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INTERVALS AND SUBLATTICES OF THE R.E. WEAK TRUTH TABLE DEGREES, PART I: DENSITY

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1. Introduction

This paper concerns itself with the structure of **W**, the upper semilattice of r.e. weak truth table (W-) degrees and to a lesser extent the collection **D** of all W-degrees below $\mathbf{0}'_W$. The upper semilattice **W** has received considerable attention recently, primarily because of its applications to the study of the structure of **R**, the r.e. T-degrees, but also for its own sake since the reducibility arises very naturally in effective mathematics. The reader should recall that $A \leqslant_W B$ means that there is a recursive function γ and a functional Γ with $\Gamma(B) = A$ and for all x the use $u(\Gamma(B;x)) \leqslant \gamma(x)$. For example, if A is an r.e. basis of an r.e. vector space V, then $A \leqslant_W V$. (In fact, in [14] it is shown that the r.e. W-degrees below W-deg(V) are exactly the W-degrees of r.e. bases of V.) We refer the reader here to, for example, [3, 4], [6], [9–11] [15], [28] and [37].

Applications to **R** usually utilize a 'structural interaction' of **R** and **W** (like contiguous degrees) coupled with the fact that constructions in **W** are smoother than in **R**. For example, various results that require infinitary methods in **R** (such as density) turn out to need only finite injury methods in **W**. Part of this smoothness stems from the fact that **W** is a distributive upper semilattice (cf. [24]). Namely, **W** satisfies

(1.1) $\quad \forall \mathbf{a}, \mathbf{b}, \mathbf{c}\ (\mathbf{a} \leqslant \mathbf{b} \cup \mathbf{c} \rightarrow \exists \mathbf{e}, \mathbf{f}\ (\mathbf{e} \leqslant \mathbf{b}\ \&\ \mathbf{f} \leqslant \mathbf{c}\ \&\ \mathbf{e} \cup \mathbf{f} = \mathbf{a}))$.

Another nice aspect of **W** is that many results from **R** have proofs that immediately give the corresponding result in **W**. One example of this is Lachlan's nondiamond theorem (c.f. [23]).

Nevertheless, despite all of these apparently helpful aspects of **W**, many of the fundamental questions concerning the structure of **W** remain open (including ones already solved for **R**). For example, properties like (1.1) mean that neither

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the Harrington–Shelah [20] nor the Harrington–Slaman (unpublished) techniques for establishing the undecidability of **R** work for **W**. Indeed the decidability — or lack thereof — of Th(**W**) remains open.

The focus of this paper is to analyse the embedding and substructure questions for intervals in **W** and **D**. Our investigations were inspired by Paul Fischer's [18] beautiful result that there exist initial segments of **W** which are lattices. Our notation will be to write **W**[**a**, **b**] for the r.e. W-degrees between **a** and **b** and [**a**, **b**] for the collection of all W-degrees between **a** and **b**. The organization of this paper is as follows. In Section 2 for completeness we review some notation and terminology we shall use. In Section 3 we begin our investigations by showing that Lachlan splitting and density can be combined for **W** preserving greatest element. Namely we show that if $\mathbf{a}<\mathbf{b}$ in **W** then there exist $\mathbf{b}_1$, $\mathbf{b}_2$ in **W** with $\mathbf{a}<\mathbf{b}_1$, $\mathbf{b}_2<\mathbf{b}$, $\mathbf{b}_1 \cup \mathbf{b}_2=\mathbf{b}$ and $\mathbf{b}_1 \cap \mathbf{b}_2$ exists. Several extensions to this result are given and a partial characterization of those lattices that can be embedded into **W**[**a**, **b**] preserving **b** is established. For example the techniques of Section 3 suffice to so embed the countable boolean algebra of finite and cofinite sets in **W**[**a**, **b**] for arbitrary $\mathbf{a}<\mathbf{b}$, preserving greatest element.

Because of Fischer's result, for arbitrary $\mathbf{a}<\mathbf{b}$ in **W** there don't necessarily exist $\mathbf{b}_1$, $\mathbf{b}_2 \in \mathbf{W}$ with $\mathbf{a}<\mathbf{b}_1$, $\mathbf{b}_2<\mathbf{b}$, $\mathbf{b}_1 \cup \mathbf{b}=\mathbf{b}$ and $\mathbf{b}_1 \cap \mathbf{b}_2$ does not exist. We begin Section 4 by showing such $\mathbf{b}_1$ & $\mathbf{b}_2$ exist if $\mathbf{b}=\mathbf{0}'_{\mathrm{W}}$. We then continue with this theme (of analysing the distribution of lattices without infimum in **W**) by extending Fischer's result to all incomplete r.e. **W**- degrees by showing that all incomplete r.e. **W**-degrees are bottoms of lattices. That is, if $\mathbf{a}\neq\mathbf{0}'$ with $\mathbf{a}\in\mathbf{W}$, then there exists $\mathbf{b}\in\mathbf{W}$ with **W**[**a**, **b**] a lattice. In this section we also establish a result — mentioned in [15] without proof — that there exist r.e. sets A and B such that the infimum of the W-degrees of A and B exists and the infimum of the T-degrees of A and B does not exist.

In Section 5 we show that lattices are dense in **W**. That is, if $\mathbf{a}<\mathbf{b}$ with $\mathbf{a}<\mathbf{b}$ in **W**, then there exist $\mathbf{c}, \mathbf{d}\in\mathbf{W}$ with $\mathbf{a}<\mathbf{c}<\mathbf{d}<\mathbf{b}$ and **W**[**c**, **d**] a lattice. As a partial result towards the classification of exactly *which* lattices can be so realized, we show that if $\mathbf{a}, \mathbf{b}\in\mathbf{W}$ with **W**[**a**, **b**] a lattice, then **W**[**a**, **b**] and [**a**, **b**] contain noncomplemented members (and thus **W**[**a**, **b**] doesn't form a boolean algebra).

Finally, in Section 6 we use a modification of the technique of Section 5 to show that the lattices that can be embedded into arbitrary **W**[**a**, **b**] with greatest element preserved, are exactly the countable distributive lattices. We establish this by so embedding the countable atomless boolean algebra. This then gives a decision procedure for the existential theory of **W**[**a**, **b**] in the language $L(\leqslant, \vee, \wedge, 1)$ for arbitrary $\mathbf{a}, \mathbf{b}\in\mathbf{W}$ with $\mathbf{a}<\mathbf{b}$ (by using the techniques of Fejer and Shore [17]).

In Part II of this paper we show that although every incomplete r.e. degree is the bottom of a lattice and although lattices are dense, these results cannot be combined. That is, not every r.e. degree bounds a nontrivial initial segment that forms a lattice. In fact, it is established that there is an r.e. set of high degree such that if B and C are r.e. nonrecursive sets with $B, C \leqslant_W A$, then there exist B_1 and C_1 with W-deg$(B_1) \cap$ W-deg(C_1) not existing and $B_1 \leqslant_W B$, $C_1 \leqslant_W C$. As a corollary, we see that there exist high r.e. W-degrees that don't W-bound minimal pairs. This stands in contrast with Coopers [8] result for **R**.

2. Notation

Notation and terminology are fairly standard. All sets, degrees, etc. are r.e. unless specifically stated otherwise. Also $\mathbf{a}, \mathbf{b}, \ldots$ are (r.e.) W-degrees unless stated otherwise. We use upper case Greek letters $(\Phi, \Gamma, \ldots)$ for functionals, and such letters with 'hats' $(\hat{\Phi}, \hat{\Gamma}, \ldots)$ as W-functionals. In the latter case they have as use functions the corresponding lower case Greek letters $(\phi, \gamma, \ldots)$. Thus $\hat{\Phi}_e(A;x)\downarrow$ only if $\phi_e(x)\downarrow$ and $\Phi_e(A;x)\downarrow$ and $u(\Phi_e(A;x)) < \phi_e(x)$.

Warning. We always assume such use functions increasing when defined.

This convention is used mercilessly throughout and saves on notation.

We let $\langle\ ,\ \rangle$ denote a standard pairing function (monotone in both variables) and let $\omega^{(e)} = \{\langle e, x\rangle : x \in \omega\}$. Many of our constructions use tree of strategies arguments. It is helpful, but not essential, if the reader is acquainted with [35, 36] or [37]. Finally, we assume all computations etc. are bounded by s at stage s.

3. Lachlan splitting and density

In his famous 'monster' paper [25], Lachlan showed that (Sacks) splitting and density [31, 32] cannot be combined in **R**. In [28], Ladner and Sasso showed that for **W**, splitting and density *could* be combined. In fact, they showed

(3.1) **Theorem** (Ladner and Sasso [28]). *If A is nonrecursive and $B <_W A$, then there exists an r.e. splitting $A_1 \sqcup A_2 = A$ of A such that $B <_W A \oplus B$, $A_2 \oplus B <_W A$.*

In his paper [27]. Lachlan improved Sacks splitting by establishing (in **R**) that

(3.2) $\quad \forall \mathbf{a} \in \mathbf{R}\ \exists \mathbf{a}_1, \mathbf{a}_2 \in \mathbf{R}\ (\mathbf{a}_1 \mid \mathbf{a}_2\ \&\ \mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a}\ \&\ \mathbf{a}_1 \cap \mathbf{a}_2 \text{ exists}).$

For our first result, we shall show that in **W**, Lachlan splitting and density may be combined. This will follow from the following result which also has several other applications.

(3.3) **Theorem.** *Suppose $\mathbf{a} \mid \mathbf{b}$. Then there exists $\mathbf{c}$ such that*

(i) $\mathbf{a} \cup \mathbf{c}, \mathbf{b} \cup \mathbf{c} < \mathbf{a} \cup \mathbf{b}$, *and*
(ii) $(\mathbf{a} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c}) = \mathbf{c}$.

Remarks. We have shown that (3.3) fails to hold in **R**. In fact, we have shown that (in **R**)

$$(3.4) \qquad \exists \mathbf{a}, \mathbf{b}\ [\mathbf{a} \mid \mathbf{b}\ \&\ \forall \mathbf{c}, \mathbf{d}((\mathbf{a} \leqslant \mathbf{c} < \mathbf{a} \cup \mathbf{b}\ \&\ \mathbf{b} \leqslant \mathbf{d} < \mathbf{a} \cup \mathbf{b}) \rightarrow \mathbf{c} \cap \mathbf{d} \text{ does not exist})].$$

The proof of (3.4) will appear elsewhere. The reader should compare (3.4) with (3.14) below.

Proof of (3.3). Let $A = \bigcup_s A_s$ and $D = \bigcup_s D_s$ be given canonical enumerations of r.e. sets with $A \mid_W D$. We shall construct an r.e. set $C = \bigcup_s C_s$ to satisfy the requirements

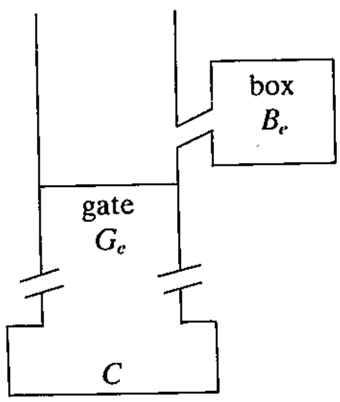
N_{2e}: $\hat{\Phi}_e(A \oplus C) \neq D$.

N_{2e+1}: $\hat{\Phi}_e(D \oplus C) \neq A$.

P_e: $\hat{\Phi}_e(A \oplus C) = \hat{\Phi}_e(D \oplus C) = f$ and f total implies $f \leqslant_W C$.

