# The Complexity of Irredundant Sets Parameterized By Size

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**Abstract**

An *irredundant* set of vertices $V' \subseteq V$ in a graph $G = (V, E)$ has the property that for every vertex $u \in V'$, $N(V' - \{u\})$ is a proper subset of $N(V')$. We investigate the parameterized complexity of determining whether a graph has an irredundant set of size $k$, where $k$ is the parameter. The interest of this problem is that while most "$k$-element vertex set" problems are *NP*-complete, several are known to be fixed-parameter tractable, and others are hard for various levels of the parameterized complexity hierarchy. Complexity classification of vertex set problems in this framework has proved to be both more interesting and more difficult. We prove that the $k$-element irredundant set problem is complete for $W[1]$, and thus has the same parameterized complexity as the problem of determining whether a graph has a $k$-clique. We also show that the "parametric dual" problem of determining whether a graph has an irredundant set of size $n - k$ is fixed-parameter tractable.

*Keywords: irredundant sets, parameterized complexity*

## 1 Introduction

For many computational problems the input consists of several parts, and it is useful to study how the different parts contribute to overall problem complexity. For example, many well-known decision problems concerning graphs including Clique, Dominating Set, Graph Genus, Min Cut Linear Arrangement, Bandwidth, Vertex Cover, Feedback Vertex Set, Perfect Code and the Irredundant Set problem that we consider here, take as input a graph $G$ and a positive integer $k$.

The parameter $k$ appears to contribute to the complexity of these problems in two qualitatively distinct ways. Graph Genus, Min Cut Linear Arrangement, Vertex Cover and Feedback Vertex Set for Undirected Graphs can all be solved in time $O(f(k)n^c)$ where $c$ is a constant independent of $k$ and $f$ is some (arbitrary) function. This "good behavior" is termed *fixed-parameter tractability* in the theory introduced by Downey and Fellows in [**?**]. As is the case with the polynomial-time complexity, the exponent $c$ is typically small.

Contrasting complexity behaviour is exhibited by the problems Clique, Dominating Set and Bandwidth, for which the best known algorithms have running times $O(n^{ck})$. These problems have been shown to be complete or hard for various levels of the $W$ hierarchy of parameterized complexity

$$W[1] \subseteq W[2] \subseteq \cdots W[P]$$

and this can be taken as evidence that they are unlikely to be fixed-parameter tractable.

As in the theory of *NP*-completeness, there are roughly two kinds of evidence. The first is that given a sufficient amount of unsuccessful effort to demonstrate tractability for various problems in a class, the knowledge that a problem is hard for the class offers a cautionary sociological message, of the sort depicted in the famous cartoon in the opening pages of [**?**]. Secondly, one may have some sort of direct intuition about why a problem complete or hard for a certain computational resource class should not be a lot easier.

For parameterized complexity, both kinds of evidence are available. Although the amount of unsuccessful effort that has been expended in attempts to show fixed-parameter tractability for $W[1]$-hard problems is much less than the total effort expended to date in attempting to develop polynomial-time algorithms for *NP*-complete problems, it is still considerable and accumulating.

Direct intuition about $W[1]$ is also available. It is shown in [**?**, **?**] that the $k$-Step Halting Problem for Nondeterministic Turing Machines is $W[1]$-complete. This is a problem so generic and opaque that it is hard to imagine that there is any algorithm for it that radically improves on simply exploring the $n$-branching depth $k$ tree of allowed transitions exhaustively.

The complexity of simple graph problems has in many cases proved to be much more difficult to settle in the parameterized framework than in the classical (*P versus NP*) framework. For example, there is presently no information about the $k$-element Feedback Vertex Set problem for directed graphs. The $k$-element Perfect Code problem is known to be in $W[2]$ and hard for $W[1]$ — does it represent a parameterized complexity degree intermediate between $W[1]$ and $W[2]$?

The Irredundant Set problem asks whether a given graph $G = (V, E)$ on $n$ vertices has a $k$-element irredundant set. The Co-Irredundant Set problem asks if $G$ has an irredundant set of size $n - k$. A set of vertices $V' \subseteq V$ is *irredundant* if for every vertex $u \in V'$, $N[V' - \{u\}]$ is a proper subset of $N[V']$. For both problems we consider the parameter to be $k$. That both are *NP*-complete (note that classically they are the same problem) was proved in [**?**]. We prove:

**Theorem 1.** Irredundant Set is $W[1]$-complete.

**Theorem 2.** Co-Irredundant Set is fixed-parameter tractable.

For general background on parameterized complexity see [?, ?]. We will assume that the reader has already this basic background concerning the formal foundations of the theory. Parameterized complexity analyses of various graph problems can be found in [?, ?, ?, ?]. A compendium of known results can be accessed on the World Wide Web.

