SOME NOTES ON THE *wtt*-JUMP

KLAUS AMBOS-SPIES, ROD DOWNEY, AND MARTIN MONATH

1. Introduction

One of the fundamental operations in classical computability is the jump operator. For a set B, $B' = \{e \mid \Phi_e^B(e)\downarrow\}$ is the canonical Σ_1^B complete set; the halting set *relative* to B. This notion goes back to Turing [17], but it is first highlighted in Kleene and Post [13]. The interplay between the jump operator and the partial ordering of the Turing degrees has provided one of the main areas of work in classical and applied computability theory.

This paper studies a notion called the *wtt*-jump. This is an operator akin to the classical Turing jump but using *wtt*-operators in place of Turing ones. The *wtt*-jump has previously been studied in one form or another by Coles, Downey and Laforte [6], Downey and Greenberg [8] and Anderson and Csima [2]. Recently, the authors have shown that properties of the *wtt*-jump can be used to characterize precisely when a c.e. set can be computed from a maximal c.e. set with a computably bounded use [1]. Earlier, Barmpalias, Downey and Greenberg [4], building on work by Chisholm, Chubb, Harizanov, Hirschfeldt, Jockusch, McNicholl and Pingrey [5] and by Afshari, Barmpalias, Cooper and Stephan [3], characterized when all c.e. members of a c.e. Turing degree are computable from a hypersimple set and from an initial segment of a scattered linear ordering via a bounded reduction; but that characterization used what are called totally ω-c.a. degrees, and concerned approximations to total functions, whereas our maximal set result involved a new hierarchy generated from *wtt*-computations. Thus we see the following. In the same way that tt-reducibility turned out to be a central unifying idea in algorithmic randomness (e.g. Downey-Hirschfeldt [9]), these studies, and others, show that rather than mere artifacts of definitions in classical computability theory, hierarchies related to strong reducibilities and bounded jump operators can give classification and unification of combinatorics in parts of computable mathematics. As a consequence, it seems we should better understand analogs of the core notions of classical computability for such hierarchies. This paper contributes to that program.

The earliest analog of a jump operator using only bounded reducibilities is the "mini-jump" hierarchy introduced by Ershov [10] as discussed in Odifreddi [15], Chapter XI.6. Ershov's hierarchy concerned a jump operator for the m-degrees involving the partial m-degrees. Also a bounded analog of the jump for *tt*-reductions was investigated by Gerla [11].

2010 *Mathematics Subject Classification.* Primary 03D25.

Downey is partially supported by Marsden Fund of New Zealand. The results of this paper were obtained whilst Downey was an Alexander von Humboldt Fellow at the University of Heidelberg.

As in Downey and Greenberg [8], we will let $\widehat{\Phi}_e$ denote the e-th *wtt*-procedure which will have use φ_e. It can be obtained from a standard listing of all pairs $\{\langle \Psi_e, \psi_e\rangle \mid e \in \mathbb{N}\}$ consisting of a partial Turing procedure and a partial computable function, by allowing $\Psi_e^A(x)\downarrow$ only if $\Psi_e^A(x)$ halts with information used $\leq \psi_e(x)\downarrow$. Note that $\{\varphi_e\}_{e\geq 0}$ is an acceptable numbering of the partial computable functions. So we may assume that the halting set is based on this numbering, i.e., $\emptyset' = \{e : \varphi_e(e)\downarrow\}$. Moreover, for any uniformly computable *wtt*-functionals $\{\widehat{\Psi}_e\}_{e\geq 0}$ (i.e., $\Psi^A(\langle e,x\rangle) = \widehat{\Psi}_e^A(x)$ is a *wtt*-functional) there are computable functions f and g such that $\widehat{\Psi}_e^A(x) = \widehat{\Phi}_{f(e)}^A(x) = \widehat{\Phi}_{g(\langle e,x\rangle)}^A(z)$ for all oracle sets A and all numbers e, x, z (see [1] or [8] for details). In the following we will tacitly use these facts.

Definition 1.1. *The* *wtt*-jump *or* bounded jump *of a set A is defined by*

$$A^\dagger = \{e \mid \widehat{\Phi}_e^A(e)\downarrow\}.$$

Clearly the usual equivalences obtained by the s-m-n theorem apply. So the *wtt*-jump of A is (up to m-degree) the same as $\{\langle e,n\rangle \mid \widehat{\Phi}_e^A(n)\downarrow\}$.

Note that $\emptyset' \equiv_m \emptyset^\dagger$, and that for a c.e. set A, $\emptyset' \leq_{wtt} A^\dagger \leq_{wtt} (\emptyset')^\dagger$. Moreover if X is Δ_2^0, $X^\dagger$ is also Δ_2^0.

The analog of the idea of lowness for the bounded jump can be defined as follows. A set A is *bounded low* if $A^\dagger \leq_{wtt} \emptyset'$ (or, equivalently, $A^\dagger \leq_{tt} \emptyset'$). Variations of bounded lowness - all of them *wtt*-equivalent to this notion - have been studied by Anderson and Csima [2], Ambos-Spies, Downey and Monath [1], and Wu and Wu [18]. It is easy to see that all classically superlow sets A (i.e. $A' \leq_{wtt} \emptyset'$) are bounded low, but in [1], we proved that there are Turing complete bounded low c.e. sets. (This result was independently obtained by Wu and Wu [18].)

2. Arithmetical characterizations

Anderson and Csima [2] proved that if $X \geq_{wtt} \emptyset'$ then there is a set A with $A^\dagger \equiv_{wtt} X$. We will be concerned with an analog of Sacks' Jump Theorem [16], and hence we will need an analog of the arithmetical hierarchy. The classical Sacks' Jump Theorem [16] says that if X is Σ_2^0 then there is a c.e. set A with $A' \equiv_T \emptyset' \oplus X$.

The problem is that relativization is not so straightforward. What should Σ_2^0 mean in this setting? Here are some candidates:

Definition 2.1. (1) *A set X is $\widehat{\Sigma}_1^B$ iff there is a partial computable function g and a relation R which is g-bounded computable with*

$$x \in X \text{ iff } \exists s R^B(x,s).$$

*Here by R being g-*bounded computable *we mean that $R^B(x,s)$ holds requires that $g(x)$ is defined and any query in the computation of $R^B(x,s)$ is bounded by $g(x)$.*

(2) *If the function g above can be chosen to be* total *computable then we say that X is $\widetilde{\Sigma}_1^B$.*

Correspondingly, we say that X is $\widehat{\Pi}_1^B$ iff $\overline{X}$ is $\widehat{\Sigma}_1^B$, and X is $\widehat{\Delta}_1^B$ iff $X \in \widehat{\Pi}_1^B \cap \widehat{\Sigma}_1^B$. Similarly, X is $\widetilde{\Pi}_1^B$ iff $\overline{X}$ is $\widetilde{\Sigma}_1^B$, and X is $\widetilde{\Delta}_1^B$ iff $X \in \widetilde{\Pi}_1^B \cap \widetilde{\Sigma}_1^B$. Obviously, $\widetilde{\Sigma}_1^B \subseteq \widehat{\Sigma}_1^B$, $\widetilde{\Pi}_1^B \subseteq \widehat{\Pi}_1^B$ and $\widetilde{\Delta}_1^B \subseteq \widehat{\Delta}_1^B$. In the following proposition we give some properties of the bounded Σ_1-, Π_1- and Δ_1-classes. In particular we show that, for some oracle sets B, the latter inclusions are proper. On the other hand note that, for the empty oracle, obviously $\widetilde{\Sigma}_1^\emptyset = \widehat{\Sigma}_1^\emptyset = \Sigma_1^\emptyset$ etc.