We shall use a pinball construction to satisfy the requirements. Strictly speaking this is unnecessary for this particular construction, but this technique provides a flexible platform for some later constructions, and furthermore we believe that this makes the current proof more perspicuous.

The pinball machine is the simple one given below. Box B_e is assigned to P_e and gate G_e to N_e. The motion of the balls is downward into pocket C (which represents the set C). We refer to the region above C but not including the box as the *track*.



We need the following definitions:

$$L(2e, s) = \max\{x : \forall y < x\ (\hat{\Phi}_{e,s}(A_s \oplus C_s; y) = D_s(y))\},$$

$$L(2e+1, s) = \max\{x : \forall y < x\ (\hat{\Phi}_{e,s}(D_s \oplus C_s; y) = A_s(y))\},$$

$$r(2e, s) = \max\{u(\hat{\Phi}_{e,s}(A_s \oplus C_s; y)) : y \leqslant L(2e, s)\},$$

$$R(2e, s) = \max\{r(2e, t) : t \leqslant s\},$$

$$r(2e+1,s)=\max\{u(\hat{\Phi}_{e,s}(D_s\oplus C_s;y)): y\le L(2e+1,s)\},$$
$$R(2e+1,s)=\max\{r(2e+1,t): t\le s\},$$
$$l(e,s)=\max\{x: \forall y<x\ (\hat{\Phi}_{e,s}(A_s\oplus C_s;y)=\hat{\Phi}_{e,s}(D_s\oplus C_s;y))\},$$
$$ml(e,s)=\max\{l(e,t): t<s\},\quad \text{and}$$
$$ls(e,s)=\begin{cases}\max\{t: t<s \text{ and } l(e,t)>ml(e,t)\} & \text{if defined,}\\ 0, & \text{otherwise.}\end{cases}$$

A stage s is called *e-expansionary* if $l(e,s)>ml(e,s)$. Also $ls(e,s)$ is referred to as the *last e-expansionary stage* (before stage s). Balls are marked. A ball y in the machine is marked $y(e,x)$ for some e,x. This will indicate it emanates from box B_e (and is thus devoted to P_e) and traces x.

The basic idea of this construction is as follows. We let $R(i,s)$ control gate i. We argue that if $R(i,s)\to\infty$ then $A\le_W D$ (or $D\le_W A$), giving a contradiction. This means that almost all balls pass gate G_i. In box B_e we monitor $l(e,s)$. Roughly speaking, if we see that 'both sides' of a computation regarding x have possibly been injured we enumerate (via the track) a trace $y(e,x)$ into C. The use of W-reductions allows us to present a collection of $y(e,x)$ in advance (at e-expansionary stages). Formal details now follow.

Rules. The machine is subject to the following rules.
Gate Rule. If a ball x is at gate G_e and $x>R(e,s)$, allow x to drop to the first gate G_j for $j<e$ with $x\le R(j,s)$. If none exist enumerate x into C.
Box Rules. Box B_e is subject to the following three rules.

Rule 1 (*Trace entourage assignment*). For any x if a stage s is e-expansionary, $l(e,s)>x$ and x has no entourage in B_e, assign $\{\langle e,x,s\rangle,\dots,\langle e,x,s+s+1\rangle\}$ as x's entourage and mark them by (e,x). By convention we note $s\le\langle e,x,s+1\rangle$ for all i. Place all the entourage in B_e.

Rule 2 (*Emission*). If there is no ball $y(e,x)$ on the track, x has an entourage in B_e, and x is *active*, then

if (i) $A_s[\phi_{e,s}(x)]\ne A_{ls(e,s)}[\phi_{e,ls(e,s)}(x)]$, and
(ii) $D_s[\phi_{e,s}(x)]\ne A_{ls(e,s)}[\phi_{e,ls(e,s)}(x)]$,

find the least member $z(e,x)$ of x's entourage still in box B_e. Allow z to drop out of B_e to the first gate G_j for $j\le e$ such that $z\le R(j,s)$. If none exists, enumerate z in C. In either case declare x as no longer active.

Rule 3 (*Activation*). If x has an entourage in B_e, x is not active and s is e-expansionary, declare x as active.

Construction, state *s*. Run the machine according to the above rules.

Verification. We argue that
(i) All gates G_e get at most finitely many permanent residents

(ii) $\lim_s R(e, s) = R(e)$ exists.
(iii) All the P_e and N_e are met, and
(iv) $C \leqslant_W A \oplus D$.

We verify this by induction on e. Let $Q \subset \{0, \ldots, e-1\}$ denote the collection of *finitely active* boxes. That is, define $j \leqslant e-1$ to be a member of Q iff there are at most finitely many j-expansionary stages. Let $R = \{j : j < e \;\&\; j \notin Q\}$. Let s_0 be a stage such that

(a) y is marked $y(j, x)$ for some $j \notin Q$ implies y has reached its final position by stage s_0,
(b) $\forall s \geqslant s_0 \quad \forall j \in Q \quad$ (s is not j-expansionary), and
(c) $\forall k < e$ (G_k has all its permanent residents by stage s_0).

Please note that Q and the permanent residents of G_k for $k < e$ and s_0 are just parameters. The crucial observation concerning s_0 is that after stage s_0 any ball entering the track below G_e must succeed in reaching C.

Now, without loss, let $e = 2i$. We show that N_e is met and hence $\lim_s R(e, s)$ exists. We first verify that

(3.5) $\quad \hat{\Phi}_i(A \oplus C) \neq D.$

(3.6) First suppose (3.5) fails. Then $L(e, s) \to \infty$ and so $R(e, s) \to \infty$. We show that this implies $D \leqslant_W A$ giving a contradiction. Let z be given. Find the least stage $s = s(z)$ with $s > s_0$ and

(i) $L(e, s) > z$,
(ii) there are no balls $q \leqslant \phi_i(z)$ currently on the track below G_e save for permanent residents of some G_j for $j < e$, and
(iii) For $q < \phi_i(z)$, if q is in box B_k at stage s, then either $k \geqslant e$ or $k \in Q$, or

(3.7) $\quad$ if $\; k \in R, \;$ then $\; A_{1s(k,s)}[q] = A[q].$

It is easy to see that such a stage must exist and is A-recursive in parameters Q, s_0, and has use $\phi_i(z)$. Now (3.7) ensures that any such q in (iii) is now a permanent resident of B_k. This means that no number below B_e can enter C below $\phi_i(z)$ and so the '$\hat{\Phi}_{i,s}(A_3 \oplus C_s; z)$' computation is final since $R(e, s)$ protects this at G_e. Thus $\hat{\Phi}_i(A \oplus C; z) = \hat{\Phi}_{i,s}(A_s \oplus C_s; z) = D_s(z) = D(z)$. Therefore $D \leqslant_W A$, a contradiction. Thus (3.5) holds.

(3.8) Next we argue that G_e gets at most finitely many permanent residents and $\lim_s R(e, s) = R(e)$ exists. But this is easy. since we are dealing with W-reductions. As $\hat{\Phi}_i(A \oplus C) \neq D$ we have that $\lim_s L(e, s) = L(e)$ exists. Therefore $\lim_s r(e, s)$ exists, and so $\lim_s R(e, s) = D(e)$ exists. Balls are only permanent residents of G_e if they are $R(e)$-restrained, and so G_e has at most $R(e)$ permanent residents.

(3.9) Now we turn to the P_e. If it is possible that P_e fails to be met, then we must have $l(e, s) \to \infty$. Let $s_1 \geqslant s_0$ be a stage such that additionally G_e has its full quota of permanent residents. Let $z > s_1$ be given. Find the least e-expansionary stage

$s_2 = s_2(z)$ such that $s_2 > s_1$ and $l(s, s_2) > z$. Then by rule 1 (of the box rules) at stage s_2 z is given an entourage

$$\{\langle e, z, s_2\rangle, \langle e, z, s_2+1\rangle, \ldots, \langle e, z, s_2+s_2+1\rangle\}.$$

By monotonicity of $\langle\ ,\ \rangle$, we know $\langle e, x, s_2+s_2+1\rangle = g$ is the largest member of this set. Now find the least e-expansionary stage s_3 such that $s_3 > s_2$ and $C_{s_3}[g] = C[g]$.

We claim that

$$(3.10)\quad \forall s > s_3 \text{ (either } \hat{\Phi}_{e,s}(A_s \oplus C_s; z) = \hat{\Phi}_{e,s_3}(A_{s_3} \oplus C_{s_3}; z) \text{ or } \hat{\Phi}_{e,s}(D_s \oplus C_s; z) = \hat{\Phi}_{e,s_3}(D_{s_3} \oplus C_{s_3}; z)).$$

If (3.10) fails there must exist a stage $t > s_3$ with

$$(A_t \oplus C_t)[\phi_e(z)] \neq (A_{1s(e,t)} \oplus C_{1s(e,t)})[\phi_e(z)] \quad \text{and}$$
$$(D_t \oplus C_t)[\phi_e(z)] \neq (D_{1s(e,t)} \oplus C_{1s(e,t)})[\phi_e(z)].$$

This follows by a simple induction and the observation that — as in a minimal pair construction — if a computation is to change *both* sides of a computation must change between e-expansionary stages. Since $C_{s_3}[g] = G[g]$ and $g > \phi_i(z)$ it must be that

$$A_t[\phi_i(z)] \neq A_{1s(e,t)}[\phi_i(z)], \quad \text{and} \quad D_t[\phi_i(z)] \neq D_{1s(e,t)}[\phi_i(z)].$$

Now since there are $>\phi_i(z)$ members of z's entourage in B_e when they are appointed, there must be some member $y < g$ of z's entourage in B_e at stage t. By choice of s_1 and z there cannot be any member of z's entourage on the track at stage s_1 and hence rule 2 will ensure that some ball $\leqslant g$ is released from B_e at stage t. But this ball must get into C by choice of s_1. Hence $C_t[g] \neq C[g]$. But this contradicts the facts that $t > s_3$ and $C_{s_3}[g] = C[g]$. Hence (3.10) holds.

(3.11) It remains to show that $C \leqslant_W A \oplus D$. Let z be given. Find the least stage s such that $A_s[z] = A[z]$ and $D_s[z] = D[z]$. It is clear that if z is not yet in C, then $z \notin C$ unless z is currently on the track. (If z is a member of a box, then it is now a permanent member.) In this construction, the restraints at B_e are monotone and so if z is blocked currently by some G_e, it is a permanent resident of G_e. Hence $C \leqslant_W A \oplus D$. □

There are several corollaries (and extensions) which use the same machinery as (3.3). First we get the promised density result for **W**.

(3.12) **Corollary.** *If* $\mathbf{a} < \mathbf{b}$, *then there exist* $\mathbf{c}, \mathbf{d}$ *such that* $\mathbf{a} < \mathbf{c}$, $\mathbf{d} < \mathbf{b}$, $\mathbf{c} \cup \mathbf{d} = \mathbf{b}$ *and* $\mathbf{c} \cap \mathbf{d}$ *exists.*

Proof. Combine (3.3) with (3.1). □

Using relatively straightforward dovetail versions of (3.1) and (3.3) we can obtain

(3.13) **Theorem.** (i) *Let* $A <_W B$. *Then there exists an infinite r.e. collection* $\{B_i\}_{i\in\omega}$ *of r.e. sets with* $A \oplus B_i \mid_W A \oplus \bigoplus_{j\neq i} B_j$ *and* $\bigoplus_i B_i \equiv_W B$.