We use the following notation. If $G = (V, E)$ is a graph and $u \in V$ is a vertex of $G$, then the *open neighborhood* $N(u)$ of $u$ is defined to be $N(u) = \{v : (u, v) \in E\}$. The *closed neighborhood* $N[u]$ of $u$ is $N[u] = N(u) \cup \{u\}$.

## 2 $W[1]$-Completeness

In this section we prove that the Irredundant Set problem parameterized by the number of vertices in the set is complete for $W[1]$. Our theorem settles a question raised in [?] where it was asked if Irredundant Set might represent a parameterized degree intermediate between *FPT* and $W[1]$.

The following is an equivalent definition of irredundance that we will use in our argument.

**Definition.** A set of vertices $J \subseteq V$ in a graph $G = (V, E)$ is *irredundant* if each vertex $u \in J$ has a *private neighbor* $\pi(u)$ in $V$ satisfying the conditions:
(1) $u$ is adjacent to $\pi(u)$, and
(2) no other vertex of $J$ is adjacent to $\pi(u)$.
If $\pi(u) = u$ then we will say that $u$ is *self-private.*

We next state a simple property about private neighbors that will be used frequently in our arguments.

**Lemma 1.** If $J$ is an irredundant set in a graph $G = (V, E)$ and if $u, v$ are distinct vertices of $J$ with $N(u) = N(v)$, then: (1) $u$ and $v$ are nonadjacent, and (2) both $u$ and $v$ are self private. □

**Theorem 1.** Irredundant Set is complete for $W[1]$.

**Proof.** Membership in $W[1]$ is proved in [?]. In order to show hardness for $W[1]$, we reduce from Clique, shown to be complete for $W[1]$ in [?]. Suppose that we are given a simple graph $G = (V, E)$ and an integer $k$. We describe an FPT transformation that produces a graph $G' = (V', E')$ and a positive integer $k'$ so that $G'$ has an irredundant set of size $k'$ if and only if $G$ has a $k$-clique. If we let $n$ denote the number of vertices in $G$, then in fact our transformation can be be computed in time polynomial in $n$ and $k$.

The integer $k'$ is described:

$$k' = k(k-1)(3k^2) + k(k-1) + 3k^2 \binom{k}{2}$$

Assume for convenience that the vertex set $V$ of $G$ is linearly ordered, and that $E$ consists of the ordered pairs of adjacent vertices $(u, v)$ with $u < v$. (Thus each edge is uniquely represented in $E$.) We will use the following set of index pairs in describing $G'$.

$$\Gamma = \{(\alpha, \beta) : 1 \leq \alpha < \beta \leq k\}$$

The vertex set $V'$ of $G'$ is next described.

$$V' = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$$

where

$$\begin{aligned}
\mathcal{A} &= \bigcup_{i=1}^{k} \mathcal{A}(i) \\
\mathcal{A}(i) &= \{a[i, j, u, r] : j \in \{1, ..., k\},\ j \neq i,\ u \in V,\ 1 \leq r \leq 3k^2\} \\
\mathcal{B} &= \bigcup_{i=1}^{k} \bigcup_{j \in \{1,...,i-1,i+1,...,k\}} \mathcal{B}(i, j) \\
\mathcal{B}(i, j) &= \{b[i, j, (v, w)] : (v, w) \in E\} \\
\mathcal{C} &= \bigcup_{(\alpha,\beta)\in\Gamma} \mathcal{C}(\alpha, \beta) \\
\mathcal{C}(\alpha, \beta) &= \{c[\alpha, \beta, (x, y), r] : (x, y) \in E,\ 1 \leq r \leq 3k^2
\end{aligned}$$

For convenience, we also define the following sets.

$$\begin{aligned}
\mathcal{A}(i, u) &= \{a[i, j, u, r] : j \in \{1, ..., k\},\ j \neq i,\ 1 \leq r \leq 3k^2\} \\
\mathcal{A}(i, j) &= \{a[i, j, u, r] : u \in V,\ 1 \leq r \leq 3k^2\} \\
\mathcal{A}(i, j, u) &= \mathcal{A}(i, u) \cap \mathcal{A}(i, j) \\
\mathcal{B}(i) &= \bigcup_{j \in \{1,...,i-1,i+1,...,k\}} \mathcal{B}(i, j) \\
\mathcal{C}(\alpha, \beta, (x, y)) &= \{c[\alpha, \beta, (x, y), r] : 1 \leq r \leq 3k^2\}
\end{aligned}$$

Sometimes for convenience we will use $e$ (or such) to denote the edge index (or coordinate) of some vertex or set, $e \in E$, and write for example, $\mathcal{C}(\alpha, \beta, e)$ or $b[i, j, e]$.