Proposition 2.2. (i) *$X \in \widehat{\Sigma}_1^B$ iff $X = dom(\widehat{\Phi}_e^B)$ for some $e \geq 0$, and $X \in \widetilde{\Sigma}_1^B$ iff $X = dom(\widehat{\Phi}_e^B)$ for some $e \geq 0$ such that φ_e is total.*
(ii) *$X \in \widehat{\Sigma}_1^B$ iff $X \leq_m B^\dagger$.*
(iii) *$X \in \widetilde{\Delta}_1^B$ iff $X \leq_{wtt} B$.*
(iv) *There is a c.e. set B such that $\widetilde{\Delta}_1^B \subset \widehat{\Delta}_1^B$, $\widetilde{\Sigma}_1^B \subset \widehat{\Sigma}_1^B$ and $\widetilde{\Pi}_1^B \subset \widehat{\Pi}_1^B$.*

Proof. (i) We prove the first part. The proof of the second part is similar.

First assume that $X \in \widehat{\Sigma}_1^B$. Fix a partial computable function g and a g-bounded computable relation R such that $x \in X$ iff $R^B(x,s)$ for some s. Then X is the domain of the wtt-functional Ψ where $\Psi^B(x)$ is defined iff there is a number s such that $R^B(x,s)$ holds (note that the use of Ψ is bounded by g). So, for any e, such that $\Psi = \widehat{\Phi}_e$, $X = dom(\widehat{\Phi}_e^B)$.

For the converse direction, assume that $X = dom(\widehat{\Phi}_e^B)$. Fix R such that $R^B(x,s)$ holds iff $\widehat{\Phi}_{e,s}^B(x)$ is defined. Then R is g-bounded computable for the partial computable function $g = \varphi_e$ and $x \in X$ iff $R^B(x,s)$ holds for some s. So X is $\widehat{\Sigma}_1^B$.

(ii) First assume that $X \in \widehat{\Sigma}_1^B$. By (i) fix e such that $X = dom(\widehat{\Phi}_e^B)$. Then, for a total computable function f such that $\widehat{\Phi}_{f(x)}^B(z) = \widehat{\Phi}_e^B(x)$ for all x and z, $X \leq_m B^\dagger$ via f.

For the converse direction, assume that $X \leq_m B^\dagger$ via f. Then $x \in X$ iff $\widehat{\Phi}_{f(x)}^B(f(x))$ is defined. So, for R such that $R^B(x,s)$ holds iff $\widehat{\Phi}_{f(x),s}^B(f(x))$ is defined and for $g(x) = \varphi_{f(x)}(f(x))$, g is partial computable, R is g-bounded computable, and $x \in X$ iff $R^B(x,s)$ holds for some s. So X is $\widehat{\Sigma}_1^B$.

(iii) First assume that $X \in \widetilde{\Delta}_1^B$, i.e., $X \in \widetilde{\Sigma}_1^B$ and $\overline{X} \in \widetilde{\Sigma}_1^B$. Fix total computable functions g_0 and g_1 and relations R_0 and R_1 such that R_0 is g_0-bounded computable and $x \in X$ iff $R_0^B(x,s)$ for some s and R_1 is g_1-bounded computable and $x \in \overline{X}$ iff $R_1^B(x,s)$ for some s. Then $X \leq_{wtt} B$ via the wtt-functional $\widehat{\Psi}$ where $\widehat{\Psi}^B(x) = 1$ if for the least s such that $R_0^B(x,s)$ or $R_1^B(x,s)$ holds, $R_0^B(x,s)$ holds, and $\widehat{\Psi}^B(x) = 0$ otherwise (note that $\widehat{\Psi}$ is g-bounded for $g(x) = \max\{g_0(x), g_1(x)\}$).

For the converse direction, assume that $X \leq_{wtt} B$, say $X = \widehat{\Phi}_e^B$ where φ_e is total. Then, for R_0 and R_1 such that $R_0(x,s)$ holds if $\widehat{\Phi}_{e,s}^B(x) = 1$ and $R_1(x,s)$ holds if $\widehat{\Phi}_{e,s}^B(x) = 0$ and for $g = \varphi_e$, g is total computable, R_0 and R_1 are g-bounded computable, $x \in X$ iff $R_0(x,s)$ holds for some s, and $x \in \overline{X}$ iff $R_1(x,s)$ holds for some s. So $X \in \widetilde{\Sigma}_1^B$ and $\overline{X} \in \widetilde{\Sigma}_1^B$ hence $X \in \widetilde{\Delta}_1^B$.

(iv) By Downey and Greenberg [8], Proposition 3.1, there is a c.e. set B and a set X such that $X \in \widehat{\Delta}^B_1$ and $X \not\leq_{wtt} B$. So, by (iii), $X \in \widehat{\Delta}^B_1 \setminus \widetilde{\Delta}^B_1$. This implies the first part of the claim. Moreover, by definition, $X \in \widehat{\Sigma}^B_1$ and $\overline{X} \in \widehat{\Sigma}^B_1$ (hence $\overline{X} \in \widehat{\Pi}^B_1$ and $X \in \widehat{\Pi}^B_1$) whereas $X \notin \widetilde{\Sigma}^B_1$ or $\overline{X} \notin \widetilde{\Sigma}^B_1$ (hence $\overline{X} \notin \widetilde{\Pi}^B_1$ or $X \notin \widetilde{\Pi}^B_1$). Obviously this implies the second and third parts of the claim. □

The most natural guess for an analog of Σ^B_1 is $\widehat{\Sigma}^B_1$. By Proposition 2.2 (i) the sets in this class are just the sets which are bounded-c.e. in B in the sense of [8]. As (ii) shows, the characterization of the classical Σ^B_1 classes as the classes of the sets which are many-one reducible to the jump of B carries over to $\widehat{\Sigma}^B_1$ if we replace the jump by the bounded jump. But (iii) together with (iv) shows a serious drawback to this definition: for some sets B, there are $\widehat{\Delta}^B_1$ sets which are not bounded computable from B. The more limited definition of $\widetilde{\Sigma}^B_1$ does not share this problem, but here the analog of (ii) fails. In particular, since $\widetilde{\Sigma}^B_1$ is downward closed under many-one reducibility, there is a set B such that the bounded jump $B^\dagger$ is not in $\widetilde{\Sigma}^B_1$ (by (ii) and (iv)).

Exactly as in the arithmetical hierarchy, we can now define $\widehat{\Sigma}^B_n$ and $\widetilde{\Sigma}^B_n$ for all $n \geq 1$. For example, X is $\widehat{\Sigma}^B_2$ means that there is a $\widehat{\Pi}^B_1$ set Q such that X is $\widehat{\Sigma}^Q_1$. If $B = \emptyset$ we say that X is $\widehat{\Sigma}^0_2$.

The following proposition is a straightforward analog of the corresponding fact in classical computability theory.