(ii) *If* $\{\mathbf{a}_i\}_{i\in\omega}$ *is an r.e. collection of r.e. W-degrees with* $\mathbf{a}_i \mid \bigcup_{j\neq i} \mathbf{a}_j$ *for all i, then there exists* $\mathbf{c} \in \mathbf{W}$ *with*

(a) $\mathbf{c} < \bigcup \mathbf{a}_i$,

(b) $\mathbf{a}_i \cup \mathbf{c} \mid \bigcup_{j\neq i} \mathbf{a}_j \cup \mathbf{c}$, *and*

(c) $(\mathbf{a}_i \cup \mathbf{c}) \cap \left(\bigcup_{j\neq i} \mathbf{a}_j \cup \mathbf{c}\right) = \mathbf{c}$.

Proof. This is left to the reader. □

We point out (3.13) mainly because it shows that we may embed a recursive presentation of the boolean algebra of cofinite and finite subsets of ω into $\mathbf{W}[\mathbf{b}, \mathbf{a}]$ for any $\mathbf{b} < \mathbf{a}$ with greatest element $\mathbf{a}$. This follows from (3.13) and the fact that $\mathbf{W}$ is a distributive lower semi-lattice. This also raises the question of exactly which lattices may be embedded into $\mathbf{W}[\mathbf{b}, \mathbf{a}]$ with greatest element $\mathbf{a}$. By the distributivity of $\mathbf{W}$, any such lattice must be distributive. In his thesis Ambos-Spies [1] showed that any countable distributive lattice can be embedded into $\mathbf{W}[\mathbf{b}, \mathbf{0}']$ preserving greatest element $\mathbf{0}'$. We shall extend this in Section 6 and show that we replace $\mathbf{0}'$ by any $\mathbf{a} > \mathbf{b}$. This result has several ramifications concerning existential theories associated with $\mathbf{W}[\mathbf{a}, \mathbf{b}]$ along the lines of Fejer and Shore [17]. We delay this proof until Section 6 because it seems to require a much more complicated technique which is introduced in Section 5.

One final result using the machinery of (3.3) concerns interactions of $\mathbf{R}$ and $\mathbf{W}$.

(3.14) **Theorem.** *Let* $A \mid_T B$. *Then there exists C with* $A \oplus C$, $B \oplus C <_T A \oplus B$ *and such that* $E \leqslant_W A \oplus C$, $B \oplus C$ *implies* $E \leqslant_W C$. *That is, the W-infinum of the W-degrees of* $A \oplus C$ *and* $B \oplus C$ *is C.*

Proof (sketch). Again we use the machine of (3.3). Our new requirements are

$$(3.15)\quad \begin{cases} N_{2e}: & \Phi_e(A \oplus C) \neq B, \\ N_{2e+1}: & \Phi_e(D \oplus C) \neq A, \end{cases}$$

and the P_e remains the same. The pinball machine and its rules are exactly the same. The only problem is to ensure that each gate gets at most finitely many permanent residents in view of the fact that we are using T-reductions in (3.15) rather than W-reductions. This is solved by the well known 'hattrick' of Lachlan.

(Specifically, for example, replace Φ_e by $\hat{\Phi}_e$ where

$$\hat{\Phi}_{e,s}(A_s \oplus C_s; y) = \begin{cases} \Phi_{e,s}(A_s \oplus C_s; y) & \text{if } a_s > u(\Phi_{e,s}(A_s \oplus C_s; y)), \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

where $a_s = \min\{s, z\colon z \in [(A_s \oplus C_s) - (A_{s-1} \oplus C_{s-1})]\}$.) We refer to Soare [34] should the reader require further details. □

4. Density towards 0′ and pairs without infimum

It is perhaps natural to conjecture that the "$(\mathbf{a} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c}) = \mathbf{c}$" of (3.3) may be replaced by "$(\mathbf{a} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c})$ does not exist", or at least to conjecture that pairs without infimum are dense in **W**. Certainly this is true for **R** since — in [2] — Ambos-Spies showed (in **R**):

(4.1) $\forall \mathbf{a}, \mathbf{b}\ (\mathbf{a} < \mathbf{b} \rightarrow \exists \mathbf{c}, \mathbf{d}\ (\mathbf{a} < \mathbf{c}, \mathbf{d} < \mathbf{b}\ \&\ \mathbf{c} \cap \mathbf{d}$ does not exist)).

Fischer [18] destroyed both of these conjectures by showing that there was an initial segment of **W** that formed a lattice. That is, he showed that

(4.2) $\exists \mathbf{a} \neq \mathbf{0}\ \forall \mathbf{c}, \mathbf{d}\ (\mathbf{c}, \mathbf{d} < \mathbf{a} \rightarrow \mathbf{c} \cap \mathbf{d}$ exists).

Fischer did, however show that upward density of pairs without infimum does work. That is, he showed

(4.3) $\forall \mathbf{a} \neq \mathbf{0}'\ \exists \mathbf{c}, \mathbf{d}\ (\mathbf{a} \leqslant \mathbf{c}, \mathbf{d}\ \&\ \mathbf{c} \cap \mathbf{d}$ does not exist).

The purpose of this section is to examine the distribution of pairs without infimum. Our first result is an improvement of (4.3) along the lines of (3.3).

(4.4) **Theorem.** $\forall \mathbf{a} \neq \mathbf{0}'\ \exists \mathbf{b}, \mathbf{c}\ (\mathbf{a} \leqslant \mathbf{b}, \mathbf{c}\ \&\ \mathbf{b} \cup \mathbf{c} = \mathbf{0}'\ \&\ \mathbf{b} \cap \mathbf{c}$ *does not exist*).

Proof (sketch). Combine Fischer's argument of (4.3) with Sacks splitting.

Specifically, we must — given r.e. sets A and K with K creative — build r.e. sets C, D and auxiliary r.e. sets $\{V_e\colon e \in \omega\}$ to satisfy

$$R_{e,i}\colon\ \hat{\Phi}_e(C \oplus A) = \hat{\Phi}_e(D \oplus A) = W_e \text{ implies}$$
$$V \leqslant_{\mathrm{W}} C \oplus A, D \oplus A \quad \text{and} \quad \hat{\Phi}_i(W_e) \neq V_e$$

and the coding requirement $C \oplus A \oplus D \equiv_{\mathrm{W}} K$. Define

(4.5) $l(e, s) = \max\{x\colon \forall y < x\ (\hat{\Phi}_{e,s}(C_s \oplus A_s; y) = \hat{\Phi}_e(D_s \oplus A_s; y) = W_{e,s}(y))\}$.

To meet the $R_{e,i}$ we have a candidate $y = \langle e, i, x\rangle$ targeted for V_e. We wait till $\hat{\Phi}_{i,s}(W_{e,s}; y) = V_{e,s}(y)$ and $x \in K_{s+1} - K_s$. We then use Jockusch's strategy of first adding y to one of C or D, raise $R_{e,i}$'s restraint, waiting till $l(e, s)$ recovers and then enumerating y into both the other set (D or C) and into V_e. This ensures that

(4.6) $\hat{\Phi}_{i,s}(W_{e,s}; y) = 0 \neq V_e(y) = 1.$

Also $V_e \leqslant_W C \oplus A$, $D \oplus A$ if $l(e, s) \to \infty$ by permitting or delayed permitting. Finally since K controls the attacks, we must meet $R_{e,i}$ with finite effect since $A <_W K$. To ensure that $C \oplus A \oplus D \equiv_W K$ it suffices, at each stage s, to code $z \in K_{s+1} - K_s$ into one of C or D. We simply choose some reasonable way to do this (e.g. use $\langle 0, 0, z \rangle$) and, as in Sacks splitting, we code z into the C or D so as not to injure the highest priority requirement threatened. For further details see [18]. □

Although not every r.e. degree is the top of a segment that is a lattice, every r.e. degree $\neq \mathbf{0}'$ is the bottom of a segment of $\mathbf{W}$ which forms a lattice.

(4.7) **Theorem.** (i) *Let* $\mathbf{a} \neq \mathbf{0}'$. *Then there exists* $\mathbf{b} \neq \mathbf{0}$ *such that* $\mathbf{b} \nleqslant \mathbf{a}$ *and* $\mathbf{W}[\mathbf{0}, \mathbf{b}]$ *is a lattice.*

(ii) *Hence* $\forall \mathbf{a} \neq \mathbf{0}' \, \exists \mathbf{c} > \mathbf{a}$ ($\mathbf{W}[\mathbf{a}, \mathbf{c}]$ *is a lattice*).

Proof of (ii). Assume (i) holds. Let $\mathbf{a} \neq \mathbf{0}'$ be given. Use (i) to get $\mathbf{b}$ with $\mathbf{b} \nleqslant \mathbf{a}$ and $\mathbf{W}[\mathbf{0}, \mathbf{b}]$ a lattice. Let $\mathbf{c} = \mathbf{b} \cup \mathbf{a}$. We claim $\mathbf{W}[\mathbf{a}, \mathbf{c}]$ is a lattice. Let $\mathbf{a} \leqslant \mathbf{e}, \mathbf{f} \leqslant \mathbf{c}$. By the distributivity of $\mathbf{W}$ we have $\mathbf{e} = \mathbf{e}_1 \cup \mathbf{e}_2$ and $\mathbf{f} = \mathbf{f}_1 \cup \mathbf{f}_2$ with $\mathbf{e}_1, \mathbf{f}_1 \leqslant \mathbf{b}$ and $\mathbf{e}_2, \mathbf{f}_2 \leqslant \mathbf{a}$. Now $\mathbf{e}_1 \cap \mathbf{f}_1$ exists by (i). Let $\mathbf{e}_1 \cap \mathbf{f}_1 = \mathbf{g}$.

Suppose $\mathbf{h} \leqslant \mathbf{e}, \mathbf{f}$. Then $\mathbf{h} \leqslant \mathbf{e}_1 \cup \mathbf{e}_2 = \mathbf{e}$ so that $\mathbf{h} = \mathbf{h}_1 \cup \mathbf{h}_2$ with $\mathbf{h}_i \leqslant \mathbf{e}_i$ for $i = 1, 2$. Also $\mathbf{h}_1 \leqslant \mathbf{f}_1 \cup \mathbf{f}_2$ and so $\mathbf{h}_1 = \mathbf{m}_1 \cup \mathbf{m}_2$ with $\mathbf{m}_i \leqslant \mathbf{f}_i$ for $i = 1, 2$. We see $\mathbf{m}_1 \leqslant \mathbf{h}_1, \mathbf{f}_1$ and so $\mathbf{m}_1 \leqslant \mathbf{e}_1, \mathbf{f}_1$. This implies $\mathbf{m}_1 \leqslant \mathbf{g}$ as $\mathbf{e}_1 \cap \mathbf{f}_1 = \mathbf{g}$. Also $\mathbf{m}_2 \leqslant \mathbf{f}_2 \leqslant \mathbf{a}$ and $\mathbf{h}_2 \leqslant \mathbf{e}_2 \leqslant \mathbf{a}$. Hence $\mathbf{h} \leqslant \mathbf{g} \cup \mathbf{a}$. This means $\mathbf{g} \cup \mathbf{a} = \mathbf{e} \cap \mathbf{f}$. □

Proof of (i). Let $A = \bigcup_s A_s$ be a given incomplete r.e. set. We shall construct $B = \bigcup_s B_s$ together with auxiliary sets $Q_e = \bigcup_s Q_{e,s}$ to satisfy:

P_e: $\hat{\Phi}_e(A) \neq B$,

$N_{e,i}$: if $\hat{\Phi}_e(A) = W_e$ and $\hat{\Gamma}_e(A) = V_e$, then

$Q_e \leqslant_W W_e$, V_e and $\hat{\Phi}_i(W_e) = \hat{\Phi}_i(V_e) = f$ total implies $f \leqslant_W Q_e$.