The edge set $E'$ of $G'$ is described as follows, where in building the sets, the indices implicitly range over all possibilities allowed by the definition of $V'$.

$$E' = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \cup \mathcal{E}_5$$

where

$$\begin{aligned}
\mathcal{E}_1 &= \{(a[i, j, u, r], a[i', j', u', r']) : i = i' \wedge u \neq u'\} \\
\mathcal{E}_2 &= \{(b[i, j, (v, w)], b[i', j', (v', w')]) : i = i' \wedge j = j'\}
\end{aligned}$$

$$\mathcal{E}_3 = \{(c[\alpha,\beta,(x,y),r], c[\alpha',\beta',(x',y'),r']) : \alpha = \alpha' \wedge \beta = \beta' \wedge (x \neq x' \vee y \neq y')\}$$
$$\mathcal{E}_4 = \{(a[i,j,u,r], b[i',j',(v,w)] : i = i' \wedge j = j' \wedge ((j < i \wedge w \neq u) \vee (i < j \wedge v \neq u))\}$$
$$\mathcal{E}_5 = \{(b[i,j,(v,w)], c[\alpha,\beta,(x,y),r] : (j < i \wedge j = \alpha \wedge i = \beta \wedge (v,w) \neq (x,y)) \vee (i < j \wedge i = \alpha \wedge j = \beta \wedge (v,w) \neq (x,y))\}$$

The overall construction may be intuitively described. The vertex sets $\mathcal{A}_i$, for $i = 1, ..., k$ form the "vertex gadgets" for representing a $k$-clique in $G$. Associated with the vertex gadget of index $i$ is a family of "edge selection gadgets", one for each index $j \neq i$. The way that the $k$-clique is to be represented can be thought of as follows. First, the $k$ vertices $v_1, ..., v_k$ of $G$ are to be selected by the vertex gadgets. Now consider a pair of selected vertices, e.g. $(v_2, v_5)$. In an edge selection gadget associated to the vertex selection gadget with index 2, an edge will be selected as "going to $v_5$". Similarly in an edge selection gadget associated to the vertex selection gadget with index 5, an edge will be selected as "going to $v_2$".

In order for the selection mechanisms to represent a clique in $G$, various consistencies must be enforced by the construction of $G'$. In particular: (1) the selected vertices must be distinct, (2) the edge selected as, e.g., "going from $v_2$ to $v_5$" must be incident on $v_2$, and similarly, the edge selected in the corresponding but distinct edge selection gadget as "going to $v_5$ from $v_2$" must be incident on $v_5$, and (3) the edge selected as "going from $v_2$ to $v_5$" must be the same as the edge selected as "going to $v_5$ from $v_2$".

It may seem to the reader that this general plan for the reduction is overly complicated. Why not just *select* $k$ vertices, and for each pair, *check* that they are adjacent? Note that this involves vertex-vertex consistency (i.e., adjacency-checking) gadgets. The overall plan here is to *select* $k$ vertices, and to *select* $\binom{k}{2}$ edges, with each edge selected twice, once in each direction, and then employ vertex-edge consistency (i.e., incidence-checking) and edge-edge consistency (i.e., equality-checking) gadgets. We remark that this more complicated architecture is not uncommon in $W[1]$ hardness proofs (e.g. [**?**]) and is in fact one of the main tricks of the trade. The seemingly simpler vertex-vertex (adjacency-checking) gadgets seem to be simply unavailable for some problems.

Let $J$ denote an irredundant set in $G'$. We will say that $J$ is *properly distributed* if it satisfies the following conditions:
(1) For all $i$, $1 \leq i \leq k$, there is a unique $u \in V$ such that $J \cap \mathcal{A}(i) = \mathcal{A}(i,u)$. It follows that $|J \cap \mathcal{A}| = k(k-1)(3k^2)$.
(2) For all $i$, $1 \leq i \leq k$, and for all $j \neq i$, there is a unique $(v,w) \in E$ such that $J \cap \mathcal{B}(i,j) = \{b[i,j,(v,w)]\}$. It follows that $|J \cap \mathcal{B}| = k(k-1)$.
(3) For all $(\alpha,\beta) \in \Gamma$ there is a unique edge index $(x,y)$ such that $J \cap \mathcal{C}(\alpha,\beta) = \mathcal{C}(\alpha,\beta,(x,y))$. It follows that $|J \cap \mathcal{C}| = 3k^2\binom{k}{2}$.

Note that $k'$ is "explained" by the notion of a properly distributed irredundant set in $G'$.

The proof of correctness for the transformation is based on the following three main claims.

*Claim A.* If $G$ has a $k$-clique then $G'$ has an irredundant set of size $k'$.

