[b, c)$ in the interval (b_4, b_5) until $n+1$ enters K; when $n+1$ enters K then we switch to building the rest of the interval $[b, c)$ in the interval (b_2, b_3). If $<_L$ eventually decides whether or not $b_2 >_L b_5$, then, since, if $n+1 \in K$, we will eventually see $n+1$ enter K, only one of the four intervals $[b_2, b_1)$, $[b_3, b_2)$, $[b_5, b_4)$, and $[b_6, b_5)$ will be infinite (we will call this interval the $(n+1)$-*active interval*), and we will have put only elements $<_L$-above this interval in our $<_L$-descending chain. (If $\mathcal{L}_{i,e}$ does not converge on whether $b_2 >_L b_5$ then $P_{i,e}$ will be finite.) Note again that from $<_L$ we can compute the element c_{n+1} of our $<_L$-descending chain. And any infinite $<_P$-descending chain must contain either elements $\leq_P b_1$ or elements $\leq_P b_4$ (but not both), and so from this and the $<_L$-ordering of the $(n+1)$-critical elements, we can compute whether $n+1 \in K$.

As mentioned in the Remark above, the above nested construction for each n is controlled by our computable approximation of how $<_L$ orders the n-critical elements. If $<_L$ does not decide the ordering of the n-critical elements, then $P_{i,e}$ will be finite; otherwise, $P_{i,e}$ will be infinite, and for each n, we will eventually settle on the correct n-active interval (while making finitely many mistakes before then). In the latter case, it is not hard to check that we are building an infinite $<_L$-computable $<_L$-descending chain $\{c_n\}_{n \in N}$; and that any infinite subset of elements of $P_{i,e}$, each of which has infinitely many elements $<_P$-below it, can compute K with an additional for oracle for $<_L$, and thus a fortiori with an additional oracle for X_i.

This concludes the proof of part (3) of Theorem 1.

3. The proof of Theorems 2 and 2A

To prove part (1) of Theorem 2, simply observe that the proof of Bonnet and Pouzet [BP69] (see also [BP82, p. 140]) can be adapted here: Given a partial ordering $\mathcal{P} = \langle P, <_P \rangle$, we say an element a of P is $<_P$-*good* if $\langle P(\leq_P a), <_P \rangle$ has a scattered linear extension; and $>_P$-*good* if $\langle P(\geq_P a), <_P \rangle$ has a scattered linear extension. Define the η-*kernel* $K(\mathcal{P})$ of $\mathcal{P}$ to be the set of all elements of $\mathcal{P}$ which are neither $<_P$-good nor $>_P$-good. (Here $P(\leq_P a)$ and $P(\geq_P a)$ are the sets of elements of P below or above a, respectively.) It is now easy to check that

$$K(\mathcal{P}) = \{a \in P \mid$$
$$\forall \text{ linear extension } \prec \text{ of } \langle P(\leq a), <_P\rangle\ \exists \prec\text{-chain } S\ (S \text{ is densely ordered})$$
$$\text{and}$$
$$\forall \text{ linear extension } \prec \text{ of } \langle P(\geq a), <_P\rangle\ \exists \prec\text{-chain } S\ (S \text{ is densely ordered})\}.$$

So $K(\mathcal{P})$ is Π^1_2-definable, and thus the η-kernel of any partial ordering $\mathcal{P}$ in our model $\mathcal{N}$ of Π^1_2-CA_0 also exists in $\mathcal{N}$.

We now observe that the collection of $<_P$-good elements, and the collection of $>_P$-good elements, form an initial segment, or a final segment, of $\mathcal{P}$, respectively. We claim that Π^1_2-CA_0 proves $\mathcal{P}$ has a scattered linear extension iff $K(\mathcal{P}) = \emptyset$. First, if $\mathcal{P}$ has a scattered linear extension, then by restriction, it is clear that every cone in $\mathcal{P}$ has a scattered linear extension. Therefore, $K(\mathcal{P}) = \emptyset$. To establish the other direction, suppose that $K(\mathcal{P}) = \emptyset$. Then, for every $a \in P$, either $P(\leq_P a)$ or $P(\geq_P a)$ has a scattered linear extension. Partition P into

$$X = \{a \in P \mid P(\leq_P a) \text{ has a scattered linear extension}\}$$

and $P \setminus X$. We use this decomposition to build a scattered linear extension of $\mathcal{P}$.