Define

$$l(e, s) = \max\{x\colon \forall y < x \ [\hat{\Phi}_{e,s}(A_s; y) = W_{e,s}(y) \ \& \Gamma_{e,s}(B_s; y) = V_{e,s}(y)]\}.$$

(Here we regard Φ_e and Γ_e as controlling the enumerations of W_e and V_e.) Now let

$$ml(e, s) = \max\{l(e, t) : t < s\}, \quad \text{and}$$

$$l(e, i, s) = \max\{x\colon \forall y < x \ [\hat{\Phi}_{i,s}(W_{e,s}; y) = \hat{\Phi}_{i,s}(V_{e,s}; y) \ \& \ \forall z \ (z < \max\{\phi_e(\phi_i(y)), \ \gamma_e(\phi_i(y))\} \to z < l(e, s))]\}.$$

The principal apparatus of this construction is a partial restraint on $\hat{\Phi}_e$ and $\hat{\Gamma}_e$ imposed like a minimal pair. We cannot stop both W_e and V_e from changing when we enumerate an element into A. But like a minimal pair, we impose restraint between 'expansionary' stages. The crucial point recognized by Fischer [18] is that *because we are using W-degrees,* it is possible to use Q_e to record any injurious changes.

It is convenient to use a tree $2^{<\omega}$ of strategies to satisfy the $N_{e,i}$ in conjunction with the P_e. We presume the reader familiar with this technique and refer him to [35, 36] for more on tree arguments. We do, however, review some notation.

For σ, $\tau \in 2^{<\omega}$ we write $\sigma \subset \tau$ if σ is an initial segment of τ. We write $\sigma \leq_{\mathrm{L}} \tau$ if $\sigma \subset \tau$ or $\exists \gamma\, (\gamma\char`^0 \subset \sigma \,\&\, \gamma\char`^1 \subset \tau)$. We refer to $\sigma \in 2^{<\omega}$ as *guesses*. Let $\mathrm{lh}(\sigma)$ denote the length of σ. We identify $2^{<\omega}$ with ω under some recursive coding and $\omega^{(\sigma)}$ denotes the σ-th column, $\omega^{(\sigma)} = \{\langle \sigma, y\rangle : y \in \omega\}$. To meet the P_e we use a coding procedure — at guess σ for $\mathrm{lh}(\sigma) = e+1$ — by coding $\{\langle \sigma, n\rangle : n \in K\}$ into B if all attacks fail. Here again, K denotes a creative set.

(4.8) **Definition.** We define the notions: *s is a σ-stage*, $l(\sigma, s)$ and $r(\sigma, s)$ by simultaneous induction as follows.

(i) Every stage s is a $\emptyset$-stage, $r(\emptyset, s) = l(\emptyset, s) = -1$.

(ii) If s is a τ-stage with $\mathrm{lh}(\tau) = \langle e, i\rangle$, then if

$$l(e, i, s) > \max\{l(e, i, t) : t < s \,\&\, t \text{ is a } \tau\text{-stage}\}, \tag{4.9}$$

we say s is a $\tau\char`^0$-stage, and define

$$l(\tau\char`^0, s) = l(e, i, s),$$
$$r(\tau\char`^0, s) = -1, \quad \text{and} \quad r(\tau\char`^1, s) = s+1.$$

If (4.9) fails, then s is a $\tau\char`^1$-stage. We define $l(\tau\char`^1, s) = l(\tau\char`^1, t)$ and $r(\tau\char`^1, s) = t$ where $t = ls(\tau\char`^0, s)$ the *last $\tau\char`^0$-stage* $< s$. This is defined by

$$ls(\tau\char`^0, s) = \begin{cases} \max\{t : t \text{ is a } \tau\char`^0\text{-stage and } t < s\}, & \text{if one defined,} \\ 0, & \text{otherwise.} \end{cases}$$

We let σ_s denote the unique guess of length s with s a σ_s-stage. During the construction we may declare P_e as *satisfied at stage s via some x with guess σ.* This satisfaction is automatically cancelled if $\exists t > s\, (\sigma_t \leq \sigma)$ or $A_t[\phi_e(x)] \neq A_s[\phi_e(x)]$. (This is well-defined since $\phi_e(x)$ will be defined if P_e has been declared satisfied via x.)

We shall say P_e *requires attention* at stage $s+1$ if P_e is unsatisfied at stage s and there exists $x \in K_s$ such that for $\sigma \subset \sigma_s$ with $\mathrm{lh}(\sigma) = e+1$:

(i) $\forall z < \langle \sigma, x\rangle \;\; (\hat{\Phi}_{e,s}(A_s; z) = B_s(z))$,

(ii) $B_s(\langle \sigma, x\rangle) = 0$, and

(iii) $\langle \sigma, x\rangle > \max\{r(\tau, s) : \tau \leq_{\mathrm{L}} \sigma\}$.

Construction

Stage 0. Set $r(\tau, 0) = -1$ for all $\tau \in 2^{<\omega}$. Set $B_0 = \emptyset$.

Stage $s+1$

Step 1. Set $r(\tau, s+1) = -1$ for all $\tau \not\leq_L \sigma_S$. For all $\tau \leq_L \sigma_s$ but $\tau \not\subset \sigma_s$ set $r(\tau, s+1) = r(\tau, s)$.

Step 2. For each e if $l(e, s) > ml(e, s)$ find the least x (if any) such that $\phi_{i,s}(x)\downarrow$, $ml(e, s) > \phi_i(x)$ and both

(i) $W_{e,s}[\phi_i(x)] \neq W_{e,ls(e,s)}[\phi_i(x)]$, and

(ii) $V_{e,s}[\phi_i(x)] \neq V_{e,ls(e,s)}[\phi_i(x)]$

hold, (where $ls(e, s)$ is the last stage $<s$ with $l(e, s) > ml(e, s)$). If x exists, find the least element $\langle e, \phi_i(x), z\rangle$ of $\omega^{\langle\langle e, \phi_i(x), z\rangle\rangle}$ not yet in $Q_{e,s}$ and enumerate $\langle e, \phi_i(x), z\rangle$ into $Q_{e,s+1} - Q_{e,s}$.

Step 3. Find the least e, if any, such that P_e requires attention. Let $\langle\sigma, x\rangle$ be least for e. Set $B_{s+1} = B_s \cup \{\langle\sigma, x\rangle\}$. Declare P_e as satisfied via x. □ End of Construction

Verification. Let β denote the leftmost path. That is, $\emptyset \subset \beta$ and $\sigma \subset \beta$ implies $\sigma\hat{\ }0 \subset \beta$ if $\exists^\infty s\ (\sigma\hat{\ }0 \subset \sigma_s)$ and $\sigma\hat{\ }1 \subset \beta$ otherwise. We must argue that

(4.10) each P_e is met and receives attention at most finitely often at γ-stages for $\gamma \subset \beta$ and $\mathrm{lh}(\gamma) = e+1$,

(4.11) all the $N_{\langle e,i\rangle}$ are met, and

(4.12) $r(\tau, s) = r(\tau)$ exists for all $\tau \leq_L \sigma$ for $\sigma \subset \beta$.

First let $\sigma \subset \beta$ with $\mathrm{lh}(\sigma) = \langle e, i\rangle + 1$ and suppose that s is a σ-stage such that for all $s > s_0$ we have

(i) $\sigma \leq_L \sigma_s$,

(ii) s is a σ-stage and $j \leq \langle e, i\rangle$ implies P_j does not receive attention at stage s, and

(iii) $r(\tau, s) = r(\tau) = r(\tau, s_0)$ for all $\tau \leq_L \sigma$ and $\tau \neq \sigma$.

To establish (4.11) suppose $l(e, s) \to \infty$. First we observe that $Q_e \leq_W W_e, V_e$ as follows: let z be given. Now $z \in Q_e$ only if $z = \langle e, q, h\rangle$ for some $h \leq q$. If h and q exist find the least stage s where $l(e, s) > ml(e, s)$ and $W_{e,s}[q] = W_e[q]$. Numbers enter Q_e only during step 2 and then only when both W_e and V_e change. It follows that $Q_e \leq_W W_e$ and $Q_e \leq_W V_e$ *mutatis mutandis*.

Now suppose $l(e, i, s) \to \infty$. Then $\sigma = \tau\hat{\ }0$ for some τ and $l(\tau\hat{\ }0, s) \to \infty$. To compute $f(x)$, find the least σ-stage $s_1 > s_0$ with $l(\tau\hat{\ }0, s_1) > x$, and $Q_{e,s_1}[\langle e, \phi_i(x), \phi_i(x)\rangle] = Q_e[\langle e, \phi_i(x), \phi_i(x)\rangle]$. Then we claim $f_s(x) = f(x)$. Indeed, we claim for all $t > s_1$, one of

(4.13) $\hat{\Phi}_{i,t}(W_{e,t}; x) = \hat{\Phi}_{i,s_1}(W_{e,s_1}; x)$, or

(4.14) $\hat{\Phi}_{i,t}(V_{e,t}; x) = \hat{\Phi}_{i,s_1}(V_{e,s_1}; x)$ holds.

To see this, we first note that after stage s_0, numbers may only enter $A_{s+1}-A_s$ below $\phi_e(\phi_i(x))$ or $\gamma_e(\phi_i(x))$ & only at σ-stages. At other than σ-stages we have set the σ-restraint $r(\tau\hat{}1, s)$ to exceed s, and so exceed $\phi_e(\phi_i(x))$ and $\gamma_e(\phi_i(x))$. $r(\tau\hat{}1, s)$ is active at σ_s-stages for $\sigma \not\leqslant_L \sigma_s$. Thus suppose $z<\phi_e(\phi_i(x))$ and $\gamma_e(\phi_i(x))$, and z enters $A_t-A_{s_1}$ for some σ-stage t. Now at stage $t+1$ we set $r(\tau\hat{}1, t+1)$ to be $t+1$. Let $t_1>t+1$ be the least stage with $l(e, t_1)>ml(e, t_1)$. Then by assumption on Q_e we know that one of

(i) $W_{e,t}[\phi_i(x)=W_{e,t_1}[\phi_i(x)]$, or

(ii) $V_{e,t}[\phi_i(x)]=V_{e,t_1}[\phi_i(x)]$ holds.

Otherwise, step 2 would cause us to change Q_e below $\langle e, \phi_i(x), \phi_i(x)\rangle$ contradicting the assumption on s_1. By induction we see both of

(i) $\hat{\Phi}_{i,t}(W_{e,t}; x)=\hat{\Phi}_{i,s_1}(W_{e,s_1}; x)$, and

(ii) $\hat{\Phi}_{i,t}(V_{e,t}; x)=\hat{\Phi}_{i,s_1}(V_{e,s_1}; x)$ hold,

since t was a σ-stage. But this means at stage t_1, one of

(i) $\hat{\Phi}_{i,t_1}(W_{e,t_1}; x)=\hat{\Phi}_{i,s_1}(V_{e,s_1}; x)$, or

(ii) $\hat{\Phi}_{i,t_1}(V_{e,t_1}; x)=\hat{\Phi}_{i,s_1}(V_{e,s_1}; x)$ holds.

But now the restraint $r(\tau\hat{}1, s)=r(\tau\hat{}1, t)=t$ will preserve this until the next $\tau\hat{}0$-stage t_2. But now at stage t_2, both of (4.13) and (4.14) hold (with t_2 in place of t). This establishes (4.11).