*Proof.* Let $v_1, ..., v_k$ be distinct vertices in $G$ forming a $k$-clique, with $v_1 < v_2 < \cdots < v_k$ in the linear ordering of $V$. The reader can verify from the definition of $G'$ that the following set $J$ of $k'$ vertices is irredundant in $G'$.

$$J = J_1 \cup J_2 \cup J_3$$

where

$$\begin{aligned} J_1 &= \bigcup_{i=1}^{k} \mathcal{A}(i, v_i) \\ J_2 &= \bigcup_{(\alpha,\beta)\in\Gamma} \{b[\alpha, \beta, (v_\alpha, v_\beta)], b[\beta, \alpha, (v_\alpha, v_\beta)]\} \\ J_3 &= \bigcup_{(\alpha,\beta)\in\Gamma} \mathcal{C}(\alpha, \beta, (v_\alpha, v_\beta)) \end{aligned}$$

Each vertex of $J$ is self-private. □

*Claim B.* If $G'$ admits a properly distributed irredundant set of cardinality $k'$ then $G$ has a $k$-clique.

*Proof.* Let $J$ denote the irredundant set. We first argue that if $J \cap \mathcal{C}(\alpha, \beta) = \mathcal{C}(\alpha, \beta, (x, y))$ and $J \cap \mathcal{B}(\alpha, \beta) = \{b[\alpha, \beta, (u, v)]\}$, then $(u, v) = (x, y)$. If not, then consider the vertex $c[\alpha, \beta, (x, y), 1] \in J$. By the definition of $E_5$, this vertex is adjacent to $b[\alpha, \beta, (u, v)] \in J$ and thus cannot be self-private. Since it has the same open neighborhood as $c[\alpha, \beta, (x, y), 2] \in J$, Lemma 1 yields a contradiction. Thus for each $(\alpha, \beta) \in \Gamma$, the edge-selection and edge-check gadgets indicate (via $J$) consistent information.

We next argue that the edges indicated by $J$ in any vertex gadget are all incident on the vertex indicated by $J$ in the gadget. Let $i \in \{1, ..., k\}$ and consider two cases: (1) $j > i$, (2) $j < i$, where $1 \leq j \leq k$. Since the argument is essentially the same, we will treat only (1).

Suppose $J \cap \mathcal{B}(i, j) = \{b[i, j, (u, v)]\}$, and for convenience let $z = b[i, j, (u, v)]$. $\pi(z) \notin \mathcal{C}(i, j)$ because, by the argument immediately above, $c[i, j, (u, v), 1] \in J$ and this vertex is adjacent to everything in $\mathcal{C}(i, j)$ that is adjacent to $z$. Thus $\pi(z) \notin \mathcal{C}$. Also, we cannot have $\pi(z) = z' \in \mathcal{B}(i, j)$ with $z' \neq z$, since $c[i, j, (u, v), 1]$ is adjacent to all such $z'$.

Since $J$ is properly distributed, we have $J \cap \mathcal{A}(i) = \mathcal{A}(i, x)$ for some $x \in V$. What we must argue is that $u = x$, and that $u$ is adjacent to $v$ in $G$.

If $u \neq x$ then $z$ is adjacent to the vertices of $J$ in $\mathcal{A}(i, x)$, and therefore cannot be self private, and furthermore $\pi(z) \notin \mathcal{A}(i, x)$. This implies that $\pi(z)$ must be in $\mathcal{A}(i, x')$ for some $x' \neq x$. This is impossible, since the vertices of $J$ in $\mathcal{A}(i, x)$ dominate $\mathcal{A}(i)$.

It follows that $u = x$ and that $z$ is self-private. The latter implies the edge index $(u, v)$ represents an edge present in $G$, by the definition of $\mathcal{E}_4$. The vertices indicated by $J$ in the vertex selection gadgets of $G'$ therefore form a $k$-clique in $G$. □

*Claim C.* Any irredundant set in $G'$ of size $k'$ must be properly distributed.

Together, Claims A,B and C yield the theorem. It remains to establish Claim C. Our argument is based on a series of lesser claims.

*Claim C.1* If $J$ is an irredundant set in $G'$, then for all $(\alpha, \beta) \in \Gamma$, there can be at most two distinct edge indices $e_1 = (x, y)$ and $e_2 = (x', y')$ such that $J$ has nontrivial intersection with $\mathcal{C}(\alpha, \beta, e_i)$.