First, we build a scattered linear extension of $(X, \leq_P)$ by arithmetical recursion over $\mathcal{N}$. Let $x_0, x_1, \ldots$ be a list of the elements of X in $<_{\mathcal{N}}$-increasing order. By Σ^1_2-choice (which is provable in Π^1_2-CA_0, see [Si98, Section VII.6]), we can fix a scattered linear extension for each cone $P(\leq_P x_s)$, $s \in \mathcal{N}$. Let $X_s = \{p \in P \mid p \leq_P x_0 \text{ or } \cdots \text{ or } p \leq_P x_s\} \subseteq X$. We define L_0 to be the fixed scattered linear extension of X_0. Assume that we have a scattered linear extension L_s of X_s. If $x_{s+1} \in X_s$, then let $L_{s+1} = L_s$. Otherwise, let $Z = P(\leq_P x_{s+1}) \setminus X_s$, and fix a scattered linear extension L_Z of Z by restricting the scattered extension of $P(\leq_P x_{s+1})$. Let L_{s+1} be the linear extension of X_{s+1} which agrees with L_s on X_s, agrees with L_Z on Z, and places elements in Z above everything from X_s. L_{s+1} is a scattered linear extension of X_{s+1} which is an end-extension of L_s. Combining these orders, $\cup L_s$ is a scattered linear extension of X.

It remains to handle $P \setminus X$. We list these elements as $y_0, y_1, \ldots$ and let $Y_s = \{p \in P \mid y_0 \leq_p p \text{ or } \cdots \text{ or } y_s \leq_P p\}$. We use a similar construction to the one above to build linear extensions L'_s of Y_s such that L'_{s+1} extends L'_s downwards. We combine $\cup L_s$ with $\cup L'_s$ by placing all elements of $P \setminus X$ above all elements of X. This gives a scattered linear extension of P, and finishes the claim that $\mathcal{P}$ has a scattered linear extension if and only if $K(\mathcal{P}) = \emptyset$.

We can now prove the extendibility of η from Π^1_2-CA_0 as follows: Fix a partial ordering $\mathcal{P}$ and assume that it does not have a scattered linear extension. Add to $\mathcal{P}$ a new least and a new greatest element, x_0 and x_1, respectively. Let $D = \{\frac{m}{2^n} \mid 0 \leq m \leq 2^n \text{ and } n \in \omega\}$ be the set of dyadic rationals in $[0,1]$. By induction, we will now construct a subset $X = \{x_d \mid d \in D \cap (0,1)\}$ of P such that $x_d <_P x_e$ iff $d < e$; so X is a dense subset of $\mathcal{P}$ as desired. Suppose x_d has been defined for all indices with denominator $< 2^n$, and fix $d = \frac{m}{2^n} \in D$ with m odd. By induction, the $\mathcal{P}$-interval $[x_{\frac{m-1}{2^n}}, x_{\frac{m+1}{2^n}}]$ has no scattered linear extension, so we can choose an element x_d in its η-kernel $K([x_{\frac{m-1}{2^n}}, x_{\frac{m+1}{2^n}}])$. By the definition of η-kernel, the intervals $[x_{\frac{m-1}{2^n}}, x_d]$ and $[x_d, x_{\frac{m+1}{2^n}}]$ have no scattered linear extension, and so the induction can continue.

To prove part (2) of Theorem 2 as well as Theorem 2A, we will show that, given a sequence $X_0 \leq_T X_1 \leq_T \dots$ of uniformly low, uniformly Δ^0_2-sets, there is a computable partial ordering $\mathcal{P} = \langle S, <_P \rangle$ which is classically scattered (i. e., there is no densely ordered $<_P$-subchain of *any* complexity) such that for any i, any X_i-computable linear extension $<_L$ of $<_P$ contains a densely ordered $<_L$-subchain Turing computable in $<_L$. Again, since, by Jockusch and Soare [JS72] and Simpson [Si98, VIII.2], there is a model of WKL_0 whose second-order part consists of all sets in the Turing ideal generated by a sequence $X_0 \leq_T X_1 \leq_T \dots$ of uniformly low, uniformly Δ^0_2-sets, this implies that WKL_0 does not imply the extendibility of η.