Proposition 2.3. (i) *X is $\widehat{\Sigma}^0_2$ iff X is $\widehat{\Sigma}^{\emptyset'}_1$.*
(ii) *X is $\widetilde{\Sigma}^0_2$ iff X is $\widetilde{\Sigma}^{\emptyset'}_1$.*

Proof. For the following recall that the classes $\widehat{\Sigma}^0_1$ and $\widetilde{\Sigma}^0_1$ coincide with the class of the c.e. sets.

(i) First suppose that X is $\widehat{\Sigma}^0_2$. Then there is a $\widehat{\Pi}^0_1$ set Q with $X \in \widehat{\Sigma}^Q_1$. Since $\widehat{\Sigma}^{\overline{Q}}_1 = \widehat{\Sigma}^Q_1$ there is a partial computable function g and a g-bounded computable relation R such that

$$x \in X \text{ iff } \exists s R^{\overline{Q}}(x, s).$$

Since $\overline{Q}$ is in $\widehat{\Sigma}^0_1$ hence c.e., $\overline{Q} \leq_m \emptyset'$; say via h. So any g-bounded query to $\overline{Q}$ can be replaced by an $h(g)$-bounded query to $\emptyset'$. So there is an $h(g)$-bounded computable relation $\hat{R}$ such that

$$x \in X \text{ iff } \exists s \hat{R}^{\emptyset'}(x, s).$$

Hence X is $\widehat{\Sigma}^{\emptyset'}_1$.

For the opposite direction, suppose that X is $\widehat{\Sigma}^{\emptyset'}_1$. Then X is $\widehat{\Sigma}^{\overline{\emptyset'}}_1$ too. Since $\emptyset'$ is c.e. and the class of the c.e. sets coincides with $\widehat{\Sigma}^0_1$, it follows that $\overline{\emptyset'}$ is in $\widehat{\Pi}^0_1$. So X is $\widehat{\Sigma}^0_2$ by definition.

(ii) The proof is similar to the proof of part (i). For the first direction it suffices to note that if g is total then, by totality of h, $h(g)$ is total too. For the second direction, it suffices to use the observation that $\widetilde{\Pi}^0_1 = \widehat{\Pi}^0_1$ hence $\overline{\emptyset'} \in \widetilde{\Pi}^0_1$. $\square$

3. Potential Analogs of the Sacks' Jump Theorem

We have pointed out that *wtt*-superlow sets are useful in classifying the internal combinatorics of constructions. In this section, we will explore how the *wtt*-jump operator operates as an analog of the normal jump operator. As mentioned above, one such result was by Anderson and Csima [2] who proved that the natural analog of the Friedberg Jump Theorem holds. There have been earlier studies looking at the relationship between the classical jump and strong reducibilities, such as Mohrherr [14] and Csima, Downey and Ng [7]. Solving an open question raised in [2], here we examine bounded versions of the Sacks' Jump Theorem [16]. Note that, by $\emptyset' \equiv_m \emptyset^\dagger$, we may replace $\emptyset'$ by $\emptyset^\dagger$ in the following results.

For any c.e. set A, $A^\dagger \in \widehat{\Sigma}^0_2$ (namely $A^\dagger \in \widehat{\Sigma}^A_1$ hence, by $A \leq_{wtt} \emptyset'$ and by Proposition 2.3, $A^\dagger \in \widehat{\Sigma}^{\emptyset'}_1 = \widehat{\Sigma}^0_2$). So, by $\emptyset' \leq_m A^\dagger$, $A^\dagger \equiv_{wtt} \emptyset' \oplus S$ for the $\widehat{\Sigma}^0_2$-set $S = A^\dagger$. This suggest that the full analog of the Sacks' Jump Theorem for the bounded jump should be the statement that, for any $\widehat{\Sigma}^0_2$-set S, there is a c.e. set A such that $A^\dagger \equiv_{wtt} \emptyset' \oplus S$. As the following theorem shows, this strong analog fails.

Theorem 3.1. *There exists $S \in \widehat{\Sigma}^0_2$ such that, for no c.e. set W, $W^\dagger \equiv_{wtt} \emptyset' \oplus S$.*

Proof. The argument is finite injury. We will build a $\widehat{\Sigma}^0_2$ set S to meet the requirements

$$R_e : \neg(\Delta_e^{W_e^\dagger} = \emptyset' \oplus S \;\&\; \Gamma_e^{\emptyset' \oplus S} = W_e^\dagger).$$

Here, in the context of this proof, $\{(W_e, \Delta_e, \Gamma_e)\}_{e \geq 0}$ is an enumeration of all triples of a c.e. set and two *wtt*-functionals. Moreover, we assume that δ_e and γ_e are partial computable bounds on the uses of Δ_e and Γ_e, respectively, and we fix computable enumerations $W_{e,s}$ etc. of W_e etc. Moreover, we let $\delta^*_e(n) = \max_{n' \leq n} \delta_e(n')$ and $\gamma^*_e(n) = \max_{n' \leq n} \gamma_e(n')$, (where $\delta^*_e(n) \uparrow$ if $\delta_e(n') \uparrow$ for some $n' \leq n$, and similarly for $\gamma^*_e(n)$) and, w.l.o.g. we assume that $\delta_e(n) \geq n$ if defined, and similarly for $\gamma_e(n)$.

Before we can describe our strategy for meeting the requirements, we have to present some of the fundamentals of the construction first.

The format of the definition of S is as follows. Besides a computable approximation $\{S_s\}_{s \geq 0}$ of S we give a computable enumeration $\{B_s\}_{s \geq 0}$ of a c.e. set B and define a partial computable function ψ such that the approximation $\{S_s\}_{s \geq 0}$ of S is ψ-bounded controlled by B via $\{B_s\}_{s \geq 0}$ in the following sense. For any number z, $S_0(z) = 0$, $\psi(z)$ is defined if $S_t(z) = 1$ for some stage t, and $B_{s+1} \restriction \psi(z) \neq B_s \restriction \psi(z)$ for any s such that z is extracted from S at stage $s+1$, i.e., $z \in S_s \setminus S_{s+1}$. Obviously this ensures that $S = dom(\Psi^B)$ for some ψ-bounded *wtt*-functional Ψ whence $S \in \widehat{\Sigma}^B_1$. Since $\widehat{\Sigma}^{\overline{B}}_1 = \widehat{\Sigma}^B_1$ and $\overline{B} \in \Pi^0_1 = \widehat{\Pi}^0_1$, this guarantees that S is in $\widehat{\Sigma}^0_2$.

In addition, we define uniformly computable approximations $\{C_{e,s}\}_{s \geq 0}$ of auxiliary sets C_e together with uniformly partial computable functions ξ_e such that

$\{C_{e,s}\}_{s\geq 0}$ is ξ_e-bounded controlled by W_e via $\{W_{e,s}\}_{s\geq 0}$ (in the above sense). So there are uniformly computable ξ_e-bounded *wtt*-functionals Ξ_e such that $C_e = dom(\Xi_e^{W_e})$. It follows that there are total uniformly computable functions f_e such that $C_e(x) = W_e^\dagger(f_e(x))$, and, by the Recursion Theorem, we may assume that the functions f_e are given in advance.