*Proof.* Suppose there were three distinct edge indices $e_i$ yielding nontrivial intersections with $J$, and let $z_i$ $(i = 1, 2, 3)$ denote three representative vertices in these intersections. Since these vertices are adjacent in $\mathcal{C}(\alpha, \beta)$ they cannot be self-private. Two of them must therefore have private neighbors in either $\mathcal{B}(\alpha, \beta)$ or $\mathcal{B}(\beta, \alpha)$. Without loss of generality, suppose the two are $z_1$ and $z_2$ and that $\pi(z_i) \in \mathcal{B}(\alpha, \beta)$ for $i = 1, 2$. It must be the case that $\pi(z_1) = b[\alpha, \beta, e_2]$, since otherwise $z_2$ would be adjacent to $\pi(z_1)$, by the definition of $\mathcal{E}_5$. But considering $z_3$, the same reasoning implies $\pi(z_1) = b[\alpha, \beta, e_3]$, a contradiction. □

*Claim C.2* Suppose $J$ is an irredundant set in $G'$ and that $J' = J \cap \mathcal{B}(i, j)$ with $i < j$ ( $j < i$ ). Then at most two vertices in $J'$ have private neighbors in $\mathcal{C}(i, j)$ ( $\mathcal{C}(j, i)$ ).

*Proof.* The argument is essentially the same as for Claim C.1. Consider $i < j$ (without loss of generality) and suppose the three vertices are $z_1, z_2, z_3$ having the distinct edge indices $e_1, e_2, e_3$. Suppose $z_1$ has a private neighbor in $\mathcal{C}(i, j)$. Then by the definition of $\mathcal{E}_5$, $\pi(z_1) \in \mathcal{C}(i, j, e_2)$ and for the same reason also, $\pi(z_1) \in \mathcal{C}(i, j, e_3)$, a contradiction. □

*Claim C.3* Suppose $J$ is an irredundant set in $G'$ and that $J' = J \cap \mathcal{B}(i, j)$. Then $|J'| \leq 4$.

*Proof.* Suppose $i < j$ and that there are five vertices in $J'$, $z_s$ for $s = 1, ..., 5$. By Claim C.2, there are three of these that must have private neighbors in $\mathcal{A}(i)$. Suppose that these three are $z_1, z_2, z_3$. Suppose that the edge coordinate of $z_s$ is $(x_s, y_s)$ for $s = 1, 2, 3$. Suppose $x_1 = x_2$. But then $z_1$ and $z_2$ would have the same neighbors in $\mathcal{A}(i)$, a contradiction. We can conclude that the $x_s$ are distinct for $s = 1, 2, 3$. But then by the definition of $\mathcal{E}_4$, we must have $\pi(z_1)$ adjacent to at least one of $z_2, z_3$. □

*Claim C.4* Suppose $J$ is an irredundant set in $G'$ and that $J' = J \cap \mathcal{C}(\alpha, \beta)$ contains two vertices $z_1$ and $z_2$ having edge coordinates $e_1$ and $e_2$, with $e_1 \neq e_2$. Then for every $e \in E$, $|J \cap \mathcal{C}(\alpha, \beta, e)| \leq 1$, and $|J'| \leq 2$.

*Proof.* Suppose there are two vertices, $z$ and $z'$ belonging to $J \cap \mathcal{C}(\alpha, \beta, e)$ for some $e \in E$. Since $z_1$ and $z_2$ dominate $\mathcal{C}(\alpha, \beta)$, $z$ and $z'$ cannot be self-private. Lemma 1 yields a contradiction. □

*Claim C.5* If $J$ is an irredundant set in $G'$, then for any $(\alpha, \beta) \in \Gamma$, $|J \cap \mathcal{C}(\alpha, \beta)| \leq 3k^2$.

*Proof.* If the size of the intersection is more than $3k^2$ then the hypotheses of Claim C.4 are satisfied, and consequently we reach a contradiction of Claim C.1. □

*Claim C.6* If $J$ is an irredundant set in $G'$, $i \in \{1, ..., k\}$, and $J' = J \cap \mathcal{A}(i)$, then $|J'| \leq 3k^2(k-1)$.

*Proof.* If $|J'| > 3k^2(k-1)$ then there are at least two distinct vertex indices $x$ and $x'$ such that $J \cap \mathcal{A}(i, x)$ and $J \cap \mathcal{A}(i, x')$ are nonempty. From this it follows that all of the private neighbors of $J'$

must be in $\mathcal{B}(i)$, since they cannot be self-private. Any two vertices in $\mathcal{A}(i,j,u)$ have the same set of neighbors, and so it must be the case, by Lemma 1, that $|J \cap \mathcal{A}(i,j,u)| \leq 1$, and therefore there must be more than $3k^2$ distinct vertex indices $u$ such that $J \cap \mathcal{A}(i,u)$ is nonempty. This implies, by the Pigeonhole Principle, that there is some index $j \in \{1, ..., k\}$, $j \neq i$, such that $|J \cap \mathcal{A}(i,j)| > 3k$. Let $z_1, z_2, z_3$ denote three vertices of $G'$ in $J \cap \mathcal{A}(i,j)$ having distinct indices $u_1, u_2, u_3 \in V$. The private neighbors of the $z_i$, $i = 1, 2, 3$, must belong to $\mathcal{B}(i,j)$. We reach a contradiction, since by the definition of $\mathcal{E}_4$, at least one of $z_1, z_2$ must be adjacent to $\pi(z_3)$. □