The construction of $\mathcal{P}$ is again a finite-injury priority argument. We construct $\mathcal{P}$ as the disjoint union of sub-partial orderings $\mathcal{P}_{i,e}$ (for $e, i \in \omega$) such that each $\mathcal{P}_{i,e}$ is a connected component of $\mathcal{P}$ (when viewed as a directed graph). Each $\mathcal{P}_{i,e}$ will be devoted to showing that if the eth binary X_i-computable relation $\mathcal{L}_{i,e} = L^{X_i}_e$ is a linear extension $<_L$ of $<_P$ then it has a densely ordered $<_L$-subchain inside $P_{i,e}$ which is computable in $\mathcal{L}_{i,e}$. At the same time, we have to show that there is no densely ordered $<_P$-subchain inside $P_{i,e}$. (Note here that any $<_P$-subchain in $\mathcal{P}$ must be completely contained in a single $P_{i,e}$.)

We can thus fix indices i and e and concentrate on the construction of the subordering $\mathcal{P}_{i,e}$. (The constructions for the various i and e can be fit together using a computable partition of ω. Since if $\mathcal{L}_{i,e}$ is not a linear extension of $<_P$ on $P_{i,e}$, $\mathcal{P}_{i,e}$ may turn out to be finite, this computable partition cannot be fixed beforehand but must be constructed simultaneously with the components $\mathcal{P}_{i,e}$.) The Remark in the proof of part (3) of Theorem 1 also applies here verbatim.

We will now construct a partial ordering $\mathcal{P}$ such that

(1) if $\mathcal{L}_{i,e} \upharpoonright P_{i,e}$ is a linear extension $<_L$ of $<_P$ on $P_{i,e}$, then we have an $\mathcal{L}_{i,e}$-computable densely ordered $<_L$-subchain C in $P_{i,e}$, and
(2) for any element x of $P_{i,e}$, there are either only finitely many elements $>_P x$, or only finitely many elements $<_P x$.

We first illustrate our construction by showing how to perform a single step: We start with three $<_P$-incomparable elements a_0, a_1 and a_2, calling them the 0-*critical elements*, and wait for $\mathcal{L}_{i,e}$ to decide the $<_L$-ordering on these three elements. Possibly relabeling them, we will assume $a_0 <_L a_1 <_L a_2$. We then place a_1 into C and call the interval $(-\infty, a_0)$ and the interval (a_2, ∞) the 0-*active intervals*. (If $\mathcal{L}_{i,e}$ does not converge on the ordering of a_0, a_1, and a_2 then $P_{i,e}$ will be finite.) Note that from $<_L$ we can compute the first element of C. And among the elements a_0, a_1, and a_2, at most one is comparable to any element of the 0-active intervals.

The full construction simply nests the above: The previous paragraph describes the definition of the 0-active intervals. Given an n-active interval (b, c), we create three new pairwise incomparable elements b_0, b_1, and b_2 in (b, c), calling them the $(n+1)$-*critical elements*, and wait for $\mathcal{L}_{e,i}$ to decide the $<_L$-ordering on these three elements. Possibly relabeling them, we will assume $b_0 <_L b_1 <_L b_2$. We then place b_1 into C and call the interval (b, b_0) and the interval (b_2, c) $(n+1)$-*active intervals*. (If $\mathcal{L}_{i,e}$ does not converge on the ordering of b_0, b_1, and b_2 then $P_{i,e}$ will be finite.) Note that from $<_L$ we can compute the first element of C in the interval (b, c). And among the elements b_0, b_1, and b_2, at most one is comparable to any element of the $(n+1)$-active intervals.

As mentioned in the Remark above, the above nested construction for each n is controlled by our computable approximation of how $<_L$ orders the n-critical elements. If $<_L$ does not decide the ordering of the n-critical elements, then $P_{i,e}$ will be finite; otherwise, $P_{i,e}$ will be infinite, and for each n, we will eventually settle on the correct n-active intervals (while making finitely many mistakes before then). In the latter case, it is not hard to check that we are building a $<_L$-computable densely $<_L$-ordered chain; and that for any element x of $P_{i,e}$, there are either only finitely many elements $>_P x$, or only finitely many elements $<_P x$.

This concludes the proof of part (2) of Theorem 2.