We work with the standard approximation $\{W_{e,s}^\dagger\}_{s\geq 0}$ of $W_e^\dagger$ where $W_{e,s}^\dagger(m) = 1$ iff $\widehat{\Phi}_{m,s}^{W_{e,s}}(m)\downarrow$. So, if m is extracted from $W_e^\dagger$ at stage $s+1$, i.e., $m \in W_{e,s}^\dagger \setminus W_{e,s+1}^\dagger$, then $\varphi_m(m)$ is defined at stage s and a number $< \varphi_m(m)$ is enumerated into W_e at stage $s+1$, i.e., $\varphi_{m,s}(m)\downarrow$ and $W_{e,s+1} \restriction \varphi_m(m) \neq W_{e,s} \restriction \varphi_m(m)$. Finally, recall that we assume that any given functional defined at stage s has use $< s$ and any given partial computable function defined at stage s has value $< s$.

Now, for a single requirement R_e, we work as follows. For readability we will drop the index "e" in the following.

In order to monitor the "controlled" length of agreement of the equations in R_e at stage s we define the length functions

$$l_\Delta(s) = \max\{n : \forall n' < n(\Delta_s^{W_s^\dagger}(n') = (\emptyset'_s \oplus S_s)(n') \ \&\ \Gamma_s^{\emptyset'_s \oplus S_s} \restriction \delta(n') = W_s^\dagger \restriction \delta(n'))\}$$

and

$$l_\Gamma(s) = \max\{n : \forall n' < n(\Gamma_s^{\emptyset'_s \oplus S_s}(n') = W_s^\dagger(n') \ \&\ \Delta_s^{W_s^\dagger} \restriction \gamma(n') = (\emptyset'_s \oplus S_s) \restriction \gamma(n'))\},$$

let

$$l(s) = \min\{l_\Delta(s), l_\Gamma(s)\},$$

and call a stage s *expansionary* if $l(s) > l(t)$ for all stages $t < s$. Note that the limit of l exists (since the functionals involved in the definition are bounded) and the limit is finite iff there are only finitely many expansionary stages iff the requirement R_e is met.

We will work with a number z and a finite sequence of numbers x_i under our control, where z is targeted for S while the numbers x_i are targeted for C (hence $f(x_i)$ is targeted for $W^\dagger$ implicitly). The numbers z and $f(x_i)$ are potential diagonalization witnesses for the first equation and the second equation, respectively, in R_e. In the first phase of the attack we define the set up to be used in the second phase of the attack.

First pick an unused number z (above the restraint of the higher priority requirements) and wait for an expansionary stage s such that $l(s) > 2z + 1$ (note that $S(z) = (\emptyset' \oplus S)(2z + 1)$). Then $\delta^*(2z + 1)$ is defined at stage s and we may pick $\delta^*(2z + 1)$ unused numbers x_i such that $z < x_1 < \cdots < x_{\delta^*(2z+1)}$. Finally, wait for the least expansionary stage $s_0 > s + 1$ (hence $s_0 \geq \delta^*(2z + 1) + 2$ and $l(s_0) > 2z + 1$) such that, for $1 \leq n \leq \delta^*(2z + 1)$, $\gamma^*(f(x_n))$ is defined at stage s_0, $s_0 > \gamma^*(f(x_n))$, and $l(s_0) > f(x_n)$. Then let $\psi(z) = s_0^3$ thereby completing the set up of the attack. (Note that if the set up cannot be completed then there are only finitely many expansionary stages. So R_e is met.)

Now in the second phase of the attack, a state is assigned to any stage $s \geq s_0$. Here the state of s is the number of $m < \delta^*(2z+1)$ such that $\varphi_m(m)$ is defined at stage s but is not yet defined at stage s_0, i.e.,

$$|\{m < \delta^*(2z+1) : \varphi_{m,s}(m)\downarrow \ \text{ and } \ \varphi_{m,s_0}(m)\uparrow\}|,$$

and we let s_n be the least expansionary stage $s \geq s_0$ of state n (if any). Note that any state is $\leq \delta^*(2z+1) < s_0 - 1$ and the state is nondecreasing in the stage. So, eventually, all stages have the same state, called the final state. Now, for any state n we have a procedure which is active at the stages of state n, i.e., it is started at stage $s_n + 1$ (if s_n is defined) and it becomes abandoned at stage s_{n+1} (if defined). Each n-procedure may change the approximative value of $S(z)$, but we guarantee that for each n the number of changes is bounded by s_0^2. Since the number of states is bounded by s_0, it follows that the total number of $S(z)$-changes is bounded by $\psi(z) = s_0^3$. So we may witness any extraction of z out of S by a change of B below $\psi(z)$. For $n \geq 1$, the n-procedure may also change the value of $C(x_n)$ hence $W^\dagger(f(x_n))$. Again we will guarantee that these changes are conform with making C uniformly bounded c.e. in W. We will show that a procedure which is never abandoned will guarantee that there are only finitely many expansionary stages. Since, for the final state n, the n-procedure is never abandoned, this shows that our strategy meets requirement R_e.

The 0*-procedure.* For any expansionary stage $t \geq s_0$, let $S_{t+1}(z) \neq S_t(z)$. If $z \in S_t$ then enumerate the least $y < \psi(z)$ which is not yet in B into B_{t+1} (we will argue below that such a y must exist).

Note that, for any two consecutive expansionary stages $t < t'$ of state 0, this ensures that $W^\dagger_{t'} \restriction \delta^*(2z+1) \neq W^\dagger_t \restriction \delta^*(2z+1)$. Since there are $\delta^*(2z+1)$ numbers m less than $\delta^*(2z+1)$ and since $W^\dagger_s(m)$ is 0-1-valued, for any set of $\delta^*(2z+1)+2$ consecutive such stages there must be at least one pair of consecutive expansionary stages $t < t'$ in the set such that $m \in W^\dagger_t \setminus W^\dagger_{t'}$ for some number $m < \delta^*(2z+1)$. So $W_{t'} \restriction \varphi_m(m) \neq W_t \restriction \varphi_m(m)$. But since t' has state 0, $\varphi_m(m)$ is defined at stage s_0 hence less than s_0. So $W_{t'} \restriction s_0 \neq W_t \restriction s_0$. It follows that, if the 0-procedure is never abandoned then the number of expansionary stages $\geq s_0$ is bounded by

$$\#_0 = (\delta^*(2z+1)+2) \cdot s_0 \leq s_0^2,$$

hence R_e is met. Moreover, the 0-procedure enumerates at most $\#_0$ many numbers (hence at most s_0^2 many numbers) $\leq \psi(z)$ into B.

The n-procedure ($n \geq 1$). At stage $s_n + 1$ (if s_n exists) let $\xi(x_n) = s_n$ be the W-use of x_n and proceed as follows (note that $x_n \notin C_{s_n}$).

(*) Wait for the next expansionary stage t such that $x_n \notin C_t$ and $C_t(x_n) = W^\dagger_t(f(x_n))$. Put x_n into C at stage $t+1$, and go to (**).

(**) At the first expansionary stage $t' > t$ such that $C_{t'}(x_n) = W^\dagger_{t'}(f(x_n))$ let $S_{t'+1}(z) \neq S_{t'}(z)$. If $z \in S_{t'}$ then enumerate the least $y < \psi(z)$ which is not yet in B into $B_{t'+1}$ (we will argue below that such a y must exist).