*Claim C.7* If $J$ is an irredundant set in $G'$ of size $k'$, then for all $i$, $1 \leq i \leq k$, $|J \cap \mathcal{A}(i)| > 3k^2$.

*Proof.* We assume $k > 1$. Suppose that the claim is contradicted for $\mathcal{A}(i)$. Then the $k'$ vertices of $J$ must be distributed as follows:
(1) There are at most $3k^2\binom{k}{2}$ in $\mathcal{C}$ by Claim C.5.
(2) There are at most $4k(k-1)$ in $\mathcal{B}$ by Claim C.3.
(3) There are at most $3k^2(k-1)^2 + 3k^2$ in $\mathcal{A}$ by Claim C.6.
This is a contradiction, since the sum is less than $k'$. □

An almost identical argument proves the following.

*Claim C.8* If $J$ is an irredundant set in $G'$ of size $k'$, then for all $(\alpha, \beta) \in \Gamma$, $|J \cap \mathcal{C}(\alpha, \beta)| \geq 3k$. □

*Claim C.9* If $J$ is an irredundant set in $G'$ of size $k'$, then for all $i \in \{1, ..., k\}$ and for all $j \in \{1, ..., i-1, i+1, ..., k\}$, $|J \cap \mathcal{B}(i,j)| \leq 2$.

*Proof.* Suppose $|J \cap \mathcal{B}(i,j)| \geq 3$, and let $z_1, z_2, z_3$ be three distinct vertices of $G'$ in this intersection. They cannot be self-private, and so by Lemma 1, they must have distinct coordinates $e_1, e_2, e_3 \in E$. Let $e_i = (x_i, y_i)$ for $i = 1, 2, 3$. Assume without loss of generality that $i < j$. Either two of the three must have private neighbors in $\mathcal{A}(i)$, or two have private neighbors in $\mathcal{C}(i,j)$. Suppose that $z_1$ and $z_2$ have private neighbors in $\mathcal{A}(i)$. If $x_1 = x_2$ then we have a contradiction, since in this case $z_1$ and $z_2$ would have the same set of neighbors in $\mathcal{A}(i)$. Thus $x_1 \neq x_2$. But then $\pi(z_1) \in \mathcal{A}(i, x_2)$, or else $z_2$ is adjacent to $\pi(z_1)$. Similarly, $\pi(z_2) \in \mathcal{A}(i, x_1)$. By Claim C.7, there is a vertex $z_3 \in J \cap \mathcal{A}(i)$. Since $x_1 \neq x_2$, $z_3$ must be adjacent to either $\pi(z_1)$ or $\pi(z_2)$, a contradiction.

¿From the above we may conclude that two of the $z_i$ have private neighbors in $\mathcal{C}(i,j)$. Suppose without loss of generality that these are $z_1$ and $z_2$. Necessarily they have distinct coordinates $e_1$ and $e_2$ in $E$, by Lemma 1. In fact, $\pi(z_1) \in \mathcal{C}(i,j,e_2)$, since otherwise $z_2$ would be adjacent to $\pi(z_1)$, and similarly $\pi(z_2) \in \mathcal{C}(i,j,e_1)$. By Claim C.8, there is a vertex $z_3 \in J \cap \mathcal{C}(i,j)$. Necessarily, $z_3$ is adjacent to at least one of $\pi(z_1), \pi(z_2)$, a contradiction. □

*Claim C.10* If $J$ is an irredundant set in $G'$ of size $k'$, then for all $(\alpha, \beta) \in \Gamma$, there is a unique $e \in E$ such that $J \cap \mathcal{C}(\alpha, \beta, e)$ is nonempty.

*Proof.* From C.1 there can be at most two $e_i$'s such that $J \cap \mathcal{C}(\alpha, \beta, e_i)$ is nonempty. If there are actually two $e_i$'s, then by Claim C.4, $|J'| \leq 2$ which contradicts Claim C.8. □

*Claim C.11* If $J$ is an irredundant set in $G'$ of size $k'$, then for all $i \in \{1, ..., k\}$, there is a unique $u \in V$ such that $J \cap \mathcal{A}(i,u)$ is nonempty.