Next we argue that (4.12) holds. But this is the standard minimal pair argument: for $\tau\leqslant_L\beta$ with $\tau\not\subset\sigma$, there are only finitely many τ-stages. Since $r(\tau, s)$ is only reset at τ-stages, $\lim_s r(\tau, s)=r(\tau)$ exists. Now with s_0 and σ as above, there are two cases. Either $\sigma=\tau\hat{}0$ or $\sigma=\tau\hat{}1$ for some τ. If $\sigma=\tau\hat{}1$, there are only finitely many $\tau\hat{}0$-stages. If t is the last $\tau\hat{}0$-stage, then $r(\sigma)=r(\sigma, t+1)=t+1$. Finally, if $\sigma=\tau\hat{}0$, then $r(\sigma, s)=-1$ for all s. This clinches (4.12).

Finally, we verify (4.10), that is that all the P_e are met, and receive attention at most finitely often. Let $\sigma\subset\beta$ with $\text{lh}(\sigma)=e+1$ and s_0 a stage such that for $\tau\leqslant\sigma$ and $s>s_0$ we have

(i) $r(\tau, s)=r(\tau, s_0)$,

(ii) $\sigma\leqslant_L\sigma_s$,

(iii) if s is a σ-stage, then P_j for $j<e$ does not receive attention at stage s, and

(iv) $x\in K_s-K_{s_0}$ implies $x>\max\{r(\tau, s)\colon \tau\leqslant_L\sigma\}$.

Now suppose that P_e fails to be met, or equivalently receives attention infinitely often. We argue as we did in the previous construction that $K\leqslant_W A$: to determine if $x\in K$ or not find the least σ-stage $s=s(x)>s_0$ such that $\forall z<\langle\sigma, x+1\rangle$ $((\hat{\Phi}_{e,s}(A_s; z)=B_s(z))$ and $A_s[\phi_e(z)]=A[\phi_e(z)])$.

Then as in (4.4) if y enters $K-K_s$ and $y<x$, we can use $\langle\sigma, y\rangle$ to kill P_e. Hence $K_s[x]=K[x]$ and so $K\leqslant_W A$, a contradiction. This concludes the proof of (4.7). □

Without much change to the construction we can also satisfy minimal pair requirements to show

(4.15) **Corollary.** $\forall \mathbf{a} \neq \mathbf{0}'\ \exists \mathbf{b}, \mathbf{c}\ (\mathbf{b}, \mathbf{c} \not\leq \mathbf{a}\ \&\ \mathbf{b} \cap \mathbf{c} = \mathbf{0}\ \&\ \mathbf{W}[\mathbf{0}, \mathbf{b} \cup \mathbf{c}]$ *a lattice*).

We mention (4.15) only because it gives a neat proof of Cohen's [7] result:

(4.16) **Corollary** (Cohen [7]). *The r.e. incomplete W-degrees are branching.*

Proof. Let $\mathbf{a} \in \mathbf{W}$ with $\mathbf{a} \neq \mathbf{0}'$. Apply (4.15) to get $\mathbf{b}$ and $\mathbf{c}$ as above. Consider $\mathbf{e} = \mathbf{a} \cup \mathbf{b}$ and $\mathbf{f} = \mathbf{a} \cup \mathbf{c}$. Let $\mathbf{h} \leq \mathbf{e}, \mathbf{f}$. Then $\mathbf{h} \leq \mathbf{a} \cup \mathbf{b}$ implies $\mathbf{h} = \mathbf{h}_1 \cup \mathbf{h}_2$ with $\mathbf{h}_1 \leq \mathbf{a}$ and $\mathbf{h}_2 \leq \mathbf{b}$. Now $\mathbf{h}_2 \leq \mathbf{f} = \mathbf{a} \cup \mathbf{c}$ so $\mathbf{h}_2 = \mathbf{h}_3 \cup \mathbf{h}_4$ with $\mathbf{h}_3 \leq \mathbf{a}$ and $\mathbf{h}_4 \leq \mathbf{c}$. Thus $\mathbf{h} = \mathbf{h}_1 \cup \mathbf{h}_3 \cup \mathbf{h}_4$ and $\mathbf{h}_1 \cup \mathbf{h}_3 \leq \mathbf{a}$, but $\mathbf{h}_4 \leq \mathbf{b}, \mathbf{c}$. But $\mathbf{b} \cap \mathbf{c} = \mathbf{0}$. Thus $\mathbf{h}_4 = \mathbf{0}$ and so $\mathbf{h} \leq \mathbf{a}$. Thus $\mathbf{e} \cap \mathbf{f} = \mathbf{a}$. Finally $\mathbf{e} \neq \mathbf{a}$ and $\mathbf{f} \neq \mathbf{a}$. For suppose (say) $\mathbf{e} = \mathbf{a}$. Then $\mathbf{a} \cup \mathbf{b} = \mathbf{a}$ but $\mathbf{b} \not\leq \mathbf{a}$, contradiction. □

We remark that (4.15) also gives the additional information that $\mathbf{W}[\mathbf{a}, \mathbf{a} \cup \mathbf{b} \cup \mathbf{c}]$ is a lattice, containing complemented members (namely $\mathbf{a} \cup \mathbf{b}$ and $\mathbf{a} \cup \mathbf{c}$).

The last conjecture suggested by this series of results is perhaps that each $\mathbf{a} \neq \mathbf{0}$ bounds an initial segment that forms a lattice. This conjecture does not hold — as we show in Part II — although we do show that segments of $\mathbf{W}$ that do form lattices are dense in $\mathbf{W}$. This last result is established in the next section.

We also point out that if $\mathbf{b} \neq \mathbf{0}$ then $[\mathbf{0}, \mathbf{b}]$ *never* forms a lattice. There are several ways to establish this result. One elegant way is to use 1-generic sets. Recall from (e.g.) Jockusch [22] that a set A is called *1-generic* if, given any r.e. set S of strings there is a string $\sigma \subset A$ such that either σ is in S or no extension of σ is in A. (Actually, the original definition is in terms of forcing, this characterization being due to Posner from his thesis.) Now a standard permitting construction shows that each nonzero $\mathbf{a} \in \mathbf{W}$ bounds a 1-generic W-degree $\mathbf{c} \in \mathbf{D}$. The details of Jockusch [22, Theorem 3.1] show that if $\mathbf{c}$ is 1-generic then $[\mathbf{0}, \mathbf{c}]$ is not a lattice.

Before we leave this section and turn to other results on the distribution of lattices in $\mathbf{W}$, there is one further result concerning pairs without infimum we would like to include. This result was stated in [15] without proof, and concerns the way infimums interrelate between $\mathbf{R}$ and $\mathbf{W}$.

(4.17) **Theorem.** *There exist r.e. sets A and B such that the wtt-degrees of A and B have an infimum, but the T-degrees of A and B do not.*

Proof. Although it is not too difficult to establish this result directly, we choose to use some results from the literature. Specifically using Fischer's result choose $\mathbf{b}$ with $W[\mathbf{0}, \mathbf{b}]$ a lattice. Now using Ladner and Sasso [28] take $\mathbf{a}$ with $\mathbf{0} < \mathbf{a} < \mathbf{b}$ and $\mathbf{a}$ of contiguous T-degree. It is easy to modify Jockusch's [21] construction of a pair of r.e. sets with no T-infimum to show that it works below any nonzero r.e. T-degree, and in particular below **a**. Let A be an r.e. set of degree **a**. By contiguity, there exist r.e. sets B, $C \leqslant_W A$ such that the infimum of the T-degrees of B and C doesn't exist. However since $\mathbf{a} < \mathbf{b}$, it must be that the inf of the W-degrees of B and C exists. □

5. Lattices are dense

Although not every r.e. degree is the top of a lattice in **W**, we have seen that each incomplete **W**-degree is the bottom of a lattice in **W**. It is natural to conjecture that

(5.1) $\quad \forall \mathbf{a} < \mathbf{b}\, \exists \mathbf{e}\, (\mathbf{a} < \mathbf{e} < \mathbf{b}$ and $\mathbf{W}[\mathbf{a}, \mathbf{e}]$ forms a lattice).

In Part II we shall show that (5.1) fails even for $\mathbf{a} = \mathbf{0}$. The goal of this section is to establish our best positive result along the lines of (5.1) by showing that *segments of* **W** *that form lattices are dense.*

(5.2) **Theorem.** $\exists \mathbf{a}, \mathbf{b}\, (\mathbf{a} < \mathbf{b} \rightarrow \exists \mathbf{e}, \mathbf{f}\, (\mathbf{a} < \mathbf{e} < \mathbf{f} < \mathbf{b}\ \&\ \mathbf{W}[\mathbf{e}, \mathbf{f}]$ *is a lattice*)).

Proof. Let $A \leqslant_W B$ be given r.e. sets. We construct $C = \bigcup_s C_s$ and $D = \bigcup_s D_s$ with $A \oplus C \oplus D \leqslant_W B$ satisfying

$$P_e\colon\quad \hat{\Phi}_e(A \oplus C) \neq D.$$

We build auxiliary sets $Q_e = \bigcup_s Q_{e,s}$ satisfying

$$\begin{aligned} N_{e,i}\colon\quad & \hat{\Phi}_e(A \oplus C \oplus D) = W_e \oplus A \oplus C \quad \text{and} \\ & \hat{\Gamma}_e(A \oplus C \oplus D) = V_e \oplus A \oplus C \quad \text{implies} \\ & Q_e \leqslant_W W_e \oplus A \oplus C, \quad V_e \oplus A \oplus C \quad \text{and if} \\ & \hat{\Phi}_i(W_e \oplus A \oplus C) = \hat{\Phi}_i(V_e \oplus A \oplus C) = f \quad \text{and} \\ & f \ \text{ is total, then } \ f \leqslant_W Q_e \oplus A \oplus C. \end{aligned}$$

For simplicity of notation, let $\tilde{W}_e = W_e \oplus A \oplus C$ and $\tilde{V}_e = V_e \oplus A \oplus C$. The basic problem of satisfying the $N_{e,i}$ in conjunction with the P_e is this. In each of the previous lattice constructions (of Section 4), a crucial characteristic of the construction is that if we put some number in D (in the notation of this construction) to satisfy some P_e, then $\tilde{W}_e$ and $\tilde{V}_e$ get essentially one chance to change. That is, roughly speaking, our restraint doesn't really 'restrain' anything when it is originally imposed (and thus 'both sides' can change). *After* the next e-expansionary stage the restraints we imposed when we attacked P_e take over. In the current construction numbers must go into D only when B-permitted. This necessitates our putting numbers into D at other than α-stages and means our restraining policy won't work. This is where C comes in. Our idea, roughly speaking, is to ensure that either our 'delayed restraint' will be successful — as in the Section 4 constructions — or C will be able to recognize that our restraint wasn't successful, because some number is enumerated into C to record this fact.

Care must be taken in selection of numbers to add to C for the sake of this strategy. Remember, the sequence will be (1) add a number to D to satisfy P_e, and (2) if this (perhaps) injures some $N_{e,i}$ restraint, add some number $y(x)$ to C to recognize this. Obviously this is useless if $y(x) < \phi_e(x) = u(\hat{\Phi}_e(A \oplus C; x))$, because then it would undo our P_e action. This is the reason for the *guessing/confirmation procedure in* the construction. That is, this procedure allows us to ensure that the enumeration of a follower x into D doesn't interfere with the entry of higher priority followers into D because of x's interaction with the $N_{j,k}$ of higher priority than x.