At the first stage $t'' > t'$ such that $W_{t''+1} \restriction s_n \neq W_{t''} \restriction s_n$ (if any) remove x_n from C at stage $t''+1$, i.e., let $C_{t''+1}(x_n) = 0$, and return to (*). While waiting for such a stage t'', at any expansionary stage $u > t'$ let $S_{u+1}(z) \neq S_u(z)$. If $z \in S_u$ then enumerate the least $y < \psi(z)$ which is not yet in B into B_{u+1} (we will argue below that such a y must exist).

The success of the n-procedure, provided that it is never abandoned, and its conformity with the required features of the definitions of $C(x_n)$ and $S(z)$ are shown as follows. First note that x_n is extracted from C only at stages $t''+1$ such that $W_{t''+1} \upharpoonright s_n \neq W_{t''} \upharpoonright s_n$. Since $\xi(x_n) = s_n$, this shows that the approximation of $C(x_n)$ is as required. This justifies the assumption that $C(x_n) = W^\dagger(f(x_n))$, and we may conclude that $C_s(x_n) = W^\dagger_s(f(x_n))$ for all sufficiently large stages s. Next observe that, for any stage t as in (*) and the corresponding stage $t' > t$ in (**), $C_t(x_n) = W^\dagger_t(f(x_n)) = 0$ whereas $C_{t'}(x_n) = W^\dagger_{t'}(f(x_n)) = 1$. Moreover, since the stages t and t' are expansionary and greater than s_0, $W^\dagger_t(f(x_n)) = \Gamma_t^{\emptyset'_t \oplus S_t}(f(x_n))$ and $W^\dagger_{t'}(f(x_n)) = \Gamma_{t'}^{\emptyset'_{t'} \oplus S_{t'}}(f(x_n))$. Since S is not changed at any stage $\hat{t}$ with $t < \hat{t} \leq t'$, this implies that $\emptyset'_{t'} \upharpoonright \gamma^*(f(x_n)) \neq \emptyset'_t \upharpoonright \gamma^*(f(x_n))$. So, by $\gamma^*(f(x_n)) < s_0$, we may conclude that the procedure (*) is called $< s_0$ times. Since (*) and (**) are run alternatingly, (**) is called $< s_0$ times, too. Moreover, for any such call, we will change the approximation of $S(z)$ at most $\delta^*(2z+1)+2$ times. Namely, just as in the analysis of the 0-procedure, we may argue that if we change $S(z)$ so many times then one of theses changes must result in a change of W below $\varphi_m(m)$ for a number m such that $\varphi_m(m)$ is defined at stage s_n hence is less than s_n. So we return to (*) at this stage. It follows that the total number of $S(z)$-changes which may be caused by the n-procedure is bounded by $s_0 \cdot (\delta^*(2z+1)+2) < s_0^2$, hence the required permittings by B will be provided. Finally, if the n-attack is not abandoned, then R_e is met since the above bounds on the calls of the subprocedures (*) and (**) and the bounds on the $S(z)$-changes imply that there are only finitely many expansionary stages.

This completes the description of the strategy to meet a single requirement R_e and the proof of its correctness. The result follows by a standard application of the finite injury method. We leave this to the reader. □

Given the above failure of the potential strong analog of the Sacks' Jump Theorem for the bounded jump, we may look for some weaker analogs. The following seems a good candidate, but this is illusory.

Proposition 3.2 (Tilde Sacks' Jump Theorem)**.** *Let S be $\widetilde{\Sigma}^0_2$. There is a c.e. set A with $A^\dagger \equiv_{wtt} \emptyset' \oplus S$.*

Originally we had a direct proof, but subsequently realized that the "tilde" version of the jump theorem is of limited interest as the jump hierarchy collapses, and Proposition 3.2 follows easily from the material below.

Proposition 3.3. (i) *For any set X, X is in $\widetilde{\Sigma}^0_2$ iff X is in $\widetilde{\Sigma}^{\emptyset'}_1$ iff X is in $\widetilde{\Delta}^{\emptyset'}_1$ iff $X \leq_{wtt} \emptyset'$.*
(ii) *For any $n \geq 2$, $\widetilde{\Sigma}^0_n = \widetilde{\Pi}^0_n = \widetilde{\Delta}^0_n = \widetilde{\Delta}^0_2 = \{X : X \leq_{wtt} \emptyset'\}$.*

Proof. (i) By Proposition 2.3 (ii) and Proposition 2.2 (iii), it suffices to show that any set in $\widetilde{\Sigma}^{\emptyset'}_1$ is wtt-reducible to $\emptyset'$. So fix $X \in \widetilde{\Sigma}^{\emptyset'}_1$. By Proposition 2.2 (i) there is an index e such that $X = dom(\widehat{\Phi}^{\emptyset'}_e)$ where the bound φ_e on the use of $\widehat{\Phi}_e$ is total.

Define the c.e. set D by

$$D = \{\langle x, \sigma \rangle : |\sigma| = \varphi_e(x) \;\&\; \widehat{\Phi}_e^{\sigma}(x)\downarrow\},$$

and fix a total computable function f such that $D \leq_m \emptyset'$ via f. Then, for any $x \geq 0$,

$$x \in X \;\Leftrightarrow\; \widehat{\Phi}_e^{\emptyset'}(x)\downarrow \Leftrightarrow\; \widehat{\Phi}_e^{\emptyset' \upharpoonright \varphi_e(x)}(x)\downarrow \Leftrightarrow\; f(\langle x, \emptyset' \upharpoonright \varphi_e(x)\rangle) \in \emptyset'.$$

Obviously, this implies $X \leq_{wtt} \emptyset'$ where the use of the reduction is bounded by the total computable function

$$g(x) = \max\{\varphi_e(x), 1 + \max_{|\sigma| = \varphi_e(x)} f(\langle x, \sigma \rangle)\}.$$

(ii) The proof is by induction on n. For $n = 2$ the claim is immediate by part (i) of the proposition since $X \leq_{wtt} \emptyset'$ iff $\overline{X} \leq_{wtt} \emptyset'$. So fix $n+1 > 2$ and assume (ii) for n. By symmetry and by (i), given $X \in \widetilde{\Sigma}^0_{n+1}$ it suffices to show that $X \leq_{wtt} \emptyset'$. Fix $Y \in \widetilde{\Pi}^0_n$ such that $X \in \widetilde{\Sigma}^Y_1$. Note that $\widetilde{\Sigma}^A_1 \subseteq \widetilde{\Sigma}^B_1$ for any sets A and B such that $A \leq_{wtt} B$. So, since $Y \leq_{wtt} \emptyset'$ by inductive hypothesis, it follows that X is in $\widetilde{\Sigma}^{\emptyset'}_1$. By (i) this implies $X \leq_{wtt} \emptyset'$. $\square$

Note that Proposition 3.3 immediately implies Proposition 3.2 above if we let $A = \emptyset$. Namely, for $S \in \widetilde{\Sigma}^0_2$, it follows by Proposition 3.3 that $S \leq_{wtt} \emptyset'$ hence $A^\dagger = \emptyset^\dagger \equiv_{wtt} \emptyset' \equiv_{wtt} \emptyset' \oplus S$. This shows that we can hardly claim that Proposition 3.2 is an analog of the Sacks' Jump Theorem for the bounded jump.