*Proof.* Suppose there are two such indices, $u$ and $u'$ that yield nonempty intersections with $J$. Then the private neighbors of the vertices in $J \cap \mathcal{A}(i)$ cannot be in $\mathcal{A}(i)$. From this it follows that for any $u \in V$, $\mathcal{A}(i,u)$ can contain at most $k-1$ vertices of $J$, since otherwise we would reach a contradiction by Lemma 1. By Claim C.7, there is therefore a set of more than $3k$ vertices of $J$ in $\mathcal{A}(i)$, each having a different vertex coordinate $u \in V$. Consequently there must be an index $j$, and three vertices $z_1, z_2, z_3$ of $J$ with $z_s \in \mathcal{A}(i,j,u_s)$ for $s = 1,2,3$, and with the vertex coordinates $u_s$ all distinct. But then we reach a contradiction, since all three must have private neighbors in $\mathcal{B}(i,j)$ and this is impossible by the definition of $\mathcal{E}_4$. □

*Claim C.12* If $J$ is an irredundant set in $G'$ of size $k'$, then for all $i \in \{1,...,k\}$ and for all $j \in \{1,...,i-1,i+1,...,k\}$, $|J \cap \mathcal{B}(i,j)| \leq 1$.

*Proof.* Our argument is based on Claims C.5, C.6 and C.9 that put upper bounds on the distribution of $J$. In particular, we already know that a set of vertices $\mathcal{B}(i,j)$ can contain at most two elements of $J$ by Claim C.9, and the bounds given by Claims C.5 and C.6 are as tight as possible. Say that $\mathcal{B}(i,j)$ is *exceptional* if it contains two elements of $J$. We argue that each exceptional $\mathcal{B}(i,j)$ implies tighter bounds on the number of vertices of $J$ in the associated sets $\mathcal{A}(i)$ and $\mathcal{C}(i,j)$. Let $z_1$ and $z_2$ be two vertices of $J$ in an exceptional $\mathcal{B}(i,j)$ (and assume without loss of generality that $i < j$). Suppose $z_s = b[i,j,(x_s,y_s)]$ for $s = 1,2$. If $x_1 = x_2$ then $z_1$ and $z_2$ have the same set of neighbors in $\mathcal{A}(i)$ and therefore they must both have private neighbors in $\mathcal{C}(i,j)$. By Claim C.8 and the arguments of the proof of Claim C.9, this is impossible. Thus $x_1 \neq x_2$, and furthermore, one must have a private neighbor in $\mathcal{A}(i)$ (suppose $z_1$), and the other must have a private neighbor in $\mathcal{C}(i,j)$ (suppose $z_2$). By Claim C.11 there is a unique vertex coordinate $u \in V$ such that $J \cap \mathcal{A}(i,u)$ is nonempty. Let $z_3$ be a vertex of $J$ in this intersection. It must be the case that $\pi(z_1) \in \mathcal{A}(i,u)$, else $z_3$ would be adjacent to $\pi(z_1)$. Consequently $\mathcal{A}(i)$ can contain at most $3k^2(k-1)-1$ vertices of $J$. A similar "displacement" can be proved for $\mathcal{C}(i,j)$ using Claim C.10. That is, since the private neighbor of $z_2$ is in $\mathcal{C}(i,j)$ the total number of vertices of $J$ that can belong to $\mathcal{C}(i,j)$ is decreased by one. Since the bounds described by C.5 and C.6 are tight, exceptions are impossible, since if there are $m$ exceptional $\mathcal{B}(i,j)$ in the distribution of $J$, then the upper bounds of Claims C.5 and C.6 together with the displacements caused by the exceptions, imply that $|J| \leq k' - m$, a contradiction. □

Claims C.5, C.6, C.10, C.11 and C.12 together establish Claim C, which completes the proof of the theorem. □

## 3 The Dual Problem

For a property $\mathcal{P}$ of vertex sets it is natural to define the *parameterized dual* of the $k$-vertex set problem for $\mathcal{P}$ to be the problem that asks whether there is a set $V' \subseteq V$ of size $k$ such that $V - V'$ has property $\mathcal{P}$. For example, the parameterized dual of Independent Set is Vertex Cover, and while Independent Set is $W[1]$-complete [**?**], Vertex Cover is fixed-parameter tractable [**?**, **?**]. In this section we show that the dual of Irredundant Set is fixed-parameter tractable.

The following observation is trivial but useful.

**Lemma 1.** If a graph $G = (V, E)$ of order $n$ has an irredundant set $J$ of size $n - k$, then at most $k$ vertices in $J$ can be non-self-private. □

We show that the following generalized problem is fixed-parameter tractable.