We now turn to the formal details of the construction. We need the following auxiliary functions

$$L(e, s) = \max\{x: \forall y < x\ (\hat{\Phi}_{e,s}(A_s \oplus C_s; y) = D_s(y))\},$$

$$mL(e, s) = \max\{L(e, t): t < s\},$$

$$l(e, s) = \max\{x: \forall y < x\ (\hat{\Phi}_{e,s}(A_s \oplus C_s \oplus D_s; y) = \tilde{W}_{e,s}(y)\ \&\ \hat{\Gamma}_{e,s}(A_s \oplus C_s \oplus D_s; y) = \tilde{V}_{e,s}(y)\},$$

$$ml(e, s) = \max\{l(e, t): t < s\}, \quad \text{and}$$

$$l(e, i, s) = \max\{x: \forall y < x\ (\hat{\Phi}_{i,s}(\tilde{W}_{e,s}; y) = \hat{\Phi}_{i,s}(\tilde{V}_{e,s}; y)\ \&\ \max\{\phi_i(y), \gamma_i(y)\} < l(e, s))\}.$$

We define the notion 'σ-stage' by induction on $\mathrm{lh}(\sigma)$:

(i) Every stage s is a $\emptyset$-stage.

(ii) If s is a τ-stage and $\mathrm{lh}(\tau) = \langle e, i\rangle$, then if

$$l(e, i, s) > \max\{l(e, i, t): t < s \text{ and } t \text{ is a } \tau\text{-stage}\},$$

we say s is a τ^0-stage. Otherwise we say s is a τ^1-stage. As usual σ_s denotes the unique path of lengths with s a σ_s-stage.

We say P_e *requires attention* at stage $s+1$ if there exists a follower $y = y(x, \sigma)$ of P_e such that one of the following options holds.

(5.3) (i) $x \in B_{s+1} - B_s$ and $c(y(x, \sigma))$ is defined.
(ii) y is τ^0-confirmed for all τ^0 $\subset \sigma$, and
(iii) $L(e, s) > y$, or

(5.4) For all followers y of P_e if $y = y(x, \sigma)$ and if $\sigma \subset \sigma_s$, then $L(e, s) > y$ and y is τ^0-confirmed for all τ^0 $\subset \sigma$.

Construction, stage $s+1$

Step 1. Cancel all followers y with guess $\sigma \not\subseteq \sigma_s$. (That is, $y = y(x, \sigma)$ for some x and $\sigma \not\subseteq \sigma_s$.)

Step 2. Find the least follower $y = y(x, \gamma)$ and $\tau^\frown 0 \subset \sigma_s$ such that

(i) $\tau^\frown 0 \subset \gamma$,

(ii) y is not yet $\tau^\frown 0$-confirmed, and

(iii) $l(f, g, s) > y$ where $\mathrm{lh}(\tau) = \langle f, g \rangle$.

Cancel all followers bigger than y. (Note: it is a characteristic of the construction that these have lower priority than y.) Declare y as $\tau^\frown 0$-confirmed for each such $\tau^\frown 0$.

Step 3. For each $e \leqslant s$ such that s is e-expansionary find the least x, if any, such that

(i) $ml(e, s) > x$, and for $t = ls(e, s)$ we have

(ii) $W_{e,s}[x] \neq W_{e,t}[x]$ and $V_{e,s}[x] \neq V_{e,t}[x]$.

If x does not exist go to step 4. If x exists, set $Q_{e,s+1} = Q_{e,s} \cup \{\langle e, x, j\rangle\}$ where j is the least number with $\langle e, x, j\rangle \notin Q_{e,s}$. (As usual we will see $j \leqslant x$.)

Step 4. Find the least e such that P_e requires attention. Adopt the first case below to hold.

Case 1: (5.3) *holds.* In this case, find the least $y = y(x, \sigma)$ which pertains. (Note: as y is alive $\sigma \leqslant_{\mathrm{L}} \sigma_s$. But we do not ask that $\sigma \subset \sigma_s$.) Cancel all followers $y(g, \tau)$ for all g with $\sigma \leqslant_{\mathrm{L}} \tau$ and $\tau \neq \sigma$. Set $D_{s+1} = D_s \cup \{y(x, \sigma)\}$. Set $C_{s+1} = C_s \cup \{\langle e, c(y)\rangle\}$. (Note: $y(x, \sigma)$ remains a 'follower' unless cancelled in steps 1 or 2.)

Case 2: (5.4) *holds.* Appoint $y = s+1$ as a follower of P_e with guess σ where $\sigma \subset \sigma_s$ and $1h(\sigma) = e+1$. Mark y as $y(x, \sigma)$ where x is least with $y(x, \sigma)$ not currently defined. Cancel all $y(g, \tau)$ for $\tau \supsetneqq \sigma$. If $x \neq 0$, define $c(y(x-1, \sigma)) = s+1$. □ End of Construction

It is easy to see (by induction) that for any follower y we have $x \leqslant y$ if $y = y(x, \sigma)$ for some σ, and that $y \leqslant c(y)$. Now since numbers which enter $D_{s+1} - D_s$ are $y = y(x, \sigma)$ for some x, σ with $x \in B_{s+1} - B_s$ we see that $D \leqslant_{\mathrm{W}} B$ by permitting. Similarly the fact that $y \leqslant c(y)$ implies that $C_{s+1}[z] \neq C_s[z]$ implies $B_{s+1}[z] \neq B_s[z]$ and hence $C \leqslant_{\mathrm{W}} B$. Also if $l(e, s) \to \infty$ our previous arguments (e.g. (4.7)) ensure that $Q_e \leqslant_{\mathrm{W}} W_e, V_e$.

First we argue that all the P_e receive attention at most finitely often at 'σ-stages' and are met. Let β denote the leftmost path. Let $\sigma \subset \beta$ with $\mathrm{lh}(\sigma) = e+1$. For an induction, let s_0 be a σ-stage such that for all $s > s_0$ we have

(a) $\sigma \leqslant_{\mathrm{L}} \sigma_s$,

(b) if s is a σ-stage, and $j < e$ then

(c) P_j does not receive attention at stage s, and if y has guess γ for $\gamma \leqslant_{\mathrm{L}} \sigma$ and $\gamma \neq \sigma$, then y does not act by confirmation at stage s.

By our cancellation procedures, we might as well suppose P_e has no followers with guess σ at stage s_0, and also that no follower of P_e with guess $\gamma \leqslant_{\mathrm{L}} \sigma$ and

$\gamma \neq \sigma$ acts. Now since P_e receives attention infinitely often at σ-stages or (equivalently) fails to be met, P_e must get an infinite set of followers

$$y(0,\sigma), y(1,\sigma), y(2,\sigma)\ldots$$

appointed after stage s_0. Each of these is confirmed and uncancellable. We claim that this implies that $B \leqslant_W A$. Let z be given. To A-recursively determine if $z \in B$ or not find the least stage $s > s_0$ such that

(i) $y(z+1,\sigma)$ is defined at stage s,

(ii) $L(e,s) > y(z+1,\sigma)$, and

(iii) $A_s[\phi_e(z+1,\sigma)] = A[\phi_e(z+1,\sigma)]$.

We claim that $z \in B$ iff $z \in B_s$. To see this, suppose that $z \notin B_s$. Let g be the least number with $g \leqslant z$ and $g \in B - B_s$. We first claim that

(5.5) $\quad C_s[\phi_e(y(g,\sigma))] = C[\phi_e(y(g,\sigma))]$.

To see that (5.5) holds, we argue as follows. Since $y(z+1,\sigma)$ is defined so is $y(g,\sigma)$ and $y(g+1,\sigma)$. When $y(g+1,\sigma)$ becomes defined, we must have $L(0,s) > y(g,\sigma)$. By our cancellation procedure when $y(g+1,\sigma)$ is appointed—say at stage t for $t < s$—the only numbers left alive which might enter $D - D_t$ are $y(q,\sigma)$ for $q \leqslant g+1$. (Everything else is fixed by choice of s_0 or cancelled.) By the way we appoint followers, this means that

(5.6) if $p \in D[y(g,\sigma)] - D_t[y(g,\sigma)]$, then $p = y(h,\sigma)$ for some $h \leqslant g$.

Now, numbers which enter C are of the form $c(y)$ for some y. Now when $y(g+1,\sigma)$ is appointed at stage t, $c(y(g+1,\sigma))$ is set. Since $L(e,t) > y(g,\sigma)$ it follows that $c(y(g+1,\sigma)) > \phi_e(y(g,\sigma))$ and furthermore by monotonicity of ϕ_e we must have that for all $p \geqslant g+1$

(5.7) $\quad c(y(p,\sigma)) > \phi_e(y(g,\sigma))$.

Now $c(y)$ enters C only when y enters D. The minimality of g means that $D_s[y(g,\sigma)-1] = D[y(g,\sigma)-1]$. Combining this with (5.6) and (5.7) will give (5.5). Let $y_1 = y(g,\sigma)$. By (5.5) we have $C_s[\phi_e(y_1)] = C[\phi_e(y_1)]$. By hypothesis (iii) $A_s[\phi_e(y_1)] = A[\phi_e(y_1)]$ and by hypothesis (ii) $L(e,s) > y_1$. This means that the computation

$$\hat{\Phi}_{e,s}(A_s \oplus C_s; y_1) = 0 = D_s(y_1)$$

Is final; that is,

(5.8) $\quad \forall s_1 \geqslant s \quad \hat{\Phi}_{e,s_1}(A_{s_1} \oplus C_{s_1}; y_1) = 0$.

But now by assumption $g \in B_{s_2+1} - B_s$ for some (least) stage $s_2 \geqslant s$. At such a stage (5.3) will pertain (since (5.8) and minimality of g mean that $L(e,s_2) > y_1$). This will create a disagreement

$$\hat{\Phi}_e(A \oplus C; y_1) \neq D(y_1).$$

It thus follows that $z \in B$ iff $z \in B_s$ since no number $\leqslant z$ can enter B after stage s. But this is a W-reduction giving $B \leqslant_W A$ and so we have a contradiction since $A \leqslant_W B$. Therefore P_e is met and receives attention at most finitely often at σ-stages.

Finally we turn to the $N_{e,i}$. Let $\sigma \subset \beta$ with $\text{lh}(\sigma) = \langle e, i\rangle + 1$. Suppose $l(e, i, s) \to \infty$. Then $\sigma = \tau\hat{\ }0$ for some τ. Let s_0 be a stage good for σ as in the verification of the P_e. We must show that $f \leqslant_W A \oplus Q_e \oplus C$, given that f is total. To see this, let $z \in \omega$ be given. Let s_1 be the least σ-stage with $s_1 > s_0$ and such that

(5.9)
(i) $l(e, i, s_1) > s$,
(ii) for all *followers* $y = y(x, \gamma)$ for $\gamma \subset \sigma$, if $y \leqslant \max\{\phi_e(\phi_i(z)), \gamma_e(\phi_i(z))\}$, then y is $\tau\hat{\ }0$-confirmed for all $\tau\hat{\ }0 \subset \sigma$.

Now let s_2 be the least σ-stage with $s_2 \leqslant s_1$ such that

(5.10)
(i) $A_{s_2}[s_1+1] = A[s_1+1]$,
(ii) $Q_{e,s_2}[\langle e, s_1+1, s_1+1\rangle] = Q_e[\langle e, s_1+1, s_1+1\rangle]$, and
(iii) $C_{s_2}[s_1+1] = C[s_1+1]$.

We then claim that at all stages $s \geqslant s_2$

(5.11)
$\hat{\Phi}_{i,s}(W_{e,s}; z) = \hat{\Phi}_{i,s_2}(W_{e,s}; z)$, or
$\hat{\Phi}_{i,s}(V_{e,s}; z) = \hat{\Phi}_{i,s_2}(V_{e,s}; z)$ holds.