4. The proof of Theorem 3

To prove that ATR_0 implies the extendibility of ζ, observe that the proof of Jullien [Ju69] (see also [BP82, p. 141], note a typo there: $\omega+\omega^*$ should be $\omega^*+\omega$, i. e., ζ) can be adapted here: Fix a partial ordering $\mathcal{P} = \langle S, <_P\rangle$ without any subchain (in our model $\mathcal{N}$ of ATR_0) of order type ζ. Call an element a of P *$<_P$-good* if $\langle P(\leq_P a), <_P\rangle$ contains no subchain of order type ω^*; and *$>_P$-good* if $\langle P(\geq_P a), <_P\rangle$ contains no subchain of order type ω. By our assumption on $\mathcal{P}$, any element of P is either $<_P$-good or $>_P$-good. Since $<_P$-goodness and $>_P$-goodness are both Π^1_1-definable, our model $\mathcal{N}$ of ATR_0 contains a set S by Σ^1_1-separation such that any element of S is $<_P$-good, and any element of $P-S$ is $>_P$-good. (We use here that ATR_0 is equivalent to Σ^1_1-separation (see Simpson [Si98, Theorem V.5.1]).)

Since the set of $<_P$-good elements is downward closed in $\mathcal{P}$, we may assume that S is downward closed in $\mathcal{P}$. Now, by part (1) of Theorem 1, ACA_0 (and thus *a fortiori* ATR_0) proves the extendibility of ω and ω^*; so we can fix linear extensions $<_{L_1}$ and $<_{L_2}$ of S and $P-S$, respectively, which have no subchains of order type ω^* and ω, respectively. We can now patch $<_{L_1}$ and $<_{L_2}$ together (by placing the elements of S left of the elements of $P-S$) to obtain a linear extension $<_L$ of $\mathcal{P}$ without a subchain of order type ζ.

We now establish the other direction of the proof of Theorem 3 by two claims:

Claim 1. *RCA_0 and the extendibility of ζ imply ACA_0.*

Claim 2. *ACA_0 and the extendibility of ζ imply ATR_0.*

Proof of Claim 1. Fix a model $\mathcal{N}$ of RCA_0 + "ζ is extendible" and an injective function $f: N \to N$ in this model. In order to establish ACA_0, we must show that the range of f is in this model (see Simpson [Si98, Lemma III.1.3]). Construct a partial ordering $\mathcal{P}$ in $\mathcal{N}$ with distinguished elements a_n and b_n (for $n \in N$) as follows: If n is not in the range of f then a_n is the first element of an ω-chain, and b_n is the last element of an ω^*-chain in $\mathcal{P}$. If $n = f(m)$ then a_n is the mth element from the end of an ω^*-chain, and b_n is the mth element of an ω-chain in $\mathcal{P}$. Since all a_n and b_n are in different connected components of $\mathcal{P}$ (when viewed as a directed graph), $\mathcal{P}$ contains no ζ-chain, so we can fix a linear extension $<_L$ without a ζ-chain. But then n is in the range of f iff $b_n <_L a_n$, the range of f is in the model $\mathcal{N}$ as desired.

Proof of Claim 2. We use again that ATR_0 is equivalent to Σ^1_1-separation. Fix a model $\mathcal{N}$ of ACA_0 + "ζ is extendible" and two disjoint sets $S_0, S_1 \subseteq N$ which are Σ^1_1-definable over this model. We will separate S_0 and S_1 as follows: Fix two sequences of trees $\mathcal{L}^0_n$ and $\mathcal{L}^1_n$ such that for all $i < 2$ and $n \in N$, $n \in S_i$ iff $\mathcal{L}^i_n$ is not well-founded. (Such sequences exist within $\mathcal{N}$; see Simpson [Si98, Theorem V.1.8].) Construct a partial ordering $\mathcal{P}$ in $\mathcal{N}$ with distinguished elements a_n and b_n (for $n \in N$) as follows:

(1) $\mathcal{P}(<_P a_n)$ is isomorphic to $\mathcal{L}^0_n$,
(2) $\mathcal{P}(>_P a_n)$ is isomorphic to $(\mathcal{L}^1_n)^*$,
(3) $\mathcal{P}(<_P b_n)$ is isomorphic to $\mathcal{L}^1_n$, and
(4) $\mathcal{P}(>_P b_n)$ is isomorphic to $(\mathcal{L}^0_n)^*$,

(where $\mathcal{L}^*$ denotes $\mathcal{L}$ under the reverse ordering). Since all a_n and b_n are in different connected components of $\mathcal{P}$ (when viewed as a directed graph) and since for each n, at most one of $\mathcal{L}^0_n$ and $\mathcal{L}^1_n$ is not well-founded, we have that $\mathcal{P}$ contains no ζ-chain; so we can fix a linear extension $<_L$ without a ζ-chain. But then $\{n \in N \mid b_n <_L a_n\}$ is in the model $\mathcal{N}$ and separates S_0 and S_1 as desired.

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