In order to get stronger and more satisfying analogs we have to consider natural extensions of the tilde classes. One possible extension to Definition 2.1 (ii), more based on the complexity of approximations akin to the hierarchies of Downey and Greenberg [8], would be again to take g to be total but to relax the requirement that g is computable by having g wtt-reducible to the oracle.

Definition 3.4. *A set X is $\breve{\Sigma}^B_1$ iff there is a function g and a relation R such that $g \leq_{wtt} B$, R is g-bounded computable and*

$$x \in X \text{ iff } \exists s R^B(x, s).$$

Unfortunately, this does little for us as the following shows.

Proposition 3.5. (1) $\widehat{\Sigma}^{\emptyset'}_1 \subseteq \breve{\Sigma}^{\emptyset'}_1$.

(2) *For all $X \in \breve{\Sigma}^{\emptyset'}_1$, there exists $Y \in \widehat{\Sigma}^{\emptyset'}_1$ such that $X \leq_{tt} Y$.*

(3) *$\widehat{\Sigma}^{\emptyset'}_1$ is a proper subclass of $\breve{\Sigma}^{\emptyset'}_1$.*

Proof. For 1. Suppose that $X \in \widehat{\Sigma}^{\emptyset'}_1$. Fix a partial computable function g and a g-bounded computable relation R such that $x \in X$ iff $\exists s R^{\emptyset'}(x, s)$. Then, for h defined by $h(x) = 0$ if $g(x)\uparrow$ and $h(x) = g(x)$ if $g(x)\downarrow$, R is h-bounded computable. So it suffices to show that $h \leq_{wtt} \emptyset'$. But this is immediate by the s-m-n Theorem: for computable f such that $g(x)\downarrow$ iff $\varphi_{f(x)}(f(x))\downarrow$ (i.e., $f(x) \in \emptyset'$), $h(x)$ can be computed from $\emptyset' \upharpoonright f(x) + 1$.

For 2. Let $X \in \breve{\Sigma}_1^{\emptyset'}$ via $g(x) = \lim_s g(x,s)$ and $R^{\emptyset'}(x,s)$. which changes at most $h(x)-1$ many times. (Recall that $g \leq_{wtt} \emptyset'$ iff g is ω-c.a. by the proof of the Limit Lemma.) We define a tt-reduction $\Gamma^Y = X$ via auxilary elements $x_1, \ldots, x_{h(x)}$ and $y_1, \ldots, y_{h(x)}$ separate elements from $(\emptyset')^{r(x)}$ whose indices are given by the Recursion Theorem, and these are controlled by $\emptyset'$ and partial computable functions $q_{\langle x,1\rangle}, \ldots, q_{\langle x,h(x)\rangle}$, whose indices are also given by the Recursion Theorem.

The reduction is that $x \in X$ iff $x_i \in Y$ where i is the least index with $y_i \in \emptyset'$ and $y_{i+1} \notin \emptyset'$. The plan is that we let $q(x_1) \downarrow [0] = g(x,0)$, and we put y_1 into $\emptyset'[0]$. While $g(x,s) = g(x,0)$ we will observe $R^{\emptyset'}(x,s)$ with this value for g, and we keep $x_1 \in Y_s$ iff $x \in x \in X_s$. If ever we see $g(x,t+1) \neq g(x,t)$ we will enumerate $y_2 \in \emptyset'[t+1]$[1] We would also define $q_{\langle x,2\rangle}(x_2) \downarrow [t+1] = g(x,t+1)$, and now begin the simulation of $R^{\emptyset'}(x,s)$ with this new value for g. Again until $g(x,u) \neq g(x,t+1)$, we would use x_2 as a representation of x, and have $x_1 \in Y_s$ iff $x \in x \in X_s$, for $s \geq t+1$. The pattern continues, and each time a new value of $g(x,u)$ is given we put the next value of y_j into $\emptyset'$ and use x_j and $q_{\langle x,j\rangle}$ for simulation of $R^{\emptyset'}(x,s)$ with this value for g. Since there are at most $h(x)-1$ many changes for $g(x,s$ we won't run out of y_i, $q_{\langle x,i\rangle}$ and x_i, and the last one we use gives the correct simulation, and validates the tt-reduction $\Gamma^Y = X$.

For 3. We build $X \in \breve{\Sigma}_1^{\emptyset'}$ such that $X \notin \widehat{\Sigma}_2^0$. To do this we will meet

$$Q_e : X \neq Y_e \text{ where } z \in Y_e \text{ iff } \varphi_e(x) \downarrow \wedge \exists s R_e^{\emptyset'}(z,s).$$

Without loss of generality we can choose R_e to be an enumeration if primitive recursive relations. we build our own computable R and ω-c.a. function $g(x,s)$. For a single Q_e we will pick a follower x. We initially have $g(x,s) = 0$ and continue with this until $\varphi_e(x) \downarrow [s]$. At this stage, we would define $g(x,s+1) = d$ a large number bigger than (e.g.) $2^{\varphi_e(x)}$ many numbers under our control for this requirement, whose indices are given by ther Recursion Theorem. We will have $x \in X_u$ (for $u > s$) iff the number of elements from this list in $\emptyset'$ is odd. Then using these elements it is easy to make sure that $x \in Y_{e,u}$ iff $x \notin X_u$.

There is no injury so this works by putting the modules together. □

4. The Low Basis Theorem

A classical result on Π_1^0-classes is the superlow basis theorem of Jockusch and Soare [12] which states that a nonempty Π_1^0 class has a member of superlow degree, and hence a member A for which $A^\dagger \equiv_{wtt} \emptyset'$ in particular. One of the corollaries to the proof is that coding can also be used. Hence, as stated in [12], if P is a special Π_1^0 class (i.e., P is nonempty and has no computable members) then, for any set S with $\emptyset' \leq_T S$, P contains a member A with $A' \equiv_T S$. We will see in this section that this generalization fails for the wtt-jump. Recall the proof of the Generalized Low Basis Theorem.

[1]Strictly speaking this would go into $\emptyset'[t']$ where $t' \geq t+1$ is given by the overhead of the Recursion Theorem, but we can pretend that this primitive recursive overhead is for the sake of readability.

Let T be a computable tree with $\mathrm{P} = [T]$. Of course T has $2^{\aleph_0}$ many paths as it has no computable paths. We inductively define computable trees T_n such that $T = T_0 \supseteq T_1 \supseteq \ldots$. In the first step, given T_{2e}, as usual let $U_{2e} = \{\sigma \in T_{2e} : \Phi_e^\sigma(e)[|\sigma|] \uparrow\}$. If U_{2e} is finite let $T_{2e+1} = T_{2e}$ else $T_{2e+1} = U_{2e}$ which also has $2^{\aleph_0}$ many paths. Note that this decision is $\leq_{wtt} \emptyset'$, which is the reason why the Low Basis Theorem actually gives a superlow member.

The second step is to find the lex-least pair (σ_0, σ_1) of incomparable strings in T_{2e+1} with $\sigma_0 <_L \sigma_1$, and let T_{2e+2} be the full subtree of T_{2e+1} above $\sigma_{S(e)}$, and then let $A \in \bigcap_e [T_e]$.