To establish (5.11) we must first observe that (5.10) (ii) means that if s and t are e-expansionary stages $>s_2$ with $t = ls(e, s)$, then

(5.12) one of $W_{e,s}[\phi_i(z)] = W_{e,t}[\phi_i(z)]$ or $V_{e,s}[\phi_i(z)] = V_{e,t}[\phi_i(z)]$ holds.

Therefore, by our reasoning in (4.7) if q and r are σ-stages with $q > r \geqslant s_2$ and

$$\text{card}(\tilde{D}_q[M] - \tilde{D}_r[M]) \leqslant 1$$

where

$$M = \max\{\phi_e(\phi_i(z)), \gamma_e(\phi_i(z))\}, \quad \text{and} \quad \tilde{D} = D \oplus A \oplus C,$$

then (5.12) means that one of $\hat{\Phi}_{i,q}(W_{e,q}; z) = \hat{\Phi}_{i,r}(W_{e,r}; z)$ or $\hat{\Phi}_{i,q}(V_{e,q}; z) = \hat{\Phi}_{i,r}(V_{e,r}; z)$ holds. Consequently, if (5.11) is to fail, this reasoning means that there must be σ-stages s_3 and s_4, and an e-expansionary stage t_1 and a least stage t_2 with $l(e, t_2) > z$ so that

(i) $s_2 \leqslant s_3 \leqslant s_4$,
(ii) $s_3 = ls(\sigma, s_4)$ (that is, $\forall n\, (s_3 < n < s_4 \to \sigma \not\subset \sigma_n)$),
(iii) $s_3 < t_1 < t_2 \leqslant s_4$,
(iv) $\text{card}(\tilde{D}_t[M] - \tilde{D}_{s_3}[M]) \geqslant 1$ ($\tilde{D}$, M as in (5.12)),
(v) $\text{card}(\tilde{D}_{t_2}[M] - \tilde{D}_{t_1}[M]) \geqslant 1$, and
(vi) $t_1 = ls(e, t_2)$.

Also, if we are to suppose (5.11) is to fail, we may choose s_3, t_1, t_2, s_4 least for (5.11)'s failure (by stage t_2) and thus know

(vii) $\hat{\Phi}_{i,s_2}(W_{e,s_2}; z) = \hat{\Phi}_{i,s_3}(W_{e,s_3}; z)$,

(viii) $\hat{\Phi}_{i,s_2}(V_{e,s_2}; z) = \hat{\Phi}_{i,s_3}(V_{e,s_3}; z)$,

(ix) $\hat{\Phi}_{i,s_3}(W_{e,s_3}; z) \neq \hat{\Phi}_{i,t_1}(W_{e,t_1}; z)$,

(x) $V_{e,s_3}[\phi_i(z)] = V_{e,t_1}[\phi_i(z)]$ and so
$\hat{\Phi}_{i,s_3}(V_{e,s_3}; z) = \hat{\Phi}_{i,t_1}(V_{e,t_1}; z)$, and

(xi) $\hat{\Phi}_{i,t_2}(V_{e,t_2}; z) \neq \hat{\Phi}_{i,t_1}(V_{e,t_1}; z)$.

We remark that (ix), (x) and (xi) may be taken without loss, by symmetry of W_e and V_e. We argue that this situation will contradict (5.9) and (5.10). By (5.10)(i), there must exist followers y_1 and y_2 (least) with

(5.13) $y_1, y_2 < M$ and $y_1 \in \tilde{D}_{t_1} - \tilde{D}_{s_3}$ and $y_2 \in \tilde{D}_{t_2} - \tilde{D}_{t_1}$.

Certainly, by the way we appoint followers, by (5.9) we know y_1 and y_2 are confirmed and present at stage s_1. The crucial claim is that neither $c(y_1)$ nor $c(y_2)$ is defined at stage s_1. For suppose $c(y_1)$ is defined at stage s_1. Then $c(y_1) \leqslant s_1$, by the way we define $c(y_1)$. But then when y_1 enters D it causes us to enumerate $c(y_1)$ into C at the same stage which would violate (5.10)(iii).

Now followers y may enter D only if $c(y)$ is defined (by construction). For any follower y, with guess $\supset\sigma$, $c(y)$ becomes defined at $1+\sigma$-stages. Consequently both $c(y_1)$ and $c(y_2)$ are defined at stage s_3, and were defined between stages s_1 and s_3. Now since y_1 and y_2 both exist at stage s_1, y_1 and y_2 must have different guesses $\gamma_1, \gamma_2 \supset \sigma$ respectively. Now if $\gamma_1 \not\subseteq \gamma_2$ and $\gamma_2 \not\subseteq \gamma_1$ then either y_1 or y_2 is cancelled at s_3. For suppose (e.g.) that $\gamma_1 \leqslant_L \gamma_2$ but $\gamma_1 \not\subseteq \gamma_2$. Now as $c(y_1)$ is defined (which only happens at $1+\gamma_1$-stages) y_2 will be cancelled at some γ_2-stage between s_1 and s_3. Finally, if $\gamma_1 \subset \gamma_2$ but $\gamma_1 \neq \gamma_2$, say, we still see that y_2 is cancelled, but now for a different reason. The point is that $c(y_1)$ is defined when P_e for $e+1 = \mathrm{lh}(\gamma_1)$ appoints a new follower. This activity automatically cancels followers with guesses $\tau \supset \gamma_1$ and $\gamma \neq \gamma_1$. Hence y_2 is cancelled. The case $\gamma_2 \subset \gamma_1$ and $\gamma_2 \neq \gamma_1$ may be taken *mutatis mutandis*. It therefore follows that both y_1 and y_2 cannot exist. This contradiction establishes (5.11) and completes our proof of (5.2). □

It is of course natural to ask exactly which lattices can be realized as segments in **W**. This question is related to the question of whether or not there exists (all?) **a**, **b** with **a** < **b** and Th(**W**[**a**, **b**]) decidable. For example, if we could construct **W**[**a**, **b**] which was complemented, then it would necessarily be the countable atomless boolean algebra, and thus Th(**W**[**a**, **b**]) would be decidable for such **a** and **b**. Unfortunately this idea fails because W[**a**, **b**] is never complemented (and nor is [**a**, **b**]).

(5.14) **Theorem.** *Let* $\mathbf{a}<\mathbf{b}$. *Then there exists* $\mathbf{c}$ *with* $\mathbf{a}<\mathbf{c}<\mathbf{b}$ *such that for all (not necessarily r.e.) degrees* $\mathbf{e}$, *if* $\mathbf{e}\cup\mathbf{c}\geqslant\mathbf{b}$, *then* $\mathbf{e}\geqslant\mathbf{b}$.

Proof. To ensure $\mathbf{c}>\mathbf{a}$ we use a standard Friedberg strategy. This is no problem combining such a strategy with the anticupping technique of [11]. We refer the reader there for further details. □

By [10] for $\mathbf{W}[\mathbf{0},\mathbf{a}]$ we can do a little better.

(5.15) **Theorem** ([10]). $\forall\mathbf{a}\neq\mathbf{0}\,\exists\mathbf{c}\,(\mathbf{c}<\mathbf{a}\;\&\;\forall\mathbf{e}\leqslant\mathbf{a}\,(\mathbf{c}\cap\mathbf{e}=\mathbf{0}\rightarrow \mathbf{e}=\mathbf{0}\;\&\;\mathbf{c}\cup\mathbf{e}=\mathbf{a}\rightarrow\mathbf{e}=\mathbf{a}))$.

We do not know if (5.18) can be improved to give the result for intervals. That is, we don't know if

$$\forall\mathbf{a}<\mathbf{b}\,\exists\mathbf{c}\,(\mathbf{a}<\mathbf{c}<\mathbf{b}\;\&\;\forall\mathbf{e}\leqslant\mathbf{b}\,(\mathbf{c}\cap\mathbf{e}\leqslant\mathbf{a}\rightarrow\mathbf{e}\leqslant\mathbf{a}\;\&\;\mathbf{c}\cup\mathbf{e}\geqslant\mathbf{b}\rightarrow\mathbf{e}\geqslant\mathbf{b}))\quad\text{holds.}\tag{5.16}$$

We conjecture that (5.16) fails.

6. Embedding the atomless boolean algebra in W[a, b]

The techniques introduced in Section 5 have other applications. The application we describe here is the promised proof that we can embed any countable distributive lattice in $\mathbf{W}[\mathbf{a},\mathbf{b}]$ preserving $\mathbf{b}$ for any $\mathbf{a}<\mathbf{b}$. this follows from

(6.1) **Theorem.** *Let* $\mathbf{a}<\mathbf{b}$. *Then there exists* $\mathbf{c}$ *with* $\mathbf{a}<\mathbf{c}<\mathbf{b}$ *such that there exists a lattice embedding of the countable atomless boolean algebra into* $\mathbf{W}[\mathbf{c},\mathbf{b}]$ *preseruing least element* $\mathbf{c}$ *and greatest element* $\mathbf{b}$.

Proof. Let $B=\bigcup_s B_s$ and $A=\bigcup_s A_s$ be canonical enumerations of r.e. sets with $A<_{\mathrm{W}}B$. We shall construct C in stages. Let $\{\alpha_i\colon i\in\omega\}$ denote a uniformly recursive sequence of recursive sets—meaning $\{\langle x,i\rangle\colon x\in\alpha_i\}$ is a recursive relation—which forms an atomless boolean algebra $\mathcal{Q}$ under $\bigcup,\bigcap$ and complementation. We construct a recursive collection of disjoint r.e. sets $\{D_i\colon i\in\omega\}$ and define $D_\alpha=\{x\colon x\in D_i \text{ and } i\in\alpha\}\oplus A\oplus C$ for each $\alpha\in\mathcal{Q}$. As in Soare [36, Ch. IX, §2], we see

$$\deg(D_\alpha)\cup\deg(D_\beta)=\deg(D_{\alpha\cup\beta}),\tag{6.2}$$

$$\alpha\subset\beta \text{ implies } D_\alpha\leqslant_{\mathrm{W}}D_\beta, \text{ and } \deg(D_{\alpha\cap\beta})\leqslant\deg(D_\alpha),\ \deg(D_\beta),\tag{6.3}$$

where, of course, 'deg' here refers to W-degree. We therefore must meet the requirements

R: $C \leqslant_W B$,

$Q_{\langle e,i\rangle}$: $\hat{\Phi}_e(A \oplus C) \neq D_i$,

$P_{\langle \alpha,\beta,e\rangle}$: $\hat{\Phi}_e(D_\alpha) = \hat{\Phi}_e(D_\beta) = f$ and f is total implies $f \leqslant_W D_{\alpha\cap\beta}$.

As in [36], we have

(6.4) $\deg(D_{\alpha\cap\beta}) = \deg(D_\alpha) \cap \deg(D_\beta)$, and

(6.5) $D_\alpha \leqslant_W D_\beta$ implies $\alpha \subset \beta$.

To see that (6.5) holds, suppose otherwise. Then $\alpha \not\subset \beta$ but $D_\alpha \leqslant_W D_\beta$. Let $i \in \alpha - \beta$. Then $D_i \leqslant_W D_\alpha \leqslant_W D_\beta$. But also $i \in \bar{\beta}$ and so $D_i \leqslant_W D_{\bar{\beta}}$. Hence $D_i \leqslant_W D_{\beta\cap\bar{\beta}} \equiv_W D_\emptyset \equiv_W C \oplus A$, contradiction.

Let $l(\alpha, \beta, e, s) = \max\{x: \forall y < x\ (\hat{\Phi}_{e,s}(D_{\alpha,s}; y) = \hat{\Phi}_{e,s}(D_{\beta,s}; y))\}$.