CONTROLLED CO-IRREDUNDANT SET
*Instance:* A triple $(G, N, k)$ where $G$ is a graph $G = V, E)$ with a distinguished set of vertices $N \subseteq V$, and $k$ is a positive integer.
*Parameter:* $k$.
*Question:* Is there a set of $k$ vertices $V' \subseteq V - N$, such that $V - V'$ is an irredundant set in $G$, with each vertex of $N$ having a private neighbor other than itself?

**Theorem 2.** CONTROLLED CO-IRREDUNDANT SET is fixed-parameter tractable. (And therefore CO-IRREDUNDANT SET is fixed-parameter tractable.)

**Proof.** By Lemma 1, the question is trivial unless $|N| \leq k$. We can also assume that every vertex has degree at least 1. We prove the theorem by induction on $k$. For $k = 0$, necessarily $|N| = 0$ and the answer is "yes" only if $V$ is irredundant, and therefore every vertex is self-private. But then $|E| = 0$, and this is easily checked in time $C \cdot n$.

Suppose $k > 0$. We consider two cases.

*Case 1.* The maximum degree of a vertex $u \in V - N$ is at most $k$.

If $|V - N| > (2k-1)2k$ then $G$ has a set of $2k+1$ independent edges $(u_i, v_i)$ for $i = 1, ..., 2k+1$. Then $V'$ can intersect at most $k$ of these edges, and among the remaining edges, for any irredundant set of size $n-k$, there must be at least one for which both endpoints are self-private, a contradiction. Consequently, the answer must be "no".

If $|V - N| \leq (2k-1)2k$ then $|G| \leq (2k-1)2k + k$ and by exhaustively analyzing all $k$-subsets of $G$ as candidates for $V'$, the problem can be solved in time $C \cdot (2k)^{2k+1}$.

*Case 2.* There is a vertex $u \in V - N$ of degree greater than $k$.

In this case, for any $V'$ of size $k$ that is a witness to the answer "yes", there are two possibilities:

(1) $u \in V'$, but since at least two of the neighbors of $u$ must then be in $V - V'$, $u$ is not a private neighbor of any vertex in $V - V'$, and therefore the answer is "yes" if and only if the answer is "yes" for the instance: $(G - u, N, k-1)$, i.e., $V'$ must extend a solution for this instance with parameter $k - 1$.

(2) $u \notin V'$, but then since it must have at least one neighbor in $V - V'$, $u$ cannot be self-private, and therefore the answer is "yes" if and only if the answer is "yes" for the instance: $(G, N \cup \{u\}, k)$, i.e., $V'$ must extend a solution for this instance with a larger control set $N$.

The above allows us to solve an instance of the problem by exploring a tree of possibilities that has at most $k$ leaves where the problem must be solved for parameter $k-1$, and one leaf involving exhaustive analysis. This yields a running time of $C \cdot (2k)^{(2k+1)} n$. □

## 4 Conclusions

Vertex set problems ("Are there $k$ vertices in $G$ having a specified property $\mathcal{P}$?") have played an important role in the development of the theory of parameterized complexity both as a source of natural problems, and in the development of proof techniques. For examples of the latter, the combinatorics of the parameterized reduction from INDEPENDENT SET to DOMINATING SET in [**?**] plays an important part in the main theorem characterizing the $W[t]$ classes in [**?**]. The VERTEX COVER problem has provided a nice example of a tractable problem for which the parameter function can be improved by various techniques [**?**, **?**]. In this paper we have shown that the parameterized complexity of IRREDUNDANT SET is precisely the all-important $W[1]$ degree, contrary to speculation that it might be a natural representative of an intractable degree between FPT and $W[1]$.

The parameterized complexity of a number of well-known vertex set problems remains unresolved, and the entire subject remains fruitful for further exploration. We mention a few of these open problems:

(1) What is the parameterized complexity of the DIRECTED FEEDBACK VERTEX SET problem? (In this problem, the input is a directed graph $G = (V, A)$ and the question is whether there is a set of $k$ vertices that covers all the directed cycles in the graph. It can be shown that this problem is FPT-equivalent to the DIRECTED FEEDBACK ARC SET problem.)

(2) Does the problem PERFECT CODE represent a degree between $W[1]$ and $W[2]$? (In this problem, the input is a graph $G = (V, E)$ and the question is whether there is a set of $k$ vertices $V' \subseteq V$ having the property that $V$ is partitioned into the sets $N[u]$, $u \in V'$. What is known is that the problem is hard for $W[1]$ and is a member of $W[2]$. It is also known that this problem is FPT-equivalent to $k$-WEIGHTED ONE-PER-CLAUSE CNF-SAT.)

(3) Are there any natural candidate vertex set problems whose parameterized complexity might be intermediate between FPT and $W[1]$?

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