It is this second step which fails to work for the bounded jump $A^\dagger$ since we seem to need the full power of a Turing reduction to $\emptyset'$ to find these two strings. This is a fatal problem:

Theorem 4.1. *There is a set S with $\emptyset' <_{wtt} S$ and a computable tree T with $[T] \neq \emptyset$ and no computable path such that, for all $A \in [T]$, $A^\dagger \not\equiv_{wtt} S$.*

Proof. The argument is finite injury. It suffices to define a Δ_2^0 set D and a computable tree T such that the following hold.

$$|[T]| = 2^{\aleph_0} \text{ (hence, in particular, } [T] \neq \emptyset) \tag{1}$$

$$\forall\, A \in [T]\ (A \text{ is not computable}) \tag{2}$$

$$\forall\, A \in [T]\ (D \not\leq_{wtt} A^\dagger) \tag{3}$$

(Then, for instance, the set $S = \emptyset'' \oplus D$ has the required properties.)

The scheme for defining T is as follows. We give a computable approximation $\iota : 2^{<\omega} \times \omega \to 2^{<\omega}$ (where $\lambda\sigma.\iota(\sigma, s)$ is defined at stage s of the construction) of a function $\iota^* : 2^{<\omega} \to 2^{<\omega}$ which induces an isomorphism from 2^ω onto $[T]$ for the desired tree T.

To be more precise, for any string σ and any stage s we guarantee

$$\forall\, i \leq 1\ (\iota(\sigma, s) \prec \iota(\sigma i, s)) \tag{4}$$

and

$$\iota(\sigma 0, s) \text{ and } \iota(\sigma 1, s) \text{ are incomparable,} \tag{5}$$

and we ensure that

$$\iota^*(\sigma) = \lim_{s \to \omega} \iota(\sigma, s) \text{ exists.} \tag{6}$$

Initially we let $\iota(\sigma, 0) = \sigma$ for any string σ. Then, at stage $s+1$, we guarantee

$$range(\lambda\sigma.\iota(\sigma, s+1)) \subseteq range(\lambda\sigma.\iota(\sigma, s)). \tag{7}$$

So, if we let

$$T_s = \{\tau : \exists\sigma(\tau \preceq \iota(\sigma, s))\}$$

then $T_0 = 2^{<\omega}$ and, by computability of ι and by (4) and (5), T_s, $s \geq 0$, are uniformly computable trees such that $[T_s]$ is isomorphic to 2^ω. Moreover, by (7), $T_{s+1} \subseteq T_s$ and, by (6),

$$T = \lim_{s \to \infty} T_s \tag{8}$$

exists and satisfies (1). Note that the tree T might be not computable. But, by (4) and (5), a node τ has an extension in $[T_s]$ iff $\tau \preceq \iota(\sigma, s)$ for some string σ of length $|\tau|$, and, by (6) and (7), τ has an extension in $[T]$ iff τ has extensions in $[T_s]$ for all $s \geq 0$. So the set of nodes which do not have extensions in $[T]$ is c.e., hence $\overline{[T]}$ is a Σ^0_1-class. It follows that there is a computable tree $\hat{T}$ with $[T] = [\hat{T}]$. So it suffices to ensure that the tree T defined by (8) satisfies (2) and (3). (Note that, as observed above already, T satisfies (1).)

For this sake it suffices to meet the requirements

$$R_{2e} : \forall\ A \in [T]\ (A \neq \varphi_e)$$

and

$$R_{2e+1} : \forall\ A \in [T]\ (\widehat{\Phi}_e^{A^\dagger} \neq D),$$

respectively ($e \geq 0$).

The requirement R_n is allowed to change the approximations to $\iota^*(\sigma)$ only for the nodes σ of length $\geq n$. Since the requirements will be finitary, this guarantees (6). The requirement R_n is split into the subrequirements $R_{n,\sigma}$ where $|\sigma| = n$ and where $R_{n,\sigma}$ guarantees that R_n is met for all sets A such that $\iota^*(\sigma)$ is an initial segment of A. By definition of T this ensures that R_n is met. Moreover, whenever $R_{n,\sigma}$ changes the approximation ι at a stage $s+1$, then it moves $\iota(\sigma, s+1)$ to $\iota(\sigma\tau, \sigma)$ for a proper extension $\sigma\tau$ of σ and simultaneously moves $\iota(\sigma', s+1)$ to the corresponding extension $\iota(\sigma'\tau, s)$ of $\iota(\sigma', s)$ for any proper extension σ' of σ (and leaves $\iota(\sigma'', s)$ unchanged for nodes σ'' not extending σ). Obviously, this ensures that ι has the required properties (4), (5) and (7).

The strategy to meet the noncomputability requirement $R_{2e,\sigma}$ for any node σ with $|\sigma| = 2e$ is as follows. Wait for a stage $s+1$, such that $\varphi_{e,s}(x)$ is defined for the least x such that $\iota(\sigma 0, s)(x) \neq \iota(\sigma 1, s)(x)$ (note that, by (5) and (4), such an x exists and $x \geq |\iota(\sigma, s)|$; also note that if there is no such stage s then, by (6), φ_e is not total and R_{2e} is trivially met). Then fix $i \leq 1$ minimal such that $\iota(\sigma i, s)(x) \neq \varphi_e(x)$ and let $\iota(\sigma\tau, s+1) = \iota(\sigma i \tau, s)$ for all strings τ (and let $\iota(\tilde{\sigma}, s+1) = \iota(\tilde{\sigma}, s)$ for any node $\tilde{\sigma}$ such that $\sigma \not\preceq \tilde{\sigma}$). If no higher priority requirement R_n ($n < 2e$) acts after stage s this will guarantee that $\iota^*(\sigma) = \iota(\sigma i, s)$ hence $A(x) \neq \varphi_e(x)$ for all sets A extending $\iota^*(\sigma)$.

The strategy to meet the jump requirement $R_{2e+1,\sigma}$ for any node σ with $|\sigma| = 2e+1$ is as follows. First we fix a number x_σ to be reserved for this subrequirement. Next we will define $\iota^*(\sigma)$ in such a way that if there is a set A such that $\iota^*(\sigma) \prec A$ and $\widehat{\Phi}_e^{A^\dagger}(x_\sigma)$ is defined then, for all such sets A, the value of $\widehat{\Phi}_e^{A^\dagger}(x_\sigma)$ depends only on $\iota^*(\sigma)$. So, finally, we will meet R_{2e+1} for A extending $\iota^*(\sigma)$ by letting $D(x_\sigma)$ diagonalize against this value. The details are as follows.