Using this 'length' function, define the notion of a σ-stage by induction on $\mathrm{lh}(\sigma)$ via

(i) every stage s is a $\emptyset$-stage, and

(ii) if s is a τ-stage and $\mathrm{lh}(\tau) = \langle \alpha, \beta, e\rangle$, then if

$$l(\alpha, \beta, e, s) > \max\{l(\alpha, \beta, e, t): t \text{ is a } \tau\text{-stage and } t < s\}$$

then s is a $\tau\hat{\ }0$-stage. Otherwise s is a $\tau\hat{\ }1$-stage. As usual let σ_s denote the unique string of length s such that s is a σ_s-stage.

Now let

$$L(e, i, s) = \max\{x: \forall y < x\ (\hat{\Phi}_{e,s}(A_s \oplus C_s; y) = D_{i,s}(y))\}.$$

We attack the $Q_{\langle e,i\rangle}$ by followers y as in Section 5, although these 'mark a position' only. These are marked $y = y(\sigma, x)$ to indicate they have guess σ with $\mathrm{lh}(\sigma) = \langle e, i\rangle$ and are 'connected to' x as their 'permitting number'. The reader should note that there will be a finite *entourage* of traces $c(y, 1), \ldots, c(y, n)$ for any follower y. We always attack the least member of the list not already attacked. The use of an entourage of traces is necessary to guarantee that we always have a follower—trace pair available to attack the $Q_{\langle e,i\rangle}$ (as in the pinball constructions of Section 3).

We say that $Q_{\langle e,i\rangle}$ *requires attention* at stage s if one of the following options holds.

(6.6) There is a follower $y = y(x, \sigma)$, say, of $Q_{\langle e,i\rangle}$ with $\mathrm{lh}(\sigma) = \langle e, i\rangle$ such that

(i) $z \leqslant y(x, \sigma)$ where $\{z\} = B_{s+1} - B_s$,

(ii) $z \nleqslant c(y(h, \gamma), j)$ for any $\gamma \leqslant_L \sigma$ and $\gamma \neq \sigma$ and any j (currently defined),

(iii) y is $\tau\hat{\ }0$-confirmed for all $\tau\hat{\ }0 \subset \sigma$, and

(iv) $c(y, 1)$ is defined, or

(6.7) for all followers $y = y(x, \sigma)$ for $\sigma \subset \sigma_s$ we have
(i) $L(e, i, s) > y$, and
(ii) y is $\tau\hat{}0$-confirmed for all $\tau\hat{}0 \subset \sigma$.

Construction, stage $s+1$

Step 1. Cancel all $y(x, \tau)$ for $\tau \nleq_{\mathrm{L}} \sigma_s$.

Step 2. Find the least follower (if any) y with $y = y(x, \gamma)$, say such that for $\tau\hat{}0 \subset \gamma$:
(i) $\tau\hat{}0 \subset \sigma_s$ and y is not $\tau\hat{}0$-confirmed, and
(ii) $l(\alpha, \beta, e, s) > y$ where $\mathrm{lh}(\tau) = \langle \alpha, \beta, e \rangle$.
Declare y as $\tau\hat{}0$-confirmed for each such $\tau\hat{}0$. Cancel all followers $y' > y$.

Step 3. Find the least $\langle e, i \rangle$ such that $Q_{\langle e,i \rangle}$ requires attention. Adopt the first case below to pertain.

Case 1: (6.6) *holds*. Find the least $y = y(x, \sigma)$, say. Cancel all $y(g, \tau)$ for $\sigma \leq_{\mathrm{L}} \tau$ but $\sigma \neq \tau$. Set $D_{i,s+1} = D_{i,s} \cup \{z\}$ where $z \in B_{s+1} - B_s$. Set $C_{s+1} = C_s \cup \{c(y, j)\}$ where $c(y, j)$ is least with $c(y, j) \notin C_s$. Note that $y(x, \sigma)$ does not become undefined here.

Case 2: (6.7) *holds*. Appoint $y = s+1$ as a follower of $Q_{\langle e,i \rangle}$ with guess $\sigma \subset \sigma_s$ where $\mathrm{lh}(\sigma) = \langle e, i \rangle$. Mark y by $y(x, \sigma)$ where x is least with $y(x, \sigma)$ currently undefined. Cancel all $y(g, \tau)$ for $\sigma \subset \tau$ and $\sigma \neq \tau$. Finally if $x \neq 0$, set

$$c(p, 1) = \langle e, i, s+1 \rangle,$$
$$c(p, 2) = \langle e, i, s+2 \rangle, \ldots, \quad c(p, s+1) = \langle e, i, 2s+1 \rangle,$$

where $p = y(x-1, \sigma)$. Note that this means that $c(y(x-1, \sigma), 1)$ is only set when $y(x, \sigma)$ is defined and so $L(e, i, s) > y(x-1, \sigma)$.

Step 4. If (6.6) did not pertain in step 3, enumerate z into $C_{s+1} - C_s$ for $z \in B_{s+1} - B_s$. □ End of Construction

The verification that all the $Q_{\langle e,i \rangle}$ receive attention finitely often 'at σ-stages' and are met, is by now familiar so we merely sketch the details. Let γ denote the leftmost path. Let $\sigma \subset \gamma$ with $\mathrm{lh}(\sigma) = \langle e, i \rangle$ and let s_0 be a stage good for σ in the sense that after stage s_0 the higher priority stuff ceases activity at σ-stages, also $\forall s > s_0\, (\sigma \leq_{\mathrm{L}} \sigma_s)$, and finally $z \in B - B_{s_0}$ implies z exceeds all $c(y, i)$ for all y with guess $\rho \leq_{\mathrm{L}} \sigma$ and $\rho \neq \sigma$.

Now suppose $Q_{\langle e,i \rangle}$ fails or, equivalently, is infinitely active at σ-stages. Then it follows that $Q_{\langle e,i \rangle}$ gets an infinite recursive list of followers

$$y(0, \sigma), \quad y(1, \sigma), \ldots$$

We claim that this implies $B \leq_{\mathrm{W}} A$. To determine if $x \in B$ or not, find the least σ-stage $s_1 > s_0$ with $y(x+1, \sigma)$ defined and $A_{s_1}[\phi_{e,s_1}(y(x, \sigma))] = A[\phi_e(y(x, \sigma))]$ and $L(e, i, s) > y(x, \sigma)$. It is claimed that $B_{s_1}[x] = B[x]$. Otherwise, let $z \leq x$ be the least number $\leq x$ to enter z after stage s_1. Then there is some least follower

$y=y(h,\sigma)\leqslant y(x,\sigma)$ with $z\leqslant y(h,\sigma)$. At the stage t when z enters $B_t-B_{s_1}$ we create a disagreement,

$$(6.8)\qquad \hat{\Phi}_e(A\oplus C;z)=0\neq D_i(z)$$

since the same reasoning as in Section 5 ensures that—by minimality of h—$C_{s_1}[c(y(h,\sigma),1)-1]=C[c(y(h,\sigma),1)-1]$ and thus

$$C_{s_1}[\phi_e(y(h,\sigma))]=C[\phi_e(y(h,\sigma))].$$

These observations combine to give (6.8).

Finally, we establish that all the $P_{\langle\alpha,\beta,e\rangle}$ are met. Let $\sigma\subset\gamma$ with $\mathrm{lh}(\sigma)=\langle\alpha,\beta,e\rangle\rangle+1$ and let s_0 be a stage good for as for the $Q_{\langle e,i\rangle}$. Let z be given. As in Section 5, find the least σ-stage $s_1>s_0$ such that

(i) $l(\alpha,\beta,e,s_1)>z$, and

(ii) all followers $y=y(x,\eta)$ with guess $\eta\supset\sigma$ with $y\leqslant\phi_e(z)$ are $\tau\hat{}0$-confirmed for all $\tau\hat{}0\subset\sigma$.

Now find the least σ-stage $s_2<s_1$ such that

$$(6.9)\qquad \begin{aligned}&\text{(i) } A_{s_2}[s_1+1]=A[s_1+1],\\ &\text{(ii) } C_{s_2}[\langle 2s_1+2,2s_1+2,2s_1+2\rangle]=C[\langle 2s_1+2,2s_1+2,2s_1+2\rangle].\\ &\text{(iii) } D_{\alpha\cap\beta,s_2}[s_1+1]=D_{\alpha\cap\beta}[s_1+1].\end{aligned}$$

Now we can use an essentially similar—but easier—argument to that of Section 5 to establish that for all $s>s_2$

$$(6.10)\qquad \begin{aligned}&\text{one of}\quad \hat{\Phi}_{e,s}(D_{\alpha,s};z)=\hat{\Phi}_{e,s_2}(D_{\alpha,s_2};z),\quad\text{or}\\ &\qquad\qquad \hat{\Phi}_{e,s}(D_{\beta,s};z)=\hat{\Phi}_{e,s_2}(D_{\beta,s_2};z)\quad\text{holds.}\end{aligned}$$

Otherwise, as in Section 5 between some least σ-stages s_3 and s_4 we have $s_3=ls(\sigma,s_4)$, (6.10) holding with $s=s_3$, and both

$$(6.11)\qquad \begin{aligned}&D_{\alpha,s_4}[\phi_e(z)]\neq D_{\alpha,s_3}[\phi_e(z)]\quad\text{and}\\ &D_{\beta,s_4}[\phi_e(z)]\neq D_{\beta,s_3}[\phi_e(z)]\quad\text{hold.}\end{aligned}$$

Now (6.9)(iii) means that we must have a follower y_1 entering $D_{\alpha,s_4}-D_{\alpha,s_3}$ and y_2 entering $D_{\beta,s_4}-D_{\beta,s_3}$, but $y_1\notin D_{\beta,s_4}-D_{\beta,s_3}$ and $y_2\in D_{\alpha,s_4}-D_{\alpha,s_3}$. By (6.9)(ii) we know that neither y_1 nor y_2 has $c(y_i)$ defined at stage s_1 and furthermore, as in Section 5 one of y_1 or y_2 will cancel the other when $c(y_i)$ becomes defined before stage s_3. □

Of course, as we remarked earlier, it is not true that in every interval **W[a, b]** we can embed (even) diamond with **b** preserved. This follows by Lachlan's nonbounding theorem [26].

We remark that (6.1) has some pleasing consequences. In particular, since (6.1) gives a complete classification of those lattices which embed into **W[a, b]** for **a** < **b** as the class of countable distributive lattices; we see:

(6.12) **Corollary.** *Let* $\mathbf{a} < \mathbf{b}$. *Then the existential theory of the semilattice* $\mathbf{W}[\mathbf{a}, \mathbf{b}]$ *in the language* $L(\leqslant, \vee, \wedge, 1)$ *is decidable.*

Proof. This follows by exactly the same decision procedure as Fejer and Shore [17]. □

We do not know about the relevant theory for the language $L(\leqslant, \vee, \wedge, 0)$. The best result we have is that the existential theory is decidable if $\mathbf{a} = \mathbf{0}$ and $\mathbf{b}$ is promptly simple. This follows by [17], (6.1), and the fact that the Lachlan–Lerman–Thomason theorem — that the countable atomless boolean algebra can be embedded into $\mathbf{W}$ preserving $\mathbf{0}$ — works below any promptly simple degree (just like a minimal pair). We remark that this also follows from a result of Ambos-Spies [4]. Concerning the classification of those lattices that can be embedded into $[\mathbf{a}, \mathbf{b}]$ for $\mathbf{a} < \mathbf{b}$ (preserving $\mathbf{a}$ and $\mathbf{b}$), Christine Haught and the author have some partial results. These will appear elsewhere.

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