Wait for a stage s_σ such that $\varphi_{e,s_\sigma}(x_\sigma)\downarrow$ (if there is no such stage then $\widehat{\Phi}_e^{A^\dagger}(x_\sigma)\uparrow$ for all sets A hence R_{2e+1} is trivially met). Then, for any stage $s \geq s_\sigma$, let

$$U(\sigma,s)=\{z : z<\varphi_e(x_\sigma)\ \&\ \varphi_{z,s}(z)\downarrow\} \text{ and } u(\sigma,s)=\max\{\varphi_z(z) : z\in U(\sigma,s)\}.$$

If $|\iota(\sigma,s)|<u(\sigma,s)$ then fix the least (in the sense of the length-lexicographic order) σ' extending σ such that $|\iota(\sigma',s)|\geq u(\sigma,s)$ and let $\iota(\sigma\tau,s+1)=\iota(\sigma'\tau,s)$ for all strings τ (and let $\iota(\tilde{\sigma},s+1)=\iota(\tilde{\sigma},s)$ for any node $\tilde{\sigma}$ such that $\sigma\not\preceq\tilde{\sigma}$). Note that there are at most $\varphi_e(x_\sigma)$ stages s at which $u(\sigma,s)$ may grow. So this action is finitary (hence compatible with (6)). Finally, define the computable approximation $D(x_\sigma,s+1)$ of $D(x_\sigma)$ at stage $s+1>s_\sigma$ as follows (and let $D(x_\sigma,s)=0$ if $s\leq s_\sigma$ or s_σ is not defined). For any set A let $A^\dagger_s=\{z : \varphi_{z,s}(z)\downarrow\ \&\ \widehat{\Phi}^A_{z,s}(z)\downarrow\}$. Then $D(x_\sigma,s+1)=1$ if $\widehat{\Phi}_{e,s}^{(\iota(\sigma,s+1))^\dagger}(x_\sigma)\downarrow=0$ and $D(x_\sigma,s+1)=0$ otherwise (where we view $\iota(\sigma,s+1)$ as the set of the numbers $y<|\iota(\sigma,s+1)|$ such that $\iota(\sigma,s+1)(y)=1$). Note that, by (6), $\widehat{\Phi}_{e,s}^{(\iota(\sigma,s+1))^\dagger}(x_\sigma)=\widehat{\Phi}_{e}^{(\iota^*(\sigma))^\dagger}(x_\sigma)$ for all sufficiently large s, hence $D(x_\sigma)=\lim_{s\to\omega}D(x_\sigma,s)$ exists.

In order to show that the above strategy succeeds to meet $R_{2e+1,\sigma}$, w.l.o.g. assume that $\varphi_e(x_\sigma)\downarrow$ and that A is a set extending $\iota^*(\sigma)$ such that $\widehat{\Phi}_e^{A^\dagger}(x_\sigma)\downarrow$. Since $D(x_\sigma)=\lim_{s\to\omega}D(x_\sigma,s)$, it suffices to show that $D(x_\sigma,s+1)\neq\widehat{\Phi}_e^{A^\dagger}(x_\sigma)$ for all sufficiently large s. So fix s large enough such that $s\geq s_\sigma$, $\iota(\sigma,s+1)=\iota^*(\sigma)$, $U(\sigma,s)=\{z<\varphi_e(x_\sigma) : \varphi_z(z)\downarrow\}$ and $\widehat{\Phi}_{e,s}^{A^\dagger}(x_\sigma)\downarrow$. Since, by $s\geq s_\sigma$ and by construction, $D(x_\sigma,s+1)\neq\widehat{\Phi}_{e,s}^{(\iota(\sigma,s+1))^\dagger}(x_\sigma)$, it suffices to show that $(\iota(\sigma,s+1))^\dagger\restriction\varphi_e(x_\sigma)=A^\dagger\restriction\varphi_e(x_\sigma)$. So fix $z<\varphi_e(x_\sigma)$. If $\varphi_z(z)\uparrow$ then $(\iota(\sigma,s+1))^\dagger(z)=A^\dagger(z)=0$. So we may assume $\varphi_z(z)\downarrow$. Then, by choice of s and by construction, $\varphi_z(z)\leq u(\sigma,s)\leq|\iota(\sigma,s+1)|$. Since $\iota(\sigma,s+1)=\iota^*(\sigma)\prec A$, it follows that $\iota(\sigma,s+1)\restriction\varphi_z(z)=A\restriction\varphi_z(z)$. So $\widehat{\Phi}_z^{\iota(\sigma,s+1)}(z)\downarrow$ iff $\widehat{\Phi}_z^{A}(z)\downarrow$, i.e., $(\iota(\sigma,s+1))^\dagger(z)=A^\dagger(z)$.

This completes the description of the individual strategies. The rest is a straightforward application of the finite injury method. $\square$

References

[1] K. Ambos-Spies, R. Downey and M. Monath, On the computational power of maximal sets, in preparation.

[2] B. Anderson and B. Csima, A bounded jump for the bounded Turing degrees, *Notre Dame J. Form. Log.* 55 (2014), no. 2, 245–264.

[3] B. Afshari, G. Barmpalias, S. B. Cooper and F. Stephan, Post's programme for the Ershov hierarchy, *J. Logic Comput.* 17 (2007), no. 6, 1025–1040.

[4] G. Barmpalias, R. Downey and N. Greenberg, Working with strong reducibilities above totally ω-c.e. and array computable degrees, *Trans. Amer. Math. Soc.* 362 (2010), no. 2, 777–813.

[5] J. Chisholm, J. Chubb, V. Harizanov, D. Hirschfeldt, C. Jockusch, Jr., T. McNicholl, and S. Pingrey, Π^0_1 classes and strong degree spectra of relations *J. Symbolic Logic* 72 (2007), no. 3, 1003–1018.

[6] R. Coles, R. Downey and G. LaForte, Notes on wtt-jump and ordinal notations. Manuscript, 1998.

[7] B. Csima, R. Downey and K. M. Ng, Limits on jump inversion for strong reducibilities, *J. Symbolic Logic* 76 (2011), no. 4, 1287–1296.

[8] R. Downey and N. Greenberg, A Hierarchy of Turing Degrees, in Series: *Annals of Mathematics Studies*, Princeton University Press, in press.

[9] R. Downey and D. Hirschfeldt, *Algorithmic Randomness and Complexity*, Springer-Verlag, 2010.
[10] Y. Ershov, A certain hierarchy of sets. I. (Russian), *Algebra i Logika* 7 (1968), no. 1, 47–74.
[11] G. Gerla, Una generalizzazione della gerarchia di Ershov, *Bolletino U.M.I.* (5) 16-B (1979), no. 2 765–778.
[12] C. Jockusch and R. Soare, Degrees of members of Π^0_1 classes, *Pacific J. Math.* 40 (1972), 605–616.
[13] S. Kleene and E. Post, The upper semi-lattice of degrees of recursive unsolvability, *Ann. of Math. (2)* 59 (1954), 379–407.
[14] J. Mohrherr, Density of a final segment of the truth-table degrees, *Pacific J. Math.* 115 (1984), no. 2, 409–419.
[15] P. Odifreddi, Classical Recursion Theory, Vol. II, North-Holland, 1999.
[16] G. Sacks, On the degrees less than $\mathbf{0}'$, *Ann. of Math. (2)* 77 (1963), no. 2, 211–231.
[17] A. Turing, Systems of logic based on ordinals, *Proc. London Math. Soc. (2)* 45 (1939), no. 3, 161–228.
[18] G. Wu and H. Wu, Bounded jump and the high/low hierarchy, in *Theory and applications of models of computation, Lecture Notes in Comput. Sci., 11436, Springer, Cham, 2019*, 647–658.

University of Heidelberg

E-mail address: ambos@math.uni-heidelberg.de

Victoria University of Wellington

E-mail address: Rod.Downey@msor.vuw.ac.nz

University of Heidelberg

E-mail address: martin.monath@posteo.de